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COMPARISON BETWEEN THE MARKOWITZ AND KONNO-YAMAZAKI MODELS FOR PORTFOLIO SELECTION

Dissertação no âmbito do Mestrado em Métodos Quantitativos em Finanças, orientada pelo Professor Doutor José Luis Esteves dos Santos e pelo Professor Doutor Helder Miguel Correia Virtuoso Sebastião, apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia e à Faculdade de Economia.

Comparison between the Markowitz and Konno-Yamazaki Models for Portfolio Selection

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Abstract

This paper aims to compare two portfolio selection models: the mean-variance (MV) model, introduced by Markowitz, and the mean absolute deviation (MAD) model, proposed by Konno and Yamazaki. This dissertation aims to remove some doubts and inconsistencies that exist in the literature. Thus, we do not intend to conclude which is the best model but under which conditions one model outperforms the other to satisfy the investor's objectives. We used a complete database for this comparison: the 500 stocks with the largest market capitalization in the United States of America with different time frames. The period used ranges from January 4, 2010, to November 30, 2021. First, we analyzed the performance of the models taking into account the computational time. The results show that the MAD model is computationally faster than the MV model, but only if we consider a rather large universe of assets and few observations simultaneously. Otherwise, the MAD model performs worse. After applying the rebalancing strategy considering the different frequencies, we performed an out-of-sample analysis of the models' performance. This analysis used different performance measures: win rate, cardinality, turnover, annualized mean return, annualized standard deviation, Sharpe ratio, Sortino ratio, CVaR and maximum drawdown. This analysis concluded that, although the minimum risk solution for MAD model shows slightly worse values than the corresponding in the MV model, they do not differ much. We also found a clear disadvantage associated with the solution obtained by MV model. MV model invests in all 500 assets in the portfolio, many of which with a very small proportion which is unrealistic in real scenarios. Whereas the case of the MAD model only suggests investing in around 40 to 50 assets of all the 500 available assets.

Keywords: portfolio selection optimization, MV model, MAD model, risk, return.

Resumo

O presente trabalho visa comparar dois modelos de seleção de carteiras: o modelo da média-variância (MV), introduzido por Markowitz, e o modelo do desvio absoluto médio (MAD), proposto por Konno e Yamazaki. O objetivo desta dissertação passa por retirar algumas dúvidas e incongruências que existem na literatura. Deste modo, não pretendemos concluir qual o melhor modelo mas sim em que condições é que um modelo supera o outro por forma a satisfazer os objetivos do investidor. Para esta comparação utilizámos uma base de dados bastante completa: as 500 ações com maior capitalização bolsista dos Estados Unidos da América com diferentes periodicidades sendo que o período utilizado engloba desde 4 de janeiro de 2010 até 30 de novembro de 2021. Primeiramente fizemos uma análise ao desempenho dos modelos tendo em conta o tempo computacional. Os resultados obtidos comprovam que, de facto, o modelo MAD é computacionalmente mais rápido que o modelo MV mas apenas se considerarmos um universo bastante elevado de ativos e, simultaneamente, poucas observações. Caso contrário, o modelo MAD apresenta pior desempenho. Após aplicarmos a estratégia de rebalanceamento tendo em conta as diferentes periodicidades, fizemos uma análise *out-of-sample* do desempenho dos modelos. Nesta análise utilizámos diferentes medidas de desempenho tais como: *win rate*, cardinalidade, *turnover*, retorno médio anualizado, desvio-padrão anualizado, rácio de Sharpe, rácio de Sortino, CVaR e *maximum drawdown*. Através desta análise concluímos que, ainda que a solução de menor risco no modelo MAD apresente ligeiramente piores valores que a correspondente solução no modelo MV, estes não diferem muito. Comprovou-se também que há uma clara desvantagem associada à solução obtida com o modelo MV no sentido em que este investe em todos os 500 ativos da carteira muitos dos quais com uma proporção ínfima, o que é pouco realista num cenário real. Ao passo que, no caso do modelo MAD este apenas sugere investir em cerca de 40 a 50 ativos de todos os 500 ativos disponíveis.

Palavras-chave: otimização em seleção de carteiras, modelo MV, modelo MAD, risco, retorno.

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Chapter 1

Introduction

Financial mathematics is the practical application of mathematics to financial problems. Through multiple tools, it allows us to manage and organize money in the best possible way to avoid losses. The importance of financial mathematics is essentially linked to the rigour and efficiency it has brought to the financial markets [1]. Financial mathematics is used to solve some problems such as risk management, portfolio optimization and asset valuation [1]. In this dissertation, we will address some financial mathematics issues related with portfolio optimization.

Investors who own an investment portfolio, which can be made up of stocks, bonds and derivatives, aim to reach the highest return and, at the same time, the lowest possible risk. However, these objectives are somewhat conflicting because higher return is usually associated with higher risk. So, investors will have to decide on which assets they should invest and on what proportions to form a portfolio with the desirable features. In this sense some models/techniques are used in financial markets to help investors in this decision making process [17].

Markowitz [25] was a pioneer on this subject when he presents a model that can help investors to build up their efficient portfolios. The Markowitz model, also known as the Mean-Variance (MV) model, uses the variance of returns as a risk measure, with the intention to minimize risk given a desired expected return.

However, some criticisms of the MV model began to emerge, and alternative proposals were put forward, such as the Konno-Yamazaki model, or Mean Absolute Deviation (MAD) model, proposed in 1991 by Konno and Yamazaki [20]. The main difference between the MV and the MAD model is that the last uses the mean absolute deviation instead of the variance as a risk measure. Konno and Yamazaki [20] pointed out strong criticisms of the MV model. However, the most important ones are that the MV problem involves a quadratic optimization problem and that it assumes the normality of the distribution of asset returns, something that is quite rare in reality. Other disadvantages are associated with the computational time involved in the MV model and a poor diversification, because through the MV model, some weights are assigned with minimal values. The main advantages associated with the Konno-Yamazaki problem are that it is related with a linear problem and does not need to compute the covariance matrix. The MAD model has been studied in the literature. However, the conclusions about the model performance have not been unanimous. There is some ambiguity in the computational results, mainly because the results are not obtained using a meaningful data set.

This dissertation aims at making a detailed comparison between these two models using real data on the 500 stocks with the largest market capitalization in the United States of America. The two models are compared not only in terms of computational time but also in terms of financial performance.

This dissertation is structured into five chapters. Chapter 2 presents a brief contextualization of the comparative studies already carried out in the literature, focusing on the geography of the data used, the measures used to compare the models and their main conclusions. Chapter 3 presents a theoretical description of MV and MAD models, accompanied with a brief illustrative example that allows the reader to become familiar with both models and to replicate them using the Matlab software. Chapter 4 presents a comparative study between the models using data on 500 USA stocks. The comparison is performed considering the computational time and the out-of-sample financial performance of portfolios with rebalancing at different frequencies. Finally, Chapter 5 discuss the main results obtained and proposes future studies.

Chapter 2

Literature Review

The modern portfolio theory was presented in 1952 by Henry Markowitz in an article of the Journal of Finance intitled "Portfolio Selection" [25]. The paper presents a Mean-Variance (MV) framework where portfolio optimal solutions are obtained. Afterwards, several authors have presented alternative models to correct some of the disadvantages of the basic MV model.

The main disadvantage associated with the MV model is that it falls into a quadratic problem, which requires a great computational effort when dealing with a large number of assets and may lead sometimes to poorly diversified portfolios, i.e, to portfolios with a large number of assets with marginal weights. Thus, new models have emerged trying to overcome these drawbacks, namely those based on linear programming, as is the case in [30, 31]. Peng et al. [27] is another example. These authors kept the quadratic programming problem but added transaction costs and proved that there was a significant improvement in relation to the Markowitz model.

In another strand of the literature, researchers address the issue by changing the risk measure. In 1991, Konno and Yamazaki [20] presented the Mean Absolute Deviation (MAD) model, which uses the absolute deviation as a risk measure instead of the quadratic deviation as in the MV model. This dissertation aims to compare the performances of the MV and MAD models. As mentioned before, there are other linear models (which will be referred throughout this chapter); however, we focus on the MAD model as it is the most referred portfolio selection linear model in the literature.

The precursors of the MAD model, Konno and Yamazaki, were the forerunners to make this comparison in [20]. They used monthly data from 1981 to 1987 on 224 stocks from Nikkei 225 (Japan's benchmark stock index). They partitioned the data into three groups, taking into account different periods. Each group's last year is used to perform an out-of-sample analysis considering portfolio rebalancing. They used several performance metrics, such as the average rate return, standard deviation of return, and Sharpe ratio. They conclude that the portfolio obtained with the MAD model was quite similar to the one obtained with the MV model. They also highlight that both models are equivalent (i.e. the mean absolute deviation corresponds to $\sqrt{\frac{2}{\pi}}$ of the standard deviation) if returns follow a multivariate normal distribution. Following Konno and Yamazaki [20], other authors have contributed to the empirical evidence on the comparison between MV and MAD models.

Simaan [33] used a random numbers generator to obtain returns for sets of stock with different cardinalities (30, 60, 90, 120), from which he estimated the mean and standard deviation of returns.

Simaan concludes that increasing the sample size in the asset space reduces the estimation error in both models. However, the estimation error is lower in the MV model.

Mansini et al. [24] compared the Markowitz model with the MAD model and other linear problems (m-MAD, Minimax (MM) Model, CVaR model and Gini Mean Difference (GMD) model). They considered weekly data on stocks from the Milan Stock Exchange in the period from 1994 to 1998. The data was divided into four sub-samples with different periods and stocks (1994-1995, 1995-1996, 1996-1997, 1997-1998). The last year of each sub-sample was then used for out-of-sample analysis, without considering portfolio rebalancing. They concluded that MAD provide more diversified portfolios than MV, and that MV and MAD provide higher return and risk out-of-sample than other linear models.

Júdice et al. [17] was the first Portuguese study to make a comparison between the MV model and the MAD model. In this work, the computational time and the stability of the solution, i.e. variation of the composition of the optimal portfolio on rebalancing, were evaluated. The authors used a universe of ninety-two European stocks (including twenty-seven Portuguese stocks) from February 2, 1998, to February 23, 2000. The last year was used to perform the out-of-sample analysis, considering portfolio rebalancing. In terms of computational time, they concluded that the Markowitz model depends on the number of stocks, while the Konno-Yamazaki model depends mainly on the number of observations.

Bower and Wentz [7] studied this issue using daily data on thirty sets of five stocks of the S&P 500 and a six-month bond, for the period from July 1 to December 31, 2004. They compared the performance of the optimal solutions of the MV and MAD models in terms of return obtained by the thirty portfolios from January 3 to June 30, 2005, where the solution of the MV achieved a higher return in sixteen of the thirty portfolios.

Karacabey [18] made a simple comparison using monthly data on 91 stocks from the Istanbul Stock Exchange. The results showed that the optimal portfolios from the MV model generally outperformed the optimal portfolios from the MAD model.

Yu et al. [35] and Hoe et al. [15] compared the MV and MAD model, and several other portfolio selection models that are seldom mentioned in the literature. Yu et al. [35] compare four models: Markowitz, Konno-Yamazaki, Cai, and Teo. Two data sets were used in this study: seventy-two months of data on thirty-three Nikkei 225 bonds, and one hundred and twenty months of data on sixty-three Nikkei 500 bonds. Yu et al. [35] concluded that MV and MAD models provided similar results in terms of expected return. They also highlighted that Teo's model is influenced by the number of stocks and the number of periods. However, there is no evidence that the number of stocks and periods impact on the computational time of MV, MAD and Cai's models. Hoe et al. [15] compared the composition and performance of portfolios using four optimization models with different risk measures: MV, MAD, MM and SV (Semi-variance). The comparison is conducted using monthly data on fifty-four stocks from the Kuala Lumpur Composite Index from January 2004 to December 2007. They concluded that the MV model and the MAD model produce portfolios with almost the same assets, however, with different weights. The MV portfolios presented higher risk than the MAD portfolios.

Bartkus and Paleviciene [5], and Kasenbacher et al. [19] also address the performances of MV and MAD models; without and with portfolio rebalancing, respectively. Bartkus and Paleviciene [5] used the 20 stocks from Vilnius Exchange Market with the highest positive skewness. They concluded that the MV model provided a more diversified portfolio, while the MAD optimal portfolio achieved a better expected return in-sample. Kasenbacher et al. [19] analyse the top 75, 150 and 200 stocks belonging to the S&P 500, over one year between 2016 and 2017. The portfolio rebalancing was activated weekly and monthly, obtaining the best expected return with the weekly rebalancing, as expected. The MAD portfolios obtained higher expected return and lower risk, hence resulting in Sharpe ratios greater than those of the MV portfolios,

Some articles, such as Angelelli et al. [2], Cesarone et al. [8] and Hunjra et al. [16] use the conditional value at risk (CVaR) as an additional metric for risk assessment when comparing portfolio performance. Angelelli et al. [2] compared only two linear models: MAD and CVaR (with different values of α). They use weekly data on four sets of stocks, with cardinalities 200, 300, 400 and 600, of three European stock exchanges: Paris, Milan and Frankfurt. The sample period was partitioned into in-sample (1999-2000) and out-of-sample (2001). The models considered transaction costs and cardinality restriction. The computational time of the CVaR model increased significantly with the number of assets included in the portfolios. Cesarone et al. [8] conducted a similar work to the previous one but also considered the MV model and equally-weighted (EW) portfolio. The authors concluded that linear models are less time consuming although producing similar results to those of the MV model. The EW portfolio does not beat the other portfolios resulting from optimization models. More recently, Hunjra et al. [16] used four models: MV, MAD, SV and CVaR on forty stocks from the Pakistan stock exchange, ninety-two stocks from the Bombay stock exchange, and thirty from the Dhaka stock exchange. The analysis considered three periods: during the economic crisis (2003-2005), the recovery period after the crisis (2006-2011) and the growth period after the crisis (2012-2015). The main conclusion was that in India, the CVaR produced the best results in all scenarios.

To give a more systematic view of the literature review, Appendix A) presents a table with a list of some studies that compare the MV and MAD models.

