mathematics

Article

Riemann–Hilbert Problem for the Matrix Laguerre Biorthogonal Polynomials: The Matrix Discrete Painlevé IV

Amílcar Branquinho [1], Ana Foulquié Moreno [2], Assil Fradi [3] and Manuel Mañas [4,5,*]

1 Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal; ajplb@mat.uc.pt
2 Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal; foulquie@ua.pt
3 Mathematical Physics Special Functions and Applications Laboratory, The Higher School of Sciences and Technology of Hammam Sousse, University of Sousse, Sousse 4002, Tunisia; assilfradi@ua.pt
4 Departamento de Física Teórica, Universidad Complutense de Madrid, 28040 Madrid, Spain
5 Instituto de Ciencias Matematicas (ICMAT), Campus de Cantoblanco UAM, 28049 Madrid, Spain
* Correspondence: manuel.manas@ucm.es

Abstract: In this paper, the Riemann–Hilbert problem, with a jump supported on an appropriate curve on the complex plane with a finite endpoint at the origin, is used for the study of the corresponding matrix biorthogonal polynomials associated with Laguerre type matrices of weights—which are constructed in terms of a given matrix Pearson equation. First and second order differential systems for the fundamental matrix, solution of the mentioned Riemann–Hilbert problem, are derived. An explicit and general example is presented to illustrate the theoretical results of the work. The non-Abelian extensions of a family of discrete Painlevé IV equations are discussed.

Keywords: Riemann–Hilbert problems; matrix Pearson equations; matrix biorthogonal polynomials; discrete integrable systems; non-Abelian discrete Painlevé IV equation

MSC: 42C05; 15A23

check for updates

Citation: Branquinho, A.; Moreno, A.F.; Fradi, A.; Mañas, M. Riemann–Hilbert Problem for the Matrix Laguerre Biorthogonal Polynomials: The Matrix Discrete Painlevé IV. *Mathematics* **2022**, *10*, 1205. https://doi.org/10.3390/math10081205

Academic Editor: Martin Bohner

Received: 25 February 2022
Accepted: 31 March 2022
Published: 7 April 2022

1. Introduction

Mark Grigorievich Krein [1,2] was the first to discuss matrix extensions of real orthogonal polynomials. Some relevant papers that appear afterwards on this subject are [3,4] and more recently [5]. The Russian mathematicians Aptekarev and Nikishin [5] made a remarkable finding: for a kind of discrete Sturm–Liouville operators they solved the scattering problem and proved that the matrix polynomials that satisfy a three-term recurrence relation with matrix coefficients

$$xP_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + A^*_{k-1} P_{k-1}(x), \qquad k = 0, 1, \dots,$$

are orthogonal with respect to a positive definite matrix measure, i.e., they derived a matrix Favard theorem. Later, it was found that matrix orthogonal polynomials (MOP) sometimes satisfy properties, as do the classical orthogonal polynomials.

For example, for matrix versions of Laguerre, Hermite and Jacobi polynomials, the scalar-type Rodrigues' formula [6,7] and a second order differential equation [8–10] has been discussed. It also has been proven in [11] that operators of the form $D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0$ have as eigenfunctions different infinite families of MOP's. In [12,13] matrix extensions of the generalized polynomials considered in [14,15] were studied. Recently, in [16], the Christoffel transformation to matrix orthogonal polynomials in the real line (MOPRL) has been extended and a new matrix Christoffel formula was obtained. Finally, in [17,18] more general transformations—of Geronimus and Uvarov type—where also considered.

Fokas, Its and Kitaev [19] found, in the context of 2D quantum gravity, that certain Riemann–Hilbert problems were solved in terms of orthogonal polynomials in the real line

(OPRL). They found that the solution of a 2×2 Riemann–Hilbert problem can be expressed in terms of orthogonal polynomials in the real line and its Cauchy transforms. Later, Deift and Zhou combined these ideas with a non-linear steepest descent analysis in a series of papers [20–23] which was the seed for a large activity in the field. To mention just a few relevant results, let us cite the study of strong asymptotic with applications in random matrix theory [20,24], the analysis of determinantal point processes [25–28], orthogonal Laurent polynomials [29,30] and Painlevé equations [31,32].

Recursion coefficients for orthogonal polynomials and their properties is a subject of current interest. See [33,34] for a review on how the form of the weight and its properties translates to the recursion coefficients. Freud [35] has studied weights in $\mathbb{R}$ of exponential variation $w(x) = |x|^{\rho} \exp(-|x|^m)$, $\rho > -1$ and $m > 0$. When $m = 2, 4, 6$ he constructed relations among them as well as determining its asymptotic behavior. The role of the discrete Painlevé I in this context was discovered later by Magnus [36]. For a weight of the form $w(\theta) = \exp(k \cos \theta)$, $k \in \mathbb{R}$, on the unit circle the discrete Painlevé II equation for the recursion relations of the corresponding orthogonal polynomials was found in [37,38] (see also [39], for a connection with the Painlevé III equation). The discrete Painlevé II was found in [40] using the Riemann–Hilbert problem given in [41], see also [42]. For a good account of the relation of these discrete Painlevé equations and integrable systems, see [43], and for a survey on the subject of differential and discrete Painlevé equations see (cf. [44]). We also mention the recent paper [45] where a discussion on the relationship between the recurrence coefficients of orthogonal polynomials with respect to a semi-classical Laguerre weight and classical solutions of the fourth Painlevé equation can be found. Moreover, in [46], the solution of the discrete alternate Painlevé equations is presented in terms of the Airy function.

The Riemann–Hilbert problem for this matrix situation appears in the paper [47] in the context of inverse scattering for the integral matrix equation, as well as in [48] where the appearance of the Riemann–Hilbert problem for this matrix situation takes place when studying non-Abelian discrete versions of Painlevé I, showing singularity confinement [49], see also [50]. The singularity analysis for a matrix discrete version of the Painlevé I equation was performed. It was found that the singularity confinement holds generically, i.e., in the whole space of parameters except possibly for algebraic subvarieties. This situation was considered in [51] for the matrix extension of the Szegő polynomials in the unit circle and corresponding non-Abelian versions of discrete Painlevé II equations.

In [52], we discussed matrix biorthogonal polynomials with matrix of weights $W(z)$ such that

- The support of $W(z)$ is a non-intersecting smooth curve on the complex plane with end points at ∞.
- Weight matrix entries were, in principle, Hölder continuous, and eventually requested to have holomorphic extensions to the complex plane.
- The matrix of weights $W(z)$ is regular, i.e., $\det \left[W_{j+k}\right]_{j,k=0,\dots n} \neq 0$, $n \in \mathbb{N} := \{0, 1, \dots\}$, where the moment of order n, W_n, associated with W is, for each $n \in \mathbb{N}$, given by, $W_n := \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_{\gamma} z^n W(z) \,\mathrm{d}\, z$.

We obtained Sylvester systems of differential equations for the orthogonal polynomials and its second kind functions, directly from a Riemann–Hilbert problem, with jumps supported on appropriate curves on the complex plane. We considered a Sylvester type differential Pearson equation for the matrix of weights. We also studied whenever the orthogonal polynomials and their second kind functions are solutions of second order linear differential operators with matrix eigenvalues. This was achieved by stating an appropriate boundary value problem for the matrix of weights. In particular, special attention was paid to non-Abelian Hermite biorthogonal polynomials in the real line, understood as those whose matrix of weights is a solution of a Sylvester type Pearson equation with given matrices of degree one polynomial coefficients. We also found nonlinear equations for the matrix coefficients of the corresponding three-term relations, which extends to the non-commutative case the discrete Painlevé I and the alternate discrete Painlevé I equations. In this paper, we conduct a similar study but with more relaxed conditions, namely of Laguerre type.

The layout of the paper is as follows. In Section 2 we present the main definitions and theorems fundamental in the theory worked on in this paper. In Section 3 we state the Riemann–Hilbert problem for matrix biorthogonal polynomials, discussing the Pearson–Laguerre matrix weights with a finite end point, introducing the constant jump fundamental matrix and the important structure matrix. We also apply these ideas to obtain first and second order matrix differential operators of Laguerre type. In Section 4, we take a Laguerre type weight as a case study and reinterpret the results just stated for this specific and general example. Then, in Section 5 we end the paper with the finding of a matrix extension of an instance of the discrete Painlevé IV equation.

2. Preliminaries

2.1. Matrix Biorthogonal Polynomials

We begin this section with the important definition of this paper.

Definition 1 (Laguerre type Matrix of weights)**.** *We say that a regular matrix of weights*

$$W=\begin{bmatrix} W^{(1,1)} & \dots & W^{(1,N)} \\ \vdots & \ddots & \vdots \\ W^{(N,1)} & \dots & W^{(N,N)} \end{bmatrix} \in \mathbb{C}^{N\times N}$$

is of Laguerre type if:

- *The support of $W(z)$ is a non self-intersecting smooth curve on the complex plane with a beginning point at 0 and an ending point at ∞, and such that it intersects the circles $|z|=R$, $R\in(0,+\infty)$, once and only once (i.e., it can be taken as a determination curve for $\arg z$).*
- *The entries $W^{(j,k)}$ of the matrix measure W can be written as*

$$W^{(j,k)}(z)=\sum_{m\in I_{j,k}} A_m(z) z^{\alpha_m}\log^{p_m} z, \qquad z\in\gamma, \tag{1}$$

  where $I_{j,k}$ denotes a finite set of indexes, $\operatorname{Re}(\alpha_m)>-1$, $p_m\in\mathbb{N}\cup\{0\}$ and $A_m(z)$ is Hölder continuous and bounded. Here, the determination of logarithm and the powers are taken along γ. We will request, in the development of the theory, that the functions A_m have a holomorphic extension to the whole complex plane.
- *The matrix of weights $W(z)$ is regular, i.e., $\det\left[W_{j+k}\right]_{j,k=0,\dots n}\neq 0$, $n\in\mathbb{N}:=\{0,1,\dots\}$, where the moment of order n, W_n, associated with W is, for each $n\in\mathbb{N}$, given by, $W_n := \dfrac{1}{2\pi\,\mathrm{i}}\displaystyle\int_\gamma z^n W(z)\,\mathrm{d}\,z$.*

In this work, for the sake of simplicity, $\gamma=(0,+\infty)$ and the finite end point of the curve γ is taken at the origin, $c=0$, with no loss of generality, as similar arguments apply for $c\neq 0$. In [10], different examples of Laguerre weights for the matrix orthogonal polynomials on the real line are studied.

Given a Laguerre type matrix of weights, as in Definition 1, we define the sequences of matrix monic polynomials, $\{P_n^{\mathsf{L}}(z)\}_{n\in\mathbb{Z}_+}$, where $\deg P_n^{\mathsf{L}}(z)=n$, $n\in\mathbb{N}$, left orthogonal and right orthogonal, $\{P_n^{\mathsf{R}}(z)\}_{n\in\mathbb{N}}$, where $\deg P_n^{\mathsf{R}}(z)=n$, $n\in\mathbb{N}$, with respect to the regular matrix of measure W, by the conditions,

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma P_n^{\mathsf{L}}(z)W(z)z^k\,\mathrm{d}\,z=\delta_{n,k}C_n^{-1}, \tag{2}$$

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma z^k W(z)P_n^{\mathsf{R}}(z)\,\mathrm{d}\,z=\delta_{n,k}C_n^{-1}, \tag{3}$$

for $k = 0, 1, \ldots, n$ and $n \in \mathbb{N}$, where C_n is a nonsingular matrix.

We can see that the sequence of monic polynomials $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ are defined by (2) with respect to a regular matrix weight, W. In fact, taking into account a representation for P_n^{L} as

$$P_n^{\mathsf{L}}(z) = p_{\mathsf{L},n}^0 z^n + p_{\mathsf{L},n}^1 z^{n-1} + \cdots + p_{\mathsf{L},n}^{n-1} z + p_{\mathsf{L},n}^n,$$

such that, for each $j = 0, 1, \ldots, n-1$

$$\int_\gamma P_n^{\mathsf{L}}(z) W(z) z^j \, \mathrm{d}z = p_{\mathsf{L},n}^0 W_{n+j} + p_{\mathsf{L},n}^1 W_{n+j-1} + \cdots + p_{\mathsf{L},n}^{n-1} W_{j+1} + p_{\mathsf{L},n}^n W_j = 0,$$

and with $j = n$

$$\int_\gamma P_n^{\mathsf{L}}(z) W(z) z^n \, \mathrm{d}z = p_{\mathsf{L},n}^0 W_{2n} + p_{\mathsf{L},n}^1 W_{2n-1} + \cdots + p_{\mathsf{L},n}^{n-1} W_{n+1} + p_{\mathsf{L},n}^n W_n = C_n^{-1}.$$

Let us notice that

$$\mathbf{U}_n = \left[W_{j+k}\right]_{j,k=0,\ldots,n} = \begin{bmatrix} W_0 & \cdots & W_n \\ \vdots & \ddots & \vdots \\ W_n & \cdots & W_{2n} \end{bmatrix} \quad \text{is such that} \quad \det \mathbf{U}_n \neq 0, \quad n \in \mathbb{N}.$$

In matrix notation, we have

$$\begin{bmatrix} p_{\mathsf{L},n}^n & p_{\mathsf{L},n}^{n-1} & \cdots & p_{\mathsf{L},n}^1 & p_{\mathsf{L},n}^0 \end{bmatrix} \mathbf{U}_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & C_n^{-1} \end{bmatrix}.$$

Since $\det \mathbf{U}_n \neq 0$, we know that the above linear system has a unique solution, i.e., there exists and are unique the matrices $p_{\mathsf{L},n}^n, p_{\mathsf{L},n}^{n-1}, \ldots, p_{\mathsf{L},n}^1, p_{\mathsf{L},n}^0$, and so the sequence $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{Z}_+}$ is uniquely defined up to a multiplicative nonsingular matrix defined by (2).