Our study aims at making a more detailed comparison between the MV and MAD models and, hence to clarify some ambiguities that persist in the empirical literature. First, empirical studies use a different numbers of stocks, data frequencies and sampling periods. We address this by considering different sets of stocks with cardinalities up to 500. We also use around 12 years of data, from January 4, 2010, to November 30, 2021, with different frequencies (daily, weekly and monthly). Second, the performance analysis is conducted out-of-sample with periodical rebalancing. Several papers, such as Simaan [33], Yu et al. [35], Peng et al. [27], Hoe et al. [15] and Bartkus et al. [5] only evaluate the in-sample performance, which is questionable from the point of view of measuring the effective performance of the models. Other studies, such as Mansini et al. [24], conduct a static out-of-sample evaluation, i.e. without applying a rebalancing strategy. Our work considers portfolio rebalancing with daily, weekly and monthly frequencies. The rebalancing strategy is a more realistic framework, because the investor updates dynamically her portfolio, considering updated data. Third, to give a more realistic and effective view of the portfolio selection problem, we introduce transaction costs into the performance analysis. We did not find many papers that consider transaction costs. The exceptions are Peng et al. [27] and Angelelli et al. [2]. So we perform a comprehensive performance evaluation of the models with and without transaction costs, therefore maintaining the comparability with most studies. Fourth, besides the computational time which is a usual feature addressed when comparing the MV and MAD models, we also use a wide range of financial performance measures, which is not usual in most of the studies, namely the Win Rate, Cardinality, Turnover, Annualized Mean Return, Annualized Standard Deviation, Annualized Sharpe Ratio, Annualized Sortino Ratio, CVaR and Maximum Drawdown. Fifth, we use the EW portfolio as a benchmark, which has been shown to provide better out-of-sample results than several portfolios based on optimization models (see DeMiguel et al. [12]) however, that paper did not address the MAD model.

Chapter 3

The Markowitz and Konno-Yamazaki Models

An investment portfolio may consist of several securities, such as equities, bonds and derivatives. Throughout this dissertation, we will mainly focuses on portfolios composed of equities. A rational risk-averse investor interested has two aims: increase the expected return whilst decreasing the risk exposure. However, these are competing features because, in general, risk increases with the expected return. Therefore, the investor has to find a solution that takes into account the trade-off between risk and return. In other words, she has to decide how much to invest in each asset, aiming at minimizing the risk and, at the same time, maximizing the expected return. The two models presented hereafter assist the investor in her decision.

This chapter provides a theoretical description of the Markowitz and the Konno-Yamazaki models and some comparative insights. This description is accompanied by illustrative examples using stocks from the S&P 500 index, the most known benchmark stock index of the United States. It also illustrates how to apply the models using the Matlab software.

3.1 Risk Measures

The assessment of risk in any financial investment may be done using different risk metrics. This dissertation considers two risk measures: the variance (or standard deviation) and the mean absolute deviation of returns, which are associated with the MV and MAD models, respectively. However, none of these metrics are coherent risk measures, taking into account Definition 1.

Let χ be the linear space of measurable functions defining the random variables X_1 and X_2 in the probability space (Ω, ψ, P). According to Artzner et al. [3], a risk measure is coherent if it satisfies the properties presented in Definition 1.

Definition 1 *A risk measure $\varphi : \chi \rightarrow \mathbb{R}$ is a coherent risk measure if it satisfies the following properties:*

1. *Translation invariance:* $\varphi(X_1 + a) = \varphi(X_1) - a$, $\forall a \in \mathbb{R}$, $\forall X_1 \in \chi$.

2. *Subadditivity:* $\varphi(X_1 + X_2) \leq \varphi(X_1) + \varphi(X_2)$, $\forall X_1, X_2 \in \chi$.

3. *Positive homogeneity:* $\varphi(\lambda X_1) = \lambda \varphi(X_1)$, $\forall \lambda \geq 0$, $\forall X_1 \in \chi$.

4. *Monotonicity: If* $X_1 \leq X_2$, $\varphi(X_2) \leq \varphi(X_1)$, $\forall X_1, X_2 \in \chi$.

As we will see, the variance is not a coherent risk measure because it does not satisfy, for instance, properties 1, 3 and 4 (see Proposition 1). The standard deviation is not a coherent risk measure either; although it already satisfies the property 3 (see Proposition 2). The mean absolute deviation is not also a coherent risk measure as it does not comply property 1 and 4.

3.2 Markowitz model

The Markowitz (or Mean-Variance (MV)) model is a quadratic optimization problem that aims to find the weights of the assets in the portfolio that minimize the risk, measured by the variance, amongst all feasible portfolios that have an expected return, at least, equal to a given threshold, ρ.

The MV model, and other basic portfolio selection models such as the Konno-Yamazaki model, is based on the assumptions of perfect, continuous and competitive markets. These assumptions are listed in Beyhaghi and Hawley [6] and Sebastião [29]. Namely:

- Assets are infinity divisible.
- There are no transaction costs, taxes, or other market frictions.
- Investors can lend or borrow at the same risk-free interest rate.
- Assets are available for transaction at any time at a known price.
- Investors are rational and risk-averse (if there are two portfolios with the same risk, the investor always prefers a portfolio with a higher expected return, if there are two different portfolios with the same expected return, the investor chooses the portfolio with lower risk).
- Investors are price-takers (their isolated actions do not influence market prices).

Let us define some notations, preliminary concepts, and definitions needed to present the MV model (Markowitz [25], Júdice et al. [17]).

Suppose that an investor has an portfolio consisting of n assets. The proportion invested into asset j, with $j \in \{1,\ldots,n\}$, is x_j. This proportion is also called the weight of asset j in the portfolio. Throughout this dissertation it is considered that $\sum_{j=1}^{n} x_j = 1$, meaning that all the wealth available to invest into equities, is used in buying the n assets.

The return of asset j is a random variable, R_j. The observation of R_j at moment t is calculated using the adjusted closing prices P_j (i.e., market prices adjusted to dividends and other management decisions such as stock splits that have an artificial impact on the stock market prices), using formulas (3.1) or (3.2) depending on if one is considering discrete or continuous returns, respectively:

$$R_j(t) = \frac{P_{j,t} - P_{j,t-1}}{P_{j,t-1}}, \tag{3.1}$$

$$R_j(t) = ln(\frac{P_{j,t}}{P_{j,t-1}}). \tag{3.2}$$

Where $P_{j,t}$ and $P_{j,t-1}$ correspond to the adjusted closing prices of asset j at time t and $t-1$, respectively. Throughout this dissertation, we mainly use logarithmic returns, which have the disadvantage of not being additive over the asset space. Whenever necessary logarithmic returns are transformed into discrete returns, which are additive across assets but not overtime. To understand the implications of using logarithmic or discrete returns, we provide a simple example below.

Example 1 *Suppose that a portfolio has two assets from the S&P500: Apple (AAPL) and Kellogg (K). The next table presents the closing prices and returns (logarithmic and discrete) in three consecutive days chosen randomly.*

Table 3.1 Comparison between discrete and logarithmic returns.

Moments	Price		Discrete returns		Log returns	
	APPL	K	APPL	K	APPL	K
t	6.86	54.42	n.a	n.a	n.a	n.a
$t+1$	7.31	52.15	0.066	-0.042	0.064	-0.043
$t+2$	8.39	53.43	0.148	0.025	0.138	0.024

Notes: This table presents the adjusted closing prices, discrete and logarithmic returns in 3 consecutive days. Returns are rounded to three decimal digits.

Logarithmic returns have two properties: **1.a)** they are non-additive in the asset space and **1.b)** are additive over time. While discrete returns are: **2.a)** additive on assets and **2.b)** non-additive over time.

1.a Applying Equation (3.2) to logarithmic returns one obtains that $R_{APPL}(t)+R_K(t)$ is different to $ln(\frac{P_{APPL,t}+P_{K,t}}{P_{APPL,t-1}+P_{K,t-1}})$. Therefore, at $t+1$, it follows that $0.021 \neq -0.030$, and at $t+2$, $0.162 \neq 0.039$, proving the non-additivity of logarithmic returns within the asset space.

2.a Applying Equation (3.1) to discrete returns one obtains that $R_{APPL}(t)+R_K(t)$ is equal to $\frac{P_{AAPL,t}-P_{AAPL,t-1}}{P_{AAPL,t-1}} + \frac{P_{K,t}-P_{K,t-1}}{P_{K,t-1}}$, hence proving the additivity of discrete returns within the asset space.

Based on Pascoal [26], we prove assertions **1.b)** and **2.b)**.

1.b) The logarithmic return in two periods, that is, between t and $t-2$ is given by

$$R_{j,2}(t) = ln\left(\frac{P_{j,t}}{P_{j,t-2}}\right) = ln\left(\frac{P_{j,t}P_{j,t-1}}{P_{j,t-1}P_{j,t-2}}\right) = ln\left(\frac{P_{j,t}}{P_{j,t-1}}\right) + ln\left(\frac{P_{j,t-1}}{P_{j,t-2}}\right) = R_j(t) + R_j(t-1).$$

Therefore, this statement may be generalized to k periods as:

$$R_{j,k}(t) = \sum_{i=0}^{k-1} R_j(t-i). \tag{3.3}$$

2.b) The discrete return for a two-period, i.e., between t and $t-2$, is given by

$$R_{j,k}(t) = \frac{P_{j,t}-P_{j,t-2}}{P_{j,t-2}} = \frac{P_{j,t}}{P_{j,t-2}} - 1 = \frac{P_{j,t}P_{j,t-1}}{P_{j,t-1}P_{j,t-2}} - 1 =$$

$$= (1+R_j(t))(1+R_j(t-1)) - 1 \neq R_j(t) + R_j(t-1).$$

Generalizing for k:

$$R_{j,k}(t) = \prod_{i=0}^{k-1}(1+R_j(t-i)) - 1. \tag{3.4}$$

The Markowitz model assumes that the return vector $R = (R_1, \ldots, R_n)$ follows a multivariate normal distribution. However, this assumption does not hold in practice (see, for instance, Konno and Yamazaki [20]). In our example, at the end of this chapter, we present two normality tests, the Kolmogorov-Smirnov test and the Jarque-Bera test, that an investor can use to assess if data are compatible with a normal distribution.

Given a portfolio, $x \in R^n$, the random variable that describes its return is $R(x) = \sum_{j=1}^{n} x_j R_j$. So, the returns of asset j contribute proportionally to the return of the portfolio, assuming that the returns on different assets are additive.

Denoting the expected return of R_j by $\mu_j = E[R_j]$, then:

$$E[R(x)] = E[\sum_{j=1}^{n} R_j x_j] = \sum_{j=1}^{n} E[R_j] x_j = \sum_{j=1}^{n} \mu_j x_j. \tag{3.5}$$

In the Markowitz framework, the risk is measured by the variance:

$$V[R(x)] = \sigma^2(R(x)) = E\left[\left\{\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right\}^2\right] =$$

$$= E\left[\left\{\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} E[R_j] x_j\right\}^2\right]$$

$$= E\left[\left\{\sum_{j=1}^{n} R_j x_j - \sum_{j=1}^{n} \mu_j x_j\right\}^2\right]$$

$$= E\left[\left\{\sum_{j=1}^{n} x_j (R_j - \mu_j)\right\}^2\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j E\left[(R_i - \mu_i)(R_j - \mu_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{i,j} \tag{3.6}$$

$$= \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} x_i x_j \sigma_{i,j}, \tag{3.7}$$

where $\sigma_{i,j}$ denotes $E[(R_i - \mu_i)(R_j - \mu_j)]$.

Notice that (3.6) can also be written as $x^T \Sigma x$, where Σ corresponds to the covariance matrix (3.8).

$$\Sigma = \begin{bmatrix} E[(R_1-\mu_1)(R_1-\mu_1)] & E[(R_1-\mu_1)(R_2-\mu_2)] & \cdots & E[(R_1-\mu_1)(R_n-\mu_n)] \\ E[(R_2-\mu_2)(R_1-\mu_1)] & E[(R_2-\mu_2)(R_2-\mu_2)] & \cdots & E[(R_2-\mu_2)(R_n-\mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ E[(R_n-\mu_n)(R_1-\mu_1)] & \cdots & \cdots & E[(R_n-\mu_n)(R_n-\mu_n)] \end{bmatrix}. \tag{3.8}$$

The covariance between the returns of two assets, $\sigma_{i,j}$, shows how they are linearly related to each other. If $\sigma_{i,j} > 0$, then assets are directly related, implying that when prices of one asset increase (decrease), in general the prices of the other also increase (decrease). If $\sigma_{i,j} < 0$, the opposite occurs, implying that when prices of one asset increase (decrease), in general the prices of the other also decrease (increase). When $\sigma_{i,j} = 0$, the assets have no linear relationship with each other (Kwon [21]). The covariance between the returns on assets i and j can be expressed as:

$$\sigma_{i,j} = Corr_{i,j}\sigma_i\sigma_j, \tag{3.9}$$

where $Corr_{ij}$ corresponds to the correlation coefficient such that $Corr_{ij} \in [-1,1]$.

The variance is a risk measure that verifies the following properties in Proposition 1[14].

Proposition 1 *Let X be a random variable. The variance of a portfolio verifies the following properties:*

1. $V(X) = E(X^2) - (E(X))^2$.
2. *If a is a constant then* $V(a+X) = V(X)$.
3. *If a is a constant then* $V(aX) = a^2V(X)$.
4. *Let Y be a random variable, so* $V(X+Y) = V(X) + V(Y) + 2cov(XY)$.

Proof:

1. $V(X) = E(X-E(X))^2 = E[X^2 - 2XE(X) + E(X)^2] = E(X^2) - 2E(X)^2 + E(X)^2 =$
   $= E(X^2) - E(X)^2$.
2. $V(X+a) = E[X+a-E(X+a)]^2 = E[X+a-E(X)-a]^2 = E(X-E(X))^2 = V(X)$.
3. $V(aX) = E[(aX)^2] - [E(aX)]^2 = a^2[E(X^2) - E(X)^2] = a^2V(X)$.
4. $V(X+Y) = E(X+Y)^2 - [E(X+Y)]^2 = E(X^2+2XY+Y^2) - [E(X^2)+2E(X)E(Y)+E(Y)^2] =$
   $= E(X^2) + 2E(XY) + E(Y^2) - E(X^2) + 2E(X)E(Y) - E(Y) =$
   $= E(X^2) - E(X)^2 + E(Y^2) - E(Y)^2 + 2[E(XY) - E(X)E(Y)] = V(X) + V(Y) + 2cov(XY)$.

From property 2 of Proposition 1 we conclude that $V(X)$ does not verify the translation invariance property from Definition 1. Property 3 indicates that $V(X)$ does not verify positive homogeneity. Finally, property 4 indicates that $V(X)$ is sub additive only when $cov(X,Y) > 0$.

Since the variance is not in the same units as the return, a new risk measure can be defined using the standard deviation $\sigma(R(x)) = \sqrt{V(R(x))}$. Proposition 2 indicates that this risk measure also has interesting properties:

Proposition 2 *Let X be a random variable. The standard deviation of a portfolio verifies the following properties.*

1. *If a is a constant then* $\sigma(a+X) = \sigma(X)$.
2. *If a is a constant then* $\sigma(aX) = a\sigma(X)$.
3. *Let Y be a random variable* $\sigma(X+Y) = \sqrt{\sigma^2(X)+\sigma^2(Y)+2cov(X,Y)}$.

The proof follows immediately from Proposition 1.

3.2.1 Formulation of the Markowitz Model

The MV model is usually formulated as

$$min \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \tag{3.10}$$

$$s.t. \sum_{j=1}^{n} \mu_j x_j \geq \rho \tag{3.11}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{3.12}$$

$$x_j \geq 0, j = 1, \ldots, n. \tag{3.13}$$

The MV model has n variables and $n+2$ constraints (n signal constraints, 1 inequality constraint and 1 equality constraint).

- The first constraint (3.11) guarantees that the expected return of the portfolio is greater than or equal to a target return, ρ, required by the investor.
- The second constraint (3.12) guarantees that the entire amount available for investment is used.
- The third constraint (3.13) implies that short selling is not allowed. Short-selling means that the investor sells an asset that is not owned, intending to buy it back later, therefore introducing an asset with negative weight in her portfolio.

It should be noted that, if risk is measured by the standard deviation, the optimal solution is the same as the one for the MV problem, making the two problems equivalent.

3.2.2 Efficient frontier

As already mentioned, a rational risk-averse investor selects, among all feasible portfolios, the one that allows the lower risk for a given return. Therefore, for each value of ρ, the investor will select the minimum risk portfolio (i.e., an efficient portfolio). These efficient portfolios range from the admissible portfolio with the minimum variance denoted by x_{min}, and the admissible portfolio with the maximum expected return, denoted by x_{max}. The expected returns associated with these portfolios correspond to ρ_{min} and ρ_{max}, respectively.

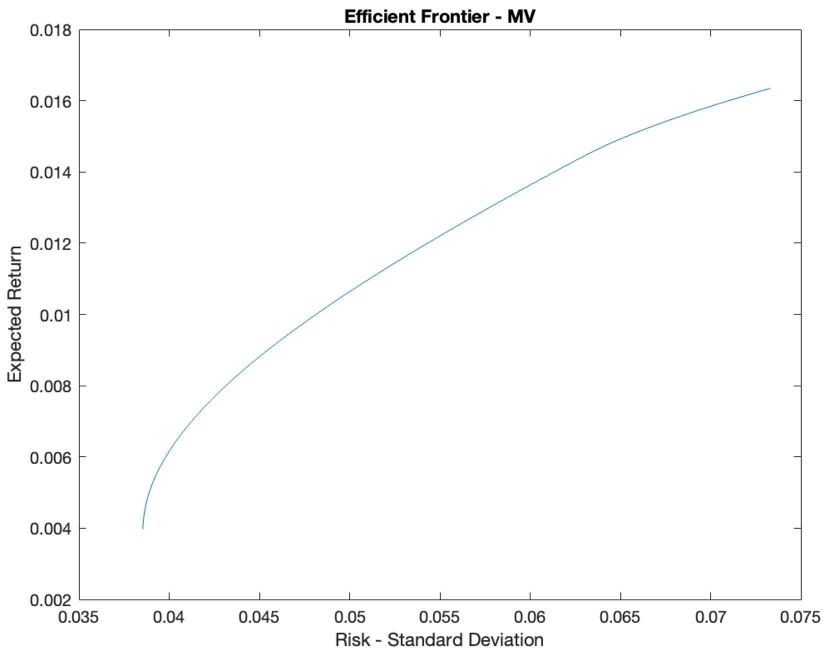


Fig. 3.1 Efficient frontier of the practical example (3.4) using monthly data.

To obtain the efficient frontier, i.e., the set of all efficient portfolios, it is necessary to calculate all solutions of the MV problem for different values of ρ. So, let us define the function $\sigma : [\rho_{min}, \rho_{max}] \to \mathbb{R}$ with standard deviation, $\sigma(\rho) = \sqrt{x_\rho^T \Sigma x_\rho}$, where x_ρ is the optimal solution associated with each value of ρ. Thus, the efficient frontier is the set $EF = \{(\sigma(\rho), \rho) : \rho \in [\rho_{min}, \rho_{max}]\}$ and corresponds to the graph of the function σ (with the coordinate axes reversed) which generally has a configuration similar to the one presented in the Figure 3.1.

The extreme points of this curve correspond to the portfolio with minimum variance (leftmost point) and the portfolio with the maximum return (rightmost point).

Over the years, several criticisms have been raised against the MV model. An important one, pointed out by Konno and Yamazaki [20], is that it falls into a quadratic optimization problem, from which results in three disadvantages: first, the calculation of an extensive covariance matrix requires non-negligible computational time when working with a large number of assets; second, in practice, returns do not follow a normal distribution; and finally, when working with a large number of assets, the optimal solution may contain very small weights, implying an increase in transaction costs.

The next section presents the linear model proposed in 1991 by Konno and Yamazaki, which according to the authors is easier to solve than quadratic problems.

3.3 Konno-Yamazaki Model

Konno and Yamazaki [20] introduced the Mean Absolute Deviation (MAD) model that uses

$$w(x) = E[|\sum_{j=1}^{n} R_j x_j - E[\sum_{j=1}^{n} R_j x_j]|] \tag{3.14}$$

as the risk measure. If returns follow a multivariate normal distribution, then the MV and MAD problems are equivalent, in the sense that minimizing $w(x)$ is equivalent to minimizing $\sigma(x)$ (Konno and Yamazaki [20], Júdice et al. [17]).

Theorem 1 *If* $(R_1,...,R_n)$ *follow a multivariate normal distribution, then*

$$w(x) = \sqrt{\frac{2}{\pi}}\sigma(x)$$

Proof: Let $(R_1,\dots,R_n)$ be a random vector following a multivariate normal distribution with mean $(\mu_1,\dots,\mu_n)$ and a covariance matrix Σ with a generic element $\sigma_{i,j}$ which represents the covariance between R_i and R_j. Let R be a weighted sum of the n variables R_j, i.e. $R(x) = \sum_{j=1}^{n} R_j x_j$, then $R(x)$ has a normal distribution with mean $\sum_{j=1}^{n} \mu_j x_j$ and standard deviation $\sigma(x) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{i,j} x_i x_j}$. Let A be the following random variable:

$$A(x) = \sum_{j=1}^{n} R_j x_j - E[\sum_{j=1}^{n} R_j x_j] = \sum_{j=1}^{n} (R_j - \mu_j) x_j. \tag{3.15}$$

Then $A(x) \sim N(0,\sigma)$ and the density function of A is $\frac{e^{-t^2/(2\sigma(x)^2)}}{\sqrt{2\pi}\sigma(x)}$, $t \in \mathbb{R}$.

So, from (3.14), we have:

$$w(x) = E(|A(x)|) = \int_{-\infty}^{+\infty} |t| \frac{1}{\sqrt{2\pi}\sigma(x)} e^{\frac{-t^2}{2\sigma(x)^2}} dt = 2\int_{0}^{+\infty} t \frac{1}{\sqrt{2\pi}\sigma(x)} e^{\frac{-t^2}{2\sigma(x)^2}} dt,$$

since the integrand function is even and $|t| = t, \forall t \geq 0$, the change in variable $y = \frac{t}{\sigma(x)}$ results in

$$w(x) = 2\int_{0}^{+\infty} \frac{y}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \sigma(x) dy = 2\frac{\sigma(x)}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\sigma(x).$$

Thus, it follows that $w(x) = \sqrt{\frac{2}{\pi}}\sigma(x)$, and therefore the MV and MAD models, under the condition of the normality of returns, are equivalent (Konno and Yamazaki [20], Cornuejols and Tutuncu [10]).

The portfolio selection problem using MAD can be formalized as:

$$min\ w(x) \tag{3.16}$$

$$s.t. \sum_{j=1}^{n} \mu_j x_j \geq \rho \tag{3.17}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{3.18}$$

$$x_j \geq 0, j = 1, ..., n. \tag{3.19}$$

Notice that this model has the same restrictions as the MV model: (3.11), (3.12) and (3.13).

Let r_{jt} be the realization of the random variable R_j during period t ($t = 1, ..., T$), which means that r_{jt} corresponds to the observed return of asset j in period t. So the mean return, μ_j, can be estimated as

$$\frac{1}{T}\sum_{t=1}^{T} r_{jt}, \tag{3.20}$$

and the mean absolute deviation can be estimated as

$$w(x) = E[|\sum_{j=1}^{n} R_j x_j - E[\sum_{j=1}^{n} R_j x_j]|] =$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt} - \mu_j)x_j\right|. \tag{3.21}$$

So, the initial MAD model may be reformulated considering (3.21). That is:

$$min\ \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt} - \mu_j)x_j\right| \tag{3.22}$$

$$s.t. \sum_{j=1}^{n} \mu_j x_j \geq \rho \tag{3.23}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{3.24}$$

$$x_j \geq 0, j = 1, ..., n. \tag{3.25}$$

The previous problem is not yet a linear one because of the modules. However, the following proposition overcomes this.

Proposition 3 *Consider the problem (P)*

$$min \sum_{i=1}^{k} |f_i(x)| \tag{3.26}$$

$$s.t.\ x \in \Omega. \tag{3.27}$$

Let us consider new auxiliary variables $y_i, i = 1, \ldots, k$, and the problem (P')

$$min \sum_{i=1}^{k} y_i \tag{3.28}$$

$$s.t.\ y_i \geq f_i(x),\ y_i \geq -f_i(x),\ x \in \Omega. \tag{3.29}$$

Then, the optimal solutions of the two problems, in variables x, coincide and the optimal solution of (P') in y verifies $y_i^* = |f_i(x^*)|$.

Proof: (By contradiction)

($\Rightarrow$) Let x^* be the optimal solution of problem (P) and $y_i^* = |f_i(x^*)|$. Then (x^*, y^*) is an admissible solution of (P'). If it is not optimal, there would be a solution $(\bar{x}, \bar{y})$ where

$$\sum_{i=1}^{k} \bar{y}_i < \sum_{i=1}^{k} y_i^*. \tag{3.30}$$

As $\bar{y}$ is admissible in (P'), then, $\bar{y}_i \geq f_i(\bar{x}_i)$ and $\bar{y}_i \geq -f_i(\bar{x}_i) \Rightarrow \bar{y}_i \geq |f_i(\bar{x})|$. Furthermore, as $\bar{x}$ is admissible in (P'), then is also admissible in (P), so, x^* cannot be an optimal solution.

($\Leftarrow$) Let (x^*, y^*) be the optimal solution of (P'). If x^* is not the optimal solution of (P), it will exist an optimal solution $(\bar{x})$ of (P) such that,

$$\sum_{i=1}^{k} |f_i(\bar{x})| < \sum_{i=1}^{k} f_i^*. \tag{3.31}$$

Let us define $\bar{y}_i = |f_i(\bar{x})|$. Then we have that $(\bar{x}, \bar{y})$ is admissible and has lower value in the objective function.

Corollary 1 *If the functions f_i in problem (P) of the previous proposition are linear and Ω is a polyhedron, then the problem (P') is a linear problem.*

Thus, the result stated in Proposition 3 can be applied to problem (3.32)-(3.37). Using Proposition 3 and Corollary 1, the problem (3.22)-(3.25) can be transformed into a linear problem by introducing a new variable y_t associated with $y_t = \left|\sum_{j=1}^{n} (r_{jt} - \mu_j) x_j\right|$, such that

$$min\ \frac{1}{T} \sum_{t=1}^{T} y_t \tag{3.32}$$

$$s.t.\ y_t + \sum_{j=1}^{n} (r_{jt} - \mu_j) x_j \geq 0, t = 1, ..., T \tag{3.33}$$

$$y_t - \sum_{j=1}^{n} (r_{jt} - \mu_j) x_j \geq 0, t = 1, ..., T \tag{3.34}$$

$$\sum_{j=1}^{n} \mu_j x_j \geq \rho \tag{3.35}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{3.36}$$

$$x_j \geq 0, j = 1, ..., n. \tag{3.37}$$

It is important to notice that there are more $2T$ constraints and T variables in relation to the competing MV problem. However, these do not depend on the number of assets. Hence, the computational time associated with the MAD model depends on the number of observations and not on the number of assets as it happens in the MV model.

Arguably, the MAD model, being a linear problem, is easier to solve than the MV model. Simaan [33] mentioned three advantages: it does not require the calculation of the covariance matrix; in large scale problems it is easier to solve a linear problem than a quadratic one and, finally, portfolios obtained by applying this model are formed by fewer assets, which implies lower transaction costs. However, if there is a large number of observations, probably there are no advantages in terms of computational time, as it will be shown in Chapter 4.

3.4 A simple example of portfolio selection

This section compares the quadratic and linear models using an illustrative example. With that example, it is explained how one can replicate these models using Matlab and then the results obtained are presented.

Quadratic problems can be solved with Matlab software using the quadprog function. The quadratic program can be described as

$$min \ \frac{1}{2} x^T H x + f^T x \tag{3.38}$$

$$s.t. \ Ax \leq b \tag{3.39}$$

$$A_{eq} x = b_{eq} \tag{3.40}$$

$$l_b \leq x \leq u_b. \tag{3.41}$$

The first constraint corresponds to the inequality constraints. Note that these have a greater or equal sign in the MV and MAD models, so they need to be converted. The second constraint represents the equality one and the last one the lower and upper bounds. Regarding the objective function, we consider $f = 0$ because this term does not exist in the MV model, and H is the covariance matrix (Kwon [21]).

Thus, these arguments are described as $[x, fval] = quadprog(H, f, A, b, Aeq, beq, lb, ub)$, where the vector x corresponds to the optimal solution, in this case, the proportions of the assets, and $fval$ is the value of the objective function, which corresponds to the variance of the optimal solution.

Considering the linear formulation of the MAD model (from equations (3.32) to (3.37)), we use Matlab's linprog function. The linear problem is described as

$$min\ f^T x \tag{3.42}$$

$$s.t.\ Ax \leq b \tag{3.43}$$

$$A_{eq}x = b_{eq} \tag{3.44}$$

$$l_b \leq x \leq u_b. \tag{3.45}$$

Hence, $[x, fval] = linprog(f, A, b, Aeq, beq, lb, ub)$, where x corresponds to the optimal solution and $fval$ to the value of the mean absolute deviation of the optimal solution.

Example 2 *Let us consider three stocks from the S&P500 Index, each one belonging to a different sector. These stocks are described in Table 3.2. The sample period is from January 4, 2010 - December 31, 2018. Data were collected from the Investing website (https://www.investing.com/). The logarithmic returns time series are constructed using three frequencies: daily, weekly and monthly.*

Table 3.2 Name, acronym and sector of each stock.

Name	**Acronym**	**Sector**
Apple	AAPL	Information Technology
Kellogg Company	K	Consumer Staples
Caterpillar Inc	CAT	Industrials

The portfolio selection problem will be mainly focused on the minimum risk portfolios, i.e. on the minimum variance and minimum mean absolute deviation portfolios. In these cases, the constraints (3.11) and (3.35) are removed from the MV and MAD models, respectively.

Table 3.3 presents some descriptive statistics of the three stocks. Apple is the stock with the highest mean return, while Kellogg is the stock with the lowest mean return. Although Apple has the highest mean return in all frequencies, its standard deviation is lower than the standard deviation of CAT at weekly and monthly frequencies.