As a direct consequence of the non-singularity of the last block of U_n^{-1}, i.e., the one in the position $(n+1), (n+1)$, of the matrix $\mathbf{U}_n^{-1}$ we find that $p_{\mathsf{L},n}^0$ is a non singular matrix. In fact, as (see for instance [53])

$$\mathbf{U}_n^{-1} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with

$$D = \left(W_{2n} - \begin{bmatrix} W_n & \cdots & W_{2n-1} \end{bmatrix} \mathbf{U}_{n-1}^{-1} \begin{bmatrix} W_n^\top & \cdots & W_{2n-1}^\top \end{bmatrix}^\top\right)^{-1},$$

and $\det D = \frac{\det \mathbf{U}_{n-1}}{\det \mathbf{U}_n}$, we get the non singularity of $p_{\mathsf{L},n}^0$. The same can be seen for $\{P_n^{\mathsf{R}}\}_{n\in\mathbb{N}}$.

The matrix of weights $W(z)$ induces a non-degenerate bilinear form in the set of matrix polynomials $\mathbb{C}^{N\times N}[z]$ given by

$$\langle P, Q \rangle_W := \frac{1}{2\pi \mathrm{i}} \int_\gamma P(z) W(z) Q(z) \, \mathrm{d}\, z, \tag{4}$$

for which $\{P_n^{\mathsf{L}}(z)\}_{n\in\mathbb{N}}$ and $\{P_n^{\mathsf{R}}(z)\}_{n\in\mathbb{N}}$ are biorthogonal

$$\langle P_n^{\mathsf{L}}, P_m^{\mathsf{R}} \rangle_W = \delta_{n,m} C_n^{-1}, \qquad n, m \in \mathbb{N}.$$

As the polynomials are chosen to be monic, we can write

$$P_n^{\mathsf{L}}(z) = I_N z^n + p_{\mathsf{L},n}^1 z^{n-1} + p_{\mathsf{L},n}^2 z^{n-2} + \cdots + p_{\mathsf{L},n}^n,$$
$$P_n^{\mathsf{R}}(z) = I_N z^n + p_{\mathsf{R},n}^1 z^{n-1} + p_{\mathsf{R},n}^2 z^{n-2} + \cdots + p_{\mathsf{R},n}^n,$$

with matrix coefficients $p^k_{\mathsf{L},n}, p^k_{\mathsf{R},n} \in \mathbb{C}^{N\times N}$, $k = 0,\dots,n$ and $n \in \mathbb{N}$ (imposing that $p^0_{\mathsf{L},n} = p^0_{\mathsf{R},n} = I_N$, $n \in \mathbb{N}$). Here, $I_N \in \mathbb{C}^{N\times N}$ denotes the identity matrix.

We define the sequence of second kind matrix functions by

$$Q^{\mathsf{L}}_n(z) := \frac{1}{2\pi\,\mathrm{i}}\int_\gamma \frac{P^{\mathsf{L}}_n(z')}{z'-z} W(z')\,\mathrm{d}\,z', \qquad Q^{\mathsf{R}}_n(z) := \frac{1}{2\pi\,\mathrm{i}}\int_\gamma W(z')\frac{P^{\mathsf{R}}_n(z')}{z'-z}\,\mathrm{d}\,z',$$

for $n \in \mathbb{N}$. From the orthogonality conditions (2) and (3), we have, for all $n \in \mathbb{N}$, the following asymptotic expansion near infinity

$$Q^{\mathsf{L}}_n(z) \sim -C_n^{-1}\big(I_N z^{-n-1} + q^1_{\mathsf{L},n} z^{-n-2} + \cdots\big), \qquad |z| \to \infty, \tag{5}$$

$$Q^{\mathsf{R}}_n(z) \sim -\big(I_N z^{-n-1} + q^1_{\mathsf{R},n} z^{-n-2} + \cdots\big)C_n^{-1}, \qquad |z| \to \infty. \tag{6}$$

2.2. Three-Term Recurrence Relation

Following standard arguments of orthogonality, we conclude that the sequence of monic polynomials $\big\{P^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{N}}$ satisfies the three-term recurrence relations

$$zP^{\mathsf{L}}_n(z) = P^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n P^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n P^{\mathsf{L}}_{n-1}(z), \qquad n \in \mathbb{N},$$

$$zQ^{\mathsf{L}}_n(z) = Q^{\mathsf{L}}_{n+1}(z) + \beta^{\mathsf{L}}_n Q^{\mathsf{L}}_n(z) + \gamma^{\mathsf{L}}_n Q^{\mathsf{L}}_{n-1}(z), \qquad n \in \mathbb{N},$$

with recursion coefficients given by $\beta^{\mathsf{L}}_n := p^1_{\mathsf{L},n} - p^1_{\mathsf{L},n+1}$ and $\gamma^{\mathsf{L}}_n := C_n^{-1}C_{n-1}$, with initial conditions,

$$P^{\mathsf{L}}_{-1} = 0_N, \quad P^{\mathsf{L}}_0 = I_N, \quad Q^{\mathsf{L}}_{-1}(z) = -C^{-1}_{-1} \quad \text{and} \quad Q^{\mathsf{L}}_0(z) = S_W(z) := \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{W(z')}{z'-z}\,\mathrm{d}z',$$

where $S_W(z)$ is the Stieltjes–Markov transformation. Analogously,

$$zP^{\mathsf{R}}_n(z) = P^{\mathsf{R}}_{n+1}(z) + P^{\mathsf{R}}_n(z)\beta^{\mathsf{R}}_n + P^{\mathsf{R}}_{n-1}(z)\gamma^{\mathsf{R}}_n, \qquad n \in \mathbb{N},$$

$$zQ^{\mathsf{R}}_n(z) = Q^{\mathsf{R}}_{n+1}(z) + Q^{\mathsf{R}}_n(z)\beta^{\mathsf{R}}_n + Q^{\mathsf{R}}_{n-1}(z)\gamma^{\mathsf{R}}_n, \qquad n \in \mathbb{N},$$

where $\beta^{\mathsf{R}}_n := C_n\beta^{\mathsf{L}}_n C_n^{-1}$, $\gamma^{\mathsf{R}}_n := C_n\gamma^{\mathsf{L}}_n C_n^{-1} = C_{n-1}C_n^{-1}$, $Q^{\mathsf{R}}_{-1}(z) = -C^{-1}_{-1}$ and $Q^{\mathsf{R}}_0(z) = S_W(z)$.

These relations could be written as,

$$Y^{\mathsf{L}}_{n+1}(z) = T^{\mathsf{L}}_n(z)Y^{\mathsf{L}}_n(z), \qquad T^{\mathsf{L}}_n(z) := \begin{bmatrix} zI_N - \beta^{\mathsf{L}}_n & C_n^{-1} \\ -C_n & 0_N \end{bmatrix}, \qquad n \in \mathbb{N},$$

where T^{L}_n denotes the left transfer matrix. For the right orthogonality, we similarly obtain,

$$Y^{\mathsf{R}}_{n+1}(z) = Y^{\mathsf{R}}_n(z)T^{\mathsf{R}}_n(z), \qquad T^{\mathsf{R}}_n(z) := \begin{bmatrix} zI_N - \beta^{\mathsf{R}}_n & -C_n \\ C_n^{-1} & 0_N \end{bmatrix}, \qquad n \in \mathbb{N},$$

where T^{L}_n denotes the right transfer matrix.

2.3. Reductions: From Biorthogonality to Orthogonality

We consider two possible reductions for the matrix of weights, the symmetric reduction and the Hermitian reduction. (i) A matrix of weights $W(z)$ with support on γ is said to be symmetric if

$$(W(z))^\top = W(z), \qquad z \in \gamma.$$

(ii) A matrix of weights $W(x)$ with support on $\mathbb{R}$ is said to be Hermitian if

$$(W(x))^\dagger = W(x), \qquad x \in \mathbb{R}.$$

These two reductions lead to orthogonal polynomials, as the two biorthogonal families are identified; i.e., for the symmetric case

$$P_n^R(z) = \left(P_n^{\mathsf{L}}(z)\right)^\top, \qquad Q_n^{\mathsf{R}}(z) = \left(Q_n^{\mathsf{L}}(z)\right)^\top, \qquad z \in \mathbb{C},$$

and for the Hermitian case, with $\gamma = \mathbb{R}$

$$P_n^{\mathsf{R}}(z) = \left(P_n^{\mathsf{L}}(\bar{z})\right)^\dagger, \qquad Q_n^{\mathsf{R}}(z) = \left(Q_n^{\mathsf{L}}(\bar{z})\right)^\dagger, \qquad z \in \mathbb{C}.$$

In both cases, biorthogonality collapses into orthogonality, that for the symmetric case reads as

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma P_n(z)W(z)(P_m(z))^\top \mathrm{d}z = \delta_{n,m}C_n^{-1}, \qquad n,m \in \mathbb{N},$$

while for the Hermitian case it can be written as follows

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} P_n(x)W(x)(P_m(x))^\dagger \mathrm{d}x = \delta_{n,m}C_n^{-1}, \qquad n,m \in \mathbb{N},$$

where $P_n = P_n^{\mathsf{L}}$.

3. Riemann–Hilbert Problem for Matrix Biorthogonal Polynomials

3.1. The Riemann–Hilbert Problem

We begin this section stating a general theorem on the Riemann–Hilbert problem for the matrix Laguerre general weights. A preliminary version of this can be found in [54].

Theorem 1. *Given a regular Laguerre type matrix of weights $W(x)$ with support on γ, we have:*

(i) *The matrix function*

$$Y_n^{\mathsf{L}}(z) := \begin{bmatrix} P_n^{\mathsf{L}}(z) & Q_n^{\mathsf{L}}(z) \\ -C_{n-1}P_{n-1}^{\mathsf{L}}(z) & -C_{n-1}Q_{n-1}^{\mathsf{L}}(z) \end{bmatrix}$$

is, for each $n \in \mathbb{N}$, the unique solution of the Riemann–Hilbert problem, which consists in the determination of a $2N \times 2N$ complex matrix function such that:

(RHL1) *$Y_n^{\mathsf{L}}(z)$ is holomorphic in $\mathbb{C} \setminus \gamma$.*

(RHL2) *Has the following asymptotic behavior near infinity,*

$$Y_n^{\mathsf{L}}(z) \sim \left(I_{2N} + \sum_{j=1}^{\infty}(z^{-j})Y_n^{j,\mathsf{L}}\right)\begin{bmatrix} I_N z^n & 0_N \\ 0_N & I_N z^{-n} \end{bmatrix}.$$

(RHL3) *Satisfies the jump condition*

$$\left(Y_n^{\mathsf{L}}(z)\right)_+ = \left(Y_n^{\mathsf{L}}(z)\right)_- \begin{bmatrix} I_N & W(z) \\ 0_N & I_N \end{bmatrix}, \qquad z \in \gamma \setminus \{0\}.$$

(RHL4) $Y_n^{\mathsf{L}}(z) = \begin{bmatrix} \mathrm{O}(1) & s_1^{\mathsf{L}}(z) \\ \mathrm{O}(1) & s_2^{\mathsf{L}}(z) \end{bmatrix}$*, as $z \to 0$, $\lim_{z\to 0} z s_j^{\mathsf{L}}(z) = 0_N$, $j = 1,2$ and the* O *conditions are understood entrywise.*

(ii) *The matrix function*

$$Y_n^{\mathsf{R}}(z) := \begin{bmatrix} P_n^{\mathsf{R}}(z) & -P_{n-1}^{\mathsf{R}}(z)C_{n-1} \\ Q_n^{\mathsf{R}}(z) & -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} \end{bmatrix}$$

is, for each $n \in \mathbb{N}$, the unique solution of the Riemann–Hilbert problem, which consists of the determination of a $2N \times 2N$ complex matrix function such that:

(RHR1) $Y_n^{\mathsf{R}}(z)$ *is holomorphic in* $\mathbb{C} \setminus \gamma$.

(RHR2) *Has the following asymptotic behavior near infinity,*

$$Y_n^{\mathsf{R}}(z) \sim \begin{bmatrix} I_N z^n & 0_N \\ 0_N & I_N z^{-n} \end{bmatrix} \Big(I_{2N} + \sum_{j=1}^{\infty} (z^{-j}) Y_n^{j,\mathsf{R}} \Big).$$

(RHR3) *Satisfies the jump condition*

$$\big(Y_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} I_N & 0_N \\ W(z) & I_N \end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)_-, \qquad z \in \gamma \setminus \{0\}.$$

(RHR4) $Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(1) \\ s_1^{\mathsf{R}}(z) & s_2^{\mathsf{R}}(z) \end{bmatrix}$, *as* $z \to 0$, $\lim_{z\to 0} z s_j^{\mathsf{R}}(z) = 0_N$, $j = 1,2$ *and the* O *conditions are understood entrywise.*

(iii) The determinant of $Y_n^{\mathsf{L}}(z)$ *and* $Y_n^{\mathsf{R}}(z)$ *are both equal to* 1*, for every* $z \in \mathbb{C}$.

Proof. Using the standard calculations from the scalar case [55], it follows that the matrices Y_n^{L} and Y_n^{R} satisfy (RHL1)–(RHL3) and (RHR1)–(RHR3), respectively.

The entries $W^{j,k}$ of the matrix measure W are given in (1). It holds (cf. [56]) that in a neighborhood of $z = 0$ the Cauchy transform

$$\phi_m(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{p(\zeta) A_m(\zeta) \zeta^{\alpha_m} \log^{p_m} \zeta}{\zeta - z} \, \mathrm{d}\,\zeta,$$

where $p(\zeta)$ denotes any polynomial in ζ, that satisfies $\lim_{z\to 0} z\phi_m(z) = 0$. Then, (RHL4) and (RHR4) are fulfilled by the matrices $Y_n^{\mathsf{L}}, Y_n^{\mathsf{R}}$, respectively. To prove the unicity of both Riemann–Hilbert problems let us consider the matrix function

$$G(z) = Y_n^{\mathsf{L}}(z) \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix}.$$

It can easily be proved that $G(z)$ has no jump or discontinuity on the curve γ and that its behavior at the end point 0 is given by

$$G(z) \sim \begin{bmatrix} \mathrm{O}(1)s_1^{\mathsf{L}}(z) + \mathrm{O}(1)s_2^{\mathsf{R}}(z) & \mathrm{O}(1)s_1^{\mathsf{L}}(z) + \mathrm{O}(1)s_1^{\mathsf{R}}(z) \\ \mathrm{O}(1)s_2^{\mathsf{L}}(z) + \mathrm{O}(1)s_2^{\mathsf{R}}(z) & \mathrm{O}(1)s_2^{\mathsf{L}}(z) + \mathrm{O}(1)s_1^{\mathsf{R}}(z) \end{bmatrix}, \qquad z \to 0,$$

so it holds that $\lim_{z\to 0} zG(z) = 0$ and we conclude that the end point 0 is a removable singularity of G. Now, from the behavior for $z \to \infty$,

$$G(z) \sim \begin{bmatrix} I_N z^n & 0_N \\ 0_N & I_N z^{-n} \end{bmatrix} \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} \begin{bmatrix} I_N z^n & 0_N \\ 0_N & I_N z^{-n} \end{bmatrix} \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix} = \begin{bmatrix} I_N & 0_N \\ 0_N & I_N \end{bmatrix},$$

hence the Liouville theorem implies that $G(z) = I_{2N}$. To prove the unicity of the solution of (RHL1)–(RHL3) and (RHR1)–(RHR3) let $\widetilde{Y}_n^{\mathsf{L}}$ be any solution of the left Riemann–Hilbert problem. Then

$$\widetilde{Y}_n^{\mathsf{L}}(z) = \left(\begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix} \right)^{-1}.$$

Hence, any solution of this left Riemann–Hilbert problem is equal to the inverse of a fixed matrix, and the uniqueness follows. We obtain the uniqueness of the solution of the right Riemann–Hilbert in a similar way.

Let us calculate the determinant of the fundamental matrix. Since

$$\det(Y^{\mathsf{L}}_{n+1}) = \det\left(\begin{bmatrix} zI_N - \beta^{\mathsf{L}}_n & C_n^{-1} \\ -C_n & 0_N \end{bmatrix}\right) \det(Y^{\mathsf{L}}_n(z)) = \det(Y^{\mathsf{L}}_n), \qquad \forall n \in \mathbb{N}.$$

Then

$$\det\left(Y^{\mathsf{L}}_n\right) = \det\left(Y^{\mathsf{L}}_0\right) = \det\begin{bmatrix} P^{\mathsf{L}}_0(z) & Q^{\mathsf{L}}_0(z) \\ -C_{-1}P^{\mathsf{L}}_{-1}(z) & -C_{-1}Q^{\mathsf{L}}_{-1}(z) \end{bmatrix} = 1.$$

A similar reasoning leads to a similar result for Y^{R}_n. □

Corollary 1. *It holds that*

$$\left(Y^{\mathsf{L}}_n(z)\right)^{-1} = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} Y^{\mathsf{R}}_n(z) \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix},$$

that entrywise reads as follows

$$Q^{\mathsf{L}}_n(z)P^{\mathsf{R}}_{n-1}(z) - P^{\mathsf{L}}_n(z)Q^{\mathsf{R}}_{n-1}(z) = C^{-1}_{n-1}, \tag{7}$$
$$P^{\mathsf{L}}_{n-1}(z)Q^{\mathsf{R}}_n(z) - Q^{\mathsf{L}}_{n-1}(z)P^{\mathsf{R}}_n(z) = C^{-1}_{n-1}, \tag{8}$$
$$Q^{\mathsf{L}}_n(z)P^{\mathsf{R}}_n(z) - P^{\mathsf{L}}_n(z)Q^{\mathsf{R}}_n(z) = 0. \tag{9}$$

3.2. Pearson–Laguerre Matrix Weights with a Finite End Point

Instead of a given matrix of weights, we consider two matrices of entire functions, say $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$, such that the following matrix Pearson equations are satisfied

$$z(W^{\mathsf{L}})'(z) = h^{\mathsf{L}}(z)W^{\mathsf{L}}(z), \qquad z(W^{\mathsf{R}})'(z) = W^{\mathsf{R}}(z)h^{\mathsf{R}}(z), \tag{10}$$

and, given solutions to them, we construct the corresponding matrix of weights as $W = W^{\mathsf{L}}W^{\mathsf{R}}$. This matrix of weights is also characterized by a Pearson equation.

Proposition 1 (Pearson differential equation)**.** *Given two matrices of entire functions $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$. A solution of the Sylvester type matrix differential equation, which we call a Pearson equation for the weight, W,*

$$zW'(z) = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z), \tag{11}$$

is of the form $W = W^{\mathsf{L}}W^{\mathsf{R}}$ where the factor matrices W^{L} and W^{R} are solutions of (10).

Proof. Given solutions W^{L} and W^{R} of (10), it follows immediately, just using the Leibniz law for derivatives, that $W = W^{\mathsf{L}}W^{\mathsf{R}}$ fulfills (11). Moreover, given a solution W of (11) we pick a solution W^{L} of the first equation in (10), then it is easy to see that $(W^{\mathsf{L}})^{-1}W$ satisfies the second equation in (10). □

We can give the following result from the literature [57].

Theorem 2 (Solution at a regular singular point)**.** *Let $h^{\mathsf{L}}(z)$ be an entire matrix function. Then, for the solutions of the Pearson Equation* (10) *we have:*

(i) *If $A^{\mathsf{L}} := h^{\mathsf{L}}(0)$ has no eigenvalues that differ from each other by positive integers then, the solution of the left matrix differential equation in (10) can be written as*

$$W^{\mathsf{L}}(z) = H^{\mathsf{L}}(z) z^{A^{\mathsf{L}}} W_0^{\mathsf{L}},$$

where $H^{\mathsf{L}}(z)$ is an entire and nonsingular matrix function such that $H^{\mathsf{L}}(0) = I_N$, and W_0^{L} is a constant nonsingular matrix.

(ii) *If the matrix function A^{L} has eigenvalues that differ from each other by positive integers, then the solution of the left matrix differential equation in (10) can be written as*

$$W^{\mathsf{L}}(z) = H^{\mathsf{L}}(z) z^{\tilde{A}^{\mathsf{L}}} W_0^{\mathsf{L}},$$

where, in this case,

$$H^{\mathsf{L}}(z) = \widetilde{S}^{\mathsf{L}}(z) \Pi^{\mathsf{L}}(z),$$

and $\widetilde{S}^{\mathsf{L}}(z)$ is a finite product of factors of the form $T_i S_i^{\mathsf{L}}(z)$, with T_i a nonsingular matrix and $S_i^{\mathsf{L}}(z)$ is a shearing matrix, i.e., a matrix given by blocks as

$$S_i^{\mathsf{L}}(z) = \begin{bmatrix} I_{n_i} & 0 \\ 0 & z I_{m_i} \end{bmatrix},$$

for some positive integers n_i, m_i, and $\Pi^{\mathsf{L}}(z)$ is an entire and non singular matrix function such that $\Pi^{\mathsf{L}}(0) = I$, $\tilde{A}^{\mathsf{L}}$ is a constant matrix built from the matrix A^{L}, where the eigenvalues of this matrix are decreased in such a way that the eigenvalues of the resulting matrix do not differ by a positive integer and W_0^{L} is a constant nonsingular matrix.

We can obtain analogous results for the right matrix differential equation in (10) and we will denote the solution as

$$W^{\mathsf{R}}(z) = W_0^{\mathsf{R}} z^{A^{\mathsf{R}}} H^{\mathsf{R}}(z).$$

Notice that given a matrix A, and the oriented curve γ, the matrix of functions $z^A = \mathrm{e}^{A \log z}$ is a matrix of holomorphic functions in $\mathbb{C} \setminus \gamma$, and

$$\left(z^A\right)_- = \left(z^A\right)_+ \mathrm{e}^{2\pi i A} = \mathrm{e}^{2\pi i A} \left(z^A\right)_+, \qquad z \in \gamma.$$

We also adopt the convention that $\left(W^{\mathsf{L}}(z) W^{\mathsf{R}}(z)\right)_+ = W(z)$, i.e., the matrix of weight is obtained from the limit behavior of the right side of the curve γ of the matrix function $W^{\mathsf{L}}(z) W^{\mathsf{R}}(z)$.

It is necessary, in order to consider the Riemann–Hilbert problem related to the matrix of weights W satisfying (11), to study the behavior of $W(z)$ around the origin. For that aim, let us consider J, the Jordan matrix similar to the matrix A, i.e., there exists an nonsingular matrix P such that $A = PJP^{-1}$. It holds $z^A = P z^J P^{-1}$ so if

$$J = (\lambda_1 I_{m_1} + N_1) \oplus (\lambda_2 I_{m_2} + N_2) \oplus \cdots \oplus (\lambda_s I_{m_s} + N_s)$$

where m_k is the order of the nilpotent matrix N_k, we have that

$$z^J = z^{\lambda_1 I_{m_1} + N_1} \oplus z^{\lambda_2 I_{m_2} + N_2} \oplus \cdots \oplus z^{\lambda_s I_{m_s} + N_s}$$

where $z^{\lambda_k I_{m_k} + N_k} = z^{\lambda_k I_{m_k}} z^{N_k}$. It is straightforward that $z^{\lambda_k I_{m_k}} = z^{\lambda_k} I_{m_k}$ and

$$z^{N_k} = \mathrm{e}^{N_k \log z} = I_{m_k} + \log z N_k + \frac{\log^2 z}{2!} N_k^2 + \cdots + \frac{\log^{m_k - 1} z}{(m_k - 1)!} N_k^{m_k - 1},$$

where we have used the nilpotency of $N_k^j = 0_N$ for $j \geq m_k$. So we can conclude that the entries of z^A are linear combinations of z^{λ_j} with polynomials coefficients in the variable $\log z$.

3.3. Constant Jump Fundamental Matrix

According to the above notation and given a regular matrix of weights as described in (11), we introduce the constant jump fundamental matrices

$$Z_n^{\mathsf{L}}(z) := Y_n^{\mathsf{L}}(z) \begin{bmatrix} W^{\mathsf{L}}(z) & 0_N \\ 0_N & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix}, \tag{12}$$

$$Z_n^{\mathsf{R}}(z) := \begin{bmatrix} W^{\mathsf{R}}(z) & 0_N \\ 0_N & (W^{\mathsf{L}}(z))^{-1} \end{bmatrix} Y_n^{\mathsf{R}}(z), \tag{13}$$

for $n \in \mathbb{N}$.

Proposition 2. *The constant jump fundamental matrices $Z_n^{\mathsf{L}}(z)$ and $Z_n^{\mathsf{R}}(z)$ satisfy, for each $n \in \mathbb{N}$, the following properties:*

(i) *They are holomorphic on $\mathbb{C} \setminus \gamma$.*

(ii) *Present the following constant jump condition on γ*

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}} & (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}} \\ 0_N & W_0^{\mathsf{R}}\,\mathrm{e}^{2\pi i A^{\mathsf{R}}}(W_0^{\mathsf{R}})^{-1} \end{bmatrix},$$

$$\big(Z_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} W_0^{\mathsf{R}}\,\mathrm{e}^{-2\pi i A^{\mathsf{R}}}(W_0^{\mathsf{R}})^{-1} & 0_N \\ W_0^{\mathsf{R}}\,\mathrm{e}^{-2\pi i A^{\mathsf{R}}}(W_0^{\mathsf{R}})^{-1} & {W_0^{\mathsf{L}}}^{-1}\,\mathrm{e}^{2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}} \end{bmatrix} \big(Z_n^{\mathsf{R}}(z)\big)_-,$$

for all $z \in \gamma$.

Proof. (i) The holomorphic properties of Z_n^{L} are inherited from those of the fundamental matrices Y_n^{L} and z^A and taking into account that $H^{\mathsf{L}}(z)$ is an entire matrix function.

(ii) From the definition of $Z_n^{\mathsf{L}}(z)$, we have

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_+ \begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)_+ & 0_N \\ 0_N & \big(W^{\mathsf{R}}(z)\big)_+^{-1} \end{bmatrix},$$

and taking into account Theorem 1 we successively obtain

$$\begin{aligned} \big(Z_n^{\mathsf{L}}(z)\big)_+ &= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} I_N & \big(W^{\mathsf{L}}(z)W^{\mathsf{R}}(z)\big)_+ \\ 0_N & I_N \end{bmatrix} \begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)_+ & 0_N \\ 0_N & \big(W^{\mathsf{R}}(z)\big)_+^{-1} \end{bmatrix} \\ &= \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)_- & 0_N \\ 0_N & \big(W^{\mathsf{R}}(z)\big)_-^{-1} \end{bmatrix} \begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)_-^{-1} & 0_N \\ 0_N & \big(W^{\mathsf{R}}(z)\big)_- \end{bmatrix} \\ &\qquad\qquad \times \begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)_+ & \big(W^{\mathsf{L}}(z)\big)_+ \\ 0_N & \big(W^{\mathsf{R}}(z)\big)_+^{-1} \end{bmatrix} \\ &= \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} \big(W^{\mathsf{L}}(z)\big)_-^{-1}\big(W^{\mathsf{L}}(z)\big)_+ & \big(W^{\mathsf{L}}(z)\big)_-^{-1}\big(W^{\mathsf{L}}(z)\big)_+ \\ 0_N & \big(W^{\mathsf{R}}(z)\big)_-\big(W^{\mathsf{R}}(z)\big)_+^{-1} \end{bmatrix} \\ &= \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}} & (W_0^{\mathsf{L}})^{-1}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,A^{\mathsf{L}}}\,W_0^{\mathsf{L}} \\ 0_N & W_0^{\mathsf{R}}\,\mathrm{e}^{2\pi\,\mathrm{i}\,A^{\mathsf{L}}}(W_0^{\mathsf{R}})^{-1} \end{bmatrix}. \end{aligned}$$

Hence, we obtain the desired constant jump condition for $Z_n^{\mathsf{L}}(z)$.