The Jarque-Bera and Kolmogorov-Smirnov tests, where the null is normality, are then used to assess if logarithmic returns follow a normal distributions. The results in Table 3.4 show that the daily returns of the three stocks do not follow a normal distribution. However, as frequency decreases the distributions approximate normality, and the tests fail to reject the null hypothesis for monthly returns. That occurs because one observation corresponds to the sum of 21 daily returns (approximately) and, for that reason the distribution tends to approach the normal.

Table 3.3 Descriptive statistics of returns.

Type	**Company**	**Mean**	**StD**	**Min**	**1st quartile**	**Median**	**3rd quartile**	**Max**
Daily	AAPL	0.0007	0.0443	-1.3891	-0.0072	0.0007	0.0095	1.3866
	K	0.0000	0.0109	-0.0930	0.0051	0.0003	0.0055	0.0603
	CAT	0.0003	0.0172	-0.2287	-0.0080	0.0002	0.0093	0.0780
Weekly	AAPL	0.0034	0.0380	-0.1283	-0.0200	0.0056	0.0267	0.1317
	K	0.0002	0.0222	-0.0933	-0.0114	0.0011	0.0131	0.0804
	CAT	0.0016	0.0387	-0.1510	-0.0188	0.0237	0.0093	0.1222
Monthly	AAPL	0.0163	0.0733	-0.2034	-0.0218	0.0157	0.0635	0.1793
	K	0.0004	0.0429	-0.1102	-0.0284	0.0026	0.0302	0.0934
	CAT	0.0083	0.0804	-0.2287	-0.0407	0.0104	0.0595	0.2463

Table 3.4 Kolmogorov-Smirnov (KS) and Jarque-Bera (JB) normality tests.

Periodicity	AAPL		K		CAT	
	KS	JB	KS	JB	KS	JB
Daily	**0.0000**	**0.0010**	**0.0000**	**0.0010**	**0.0000**	**0.0010**
Weekly	0.6402	**0.0096**	**0.0446**	**0.0010**	0.0681	**0.0010**
Monthly	0.7606	0.3430	0.9795	0.5	0.9839	0.2950

Notes: The values in bold represent the rejection of the null of normality at the 1% level.

Table 3.5 reports the composition of the minimum risk MV and MAD portfolios and its expected return and risk. Both models, independently of the frequency, suggest investing a higher proportion in Kellogg. Since we are dealing with the minimum risk portfolio in both models, it is expected that the optimal solution proposes an higher weight for the less risky (Kellogg). The expected returns of the minimum risk MV and MAD portfolios are close, and its closeness increases when the frequency decreases. This example also illustrates Theorem 1. In the case of daily data, which do not follow a normal distribution, one may observe that $0.0001 \neq \sqrt{(\frac{2}{\pi})}\sqrt{0.0001}$. However, in the case of monthly data, when returns follow a distribution closed to the normal (see Table 3.4), $0.0307 \approx \sqrt{(\frac{2}{\pi})}\sqrt{0.0014}$.

Other insights can be obtained considering how the portfolios' composition change with different values of ρ. That is, how the distribution of weights change according to the desired return. To construct the efficient frontier, we consider an expected return in the interval $[\rho_{min}, \rho_{max}]$, as explained in section (3.2.2), divided into 100 points .

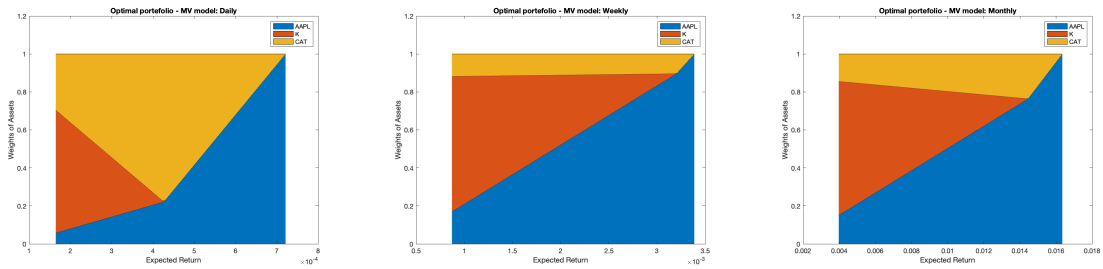
Figures 3.2 and 3.3 show the distribution of weights as a function of ρ for the MV and MAD models, respectively. In these figures, we can note that the minimum risk portfolio is the most diversified portfolio. Also as the portfolio's required minimum expected return increases, the investment in the asset with the highest return increases.

This illustrative example allow us to draw some conclusions which are inline with previous studies, namely [20]. First, the normality of returns usually does not occur except for monthly returns. Second, if normality holds, then the risk of the two models agree according to Theorem 1 and the weights and

Table 3.5 MV and MAD minimum risk portfolios.

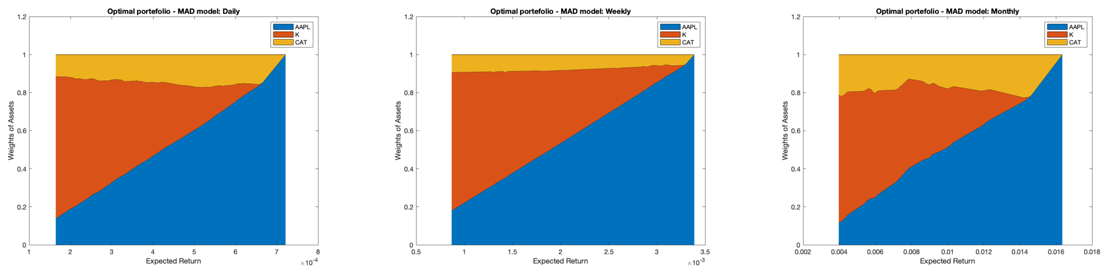
Period	**Model**	**Expected Return (%)**	Weight AAPL	Weight K	Weight CAT	**Risk**
Daily	**MV**	0.0120	0.0303	0.7560	0.2138	0.0001
	MAD	0.0165	0.1391	0.7467	0.1141	0.0001
Weekly	**MV**	0.0869	0.1689	0.7134	0.1177	0.0003
	MAD	0.0842	0.1715	0.7349	0.0936	0.0149
Monthly	**MV**	0.3891	0.1462	0.7103	0.1435	0.0014
	MAD	0.3970	0.1184	0.6718	0.2098	0.0307

Notes: This table presents the weights of the MV and MAD minimum risk portfolios. It also shows the expected returns and risk (measured by the variance and mean absolute deviation for the MV and MAD portfolios, respectively).



(a) Daily data. (b) Weekly data. (c) Monthly data.

Fig. 3.2 Composition of efficient portfolios of MV model.



(a) Daily data. (b) Weekly data. (c) Monthly data.

Fig. 3.3 Composition of efficient portfolios of MAD model.

returns obtained by the two models are similar. Third, the portfolio with the lowest risk is the most diversified one.

3.5 Equally-Weighted portfolio

In the study carried out in Chapter 4, we compare the optimization models, MV and MAD, with a simple heuristics - the Equally-Weighted (EW) portfolio. DeMiguel et al. [12] call this investing strategy the "naive portfolio", which consists in assigning the same weight to all assets in the investment universe, that is, if the portfolio has n assets, then

$$x_j = \frac{1}{n} \tag{3.46}$$

for each asset j, $j = 1, ..., n$.

This is a straightforward and intuitive investing strategy, because it does not require the estimation of the moments of returns or the application of any optimization procedure. Hence, there are no measurement or estimation errors. In the one hand, DeMiguel et al. [12] showed that the EW portfolio may obtain better out-of-sample performances than portfolios based on optimization techniques. On the other hand, this portfolio may be seen as a proxy for the overall market, as it is formed by all assets in the considered stock universe. Hence we use the EW portfolio as a benchmark portfolio in our comparative analysis.

Chapter 4

Comparative Study

This chapter compares the performance of the MV and MAD models considering the following aspects: the computational time regarding the dimensionality of the problem (number of stocks and number of observations collected over time); sensitivity to portfolio rebalancing, and portfolio financial performance. The software used in this chapter was Matlab R2021b. The Matlab codes were written with the help of the supervisors.

4.1 Data and Preliminary Analysis

Data was collected from the Refinitiv Eikon (https://www.refinitiv.com/en), also known as Thomson Reuters Eikon. The data covers the period from January 4, 2010, to November 30, 2021. US stocks were sorted in descending order of market capitalization (overall share value of a company) on the first day of the sample. Then, the adjusted daily closing prices were obtained for the first 500 stocks. A filtering procedure was then applied: if a given stock does not have data for the entire period, it was discarded, and a new stock, with the highest market capitalization, is added. We end up the procedure when a set of 500 stocks with full data was achieved. More specifically, we selected the stocks in the Refinitiv Eikon database as follows:

1. In the Static Request section, in Series/List, the search was redefined with the following constraints: category - equities, exchange - NASDAQ and NYSE, market - the United States, currency - United States Dollar, type - equity, activity - active, and base date - 2010. Then in Datatypes/Expressions 'WC08001' was selected, which is the ticket corresponding to market capitalization. Before submitting the request, it was essential to click on the primary sort and ask for the data to be sorted by descending order of market capitalization and in the date insert the first day of the sample with data, which, in this case, corresponded to January 4, 2010.

2. After having the data sorted, in the Times Series Request section, in Series, the first 500 stocks were chosen by selecting the cells in Excel. Finally, in Datatypes, the 'P' ticket was inserted, corresponding to the adjusted closing price, not forgetting to put the period to be observed.

Most of the selected stocks (305) were constituents of the S&P500 index on the first day of the sample, and this number increased to 420 on the last day of the sample. Information on the components of the S&P500 index in 2010 is available at [4], and that information for the year 2021 is available in Refinitiv Eikon, using the symbol "LS&PCOMP".

Since we are only interested in days when the market is open, the days corresponding to New York holidays were removed (e.g., January 1, third Monday in January and February, last Monday in May, July 4, first Monday in September, second Monday in October, November 11, last Thursday in November and December 25). The daily data was then converted into weekly data, using the prices on Wednesdays, which is the day-of-the-week less subjected to the weekend effect. If on a given Wednesday the market was closed then the previous business day was considered. Monthly data was also obtained by using prices every four weeks. So, the sample contains 3002 daily observations, 621 weekly observations and 156 monthly observations.

The period under study was divided into two sub-periods: "in-sample" (January 4, 2010 - December 31, 2018) and "out-of-sample" (January 2, 2019 - November 30, 2021).

As most stocks in our database belong to the S&P500 index, we present in Figure 4.1 the evolution of this index during the sample period. The S&P500 index presented a positive trend during the overall period, following the aftermath of the 2007-2008 crisis. In March 2020 there was a sudden fall down originated by the Covid-19 crisis. Despite the Covid-19 crisis, the price average rate of increase was higher in the out-of-sample period than in the in-sample period.

Table 4.1 shows some descriptive statistics of daily, weekly and monthly logarithmic returns of the S&P 500 in the entire sample, while Table 4.2 shows these statistics for the in-sample and out-of-sample periods. All the statistics increase when the periodicity of the data decreases, except the kurtosis, which is higher in daily data. This is expected, as weekly data results from the aggregation of daily data, while monthly data results form the aggregation of weekly data. Hence, the monthly average corresponds approximately to four times the weekly average and the monthly standard deviation is twice ($\sqrt{4}$) the weekly standard deviation. The weekly average is approximately five times the daily average returns and the weekly standard deviation is $\sqrt{5}$ times the daily standard deviation.

The mean and median of returns are higher in the out-of-sample period than in the in-sample period, this supported what previously has been said about Figure 4.1. The variability of returns is also higher in the out-of-sample period (higher standard deviation, minimum and maximum range and inter-quartile range).

The kurtosis presents for all series values are greater than three which implies that the distributions have heavier "tails" than the normal distribution. The periodicity of the data affects the weight of the tails, being heavier when the frequency of the data increases. Moreover, it is observed that the distribution of returns in the out-of-sample period has heavier tails than in the in-sample period, being in accordance with an increase in extreme values reported earlier.

4.2 In-Sample Performance

This section analyzes the in-sample computational effort and performance of models. The in-sample portfolio optimal solutions can be seen as estimations of the optimal solutions in the out-of-sample period, when the portfolio is actually retained by the investor.

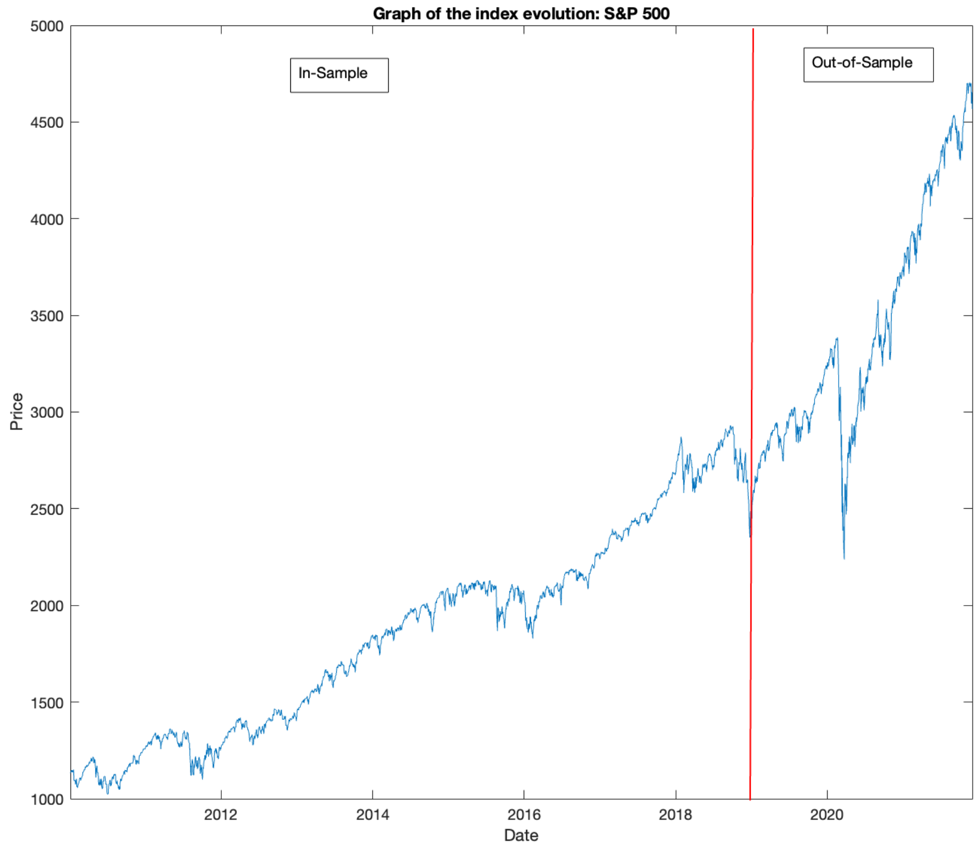


Fig. 4.1 Evolution of the S&P 500 index from January 4, 2010, to November 30, 2021. The vertical line separates the in-sample and out-of-sample periods.