To complete the proof we only have to check that

$$Z^{\mathsf{R}}_n(z) = \begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix} (Z^{\mathsf{L}}_n(z))^{-1} \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}, \tag{14}$$

which is a consequence of (13). □

3.4. Structure Matrix and Zero Curvature Formula

In parallel to the matrices $Z^{\mathsf{L}}_n(z)$ and $Z^{\mathsf{R}}_n(z)$, for each factorization we introduce what we call structure matrices given in terms of the left, respectively right, logarithmic derivatives by,

$$M^{\mathsf{L}}_n(z) := (Z^{\mathsf{L}}_n)'(z)(Z^{\mathsf{L}}_n(z))^{-1}, \qquad M^{\mathsf{R}}_n(z) := (Z^{\mathsf{R}}_n(z))^{-1}(Z^{\mathsf{R}}_n)'(z). \tag{15}$$

It is not difficult to prove that

$$M^{\mathsf{R}}_n(z) = -\begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix} M^{\mathsf{L}}_n(z) \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}, \qquad n \in \mathbb{N}. \tag{16}$$

Proposition 3 ([52])**.** *We have the following properties:*

(i) *The transfer matrices satisfy*

$$T^{\mathsf{L}}_n(z)Z^{\mathsf{L}}_n(z) = Z^{\mathsf{L}}_{n+1}(z), \qquad Z^{\mathsf{R}}_n(z)T^{\mathsf{R}}_n(z) = Z^{\mathsf{R}}_{n+1}(z), \qquad n \in \mathbb{N}.$$

(ii) *The zero curvature formulas*

$$\begin{bmatrix} I_N & 0_N \\ 0_N & 0_N \end{bmatrix} = M^{\mathsf{L}}_{n+1}(z)T^{\mathsf{L}}_n(z) - T^{\mathsf{L}}_n(z)M^{\mathsf{L}}_n(z),$$
$$\begin{bmatrix} I_N & 0_N \\ 0_N & 0_N \end{bmatrix} = T^{\mathsf{R}}_n(z)\, M^{\mathsf{R}}_{n+1}(z) - M^{\mathsf{R}}_n(z)T^{\mathsf{R}}_n(z),$$

$n \in \mathbb{N}$, are fulfilled.

Now, we discuss the holomorphic properties of the structure matrices just introduced.

Theorem 3. *The structure matrices $M^{\mathsf{L}}_n(z)$ and $M^{\mathsf{R}}_n(z)$, cf. (15), are, for each $n \in \mathbb{N}$, meromorphic on $\mathbb{C}$, with singularity located at $z = 0$, which happens to be a removable singularity or a simple pole.*

Proof. Let us prove the statement for $M^{\mathsf{L}}_n(z)$, for $M^{\mathsf{R}}_n(z)$ one should proceed similarly. From (15) it follows that $M^{\mathsf{L}}_n(z)$ is holomorphic in $\mathbb{C} \setminus \gamma$. Due to the fact that $Z^{\mathsf{L}}_n(z)$ has a constant jump on the curve γ, the matrix function $(Z^{\mathsf{L}}_n)'$ has the same constant jump on the curve γ, so the matrix $M^{\mathsf{L}}_n(z)$ has no jump on the curve γ, and it follows that, at the origin, $M^{\mathsf{L}}_n(z)$ has an isolated singularity. From (15) and (12), it holds

$$\begin{aligned} M^{\mathsf{L}}_n(z) &= (Z^{\mathsf{L}}_n)'(z)(Z^{\mathsf{L}}_n(z))^{-1} \\ &= (Y^{\mathsf{L}}_n)'(z)(Y^{\mathsf{L}}_n(z))^{-1} + \frac{1}{z} Y^{\mathsf{L}}_n(z) \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z) \end{bmatrix} (Y^{\mathsf{L}}_n(z))^{-1}, \end{aligned}$$

where

$$Y^{\mathsf{L}}_n(z) = \begin{bmatrix} P^{\mathsf{L}}_n(z) & Q^{\mathsf{L}}_n(z) \\ -C_{n-1}P^{\mathsf{L}}_{n-1}(z) & -C_{n-1}Q^{\mathsf{L}}_{n-1}(z) \end{bmatrix}.$$

Each entry of the matrix $Q^{\mathsf{L}}_n(z)$ is the Cauchy transform of certain function $f(z)$, where $f(z) = \sum_{i\in I} \phi_i(z) z^{\alpha_i} \log^{p_i} z$, $\phi_i(z)$ is an entire function, $\operatorname{Re}(\alpha_i) > -1$, $p_i \in \mathbb{N}\cup\{0\}$, and I is a finite set of indices.

It is clear that $\lim_{z\to 0} z f(z) = 0$. Now, (see [56] Sections 8.3–8.6) and [58], its Cauchy transform $g(z) = \dfrac{1}{2\pi\,\mathrm{i}} \displaystyle\int_\gamma \frac{f(t)}{t-z}\,\mathrm{d}\,t$ also satisfies the same property $\lim_{z\to 0} z g(z) = 0$. We can also see that $\lim_{z\to 0} z^2 g'(z) = 0$. Indeed,

$$
\begin{aligned}
z g'(z) &= \int_\gamma \frac{z f(t)}{(t-z)^2}\,\mathrm{d}\,t = \int_\gamma \frac{(z-t) f(t)}{(t-z)^2}\,\mathrm{d}\,t + \int_\gamma \frac{t f(t)}{(t-z)^2}\,\mathrm{d}\,t,\\
&= -\int_\gamma \frac{f(t)}{t-z}\,\mathrm{d}\,t - \left.\frac{t f(t)}{t-z}\right|_{\partial\gamma} + \int_\gamma \frac{(t f(t))'}{t-z}\,\mathrm{d}\,t\\
&= -\left.\frac{t f(t)}{t-z}\right|_{\partial\gamma} + \int_\gamma \frac{t f'(t)}{t-z}\,\mathrm{d}\,t.
\end{aligned}
$$

From the boundary conditions, the first term is zero and we obtain $z g'(z) = \displaystyle\int_\gamma \frac{t f'(t)}{t-z}\,\mathrm{d}\,t$, and from the definition of f we obtain that $t f'(t)$ is a function in the class of f, which we denote by v and, consequently, $\lim_{z\to 0} z^2 g'(z) = 0$. From these considerations it follows,

$$
\big(Y^{\mathsf{L}}_n\big)'(z) = \begin{bmatrix} \mathrm{O}(1) & r^{\mathsf{L}}_1(z) \\ \mathrm{O}(1) & r^{\mathsf{L}}_2(z) \end{bmatrix}, \qquad \big(Y^{\mathsf{L}}_n(z)\big)^{-1} = \begin{bmatrix} r^{\mathsf{L}}_3(z) & r^{\mathsf{L}}_4(z) \\ \mathrm{O}(1) & \mathrm{O}(1) \end{bmatrix}, \qquad z \to 0,
$$

where $\lim_{z\to 0} z^2 r^{\mathsf{L}}_i(z) = 0_N$, for $i = 1,2$, and $\lim_{z\to 0} z r^{\mathsf{R}}_i(z) = 0_N$, for $i = 3,4$, so it holds that

$$
\lim_{z\to 0} z^2 \big(Y^{\mathsf{L}}_n\big)'(z)\big(Y^{\mathsf{L}}_n\big)^{-1} = \lim_{z\to 0} z^2 \begin{bmatrix} \mathrm{O}(1) r^{\mathsf{L}}_1(z) + \mathrm{O}(1) r^{\mathsf{L}}_3(z) & \mathrm{O}(1) r^{\mathsf{L}}_1(z) + \mathrm{O}(1) r^{\mathsf{L}}_4(z) \\ \mathrm{O}(1) r^{\mathsf{L}}_2(z) + \mathrm{O}(1) r^{\mathsf{L}}_3(z) & \mathrm{O}(1) r^{\mathsf{L}}_2(z) + \mathrm{O}(1) r^{\mathsf{L}}_4(z) \end{bmatrix} = 0_{2N}.
$$

Similar considerations leads us to the result that

$$
\lim_{z\to 0} z Y^{\mathsf{L}}_n(z) \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z) \end{bmatrix} \big(Y^{\mathsf{L}}_n(z)\big)^{-1} = 0_{2N},
$$

so we obtain that $\lim_{z\to 0} z^2 M^{\mathsf{L}}_n(z) = 0_{2N}$, and hence the matrix function $M^{\mathsf{L}}_n(z)$ has at most a simple pole at the point $z = 0$. □

3.5. Differential Relations from the Riemann–Hilbert Problem

We are interested in the differential equations fulfilled by the biorthogonal matrix polynomials determined by Laguerre type matrices of weights. Here we use the Riemann–Hilbert problem approach in order to derive these differential relations. We use the notation for the structure matrices

$$
\widetilde{M}^{\mathsf{L}}_n(z) = z M^{\mathsf{L}}_n(z), \qquad \widetilde{M}^{\mathsf{R}}_n(z) = z M^{\mathsf{R}}_n(z),
$$

with $\widetilde{M}^{\mathsf{L}}_n(z)$ and $\widetilde{M}^{\mathsf{R}}_n(z)$ matrices of entire functions.

Proposition 4 (First order differential equation for the fundamental matrices $Y^{\mathsf{L}}_n(z)$ and $Y^{\mathsf{R}}_n(z)$)**.** *It holds that*

$$
z\big(Y^{\mathsf{L}}_n\big)'(z) + Y^{\mathsf{L}}_n(z) \begin{bmatrix} h^{\mathsf{L}}(z) & 0_N \\ 0_N & -h^{\mathsf{R}}(z) \end{bmatrix} = \widetilde{M}^{\mathsf{L}}_n(z) Y^{\mathsf{L}}_n(z), \tag{17}
$$

$$z\left(Y_n^{\mathsf{R}}\right)'(z)+\begin{bmatrix} h^{\mathsf{R}}(z) & 0_N \\ 0_N & -h^{\mathsf{L}}(z)\end{bmatrix} Y_n^{\mathsf{R}}(z)=Y_n^{\mathsf{R}}(z)\widetilde{M}_n^{\mathsf{R}}(z). \tag{18}$$

Proof. Equations (17) and (18) follow immediately from the definition of the matrices $M_n^{\mathsf{L}}(z)$ and $M_n^{\mathsf{R}}(z)$ in (15). □

Corollary 2. *Let $h^{\mathsf{L}}(z)=A^{\mathsf{L}}z+B^{\mathsf{L}}$ and $h^{\mathsf{R}}(z)=A^{\mathsf{R}}z+B^{\mathsf{R}}$ be two first degree matrix polynomials. The left and right fundamental matrices are given, respectively, by*

$$M_n^{\mathsf{L}}(z)=\frac{1}{z}\begin{bmatrix} A^{\mathsf{L}}z+[p^1_{\mathsf{L},n},A^{\mathsf{L}}]+nI_N+B^{\mathsf{L}} & A^{\mathsf{L}}C_n^{-1}+C_n^{-1}A^{\mathsf{R}} \\ -C_{n-1}A^{\mathsf{L}}-A^{\mathsf{R}}C_{n-1} & -A^{\mathsf{R}}z+[p^1_{\mathsf{R},n},A^{\mathsf{R}}]-nI_N-B^{\mathsf{R}}\end{bmatrix}, \tag{19}$$

$$M_n^{\mathsf{R}}(z)=\frac{1}{z}\begin{bmatrix} A^{\mathsf{R}}z-[p^1_{\mathsf{R},n},A^{\mathsf{R}}]+nI_N+B^{\mathsf{R}} & -C_{n-1}A^{\mathsf{L}}-A^{\mathsf{R}}C_{n-1} \\ A^{\mathsf{L}}C_n^{-1}+C_n^{-1}A^{\mathsf{R}} & -A^{\mathsf{L}}z-[p^1_{\mathsf{L},n},A^{\mathsf{L}}]-nI_N-B^{\mathsf{L}}\end{bmatrix}. \tag{20}$$

Proof. By considering (17), Theorem 3, the asymptotic expansion at infinity of the fundamental matrix Y_n^{L} and (5), the asymptotic behavior of the second kind matrix functions at infinity, and using the identities $p^1_{\mathsf{R},n}=-q^1_{\mathsf{L},n-1}$ and $p^1_{\mathsf{L},n}=-q^1_{\mathsf{R},n-1}$ (19) follows. The relation (16) leads to (20). □

We introduce the $\mathcal{N}$ transform, $\mathcal{N}(F(z))=F'(z)+\dfrac{F^2(z)}{z}$.

Proposition 5 (Second order differential equation for the fundamental matrices)**.** *It holds*

$$z\left(Y_n^{\mathsf{L}}\right)''+\left(Y_n^{\mathsf{L}}\right)'\begin{bmatrix} 2h^{\mathsf{L}}+I_N & 0_N \\ 0_N & -2h^{\mathsf{R}}+I_N\end{bmatrix}+Y_n^{\mathsf{L}}(z)\begin{bmatrix} \mathcal{N}(h^{\mathsf{L}}) & 0_N \\ 0_N & \mathcal{N}(-h^{\mathsf{R}})\end{bmatrix}=\mathcal{N}(\widetilde{M}_n^{\mathsf{L}})Y_n^{\mathsf{L}}, \tag{21}$$

$$z\left(Y_n^{\mathsf{R}}\right)''+\begin{bmatrix} 2h^{\mathsf{R}}+I_N & 0_N \\ 0_N & -2h^{\mathsf{L}}+I_N\end{bmatrix}\left(Y_n^{\mathsf{R}}\right)'+\begin{bmatrix} \mathcal{N}(h^{\mathsf{R}}) & 0_N \\ 0_N & \mathcal{N}(-h^{\mathsf{L}})\end{bmatrix}Y_n{}^{\mathsf{R}}(z)=Y_n^{\mathsf{R}}\mathcal{N}(\widetilde{M}_n^{\mathsf{R}}). \tag{22}$$

Proof. Differentiating in (15), we obtain

$$\left(Z_n^{\mathsf{L}}\right)''\left(Z_n^{\mathsf{L}}\right)^{-1}=\frac{\left(\widetilde{M}_n^{\mathsf{L}}\right)'}{z}-\frac{\widetilde{M}_n^{\mathsf{L}}}{z^2}+\frac{\left(\widetilde{M}_n^{\mathsf{L}}\right)^2}{z^2}.$$

so that

$$z\left(Z_n^{\mathsf{L}}\right)''\left(Z_n^{\mathsf{L}}\right)^{-1}+\left(Z_n^{\mathsf{L}}\right)'\left(Z_n^{\mathsf{L}}\right)^{-1}=\left(\widetilde{M}_n^{\mathsf{L}}\right)'+\frac{\left(\widetilde{M}_n^{\mathsf{L}}\right)^2}{z}.$$

Now, using (12) and (10), we obtain the stated result (21). Equation (22) follows in a similar way from definition of M_n^{R} in (15). □

We introduce the following $\mathbb{C}^{2N\times 2N}$ valued functions

$$\mathsf{H}_n^{\mathsf{L}}=\begin{bmatrix} \mathsf{H}^{\mathsf{L}}_{1,1,n} & \mathsf{H}^{\mathsf{L}}_{1,2,n} \\ \mathsf{H}^{\mathsf{L}}_{2,1,n} & \mathsf{H}^{\mathsf{L}}_{2,2,n}\end{bmatrix}:=\mathcal{N}(\widetilde{M}_n^{\mathsf{L}}), \qquad \mathsf{H}_n^{\mathsf{R}}=\begin{bmatrix} \mathsf{H}^{\mathsf{R}}_{1,1,n} & \mathsf{H}^{\mathsf{R}}_{1,2,n} \\ \mathsf{H}^{\mathsf{R}}_{2,1,n} & \mathsf{H}^{\mathsf{R}}_{2,2,n}\end{bmatrix}:=\mathcal{N}(\widetilde{M}_n^{\mathsf{R}}).$$

It holds that the second order matrix differential Equations (21) and (22) split in the following differential relations

$$z\left(P_n^{\mathsf{L}}\right)''+\left(P_n^{\mathsf{L}}\right)'(2h^{\mathsf{L}}+I_N)+P_n^{\mathsf{L}}\mathcal{N}(h^{\mathsf{L}})=\mathsf{H}^{\mathsf{L}}_{1,1,n}P_n^{\mathsf{L}}-\mathsf{H}^{\mathsf{L}}_{1,2,n}C_{n-1}P^{\mathsf{L}}_{n-1}, \tag{23}$$

$$z\left(Q_n^{\mathsf{L}}\right)''+\left(Q_n^{\mathsf{L}}\right)'\left(-2h^{\mathsf{R}}+I_N\right)+Q_n^{\mathsf{L}}\mathcal{N}(-h^{\mathsf{R}})=\mathsf{H}^{\mathsf{L}}_{1,1,n}Q_n^{\mathsf{L}}-\mathsf{H}^{\mathsf{L}}_{1,2,n}C_{n-1}Q^{\mathsf{L}}_{n-1}, \tag{24}$$

$$z\left(P_n^{\mathsf{R}}\right)''+\left(2h^{\mathsf{R}}+I_N\right)\left(P_n^{\mathsf{R}}\right)'+\mathcal{N}(h^{\mathsf{R}})P_n^{\mathsf{R}}=P_n^{\mathsf{R}}\mathsf{H}^{\mathsf{R}}_{1,1,n}-P^{\mathsf{R}}_{n-1}C_{n-1}\mathsf{H}^{\mathsf{R}}_{2,1,n}, \tag{25}$$

$$z\left(Q_n^{\mathsf{R}}\right)''+\left(-2h^{\mathsf{L}}+I_N\right)\left(Q_n^{\mathsf{R}}\right)'+\mathcal{N}(-h^{\mathsf{L}})Q_n^{\mathsf{R}}=Q_n^{\mathsf{R}}\mathsf{H}^{\mathsf{R}}_{1,1,n}-Q^{\mathsf{R}}_{n-1}C_{n-1}\mathsf{H}^{\mathsf{R}}_{2,1,n}. \tag{26}$$

4. A Class of Laguerre Matrix Weights

Let us consider the matrix weight $W(z) = \mathrm{e}^{A_1 z} z^{\alpha} \mathrm{e}^{A_2 z}$, $z \in \mathbb{C}$, defined in $\mathbb{C} \setminus [0, +\infty)$ with support on $\gamma = [0, +\infty)$. Here α, A_1, $A_2 \in \mathbb{C}^{N\times N}$ are matrices such that $[\alpha, A_1] = [\alpha, A_2] = 0_N$, with spectrum $\sigma(\alpha)$, $\operatorname{Re}(\sigma(\alpha)) \subset (-1, +\infty)$. This class of weights contains in the Hermitian case some of the cases studied in the literature [6,7,11,59].

For this class of Laguerre weights, we obtain, using analytic arguments, an alternative formula for the residue matrix with the simple pole at $z = 0$ of the left fundamental matrix. In a similar manner, we could obtain the result for the right fundamental matrix. Notice that the fundamental matrix is completely determined in the previous section (19), where A^{L}, A^{R}, is substituted, respectively, by A_1, A_2, and B^{L}, B^{R} by $\frac{\alpha}{2}$. This alternative formula enables us to make an important simplification in the Equation (21) previously obtained.

Accordingly, we choose

$$W^{\mathsf{L}}(z) = \mathrm{e}^{A_1 z} z^{\frac{\alpha}{2}}, \qquad\qquad W^{\mathsf{R}}(z) = z^{\frac{\alpha}{2}} \mathrm{e}^{A_2 z}.$$

Straightforward calculation shows that h^{L} and h^{R} appearing in (11) are given by

$$h^{\mathsf{L}}(z) = A_1 z + \frac{\alpha}{2}, \qquad\qquad h^{\mathsf{R}}(z) = A_2 z + \frac{\alpha}{2}.$$

Proposition 6. *The structure matrix M_n^{L} defined in (15) has a simple pole given by the following expression:*

(1) *If $\operatorname{Re}(\sigma(\alpha)) \subset (-1, +\infty)$ and $\sigma(\alpha) \cap \mathbb{N} = \varnothing$, then*

$$M_n^{\mathsf{L}}(z) = \frac{1}{z} F_n^{\mathsf{L}}(0) \begin{bmatrix} \frac{\alpha}{2} & 0_N \\ 0_N & -\frac{\alpha}{2} \end{bmatrix} \big(F_n^{\mathsf{L}}(0)\big)^{-1} + \mathrm{O}(1), \qquad z \to 0,$$

where

$$F_n^{\mathsf{L}}(0) = \hat{Y}_n^{\mathsf{L}}(0) \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix} \left[\begin{array}{cc|cc} I_{N^+} & 0_{N^+\times N^-} & 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & 0_{N^-} & 0_{N^-\times N^+} & \mathrm{e}^{-\mathrm{i}\pi J^-} \\ \hline 0_{N^+} & 0_{N^+\times N^-} & \mathrm{e}^{\mathrm{i}\pi J^+} - \mathrm{e}^{-\mathrm{i}\pi J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\mathrm{i}\pi J^-} & 0_{N^-\times N^+} & 0_{N^-} \end{array}\right] \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix}^{-1},$$

α has the yielding canonical Jordan form, $\alpha = PJP^{-1}$ with

$$J = \begin{bmatrix} J^+ & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & J^- \end{bmatrix},$$

and N^+ (respectively, N^-) being the sum of the algebraic multiplicities associated with eigenvalues with positive (respectively, negative) real parts and in J^+ (respectively, J^-), we gather together the Jordan blocks of all eigenvalues with positive (respectively, negative) real parts, and $\hat{Y}_n^{\mathsf{L}}(z)$ being given by

$$\hat{Y}_n^{\mathsf{L}}(z) := Y_n^{\mathsf{L}}(z) \begin{bmatrix} z^{-\alpha} & 0_N \\ I_N - \mathrm{e}^{2\mathrm{i}\pi\alpha} & z^{\alpha} \end{bmatrix}^{-1}.$$

(2) *If $\alpha = mI_N, m \in \mathbb{N}$*

$$M_n^{\mathsf{L}}(z) = \frac{1}{z} F_n^{\mathsf{L}}(0) \begin{bmatrix} \frac{m}{2} I_N & -\frac{z^m}{2\pi i} I_N \\ 0_N & -\frac{m}{2} I_N \end{bmatrix} \big(F_n^{\mathsf{L}}(0)\big)^{-1} + \mathrm{O}(1), \qquad z \to 0,$$

where $F_n^{\mathsf{L}}(0) = \hat{Y}_n^{\mathsf{L}}(0)\begin{bmatrix} 0_N & \frac{1}{2\pi i} I_N \\ -2\pi\,\mathrm{i}\, I_N & 0_N \end{bmatrix}$, *with*

$$\hat{Y}_n^{\mathsf{L}}(z) := Y_n^{\mathsf{L}}(z)\begin{bmatrix} (\log z)^{-1} I_N & 0_N \\ -2\pi\,\mathrm{i}\, I_N & \log z I_N \end{bmatrix}^{-1}.$$

Remark 1. *In the first case,* $F_n^{\mathsf{L}}(0)$ *have a simpler form if* $\mathrm{Re}(\sigma(\alpha))$ *are all positive or all negative*

- *If* $\mathrm{Re}(\sigma(\alpha)) \subset (0, +\infty)$, *then* $F_n^{\mathsf{L}}(0) = Y_n^{\mathsf{L}}(0)\begin{bmatrix} I_N & 0_N \\ 0_N & \mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \end{bmatrix}$.
- *If* $\mathrm{Re}(\sigma(\alpha)) \subset (-1, 0)$, *then* $F_n^{\mathsf{L}}(0) = \lim_{z\to 0} Y_n^{\mathsf{L}}(z)\begin{bmatrix} 0_N & z^{\alpha}\,\mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ z^{\alpha}\left(I_N - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha}\right) & \mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \end{bmatrix}$.

Proof. It can be seen that the matrix function Z_n^{L} defined by

$$Z_n^{\mathsf{L}}(z) = Y_n^{\mathsf{L}}(z)\mathcal{C}(z), \qquad \text{where} \qquad \mathcal{C}(z) = \begin{bmatrix} W^{\mathsf{L}}(z) & 0 \\ 0 & (W^{\mathsf{R}}(z))^{-1} \end{bmatrix},$$

with $W^{\mathsf{L}}(z)W^{\mathsf{R}}(z) = W(z)$, satisfies the following conditions:

- Z_n^{L} is holomorphic in $\mathbb{C} \setminus [0, +\infty)$.
- $\left(Z_n^{\mathsf{L}}(z)\right)_+ = \left(Z_n^{\mathsf{L}}(z)\right)_- \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix}$ over $(0, +\infty)$.

Let us start with the first case: $\mathrm{Re}(\sigma(\alpha)) \subset (-1, +\infty)$ and $\sigma(\alpha) \cap \{\mathbb{N}\} = \emptyset$.

In this case, the constant jump matrix $\begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix}$ can be block diagonalized. For that aim, we consider the matrix

$$P = \begin{bmatrix} I_N & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \end{bmatrix} \qquad \text{such that} \qquad \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix} P = P \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix}.$$

So, over the interval $(0, +\infty)$, we have

$$\left(Z_n^{\mathsf{L}}(z)P\right)_+ = \left(Z_n^{\mathsf{L}}(z)P\right)_- \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix}.$$

For $z \in \mathbb{C} \setminus [0, +\infty)$, let us define the matrix

$$\psi(z) := \begin{bmatrix} z^{\frac{\alpha}{2}} & 0 \\ 0 & z^{-\frac{\alpha}{2}} \end{bmatrix}, \tag{27}$$

which satisfies, over $(0, +\infty)$, the following jump condition

$$\left(\psi(z)\right)_+ = \left(\psi(z)\right)_- \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi\alpha} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\,\pi\alpha} \end{bmatrix}.$$

Consequently, the matrix

$$F_n^{\mathsf{L}}(z) := Z_n^{\mathsf{L}}(z)P\psi^{-1}(z),$$

has no jump in the interval $(0,+\infty)$. The matrix function F^{L}_n has an isolated singularity at the origin which, as we will show now, is a removable singularity, i.e., $\lim_{z\to 0} zF^{\mathsf{L}}_n(z) = 0_{2N}$. From its definition, we have that

$$\begin{aligned} zF^{\mathsf{L}}_n(z) &= \begin{bmatrix} \mathrm{O}(z) & zs^{\mathsf{L}}_1(z) \\ \mathrm{O}(z) & zs^{\mathsf{L}}_2(z) \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z}\, z^{\frac{\alpha}{2}} & 0_N \\ 0_N & \mathrm{e}^{-A_2 z}\, z^{-\frac{\alpha}{2}} \end{bmatrix} \begin{bmatrix} I_N & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ 0_N & \mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \end{bmatrix} \begin{bmatrix} z^{-\frac{\alpha}{2}} & 0 \\ 0 & z^{\frac{\alpha}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{O}(z) & zs^{\mathsf{L}}_1(z) \\ \mathrm{O}(z) & zs^{\mathsf{L}}_2(z) \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\, \mathrm{e}^{-\mathrm{i}\,\pi\alpha}\, z^{\alpha} \\ 0_N & \mathrm{e}^{-A_2 z}(\mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha}) \end{bmatrix}, & z \to 0, \end{aligned}$$

and as $zs^{\mathsf{L}}_1, zs^{\mathsf{L}}_2 \to 0_N$ as $z \to 0$ and $\mathrm{O}(z)z^{\alpha} \to 0_N$, as $z \to 0$ (because the eigenvalues of α are bounded from below by -1), we conclude that $zF^{\mathsf{L}}_n(z) \to 0_{2N}$ for $z \to 0$. Hence, $F^{\mathsf{L}}_n(z)$ is a matrix of entire functions.