Table 4.1 Descriptive statistics of logarithmic returns (full sample).

Descriptive statistics	Return (Full Sample)		
	Daily	Weekly	Monthly
Observations	3002	621	156
Mean	0.0004	0.0024	0.0096
Median	0.0006	0.0036	0.0136
Std.Deviation	0.0191	0.0407	0.0834
1st Quartile	-0.0082	-0.0170	-0.0297
3rd Quartile	0.0096	0.0235	0.0539
Minimum	-0.1997	-0.2850	-0.4794
Maximum	0.1676	0.2070	0.2760
Kurtosis	19.4526	13.3193	13.7458

4.2.1 Analysis of portfolio performance

The 500 stocks in the database were then sorted alphabetically and divided into different groups with equal cardinality. For example, in the 2 groups case, each group has 250 stocks, in the 5 group

Table 4.2 Descriptive statistics logarithmic returns (in-sample and out-of-sample).

Descriptive statistics	In-Sample			Out-of-Sample		
	Daily	**Weekly**	**Monthly**	**Daily**	**Weekly**	**Monthly**
Observations	2268	469	118	734	152	38
Mean	0.0004	0.0020	0.0079	0.0006	0.0036	0.0132
Median	0.0005	0.0032	0.0118	0.0010	0.0050	0.0188
Std.Deviation	0.0169	0.0357	0.0665	0.0242	0.0521	0.1192
1st Quartile	-0.0079	-0.0167	-0.0298	-0.0095	-0.0183	-0.0310
3rd Quartile	0.0091	0.0223	0.0496	0.0115	0.0284	0.0686
Minimum	-0.1343	-0.1729	-0.2123	-0.1833	-0.2685	-0.4675
Maximum	0.1179	0.1416	0.1849	0.1549	0.1946	0.2634
Kurtosis	12.2672	6.4841	4.2936	16.5352	11.0982	9.4412

case, each group has 100 stocks, and so on. The weights of stocks are equally assigned within each group. For instance, in the 2 group case, if the weight of the first group is 30%, then the final weight of any stock in that group is $\frac{0.3}{250} = 0.12\%$. We consider eleven partitions of the stocks into 1, 2, 5, 10, 20, 25, 50, 100, 125, 250 and 500 groups.

Table 4.3 Expected return and standard deviation of portfolios selected from different groups of stocks.

Number of groups	MV		MAD		EW	
	ER	**StD**	**ER**	**StD**	**ER**	**StD**
1	0.0520	1.0049	0.0520	1.0049	0.0520	1.0049
2	0.0506	0.9760	0.0506	0.9760	0.0520	1.0049
5	0.0539	0.9539	0.0539	0.9539	0.0520	1.0049
10	0.0536	0.9533	0.0535	0.9540	0.0520	1.0049
20	0.0487	0.9265	0.0488	0.9267	0.0520	1.0049
25	0.0543	0.9089	0.0542	0.9105	0.0520	1.0049
50	0.0542	0.8705	0.0537	0.8762	0.0520	1.0049
100	0.0497	0.7962	0.0489	0.8005	0.0520	1.0049
125	0.0503	0.7602	0.0492	0.7682	0.0520	1.0049
250	0.0444	0.6782	0.0436	0.6833	0.0520	1.0049
500	0.0319	0.6063	0.0322	0.6117	0.0520	1.0049

Notes: This table shows the expected return (ER) and risk, measured by the standard deviation (StD) of minimum risk portfolios selected in-sample using the Mean-Variance model (MV) and the Mean Absolute Deviation model (MAD). It also shows these statistics for the Equally Weighted (EW) portfolio. These portfolios are constructed considering different numbers of groups with equal cardinality. In each group, stocks are equally weighted. The analysis is conducted in-sample (January 4, 2010 - December 31, 2018) using daily data. The results are shown in percentage.

Table 4.3 shows the expected return and risk for the minimum risk MV and MAD portfolios, and EW portfolios considering different numbers of groups. The expected return and standard deviation of the EW portfolios do not change, has in fact these correspond to a unique portfolio with equal weights (also notice that this portfolio corresponds, by construction, to the MV and MAD portfolios when only

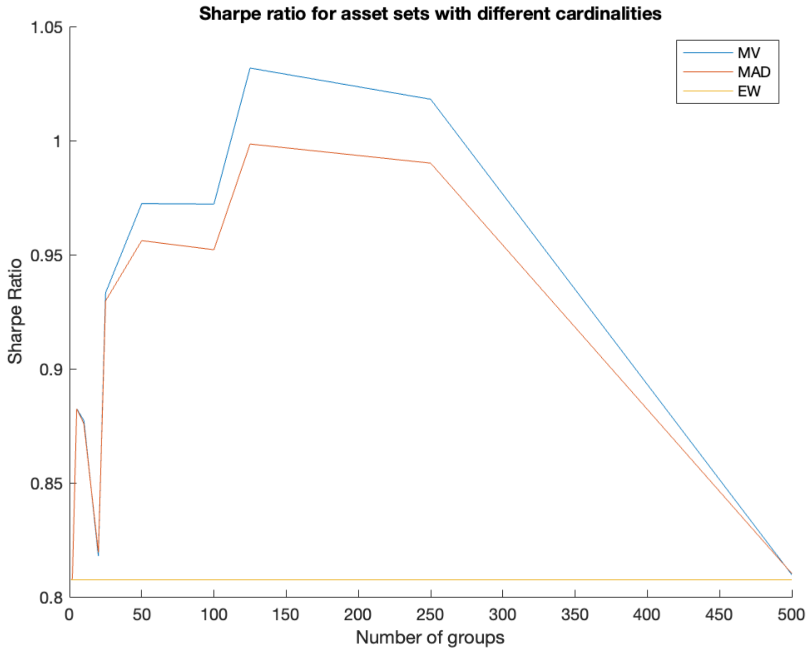


Fig. 4.2 Sharpe ratio as a function of the number of groups using daily data.

one group is used). The EW portfolio presents a good performance in terms of expected return but shows the worst performance in terms of risk. The MV and MAD portfolios present similar results, however, the risk of the MV portfolio is always equal to or lower than the risk of the MAD portfolio. This is expected because the MV minimizes the risk measured by the variance (or equivalently the standard deviation) which is the metric used to assess the risk of MV and MAD portfolios. The mean return of the minimum risk portfolios shows a tendency to increase with the number of groups until 25 and then decreases. The risk tends always to decrease with the number of groups. This is expected because we are dealing with minimum risk portfolios, and the optimization procedure is more effective with an increase in the number of groups which imply a decrease in the number of superimposed restrictions. In Appendix B, Figure B.1 and Figure B.2 show graphically the expected return and risk as a function of the number of groups, respectively, where these patterns are visible. Figure 4.2 shows the annualized Sharpe ratio (see Equation (5)) over the different groups. As expected from the analysis of the expected return and standard deviation, the Sharpe ratio of MV portfolios is never lower than the corresponding Sharpe ratio for the MAD portfolios. The Sharpe ratio increases for both MV and MAD models until 125 groups (4 assets per group) and then decreases reaching the minimum of approximately 0.8 when considering 500 groups.

The same analysis is performed for weekly and monthly data (see Appendix B). Basically, the same patterns reported for daily data are also visible for weekly and monthly data.

4.2.2 Analysis of computational time

This section assesses the impact of the problem's dimensionality on the computational time of the MV and MAD models. As already mentioned in Chapter 3, Markowitz's main disadvantage is that it falls on a quadratic problem, which arguably needs a greater computational effort to compute the optimal solution than the linear model. The MV model is strongly dependent on the number of stocks, in contrast, the MAD model depends on the number of stocks and the number of observations (see, for instance, Júdice et al. [17]). In that paper, the authors only use a maximum of 92 stocks and 300 observations, which are small numbers to reach accurate conclusions.

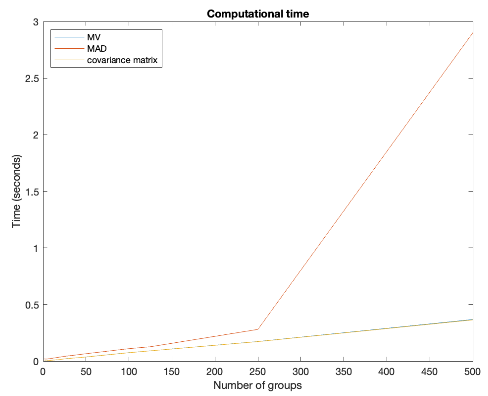


Fig. 4.3 Computational time using daily data.

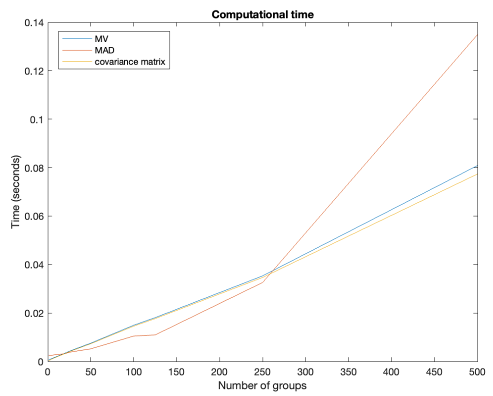


Fig. 4.4 Computational time using weekly data.

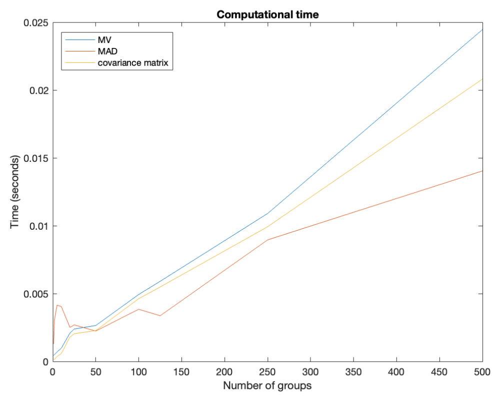


Fig. 4.5 Computational time using monthly data.

To analyse the computational time we use the previous partition of stocks into groups, and consider the minimum risk portfolios of the MV and MAD models for the three frequencies (daily, weekly and monthly). The daily, weekly, and monthly data consist of 2268, 469 and 118 observations, respectively, as mentioned in Table 4.2. Since the computational time varies significantly from run to run, the portfolios for each group is computed ten times and then their computational times are averaged. The average computational times for the three frequencies as a function of the number of groups are shown in Figure 4.3, Figure 4.4, and Figure 4.5. Notice that the computational times for the MV model include the time needed to calculate the covariance matrix. The MAD model needs less time only when using monthly data (i.e., when the number of time points is small). These results are in line with those presented in several papers [2, 8, 17]. So, the MAD model has $n \times T$ decision variables and, consequently, it is T times greater (in terms of dimensionality) than the MV model.

To have a more clear view of the dependence of the computational time with respect to the number of stocks and the number of time observations, we resort to simulated data. Returns, following the standard normal distribution, were simulated for 5,000 stocks and 200 time observations.

Figure 4.6 represents the computational time for 100 time observations and 5,000 stocks. Figure C.1(a), Figure C.1(b), and Figure C.1(c), show the differences in computational time using 50, 100 and 200 time points. All these figures highlight that the computational time increases quadratically and linearly with the number of stocks for the MV model and MAD models, respectively. However, the inclination of the straight line that describes the evolution of computational time in the MAD model increases with the number of observations.

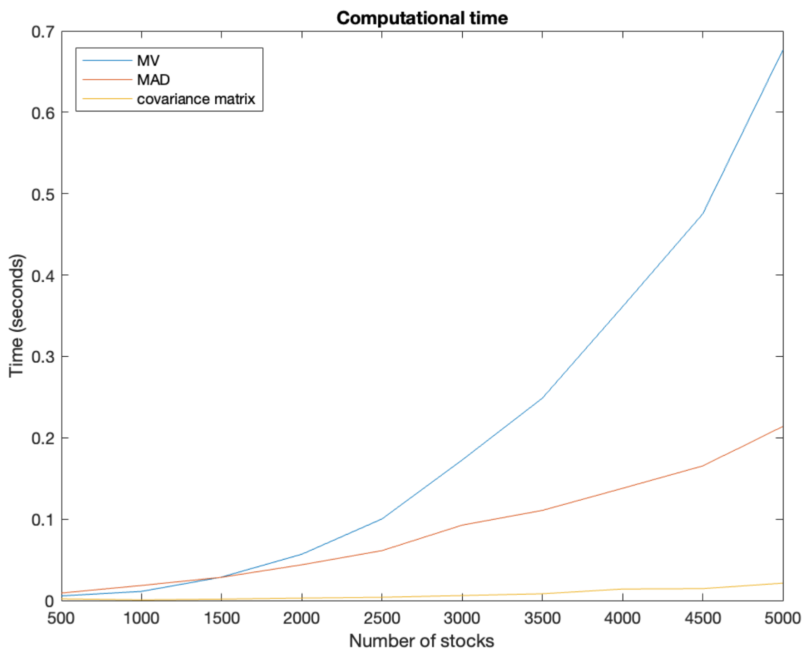


Fig. 4.6 Computational time using random data.

So, from Figure 4.6, we may conclude that when there are many assets with low frequency data, the MAD model performs much better than the MV model. It is essential to mention that in our computational experience, the covariance matrix was computed using an optimized routine available for Matlab, so we believe that this may also have influenced the computational time to solve the MV problem.

It seems that the MAD model is better when the number of observations is lower than 1/15 of the stocks number (this is the point where the two curves intersect). This pattern is noticeable if one computes the ratio $\frac{T}{n}$, where T is the number of time points and n is the number of assets. For instance for $T = 50$, $50/1{,}000 = 0.05 = 1/15$, for $T = 100$, $100/1{,}500 = 0.05 = 1/15$, and for $T = 200$ $200/3{,}500 = 0.05 = 1/15$. However, in practice, to obtain accurate estimates of the models' inputs, the number of observations must be at least equal to the number of stocks. This raises some questions on the usefulness of the MAD model.

One should notice that in reality, investors seldom intend to invest in such a large number of assets. However, if the investment strategy aims at creating a widely diversified portfolio, the investor may invest in ETFs (Exchange Traded Funds), which are traded assets that replicate the behaviour of stock indexes. Let us suppose that an investor wants to pursue an international diversification strategy, then she can buy ETFs on, for instance, the S&P500, Nikkei 250, and FTSE 100. Using just these three assets the investor will have a market position on 850 equities from three continents.

4.3 Out-of-Sample Performance

4.3.1 Portfolio rebalancing

The portfolio rebalancing strategy is a common portfolio management practice. By rebalancing her portfolio, the investor changes the composition of the portfolio according to updated data, and also enables the investor to manage his risk exposure and to pick up the best stocks in the recent past.