Now, we want to compute $F^{\mathsf{L}}_n(0) = \lim_{z\to 0} F^{\mathsf{L}}_n(z)$. For this fact, we will elaborate with respect to the sign of the real part of spectrum of α. Notice that,

$$F^{\mathsf{L}}_n(0) = \lim_{z\to 0} Y^{\mathsf{L}}_n(z) \begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\, \mathrm{e}^{-\mathrm{i}\,\pi\alpha}\, z^{\alpha} \\ 0_N & \mathrm{e}^{-A_2 z}(\mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha}) \end{bmatrix},$$

where the limit of each factor does not need to exist.

We separately compute $F^{\mathsf{L}}_n(0)$ in the cases, when $\operatorname{Re}(\sigma(\alpha)) \subset (0,+\infty)$ and when $\operatorname{Re}(\sigma(\alpha)) \subset (-1,0)$, and then we give $F^{\mathsf{L}}_n(0)$ in general.

Case $\operatorname{Re}(\sigma(\alpha)) \subset (0,+\infty)$ **and** $\operatorname{Re}(\sigma(\alpha)) \cap \{\mathbb{N}\} = \emptyset$.

When the real part of all the eigenvalues of α are strictly positive then each limit exists and

$$F^{\mathsf{L}}_n(0) = Y^{\mathsf{L}}_n(0) \begin{bmatrix} I_N & 0_N \\ 0_N & \mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \end{bmatrix}.$$

Case $\operatorname{Re}(\sigma(\alpha)) \subset (-1,0)$ **and** $\sigma(\alpha) \cap \{\mathbb{N}\} = \emptyset$.

We cannot proceed as before. However, as the limit exists, if we are able to rewrite

$$Y^{\mathsf{L}}_n(z) \begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\, \mathrm{e}^{-\mathrm{i}\,\pi\alpha}\, z^{\alpha} \\ 0_N & \mathrm{e}^{-A_2 z}(\mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha}) \end{bmatrix} = \hat{Y}^{\mathsf{L}}_n(z) f(z),$$

in terms of two matrix factors $\hat{Y}^{\mathsf{L}}_n(z)$ and $f(z)$, a non singular matrix, with f having a well defined limit for $z \to 0$, also being a non-singular matrix, we can ensure the existence of $\lim_{z\to 0} \hat{Y}^{\mathsf{L}}_n(z)$, and $F^{\mathsf{L}}_n(0) = \big(\lim_{z\to 0} \hat{Y}^{\mathsf{L}}_n(z)\big)\big(\lim_{z\to 0} f(z)\big)$. This can be achieved by considering

$$\begin{aligned} \hat{Y}^{\mathsf{L}}_n(z) &:= Y^{\mathsf{L}}_n(z) \begin{bmatrix} z^{-\alpha} & 0_N \\ I_N - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha} & z^{\alpha} \end{bmatrix}^{-1}, \\ f(z) &:= \begin{bmatrix} z^{-\alpha} & 0_N \\ I_N - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha} & z^{\alpha} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\, \mathrm{e}^{-\mathrm{i}\,\pi\alpha}\, z^{\alpha} \\ 0_N & \mathrm{e}^{-A_2 z}(\mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha}) \end{bmatrix} \\ &= \begin{bmatrix} z^{-\alpha}\, \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\, \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ (I_N - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha})\, \mathrm{e}^{A_1 z} & (-\,\mathrm{e}^{A_1 z} + \mathrm{e}^{-A_2 z})(\mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha}) z^{\alpha} \end{bmatrix}. \end{aligned}$$

So that,

$$\lim_{z\to 0} f(z) = \begin{bmatrix} 0_N & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ I_N - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha} & 0_N \end{bmatrix}, \qquad F^{\mathsf{L}}_n(0) = \hat{Y}^{\mathsf{L}}_n(0) \begin{bmatrix} 0_N & \mathrm{e}^{-\mathrm{i}\,\pi\alpha} \\ I_N - \mathrm{e}^{2\,\mathrm{i}\,\pi\alpha} & 0_N \end{bmatrix}.$$

General case $\operatorname{Re}(\sigma(\alpha)) \subset (-1, +\infty)$ **and** $\sigma(\alpha) \cap \{\mathbb{N}\} = \emptyset$.

Recalling the canonical Jordan form, we can write $\alpha = PJP^{-1}$ with

$$J = \begin{bmatrix} J^+ & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & J^- \end{bmatrix},$$

and N^+ (respectively, N^-) being the sum of the algebraic multiplicities associated with positive (respectively, negative) eigenvalues and in J^+ (respectively, J^-), we gather together the Jordan blocks of all positive (respectively, negative) eigenvalues. Hence,

$$\begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{A_1 z}\,\mathrm{e}^{-\mathrm{i}\,\pi\alpha}\, z^{\alpha} \\ 0_N & \mathrm{e}^{-A_2 z}(\mathrm{e}^{\mathrm{i}\,\pi\alpha} - \mathrm{e}^{-\mathrm{i}\,\pi\alpha}) \end{bmatrix} = \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z} & \mathrm{e}^{\tilde{A}_1 z}\,\mathrm{e}^{-\mathrm{i}\,\pi J}\, z^{J} \\ 0_N & \mathrm{e}^{-\tilde{A}_2 z}(\mathrm{e}^{\mathrm{i}\,\pi J} - \mathrm{e}^{-\mathrm{i}\,\pi J}) \end{bmatrix} \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix}^{-1}$$

with $\tilde{A}_j = P^{-1}A_jP$, $j = 1, 2$.

Now, as we did in the previous case, with negative eigenvalues only, we left multiply by the following nonsingular matrix

$$S(z) := \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix} \left[\begin{array}{cc|cc} I_{N^+} & 0_{N^+\times N^-} & \multicolumn{2}{c}{\multirow{2}{*}{0_N}} \\ 0_{N^-\times N^+} & z^{-J^-} & & \\ \hline 0_{N^+} & 0_{N^+\times N^-} & I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} & 0_{N^-\times N^+} & z^{J^-} \end{array}\right] \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix}^{-1},$$

to obtain

$$\begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix} \left[\begin{array}{c|c} \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{-J^-} \end{bmatrix} \mathrm{e}^{\tilde{A}_1 z} & \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{-J^-} \end{bmatrix} \mathrm{e}^{\tilde{A}_1 z} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi J^+} z^{J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{-\mathrm{i}\,\pi J^-} z^{J^-} \end{bmatrix} \\ \hline \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} \end{bmatrix} \mathrm{e}^{\tilde{A}_1 z} & \begin{array}{l} \begin{bmatrix} 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} \end{bmatrix} \mathrm{e}^{\tilde{A}_1 z} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\,\pi J^+} z^{J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{-\mathrm{i}\,\pi J^-} z^{J^-} \end{bmatrix} \\ + \begin{bmatrix} I_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & z^{J^-} \end{bmatrix} \mathrm{e}^{-\tilde{A}_2 z} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,\pi J^+} - \mathrm{e}^{-\mathrm{i}\,\pi J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & \mathrm{e}^{\mathrm{i}\,\pi J^-} - \mathrm{e}^{-\mathrm{i}\,\pi J_-} \end{bmatrix} \end{array} \end{array}\right] \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix}^{-1}$$

which for $z \to 0$ has a well defined limit, being a non-singular matrix, given by

$$\begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix} \left[\begin{array}{cc|cc} I_{N^+} & 0_{N^+\times N^-} & 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & 0_{N^-} & 0_{N^-\times N^+} & \mathrm{e}^{-\mathrm{i}\,\pi J^-} \\ \hline 0_{N^+} & 0_{N^+\times N^-} & \mathrm{e}^{\mathrm{i}\,\pi J^+} - \mathrm{e}^{-\mathrm{i}\,\pi J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} & 0_{N^-\times N^+} & 0_{N^-} \end{array}\right] \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix}^{-1}.$$

Thus,

$$F^{\mathsf{L}}_n(0) = \hat{Y}^{\mathsf{L}}_n(0) \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix} \left[\begin{array}{cc|cc} I_{N^+} & 0_{N^+\times N^-} & 0_{N^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & 0_{N^-} & 0_{N^-\times N^+} & \mathrm{e}^{-\mathrm{i}\,\pi J^-} \\ \hline 0_{N^+} & 0_{N^+\times N^-} & \mathrm{e}^{\mathrm{i}\,\pi J^+} - \mathrm{e}^{-\mathrm{i}\,\pi J^+} & 0_{N^+\times N^-} \\ 0_{N^-\times N^+} & I_{N^-} - \mathrm{e}^{2\,\mathrm{i}\,\pi J^-} & 0_{N^-\times N^+} & 0_{N^-} \end{array}\right] \begin{bmatrix} P & 0_N \\ 0_N & P \end{bmatrix}^{-1}.$$

By definition,

$$M^{\mathsf{L}}_n = \big(Z^{\mathsf{L}}_n\big)'\big(Z^{\mathsf{L}}_n\big)^{-1} = \big(F^{\mathsf{L}}_n\big)'\big(F^{\mathsf{L}}_n\big)^{-1} + F^{\mathsf{L}}_n \psi' \psi^{-1}\big(F^{\mathsf{L}}_n\big)^{-1},$$

as $\det F^{\mathsf{L}}_n(z) \neq 0$, we know that $\big(F^{\mathsf{L}}_n\big)'\big(F^{\mathsf{L}}_n\big)^{-1}$ has no singularities, while

$$F^{\mathsf{L}}_n \psi' \psi^{-1}\big(F^{\mathsf{L}}_n\big)^{-1} = \frac{1}{z} F^{\mathsf{L}}_n \begin{bmatrix} \frac{\alpha}{2} & 0_N \\ 0_N & -\frac{\alpha}{2} \end{bmatrix} \big(F^{\mathsf{L}}_n\big)^{-1}.$$

Consequently, $M^{\mathsf{L}}_n(z)$ has a simple pole at the origin with

$$M^{\mathsf{L}}_n(z) = \frac{1}{z} F^{\mathsf{L}}_n(0) \begin{bmatrix} \frac{\alpha}{2} & 0_N \\ 0_N & -\frac{\alpha}{2} \end{bmatrix} \big(F^{\mathsf{L}}_n(0)\big)^{-1} + \mathrm{O}(1), \qquad z \to 0.$$

Let us move to the proof of second case, i.e., $\alpha = mI_N$, $m \in \mathbb{N}$.

It can be seen that the matrix function Z_n^{L} satisfies over $(0, +\infty)$ the following jump condition

$$\big(Z_n^{\mathsf{L}}(z)\big)_+ = \big(Z_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} (-1)^m I_N & (-1)^m I_N \\ 0 & (-1)^m I_N \end{bmatrix}.$$

For $z \in \mathbb{C} \setminus [0, +\infty)$, instead of (27), let us define the matrix

$$\psi(z) := \begin{bmatrix} z^{\frac{m}{2}} I_N & -\frac{1}{2\pi i} z^{\frac{m}{2}} \log z I_N \\ 0 & z^{-\frac{m}{2}} I_N \end{bmatrix},$$

where we take the branch of the logarithmic function defined in $\mathbb{C} \setminus [0, +\infty)$, which satisfies, over $(0, +\infty)$, the same jump condition

$$\big(\psi(z)\big)_+ = \big(\psi(z)\big)_- \begin{bmatrix} (-1)^m I_N & (-1)^m I_N \\ 0 & (-1)^m I_N \end{bmatrix}.$$

Consequently, the matrix

$$F_n^{\mathsf{L}}(z) := Z_n^{\mathsf{L}}(z) \psi^{-1}(z)$$

has no jump in the interval $(0, +\infty)$. The matrix function F_n^{L} has an isolated singularity at the origin which, as we will show now, is a removable singularity, i.e.,

$$\begin{aligned} zF_n^{\mathsf{L}}(z) &= \begin{bmatrix} \mathrm{O}(z) & zs_1^{\mathsf{L}}(z) \\ \mathrm{O}(z) & zs_2^{\mathsf{L}}(z) \end{bmatrix} \begin{bmatrix} \mathrm{O}(1) & 0_N \\ 0_N & \mathrm{O}(1) \end{bmatrix} \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(\log z) \\ \mathrm{O}(1) & \mathrm{O}(1) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{O}(z) + zs_1^{\mathsf{L}}(z) & \mathrm{O}(z \log z) + zs_1^{\mathsf{L}}(z) \\ \mathrm{O}(z) + zs_2^{\mathsf{L}}(z) & \mathrm{O}(z \log z) + zs_2^{\mathsf{L}}(z) \end{bmatrix}, & z \to 0, \end{aligned}$$

and as $zs_1^{\mathsf{L}}, zs_2^{\mathsf{L}} \to 0_N$ as $z \to 0$, we conclude that $zF_n^{\mathsf{L}}(z) \to 0_{2N}$, as $z \to 0$. Hence, $F_n^{\mathsf{L}}(z)$ is a matrix of entire functions. To compute $F_n^{\mathsf{L}}(0)$ we notice that,

$$F_n^{\mathsf{L}}(0) = \lim_{z \to 0} Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathrm{e}^{A_1 z} & \frac{1}{2\pi i} z^m \log z \, \mathrm{e}^{A_1 z} \\ 0_N & \mathrm{e}^{-A_2 z} \end{bmatrix}.$$