Considering that the portfolios are rebalanced periodically, allows a proper assess the out-of-sample performance of the portfolio selection models. Usually, rebalancing is performed using the information in a rolling window with fixed length (see DeMiguel et al. [12]). Here we consider rebalancing at daily, weekly and monthly frequencies considering a rolling window with 2268 days, 469 weeks and 118 months respectively, i.e. corresponding to the in-sample period. The investment universe is formed by the 500 stocks in the database. Using this strategy implies obtaining 734 daily returns, 152 weekly and 38 monthly returns corresponding to the out-of-sample period. A simple example clarifies this. Suppose that one has a sample consisting of 1,000 daily observations, 700 of which are in-sample and 300 out-of-sample. First, using the in-sample data from day 1 to day 700, one computes the optimal solutions applying both models. Then, with these vectors of weights, one computes the effective return of the portfolio for the first day out-of-sample, i.e., day 701. Then, the optimal solutions are calculated using data day 2 to day 701, which are then used to compute the portfolio return on day 702, and so on until all days in the out-of-sample are covered. One ends up with 300 portfolio returns produced by each of the models under study. The reasoning is identical if we have weekly and monthly data.

The out-of-sample analysis is performed considering the minimum risk portfolio, hence the minimum expected return constraints present in both models, namely Equation 3.11 and Equation 3.35 are not considered.

In the out-of-sample analysis, one must take into account the way that returns are computed. The optimal solutions were estimated using logarithmic returns, but they are not additive in the asset space. So, one must convert them into discrete returns using the operation $exp(r_t) - 1$, where r_t are logarithmic returns. To obtain the vector portfolio returns, one has to apply the operation $ln(1+R_t)$, where R_t are the discrete effective portfolio returns.

Figure 4.7 shows the daily cumulative return (with basis equal to 100), i.e., the path of $100e^{\sum_{t=1}^{T} R_t}$, where R_t is the out-of-sample portfolio logarithmic return at time t after rebalancing (weekly and monthly rebalancing provide similar figures). This figure shows that the EW almost always dominates the other portfolios. This does not happen during the second semester of 2019, when there was an increase in the trade war between the USA and China [23], uprising the uncertainties on the USA economic growth, and in March 2020, due to the market crash caused by the COVID-19 pandemic. Therefore, the EW portfolio appears to perform worse when there is is a lot of uncertainty in the financial market.

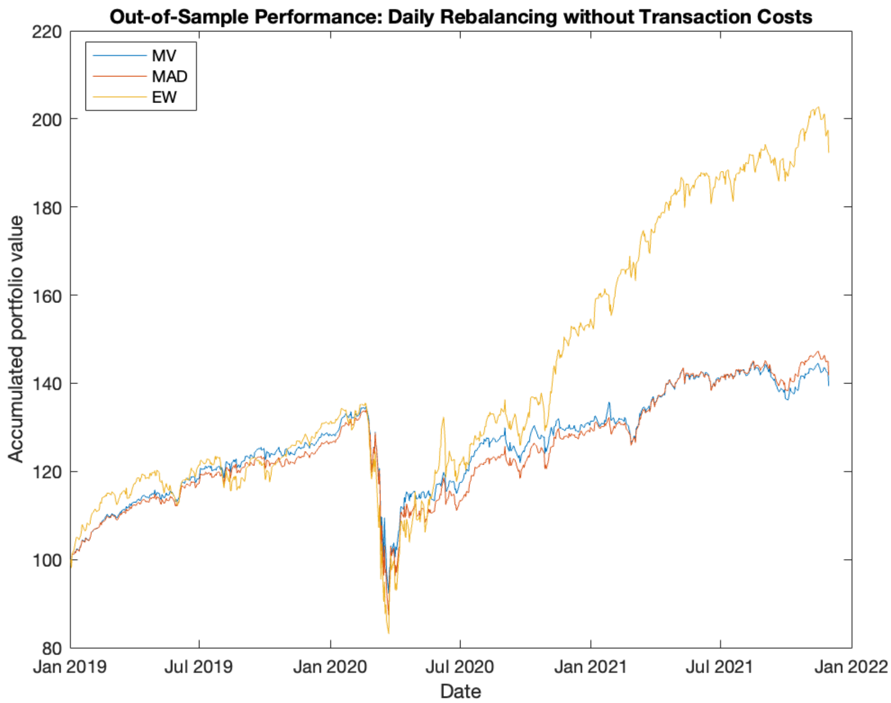


Fig. 4.7 Portfolio cumulative returns with daily rebalancing

Table 4.4 shows the final values obtained by applying different rebalancing strategies to the different solutions analyzed. Comparing the optimal solutions of the MV and MAD models, the best value obtained was with the weekly MV model. However, the MAD model exceeds the value obtained by the MV model. MAD model performs poorly when transaction costs are taken into account.

4.3.2 Metrics of financial performance

This section evaluates the financial performance of the models using different metrics. These measures are described below. The analysis is conducted with and without transaction costs. For

Table 4.4 Final values of an investment of 100.

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Model	Type of reb.	Final Value (without TC)	Final Value (with TC)	Loss (%)
MV	**Daily**	139.3884	129.6288	7.0017
	Weekly	148.5556 (+6.5767)	140.9601 (+8.7413)	5.1129
	Monthly	119.7246 (-14.1072)	115.6821 (-10.7590)	3.3765
MAD	**Daily**	141.8270	122.3748	13.7154
	Weekly	145.4904 (+2.5830)	130.7066 (+6.8084)	10.1614
	Monthly	123.9912 (-12.5757)	113.2313 (-7.4717)	8.6780
EW	**Daily**	192.2909	183.7471	4.4432
	Weekly	204.6490 (+6.4268)	200.3382 (+9.0293)	2.1064
	Monthly	191.5407 (-0.3901)	189.3973 (+3.0750)	1.1190

Notes: In parentheses there are the difference in percentage in relation to daily rebalancing. The final value is computed without transaction costs (without TC) and with proportional costs of 0.5% (with TC). The last column shows the relative difference when transaction costs are considered.

these analyses one needs the matrix of asset weights and the vector of out-of-sample effective returns. The evaluation takes into account the three frequencies (daily, weekly and monthly). Inclusion of transaction costs gives a more realistic view on the investment differences. According to Dybvig and Pezzo [13] and Sebastião [29], transaction costs may be associated with brokerage costs, taxes, bid-ask spreads, time and effort to calculate the optimal portfolios, and costs to study the assets in the investor's portfolio.

Based on DeMiguel et al. [12], we use a proportional transaction cost of $c = 0.5\%$. So, the overall transaction cost at time t is computed as:

$$cost_t = 0.5\% \sum_{i=1}^{n} \left| (x_{j,t} - x^h_{j,t}) \right| . \tag{4.1}$$

The investor's wealth (W_t) at each moment t, considering that at the initial moment, $t = 0$, wealth equals to $W_0 = 100$ is given by (4.2), where $R_{p,t}$ corresponds to the discrete return of the portfolio at the moment t.

$$W_t = W_{t-1}(1 + R_{p,t})(1 - cost_t). \tag{4.2}$$

1. **Win Rate:** The win rate corresponds to the percentage of periods out-of-sample with positive returns.

$$WR = \frac{n_{positive\_returns}}{\#periods} \tag{4.3}$$

   where $n_{positive\_returns}$ corresponds to the number of periods that have positive return.

2. **Cardinality:** The cardinality allows us to observe the evolution of the number of assets in the portfolio. With the rebalancing strategy, assets weights are updated every day (week or month), hence cardinality is a variable that counts the number of assets that have a more weight above a certain threshold and gives an idea on the level of diversification of the portfolio. We first

used three thresholds: 0%, $1/n$ (where $n = 500$ is the number of assets) and 1%. However, we observed that with a 0% threshold the optimal solution of the Markowitz model invests in all assets, although the investment in more than 450 assets is less than 0.2% . Hence, we chosen to evaluate the cardinality using the following thresholds: $1/n$, 1% and 5%.

3. **Turnover:** This measure corresponds to the turnover presented by DeMiguel et al. [12] given by the formula (4.4). It gives an idea of the amount of trading executed by a particular investment strategy. The turnover corresponds to the average, over all periods, of the absolute changes in the weights of the assets.

$$Turnover = \frac{1}{\#periods} \sum_{t=1}^{\#periods} \sum_{j=1}^{n} \left|(x_{j,t} - x^h_{j,t})\right|. \tag{4.4}$$

$x^h_{j,t}$ and $x_{j,t}$ correspond to the portfolio weights before and after rebalancing at t. One should notice that the amount of trading of each asset is not equal to the difference between the two consecutive optimal weights, because the weight changes simply because the asset's price changes. Hence, the price impact must be considered. Let $R_{j,t}$ and and $R_{P,t}$ be the out-of-sample returns of asset j and portfolio P at time t, it follows that $x^h_{j,t} = x_{j,t-1}\frac{1+R_{j,t}}{1+R_{P,t}}$.

4. **Annualized Mean Return and Standard Deviation:** The annualized mean is obtained by multiplying the average of the out-of-sample returns by the number of periods in the year of, i.e. 254, 52 and 12 for daily, weekly and monthly data, respectively. The logic is the same for the annualized standard deviation, for which the multipliers are the square roots of 254, 52 and 12 for daily, weekly and monthly data, respectively.

5. **Sharpe Ratio:** William Sharpe [32], in 1966, presents a measure to evaluate the performance of investment portfolios called reward to variability ratio, better known as the Sharpe Ratio. It describes how much the investor is rewarded per unit of risk.

   To compute the Sharpe ratio one needs to firstly define the vector of excess returns, $R_{ER} = R_P - R_f$, where R_P is the portfolio return and R_f is the risk-free interest rate. In this study, the risk-free interest rate is proxied by the yield-to-maturity of the 3-month Treasury bills collected from the US Federal Reserve's database [11]. This interest rates are presented in percentages and are annualized, hence, were converted into daily, weekly and monthly rates, by dividing per 36,000, 5,200, 1,200, respectively.

   The Sharpe ratio corresponds to the quotient between the average excess return and the standard deviation of the excess return, i.e,

$$SR = \frac{mean(R_{ER})}{\sigma_(R_{ER})}. \tag{4.5}$$

   This ratio (4.5) is then annualized by multiplying it by the square root of 254, 52, 12 for daily, weekly and monthly data, respectively.

6. **Sortino Ratio:** In 1994, Sortino and Price [34] introduced a variation of the Sharpe ratio. The Sortino ratio has the same numerator as the Sharpe ratio; however, the denominator only

accounts for the downside risk, i.e. only considers the returns lower than a given threshold, B. The downside risk is defined as,

$$DR = \sqrt{\frac{1}{T}\sum_{j=1}^{T} min(0, R_{P_j} - B)^2}, \tag{4.6}$$

The usual thresholds are the risk-free rate or zero. In this study we chose $B = 0$. Thus, the Sortino ratio is defined as

$$SR = \frac{mean(R_{ER})}{DR} \tag{4.7}$$

The annualized Sortino ratio is obtained multiplying (4.7) per the square root of 254, 52, 12 for daily, weekly and monthly data, respectively.

7. **CVaR:** Rockafellar and Uryasev [28] proposed the Conditional Value at Risk (CVaR), which is related to other risk measure, the Value at Risk (VaR). The CVaR measures the expected loss below the VaR, i.e. it measures the extreme losses when the VaR value is exceeded. The VaR is the α quantile of the loss distribution and represents the smallest possible loss with probability greater than, or equal to, α. The usual values for α are 0.90, 0.95 and 0.99.

   Figure 4.8 presents the histogram of the returns of the Markowitz out-of-sample portfolio applied the daily rebalancing. One can observe the VaR value with a confidence level of 99%. In red are present the values below the VaR, that is, the values corresponding to the CVaR (the CVaR is usually presented as the loss value, that is, the symmetric of negative returns).

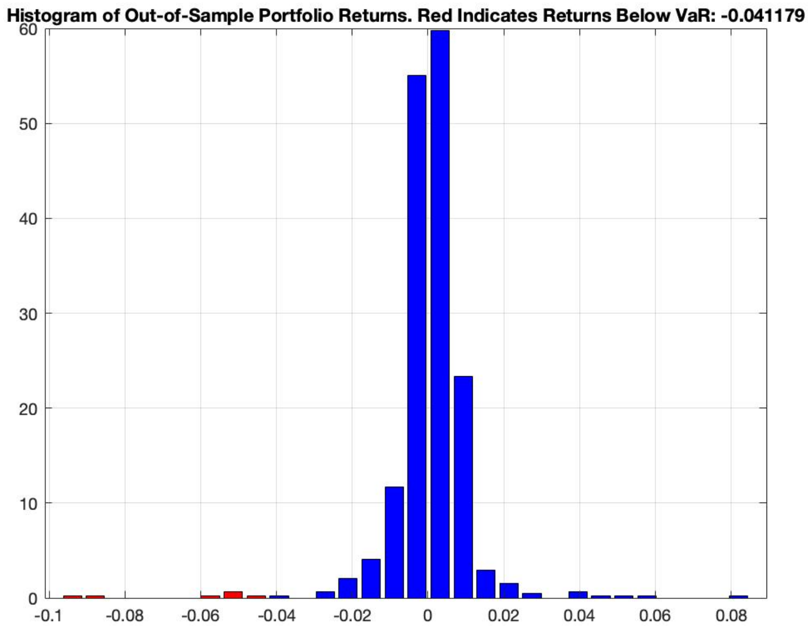

Fig. 4.8 Histogram of out-of-sample portfolio return.

8. **Maximum Drawdown:** A drawdown is an essential tool for investors, as it allows them to understand the instability of a given investment better. The Maximum Drawdown measures the maximum loss observed from the highest accumulated value (peak) to the lowest accumulated value (valley or trough) before a new peak is reached, relative to the initial peak. As stated by Choi [9], investors will prefer a portfolio with a lower maximum drawdown value.

$$MDD = \frac{Trough\_Value - Peak\_Value}{Peak\_Value} \tag{4.8}$$

4.3.3 Financial performance of the minimum risk portfolios

Table 4.5 shows the performance of the minimum risk MV and MAD portfolios, and EW portfolios with and without transaction costs. The EW portfolio outperforms the MV and MAD portfolios in terms of win rate, turnover, mean return, Sharpe ratio, and Sortino ratio, with and without transaction costs. However, these portfolios are more risky (higher standard deviation) and are more susceptible to extreme losses (higher CVaR and Maximum Drawdown).

Except for the weekly rebalancing, the MAD model outperforms the MV model in terms of mean return. However this advantage is lost when transaction costs are included. The impact of transaction costs is higher for the MAD portfolios than for the other two portfolios. For the other metrics, and independently of the rebalancing frequency, the MV model outperforms the MAD model.

As for the turnover value, we did not expect to obtain such a high value for the MAD model. However, for instance when analysing in more detail the weights of the MAD portfolio with monthly frequency, we noticed that the weights fluctuate a lot over time, in many cases excluding the assets that previously were in the portfolio and including new ones, hence increasing the turnover significantly.

So, the results undoubtedly show that the EW is the best portfolio, followed by the MV portfolio, while the MAD portfolio present the worst performance, especially if transaction costs are taken into account.

Table 4.6 shows the bootstrap p-values of the test on the difference of Sharpe ratios. The test is conducted pairwise and the null hypothesis is that the difference is equal to zero. The bootstrap p-values were calculated according to the Ledoit and Wold [22] methodology. Table 4.6 highlights that in most of the cases the differences between the Sharpe ratios are not statistically different from zero. However, for monthly rebalancing, without transation costs, the MAD portfolio has a Sharpe ratio significantly lower (at the 5% level) than the EW portfolio. At monthly frequency, the consideration of transaction costs, reinforces the superiority of the EW portfolio, which has a Sharpe ratio significantly higher than the MV and MAD portfolios.

We now focus on the cardinality of the portfolios.

Table 4.7 shows the mean, standard deviation, maximum and minimum values of the number of assets in the portfolios with weights higher than a given threshold. On average the MAD portfolios with daily and weekly rebalancing have more assets (thresholds $1/n$ and 1%), and

Table 4.5 Performance measures.

Metric	Models	Without transaction costs Daily	Weekly	Monthly	With transaction costs Daily	Weekly	Monthly
WR (%)	**MV**	55.722	67.105	76.316	55.313	65.790	76.316
	MAD	56.540	65.132	68.421	55.041	65.132	65.790
	EW	57.493	67.105	78.947	57.357	67.105	78.947
Tr (%)	**MV**	1.980	6.934	18.600	1.980	6.934	18.600
	MAD	4.026	14.168	49.107	4.026	14.168	49.107
	EW	1.241	2.819	6.042	1.241	2.819	6.042
AMR (%)	**MV**	11.317	13.801	6.944	8.802	11.998	5.827
	MAD	11.923	13.174	8.347	6.809	9.489	5.396
	EW	22.570	24.954	22.950	20.994	24.221	22.588
AStd (%)	**MV**	17.313	15.857	19.665	17.400	16.034	19.940
	MAD	19.291	15.732	24.860	19.317	15.842	25.070
	EW	24.903	23.426	30.103	24.906	23.439	30.112
SR (%)	**MV**	61.872	81.607	31.100	47.109	69.454	25.049
	MAD	58.672	78.789	30.238	32.117	54.463	18.192
	EW	88.200	102.850	73.456	81.862	99.621	72.230
Sort.R (%)	**MV**	82.706	108.755	35.231	62.119	91.030	28.147
	MAD	76.846	100.789	34.566	41.675	69.001	20.549
	EW	119.555	139.281	95.023	110.652	134.501	93.334
CVaR (%)	**MV**	-6.109	-10.919	-30.200	-6.240	-11.246	-30.854
	MAD	-6.924	-12.553	-37.954	-6.978	-12.781	-38.608
	EW	-7.932	-17.109	-40.686	-7.949	-17.179	-40.734
MD (%)	**MV**	-31.542	-28.870	-26.066	-32.659	-29.628	-26.545
	MAD	-34.762	-26.763	-31.582	-35.335	-27.318	-32.028
	EW	-38.625	-34.663	-33.427	-38.841	-34.795	-33.458

Notes: This table presents the performance metrics for the minimum risk MV and MAD portfolios, and Equally Weighted portfolios with and without transaction costs. These portfolios were constructed using daily, weekly and monthly data, and were rebalanced at those frequencies. The metrics are: WR - Win Rate; Tr - Turnover; AMR - Annualized Mean Return; AStd - Annualized Standard Deviation; SR - Sharpe Ratio; Sort.R - Sortino Ratio; MD - Maximum Drawdown.

show less concentration (threshold 5%) than the MV portfolios. However the variability of the cardinality is higher for the MAD portfolios (except when the threshold is 5%). This supports the previous results that show that turnover and hence transaction costs have an higher impact on the MAD portfolios than in the MV portfolios.

Figure 4.9 and Figure 4.10 shown the time evolution of cardinality for the MV and MAD portfolios with daily rebalancing, respectively. These figures clearly show that although the

Table 4.6 Bootstrap p-values for the difference in the Sharpe ratios.

	Bootstrap p-values			
	Period	**MV vs. MAD**	**MAD vs. EW**	**EW vs. MV**
without TC	**Daily**	0.8322 (MV)	0.2886 (EW)	0.4962 (EW)
	Weekly	0.8465 (MV)	0.4289 (EW)	0.5756 (EW)
	Monthly	0.8202 (MV)	**0.0313** (EW)	0.0605 (EW)
with TC	**Daily**	0.3646 (MV)	0.0674 (EW)	0.3644 (EW)
	Weekly	0.8465(MV)	0.1182 (EW)	0.4076 (EW)
	Monthly	0.3779 (MV)	**0.0146** (EW)	**0.0383** (EW)

Notes: This table shows the p-values of the pairwise test on the difference between Sharpe ratio. The rejection of the null hypothesis of no difference between the Sharpe ratios at the 5% significance level is n bold. In parentheses are the models that in each pair presented the higher Sharpe ratios.

MAD portfolios are more diversified, the cardinality of MAD portfolios changes more frequently than the cardinality of MV portfolios.

Interestingly, the MV model suggest forming portfolios that include all 500 assets at all times, while the MAD model suggests using portfolios with as much as 50 assets. Most weights of the MV portfolios are in fact infinitesimal, which raises some questions about their practicability in real markets.

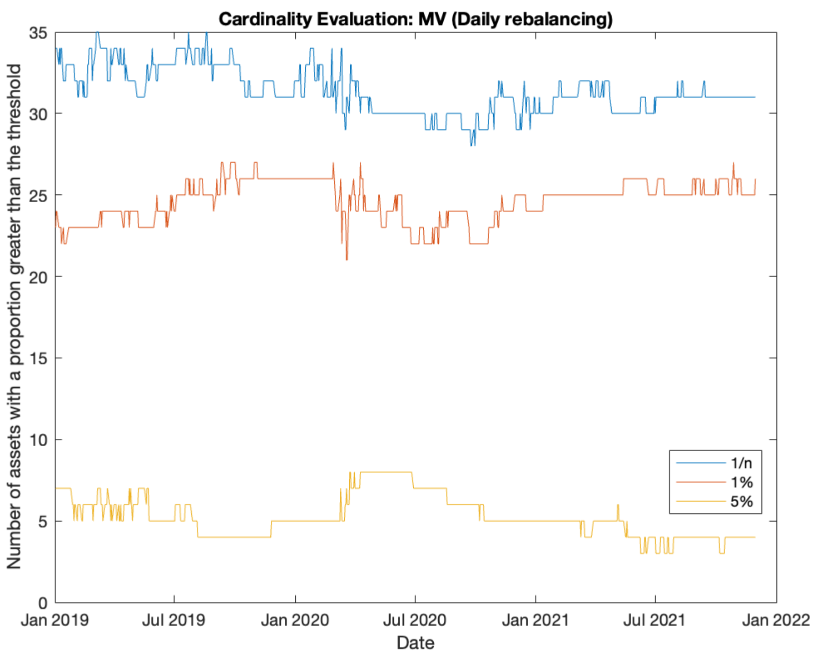

Fig. 4.9 Time path of cardinality of MV portfolios with daily rebalancing.

Table 4.7 Statistical analysis of the portfolios' cardinality.

Threshold	**Cardinality**				
	Models	**Statistic**	**Daily**	**Weekly**	**Monthly**
1/n	**MV**	Mean	31.350	32.993	29.447
		Std Deviation	1.466	3.866	2.446
		Maximum	35	41	34
		Minimum	28	25	25
	MAD	Mean	41.928	39.921	31.632
		Std Deviation	3.644	3.702	3.483
		Maximum	50	50	38
		Minimum	33	29	25
1%	**MV**	Mean	24.583	24.276	23.605
		Std Deviation	1.245	2.373	1.516
		Maximum	27	29	26
		Minimum	21	20	20
	MAD	Mean	26.809	25.559	24.921
		Std Deviation	1.787	3.624	2.530
		Maximum	32	32	31
		Minimum	23	17	21
5%	**MV**	Mean	5.275	6.632	6.184
		Std Deviation	1.283	1.021	0.896
		Maximum	8	10	8
		Minimum	3	5	5
	MAD	Mean	5.228	6.132	5.658
		Std Deviation	1.243	0.961	1.744
		Maximum	8	8	10
		Minimum	3	4	3

Notes: This table shows some statistics of the number of assets in the MV and MAD portfolios, with weights above given threshold. These portfolios were rebalanced daily, weekly and monthly. The thresholds are $1/n = 1/500$, 1% and 5%.

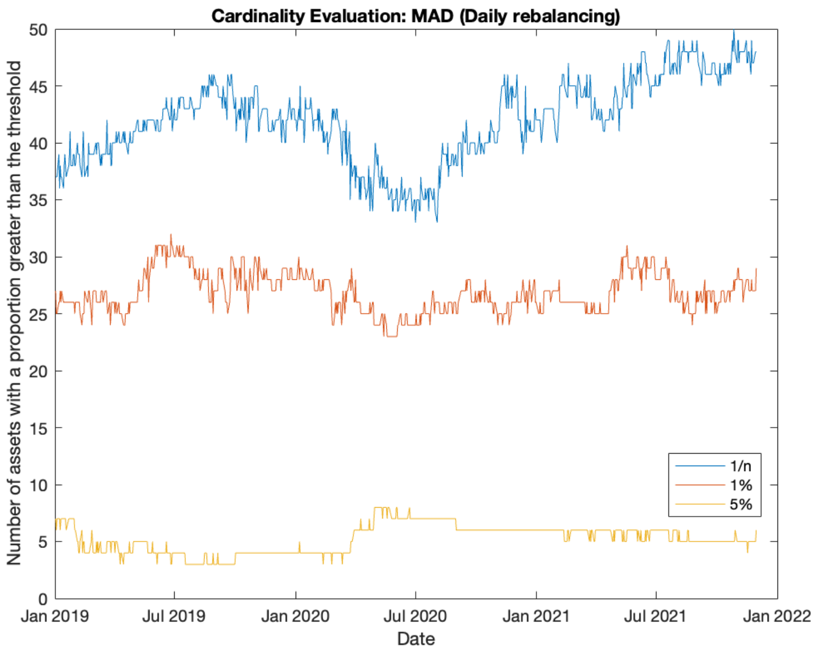

Fig. 4.10 Time path of cardinality of MAD portfolios with daily rebalancing.

Chapter 5

Conclusions

The Markowitz and Konno-Yamazaki models assist the investors in choosing their portfolios. The main objective of this dissertation is to present a comparative study between those two portfolio selection models, highlighting their strengths and weaknesses. In this way, this dissertation presents some insights that help the investor to choose the model that best fits their preferences.

Computational time is an essential aspect that should be considered when choosing the model to be used. The MV model computational time increases quadratically with n, while the MAD model computational time increases linearly with n and T. Thus, the computational time advantage of the MAD model over the MV model decreases as T increases. In our study, we observe that the MAD model is faster than the MV model when the ratio n/T is greater than 15. However, in practice, to obtain results with any accuracy, the number of observations must be at least equal to the number of stocks, put into question the usefulness of the MAD model.

The out-of-sample analysis of portfolios with periodical rebalancing showed that the EW portfolio outperforms the MV and MAD models, confirming the main claim of DeMiguel et al. [12], according to which this simple heuristic is most of the times better than portfolio optimization techniques. Similar results were obtained for the minimum risk MV and MAD portfolios. However, the MAD model performs slightly worse in terms of performance measures than the MV model.

We also highlighted the results on the cardinality of the portfolios. The MV model presents a clear disadvantage in this issue. While the MAD model suggests investing in about 40/50 assets, the MV model proposes an investment in all assets, and most weights are very small. These small weights may lead to an increase in transaction costs and may even be are unpractical in real markets.

In future works, we intend to continue this study and publish our conclusions in an international journal on this topic by. First, we intend to compare the MAD and MV models with other portfolio selection models, such as the CVaR, Minimax, and Semi-Variance models. Second, we intend to insert the transaction costs directly in the models' formulations. In this way, the weights attributed to each asset would already be calculated considering transaction costs. Third, we also intend to analyse other portfolios besides the minimum risk portfolio (which we have address in the present work).

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Appendix A

Literature Review

Authors/Year	Models	In-Sample (data, periodicity, where)?	Out-of-sample analysis?	Performance metrics	Rebalancing?	TC?	Conclusions
Konno and Yamazaki [20] 1991	MV MAD Sharpe single factor QP model.	224 stocks from NIKKEI 225. Period: 1981-1987 (monthly data)	Yes.	Average rate return, standard deviation of rate of return and sharpe ratio.	Yes.	No.	Portfolio obtained with MAD Model is quite similar to the one that it was obtained with Markowitz. The number of stocks in each portfolio is practically the same.
Simaan [33] 1997	MV MAD	Random values for returns of n stocks, normal distribution. Mean between [1.08,1.25] and standard deviation between [0.15,0.45]. The number of stocks n can be: 30, 60, 90 and 120.	No.	Mean and standard deviation of portfolio.	No.	No.	Increasing the sample size generally reduces the estimation error in both models. The estimation error of MV model is lower than the estimation error of MAD.
Mansini et al. [24] 2003	MAD m-MAD MM CVaR GMD MV	Stocks from Milan Stock Exchange. (weekly data)	Yes	Average return. Portfolio diversification.	No.	No.	Linear problems provide a reasonable diversification than that given by Markowitz. In out-of-sample analysis MAD and Markowitz generate the portfolios with the largest return but also with the largest risk.
Júdice et al. [17] 2003	MV MAD	92 European stocks (23 portuguese plus 65 international) Data period: 2/02/1998 – 23/05/2000 (daily data)	Yes.	Computational time.	Yes.	No.	Computational time: Markowitz depends of the number of stocks and Konno – number of observations. The greater the number of observations the more stable is the solution obtained (in both models).

Continues on the next page

Authors/Year	Models	In-Sample (data, periodicity, where)?	Out-of-sample analysis?	Performance metrics	Rebalancing?	TC?	Conclusions
Bower and Wentz [7] 2005	MV MAD	150 stocks from S&P500 (groups of 5 stocks) Data period: july, 1 - december, 31, 2004 (daily data)	Yes.	Expected return.	No.	No.	MV model gives a better expected return in 16 from 30 portfolios.
Karacabey [18] 2005	MV MAD	91 stocks from Istanbul Stock Exchange. Period: january 2000 - december 2004 (monthly data)	No.	Mean. Standard Deviation. Sharpe ratio.	No.	No.	MV portfolios generally over performed the MAD portfolios.
Angelelli et al. [2] 2007	MAD CVaR	Four different sets constituted by 200, 300, 400, 600 stocks, respectively. They used from Italy (152), France (224) and Germany (224) stock exchanges. Period: 1999-2001 (weekly data)	Yes.	Expected return. Standard deviation. Computational time.	No.	Yes.	Large computational time required to solve the solution of the CVaR model. (even in small instances)
Yu et al. [35] 2010	MV MAD Cai Teo	33 stocks from NIKKEI 225. Period: january 1995 - december 2000 (monthly data). 63 stocks from NIKKEI 225. Period: january 1991 - december 2000 (monthly data).	No.	Average expected returns. Computational time.	No.	No.	MV and MAD have similar values for expected return. Teo's model is influenced by the number of stocks and the number of periods we choose. However, the other three models do not show evidence that the number of stocks and periods influence the computational time to obtain the solution.