For $m = 1, 2 \ldots$ it holds that $F_n^{\mathsf{L}}(0) = Y_n^{\mathsf{L}}(0)$. For $m = 0$, the limit of each factor inside the limit does not need to exist. As the limit exists, let us write

$$Y_n^{\mathsf{L}}(z) \begin{bmatrix} \mathrm{e}^{A_1 z} & \frac{1}{2\pi i} \log z \, \mathrm{e}^{A_1 z} \\ 0_N & \mathrm{e}^{-A_2 z} \end{bmatrix} = \hat{Y}_n^{\mathsf{L}}(z) f(z),$$

with

$$\begin{aligned} \hat{Y}_n^{\mathsf{L}}(z) &:= Y_n^{\mathsf{L}}(z) \begin{bmatrix} (\log z)^{-1} I_N & 0_N \\ -2\pi \mathrm{i} \, I_N & \log z I_N \end{bmatrix}^{-1}, \\ f(z) &:= \begin{bmatrix} (\log z)^{-1} I_N & 0_N \\ -2\pi \mathrm{i} \, I_N & \log z I_N \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A_1 z} & \frac{1}{2\pi \mathrm{i}} \log z \, \mathrm{e}^{A_1 z} \\ 0_N & \mathrm{e}^{-A_2 z} \end{bmatrix} \\ &= \begin{bmatrix} (\log z)^{-1} \mathrm{e}^{A_1 z} & \frac{1}{2\pi \mathrm{i}} \mathrm{e}^{A_1 z} \\ -2\pi \mathrm{i} \, \mathrm{e}^{A_1 z} & -\log z (\mathrm{e}^{A_1 z} - \mathrm{e}^{-A_2 z}) \end{bmatrix}. \end{aligned}$$

So that,

$$\lim_{z\to 0} f(z) = \begin{bmatrix} 0_N & \frac{1}{2\pi\,\mathrm{i}} I_N \\ -2\pi\,\mathrm{i}\, I_N & 0_N \end{bmatrix}, \qquad F_n^{\mathsf{L}}(0) = \hat{Y}_n^{\mathsf{L}}(0) \begin{bmatrix} 0_N & \frac{1}{2\pi i} I_N \\ -2\pi\,\mathrm{i}\, I_N & 0_N \end{bmatrix}.$$

Using the same kind of reasoning as above, we find that $M_n^{\mathsf{L}}(z)$ has a simple pole at the origin with

$$M_n^{\mathsf{L}}(z) = \frac{1}{z} F_n^{\mathsf{L}}(0) \begin{bmatrix} \frac{m}{2} I_N & -\frac{z^m}{2\pi i} I_N \\ 0_N & -\frac{m}{2} I_N \end{bmatrix} \big(F_n^{\mathsf{L}}(0)\big)^{-1} + \mathrm{O}(1), \qquad z \to 0,$$

which ends the proof. □

Proposition 7. *The structure matrix M_n^{L} has the yielding expression*

$$M_n^{\mathsf{L}}(z) = \frac{1}{z} \begin{bmatrix} A_1 z + [p^1_{\mathsf{L},n}, A_1] + nI_N + \frac{\alpha}{2} & A_1 C_n^{-1} + C_n^{-1} A_2 \\ -C_{n-1} A_1 - A_2 C_{n-1} & -A_2 z + [p^1_{\mathsf{R},n}, A_2] - nI_N - \frac{\alpha}{2} \end{bmatrix}.$$

Proof. Substitute A^{L}, A^{R}, respectively, by A_1, A_2, and B^{L}, B^{R} by $\frac{\alpha}{2}$ in (19) and (20). □

Proposition 8. *Let α, A_1 and A_2, such that $[\alpha, A_1] = [\alpha, A_2] = 0_N$, and the real part of spectrum of α, $\sigma(\alpha)$, is contained on $(-1, +\infty)$ with $\sigma(\alpha) \cap \{\mathbb{N}\} = \varnothing$. If there exists $\lambda \in (0, +\infty)$ such that $\alpha^2 = \lambda I_N$, or $\alpha = mI_N$, for some $m \in \{0, 1, 2, \ldots\}$, then the second order differential equation is simplified to*

$$\begin{aligned} z\big(Y_n^{\mathsf{L}}\big)'' + \big(Y_n^{\mathsf{L}}\big)' &\begin{bmatrix} \alpha + I_N + 2A_1 z & 0_N \\ 0_N & I_N - \alpha - 2A_2 z \end{bmatrix} \\ &+ Y_n^{\mathsf{L}} \begin{bmatrix} A_1 + A_1\alpha + A_1{}^2 z & 0_N \\ 0_N & -A_2 + A_2\alpha + A_2{}^2 z \end{bmatrix} \\ = \begin{bmatrix} A_1 + [p^1_{\mathsf{L},n}, A_1^2] + (nI_N + \alpha) A_1 + A_1^2 z & A_1^2 C_n^{-1} - C_n^{-1} A_2^2 \\ -C_{n-1} A_1^2 + A_2^2 C_{n-1} & -A_2 - [p^1_{\mathsf{R},n}, A_2^2] + (nI_N + \alpha) A_2 + A_2^2 z \end{bmatrix} &Y_n^{\mathsf{L}}(z). \end{aligned}$$

Proof. If we take into account that $\widetilde{M}_n^{\mathsf{L}}(z) = \widetilde{M}_n^{\mathsf{L}}(0) + z(\widetilde{M}_n^{\mathsf{L}})'(0)$ and that

$$\begin{aligned} \mathcal{N}(\widetilde{M}_n^{\mathsf{L}}(z)) = (\widetilde{M}_n^{\mathsf{L}})'(0) + (\widetilde{M}_n^{\mathsf{L}}(0))^2 \frac{1}{z} + (\widetilde{M}_n^{\mathsf{L}})'(0) \widetilde{M}_n^{\mathsf{L}}(0) \\ + \widetilde{M}_n^{\mathsf{L}}(0) (\widetilde{M}_n^{\mathsf{L}})'(0) + \big((\widetilde{M}_n^{\mathsf{L}})'(0)\big)^2 z, \end{aligned}$$

we find that (21), the second order differential equation that the fundamental matrix satisfies, can be written as

$$\begin{aligned} &z\big(Y_n^{\mathsf{L}}\big)'' + \big(Y_n^{\mathsf{L}}\big)' \begin{bmatrix} \alpha + I_N + 2A_1 z & 0_N \\ 0_N & I_N - \alpha - 2A_2 z \end{bmatrix} \\ &+ Y_n^{\mathsf{L}} \begin{bmatrix} A_1 + \frac{1}{2} A_1 \alpha + \frac{1}{2} \alpha A_1 + z A_1{}^2 & 0_N \\ 0_N & -A_2 + \frac{1}{2} A_2 \alpha + \frac{1}{2} \alpha A_2 + z A_2{}^2 \end{bmatrix} + \frac{1}{z} Y_n^{\mathsf{L}} \begin{bmatrix} (\frac{\alpha}{2})^2 & 0_N \\ 0_N & (\frac{\alpha}{2})^2 \end{bmatrix} \\ &= \Big((\widetilde{M}_n^{\mathsf{L}})'(0) + (\widetilde{M}_n^{\mathsf{L}}(0))^2 \frac{1}{z} + (\widetilde{M}_n^{\mathsf{L}})'(0) \widetilde{M}_n^{\mathsf{L}}(0) + \widetilde{M}_n^{\mathsf{L}}(0) (\widetilde{M}_n^{\mathsf{L}})'(0) + \big((\widetilde{M}_n^{\mathsf{L}})'(0)\big)^2 z \Big) Y_n^{\mathsf{L}}(z). \end{aligned}$$

Under the restriction that the real part of the spectrum of α is contained on $(-1,+\infty)$ and $\sigma(\alpha)\cap\{\mathbb{N}\}=\emptyset$, the matrix $M_n^{\mathsf{L}}=\left(Z_n^{\mathsf{L}}\right)'\left(Z_n^{\mathsf{L}}\right)^{-1}$ has a pole of order 1 at $z=0$, with residue given by

$$\widetilde{M}_n^{\mathsf{L}}(0)=F_n^{\mathsf{L}}(0)\begin{bmatrix}\frac{\alpha}{2} & 0_N\\ 0_N & -\frac{\alpha}{2}\end{bmatrix}\left(F_n^{\mathsf{L}}(0)\right)^{-1}.$$

If we now also assume on the matrix α that $\alpha^2=\lambda I_N$, we obtain

$$(\widetilde{M}_n^{\mathsf{L}}(0))^2=F_n^{\mathsf{L}}(0)\begin{bmatrix}\left(\frac{\alpha}{2}\right)^2 & 0_N\\ 0_N & \left(\frac{\alpha}{2}\right)^2\end{bmatrix}\left(F_n^{\mathsf{L}}(0)\right)^{-1}=\frac{\lambda}{4}I_{2N}.$$

In the case that $\alpha=mI_N$, for some $m\in\{0,1,2,\dots\}$, we find that

$$(\widetilde{M}_n^{\mathsf{L}}(0))^2=F_n^{\mathsf{L}}(0)\begin{bmatrix}\left(\frac{m}{2}\right)^2 & 0_N\\ 0_N & \left(\frac{m}{2}\right)^2\end{bmatrix}\left(F_n^{\mathsf{L}}(0)\right)^{-1}=\frac{m^2}{4}I_{2N}.$$

In both cases, we have

$$\begin{aligned}z\left(Y_n^{\mathsf{L}}\right)''+\left(Y_n^{\mathsf{L}}\right)'&\begin{bmatrix}\alpha+I_N+2A_1z & 0_N\\ 0_N & I_N-\alpha-2A_2z\end{bmatrix}\\&+Y_n^{\mathsf{L}}\begin{bmatrix}A_1+A_1\alpha+{A_1}^2z & 0_N\\ 0_N & -A_2+A_2\alpha+{A_2}^2z\end{bmatrix}\\&=\Big((\widetilde{M}_n^{\mathsf{L}})'(0)+(\widetilde{M}_n^{\mathsf{L}})'(0)\widetilde{M}_n^{\mathsf{L}}(0)+\widetilde{M}_n^{\mathsf{L}}(0)(\widetilde{M}_n^{\mathsf{L}})'(0)+\left((\widetilde{M}_n^{\mathsf{L}})'(0)\right)^2z\Big)Y_n^{\mathsf{L}}(z).\end{aligned}$$

and substituting

$$\widetilde{M}_n^{\mathsf{L}}(z)=\begin{bmatrix}A_1z+[p^1_{\mathsf{L},n},A_1]+nI_N+\frac{\alpha}{2} & A_1C_n^{-1}+C_n^{-1}A_2\\ -C_{n-1}A_1-A_2C_{n-1} & -A_2z+[p^1_{\mathsf{R},n},A_2]-nI_N-\frac{\alpha}{2}\end{bmatrix}.$$

The result follows. □

Remark 2. *If the spectrum of α is contained in $(-1,+\infty)\setminus\mathbb{N}$ then:*

- *when $|\lambda|<1$, then the $\pm\lambda$ are admissible eigenvalues for α,*
- *when $|\lambda|>1$, then only positive and bigger than 1 eigenvalues are admissible for α, and we have $\alpha=\lambda I_N$.*

Corollary 3. *Let us consider $N=1$ (i.e., the scalar case). If $A_1=A_2=-\frac{1}{2}$, and $\alpha>-1$, then the second order equation for $\{P_n^L\}_{n\in\mathbb{N}}$ and $\{Q_n^L\}_{n\in\mathbb{N}}$ is given by*

$$\begin{aligned}zP_n''(z)-(z-\alpha-1)P_n'(z)&=-nP_n(z),\\ zQ_n''(z)+(z-\alpha+1)Q_n'(z)&=-(n+1)Q_n(z).\end{aligned}$$

Proof. In the scalar case, this equation reduces to

$$\begin{aligned}z\left(Y_n^{\mathsf{L}}\right)''+\left(Y_n^{\mathsf{L}}\right)'&\begin{bmatrix}\alpha+1+2A_1z & 0\\ 0 & 1-\alpha-2A_1z\end{bmatrix}\\&+Y_n^{\mathsf{L}}\begin{bmatrix}A_1+A_1\alpha+{A_1}^2z & 0\\ 0 & -A_1+A_1\alpha+{A_1}^2z\end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} A_1 + (n+\alpha)A_1 + A_1^2 z & 0 \\ 0 & -A_1 + (n+\alpha)A_1 + A_1^2 z \end{bmatrix} Y_n^{\mathsf{L}}(z),$$

as $A_1^2 C_n^{-1} = C_n^{-1} A_1^2$ and $A_1 = A_2 = -\frac{1}{2}$, and so

$$z(Y_n^{\mathsf{L}})'' + (Y_n^{\mathsf{L}})' \begin{bmatrix} \alpha+1-z & 0 \\ 0 & 1-\alpha+z \end{bmatrix} + Y_n^{\mathsf{L}} \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{n+1}{2} & 0 \\ 0 & -\frac{n-1}{2} \end{bmatrix} Y_n^{\mathsf{L}}(z),$$

now, considering the $(1,1)$ and the $(1,2)$ entry of this differential matrix equation the result follows. □