Continues on the next page

Authors/Year	Models	In-Sample (data, periodicity, where)?	Out-of-sample analysis?	Performance metrics	Rebalancing?	TC?	Conclusions
Peng et al. [27] 2010	MV	Two particular stocks from a stock market and stocks are chosen as indexes. Period: December 1979 to April 2001 (monthly data)	No.	Mean yearly rate of return. Gross assets.	No.	Yes.	MV model with transaction costs provides a significant improvement over MV model.
Hoe et al. [15] 2010	MV MAD MM SV	54 stocks from Kuala Lumpur Composite Index (Bursa Malaysia Index). Period: January 2004 - December 2007 (monthly data).	No.	Expected return. Risk. Portfolio performance (mean return/risk).	No.	No.	MV model shows a higher risk than MAD model. Mimimax was the best model compared to the other models in terms of mean return and risk.
Cesarone et al. [8] 2013	MV MSAD CVaR EW	Data from Beasley's OR-Library that include stocks from Hang Seng (31), DAX 100 (85), FTSE 100 (89), S&P100 (98), Nikkei (225), S&500 (457), Russel 2000 (1318) and Russel 3000 (2151). The authors added more data from EuroStoxx50 (47), FTSE100 (76), MIBTEL (221), S&P 500 (476) and NASDAQ (2191). Period: first data - March 1992 to September 1997; second data - March 2003 to March 2008.	Yes	Excess return. Sharpe, Rachev and Sortino ratios. Computational time.	Yes.	No.	Linear models are better than MV model in terms of computational time. MV model can be very similar in terms of efficiency of solution. The equally-weighted portfolio does not seem to have any advantage over the other three models.

Continues on the next page

Authors/Year	Models	In-Sample (data, periodicity, where)?	Out-of-sample analysis?	Performance metrics	Rebalancing?	TC?	Conclusions
Bartkus et al. [5] 2013	MV MAD MM	20 stocks from Vilnius Exchange Market (Lithuania) with the highest positive skewness. Period: January, 2, 2006 to April, 1, 2011 (daily data)	No.	Mean. Standard deviation. Skewness. Kurtosis.	No.	No.	MV portfolio is more diversified than MAD portfolio. MAD model gives the highest return.
Kasenbacher et at. [19] 2017	MV MAD	Top stocks from S&P500 divided into different groups (75, 150 and 200). Period: January, 1, 2016 to January, 1, 2017 (daily data).	Yes.	Expected return. Risk. Sharpe ratio.	Yes.	No.	MAD model gives better expected returns and lower risk. Sharpe ratio values are better in MAD model than in MV model.
Hunjra et al. [16] 2020	MV MAD SV CVaR	40 stocks from Pakistan, 92 from Bombay and 30 from Dhaka. Period: 3 different periods (crisis: 2003-2005; recovery: 2006-2011; growth period: 2012-2015). (monthly data)	No.	Mean. Variance. Skewness.	No.	No.	In Pakistan all risk measures perform better in the growth period. In Bangladesh the models perform better in the recovery period. In India, better performance appears during the crisis period. CVaR model obtain better results in all scenario.

Table A.4 Literature review summary table.

Appendix B

In-Sample performance (more details)

B.1 Expected Return, Risk and Sharpe Ratio for different groups of stocks.

B.1.1 Daily Data

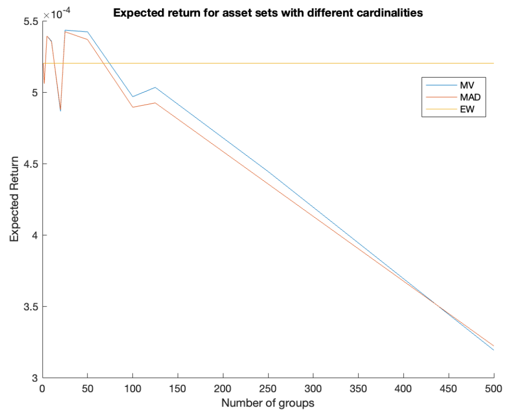


Fig. B.1 Expected return as a function of the number of groups using daily data.

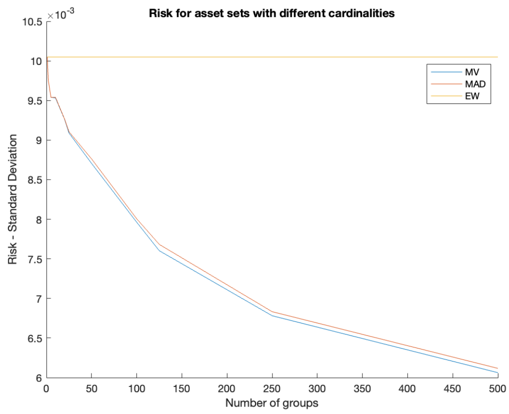


Fig. B.2 Risk as a function of the number of groups using daily data.

B.1.2 Weekly Data

Table B.1 Expected return and standard deviation of portfolios selected from different groups of stocks.

Number of groups	MV		MAD		EW	
	Exp.Return	Risk	Exp.Return	Risk	Exp.Return	Risk
1	0.2457	2.0534	0.2457	2.0534	0.2457	2.0534
2	0.2385	1.9902	0.2385	1.9902	0.2457	2.0534
5	0.2554	1.9188	0.2554	1.9188	0.2457	2.0534
10	0.2496	1.9011	0.2523	1.9112	0.2457	2.0534
20	0.2296	1.8298	0.2314	1.8318	0.2457	2.0534
25	0.2536	1.8064	0.2524	1.8107	0.2457	2.0534
50	0.2545	1.6828	0.2434	1.6937	0.2457	2.0534
100	0.2328	1.5185	0.2181	1.5410	0.2457	2.0534
125	0.2531	1.4660	0.2309	1.4985	0.2457	2.0534
250	0.2140	1.3261	0.2132	1.3637	0.2457	2.0534
500	0.1608	1.1366	0.1691	1.1751	0.2457	2.0534

Notes: See the notes of Table 4.3. The only difference is the use here of weekly data.

B.1.3 Monthly Data

Table B.2 Expected return and standard deviation of portfolios selected from different groups of stocks.

Number of groups	MV		MAD		EW	
	Exp.Return	Risk	Exp.Return	Risk	Exp.Return	Risk
1	0.9734	3.4487	0.9734	3.4487	0.9734	3.4487
2	0.9439	3.3302	0.9439	3.3302	0.9734	3.4487
5	1.0101	3.1450	1.0101	3.1450	0.9734	3.4487
10	0.9798	3.0848	0.9794	3.0847	0.9734	3.4487
20	0.9325	2.9636	0.9415	2.9707	0.9734	3.4487
25	0.9778	2.9636	0.9514	2.9919	0.9734	3.4487
50	1.0095	2.7689	0.9910	2.7755	0.9734	3.4487
100	0.9419	2.4710	1.0005	2.5152	0.9734	3.4487
125	1.0690	2.3773	1.0492	2.4722	0.9734	3.4487
250	0.9510	2.0592	0.9658	2.1404	0.9734	3.4487
500	0.8733	1.7414	0.8865	1.8559	0.9734	0.0345

Notes: See the notes of Table 4.3. The only difference is the use here of monthly data.

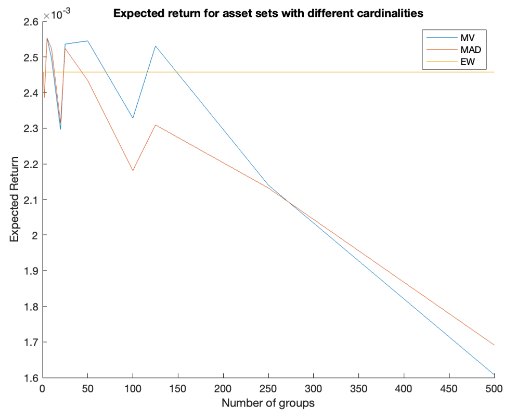


Fig. B.3 Expected return as a function of the number of groups using weekly data.

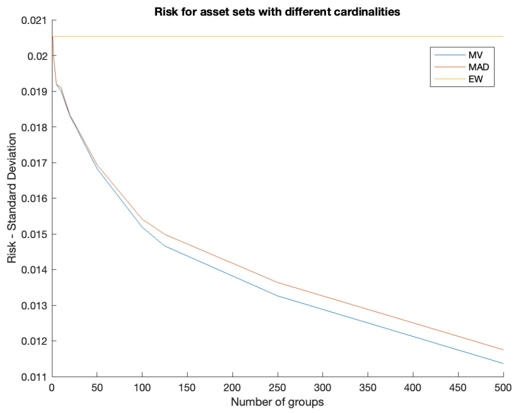


Fig. B.4 Risk as a function of the number of groups using weekly data.

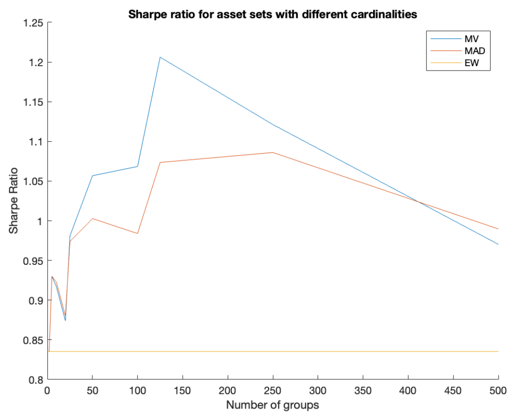


Fig. B.5 Sharpe ratio as a function of the number of groups using weekly data.

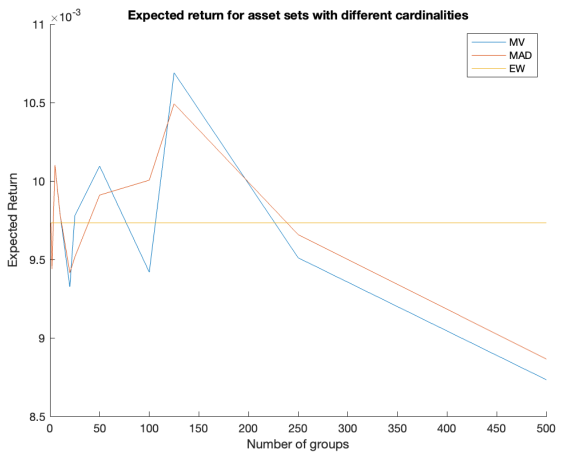


Fig. B.6 Expected return as a function of the number of groups using monthly data.

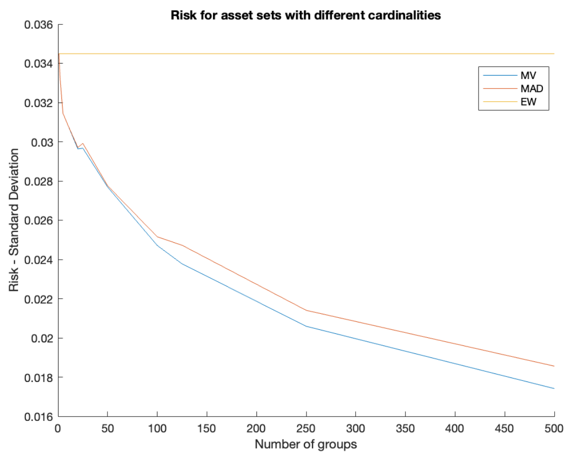


Fig. B.7 Risk as a function of the number of groups using monthly data.

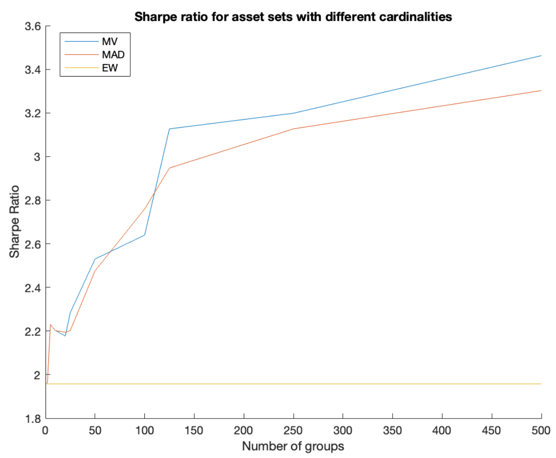
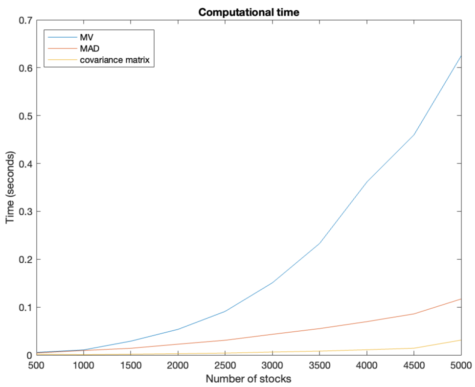


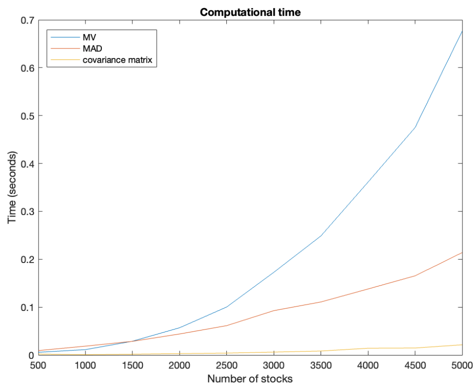
Fig. B.8 Sharpe ratio as a function of the number of groups using monthly data.

Appendix C

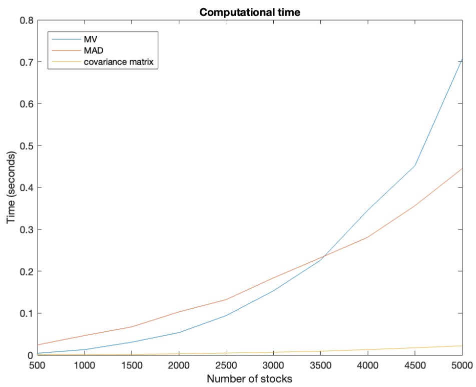
Computational time using random values



(a) 50 observations.



(b) 100 observations.



(c) 200 observations.

Fig. C.1 Computational time using 5000 random values and with different numbers of observations.