5. Matrix Discrete Painlevé IV

We can consider, using the notation introduced before, the matrix weight measure $W = W_{\mathsf{L}} W_{\mathsf{R}}$ such that

$$z(W^{\mathsf{L}})'(z) = (h_0^{\mathsf{L}} + h_1^{\mathsf{L}} z + h_2^{\mathsf{L}} z^2) W^{\mathsf{L}}(z), \qquad z(W^{\mathsf{R}})'(z) = W^{\mathsf{R}}(z)(h_0^{\mathsf{R}} + h_1^{\mathsf{R}} z + h_2^{\mathsf{R}} z^2).$$

From Proposition 5 we find that the matrix

$$\widetilde{M}_n = z M_n^{\mathsf{L}}$$

is given explicitly by

$$\begin{aligned}
(\widetilde{M}_n^{\mathsf{L}})_{11} &= C_n^{-1} h_2^{\mathsf{R}} C_{n-1} + (h_0^{\mathsf{L}} + h_1^{\mathsf{L}} z + h_2^{\mathsf{L}} z^2) + h_1^{\mathsf{L}} q_{\mathsf{R},n-1}^1 + p_{\mathsf{L},n}^1 h_1^{\mathsf{L}} \\
&\quad + z(h_2^{\mathsf{L}} q_{\mathsf{R},n-1}^1 + p_{\mathsf{L},n}^1 h_2^{\mathsf{L}}) + h_2^{\mathsf{L}} q_{\mathsf{R},n-1}^2 + p_{\mathsf{L},n}^2 h_2^{\mathsf{L}} + p_{\mathsf{L},n}^1 h_2^{\mathsf{L}} q_{\mathsf{R},n-1}^1 + n I_N, \\
(\widetilde{M}_n^{\mathsf{L}})_{12} &= (h_1^{\mathsf{L}} + h_2^{\mathsf{L}} z + h_2^{\mathsf{L}} q_{\mathsf{R},n}^1 + p_{\mathsf{L},n}^1 h_2^{\mathsf{L}}) C_n^{-1} + C_n^{-1}(h_1^{\mathsf{R}} + h_2^{\mathsf{R}} z + h_2^{\mathsf{R}} p_{\mathsf{R},n}^1 + q_{\mathsf{L},n}^1 h_2^{\mathsf{R}}), \\
(\widetilde{M}_n^{\mathsf{L}})_{21} &= -C_{n-1}(h_1^{\mathsf{L}} + h_2^{\mathsf{L}} z + h_2^{\mathsf{L}} q_{\mathsf{R},n-1}^1 + p_{\mathsf{L},n-1}^1 C_{\mathsf{L}}) \\
&\quad - (h_1^{\mathsf{R}} + h_2^{\mathsf{R}} z + h_2^{\mathsf{R}} p_{\mathsf{R},n-1}^1 + q_{\mathsf{L},n-1}^1 h_2^{\mathsf{R}}) C_{n-1}, \\
(\widetilde{M}_n^{\mathsf{L}})_{22} &= -C_{n-1} h_2^{\mathsf{L}} C_n^{-1} - (h_0^{\mathsf{R}} + h_1^{\mathsf{R}} z + h_2^{\mathsf{R}} z^2) - h_1^{\mathsf{R}} p_{\mathsf{R},n}^1 - q_{\mathsf{L},n-1}^1 h_1^{\mathsf{R}} \\
&\quad - z(h_2^{\mathsf{R}} p_{\mathsf{R},n}^1 + q_{\mathsf{L},n-1}^1 h_2^{\mathsf{R}}) - h_2^{\mathsf{R}} p_{\mathsf{R},n}^2 - q_{\mathsf{L},n-1}^2 h_2^{\mathsf{R}} - q_{\mathsf{L},n-1}^1 h_2^{\mathsf{R}} p_{\mathsf{R},n}^1 - n I_N.
\end{aligned}$$

From the three-term recurrence relation for $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ we find that $p_{\mathsf{L},n}^1 - p_{\mathsf{L},n+1}^1 = \beta_n^{\mathsf{L}}$ and $p_{\mathsf{L},n}^2 - p_{\mathsf{L},n+1}^2 = \beta_n^{\mathsf{L}} p_{\mathsf{L},n}^1 + \gamma_n^{\mathsf{L}}$ where $\gamma_n^{\mathsf{L}} = C_n^{-1} C_{n-1}$. Consequently,

$$p_{\mathsf{L},n}^1 = -\sum_{k=0}^{n-1} \beta_k^{\mathsf{L}}, \qquad\qquad p_{\mathsf{L},n}^2 = \sum_{i,j=0}^{n-1} \beta_i^{\mathsf{L}} \beta_j^{\mathsf{L}} - \sum_{k=0}^{n-1} \gamma_k^{\mathsf{L}}.$$

In the same manner, from the three-term recurrence relation for $\{Q_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$, we deduce that $q_{\mathsf{L},n}^1 - q_{\mathsf{L},n-1}^1 = \beta_n^{\mathsf{R}} := C_n \beta_n^{\mathsf{L}} C_n^{-1}$ and $q_{\mathsf{L},n}^2 - q_{\mathsf{L},n-1}^2 = \beta_n^{\mathsf{R}} q_{\mathsf{L},n}^1 + \gamma_n^{\mathsf{R}}$, where $\gamma_n^{\mathsf{R}} = C_n C_{n+1}^{-1}$.

If we consider that $W = W^{\mathsf{L}}$ and $W^{\mathsf{R}} = I_N$, and use the representation for $\{P_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ and $\{Q_n^{\mathsf{L}}\}_{n\in\mathbb{N}}$ in z powers, the $(1,2)$ and $(2,2)$ entries in (17) read

$$\begin{aligned}
&(2n+1)I_N + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}} + (\beta_n^{\mathsf{L}})^2) + h_1^{\mathsf{L}} \beta_n^{\mathsf{L}} \\
&\qquad\qquad = [p_{\mathsf{L},n}^1, h_2^{\mathsf{L}}] p_{\mathsf{L},n+1}^1 - [p_{\mathsf{L},n}^2, h_2^{\mathsf{L}}] - [p_{\mathsf{L},n}^1, h_1^{\mathsf{L}}], \\
&\beta_n^{\mathsf{L}} = \gamma_n^{\mathsf{L}}(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}}) + [p_{\mathsf{L},n-1}^1, h_2^{\mathsf{L}}] + h_1^{\mathsf{L}}) - (h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}}) + [p_{\mathsf{L},n}^1, h_2^{\mathsf{L}}] + h_1^{\mathsf{L}})\gamma_{n+1}^{\mathsf{L}}.
\end{aligned}$$

We can write these equations as follows

$$(2n+1)I_N + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}})) + (h_2^{\mathsf{L}} \beta_n^{\mathsf{L}} + h_1^{\mathsf{L}}) \beta_n^{\mathsf{L}}$$

$$= \Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big]\sum_{k=0}^{n}\beta_k^{\mathsf{L}} - \Big[\sum_{i,j=0}^{n-1}\beta_i^{\mathsf{L}}\beta_j^{\mathsf{L}} - \sum_{k=0}^{n-1}\gamma_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big] - \Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_1^{\mathsf{L}}\Big], \quad (28)$$

$$\beta_n^{\mathsf{L}} - \gamma_n^{\mathsf{L}}\big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}}) + h_1^{\mathsf{L}}\big) + \big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}}) + h_1^{\mathsf{L}}\big)\gamma_{n+1}^{\mathsf{L}}$$
$$= -\gamma_n^{\mathsf{L}}\Big[\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big] + \Big[-\sum_{k=0}^{n-1}\beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big]\gamma_{n+1}^{\mathsf{L}}. \quad (29)$$

We will show now that this system contains a noncommutative version of an instance of discrete Painlevé IV equation, as happens in the analogous case for the scalar scenario.

We see, on the right hand side of the nonlinear discrete Equations (28) and (29) nonlocal terms (sums) in the recursion coefficients β_n^{L} and γ_n^{L}, all of them inside commutators. These nonlocal terms vanish whenever the three matrices $\{h_0^{\mathsf{L}}, h_1^{\mathsf{L}}, h_2^{\mathsf{L}}\}$ conform an Abelian set. Moreover, $\{h_0^{\mathsf{L}}, h_1^{\mathsf{L}}, h_2^{\mathsf{L}}, \beta_n^{\mathsf{L}}, \gamma_n^{\mathsf{L}}\}$ is also an Abelian set. In this commutative setting, we have

$$(2n+1)I_N + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}})) + (h_2^{\mathsf{L}}\beta_n^{\mathsf{L}} + h_1^{\mathsf{L}})\beta_n^{\mathsf{L}} = 0_N,$$
$$\beta_n^{\mathsf{L}} - \gamma_n^{\mathsf{L}}\big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}}) + h_1^{\mathsf{L}}\big) + \big(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}}) + h_1^{\mathsf{L}}\big)\gamma_{n+1}^{\mathsf{L}} = 0_N.$$

In terms of $\xi_n := \dfrac{h_0^{\mathsf{L}}}{2} + nI_N + h_2^{\mathsf{L}}\gamma_n$ and $\mu_n := h_2^{\mathsf{L}}\beta_n^{\mathsf{L}} + h_1^{\mathsf{L}}$ the above equations are

$$\beta_n^{\mathsf{L}}\mu_n = -(\xi_n + \xi_{n+1}), \qquad \beta_n^{\mathsf{L}}(\xi_n - \xi_{n+1}) = -\gamma_n\mu_{n-1} + \gamma_{n+1}\mu_{n+1}.$$

Now, we multiply the second equation by μ_n and taking into account the first one we arrive at

$$-(\xi_n + \xi_{n+1})(\xi_n - \xi_{n+1}) = -\gamma_n\mu_{n-1}\mu_n + \gamma_{n+1}\mu_n\mu_{n+1}$$

and so

$$\xi_{n+1}^2 - \xi_n^2 = \gamma_{n+1}\mu_n\mu_{n+1} - \gamma_n\mu_{n-1}\mu_n.$$

Hence,

$$\xi_{n+1}^2 - \xi_0^2 = \gamma_{n+1}\mu_n\mu_{n+1} \qquad \text{and} \qquad \beta_n^{\mathsf{L}}\mu_n = -(\xi_n + \xi_{n+1}) \quad (30)$$

coincide to the ones presented in [60] as a discrete Painlevé IV (dPIV) equation. In fact, taking $\nu_n = \mu_n^{-1}$ we finally arrive at

$$\nu_n\nu_{n+1} = \frac{h_2^{\mathsf{L}}(\xi_{n+1} - h_0^{\mathsf{L}}/2 - nI_N)}{\xi_{n+1}^2 - \xi_0^2} \quad \text{and} \quad \xi_n + \xi_{n+1} = \Big(\big(h_2^{\mathsf{L}}\big)^{-1}h_1^{\mathsf{L}} - \big(h_2^{\mathsf{L}}\big)^{-1}\nu_n^{-1}\Big)\nu_n^{-1}.$$

If we take $h_1^{\mathsf{L}} = 0$ in (30), then $\mu_n = h_2^{\mathsf{L}}\beta_n^{\mathsf{L}}$, and so

$$(\beta_n^{\mathsf{L}})^2 h_2^{\mathsf{L}} = -(\xi_n + \xi_{n+1}).$$

Now, taking square in the first equation in (30) we obtain

$$(\xi_n + \xi_{n+1})(\xi_{n+1} + \xi_{n+2}) = \Big(\big(\xi_{n+1} - \frac{h_0^{\mathsf{L}}}{2} - nI_N\big)^{-1}(\xi_{n+1}^2 - \xi_0^2)\Big)^2,$$

which is an instance of dPIV by Grammaticos, Hietarinta, and Ramani (cf. [61]).

Thus, (28) and (29) for $B_{\mathsf{L}} = 0_N$ may be considered as a non-Abelian extension of this instance of dPIV.

We have just seen that,

Theorem 4 (**Non-Abelian extension of the dPIV**)**.** *When $B_{\mathsf{L}} = 0_N$, the following nonlocal nonlinear non-Abelian system for the recursion coefficients is fulfilled*

$$
\begin{aligned}
&(2n+1)I_N + h_0^{\mathsf{L}} + h_2^{\mathsf{L}}(\gamma_{n+1}^{\mathsf{L}} + \gamma_n^{\mathsf{L}})) + h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}})^2 \\
&\qquad = \Big[\sum_{k=0}^{n-1} \beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big] \sum_{k=0}^{n} \beta_k^{\mathsf{L}} - \Big[\sum_{i,j=0}^{n-1} \beta_i^{\mathsf{L}}\beta_j^{\mathsf{L}} - \sum_{k=0}^{n-1} \gamma_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big], \\
&\beta_n^{\mathsf{L}} - \gamma_n^{\mathsf{L}}(h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n-1}^{\mathsf{L}})) + (h_2^{\mathsf{L}}(\beta_n^{\mathsf{L}} + \beta_{n+1}^{\mathsf{L}}))\gamma_{n+1}^{\mathsf{L}} \\
&\qquad = -\gamma_n^{\mathsf{L}}\Big[\sum_{k=0}^{n-1} \beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big] + \Big[-\sum_{k=0}^{n-1} \beta_k^{\mathsf{L}}, h_2^{\mathsf{L}}\Big]\gamma_{n+1}^{\mathsf{L}}.
\end{aligned}
$$

Moreover, this system reduces in the commutative context to the standard dPIV equation.

6. Conclusions and Future Work

In this paper, using the Riemann–Hilbert problem for the Laguerre-type weight matrices, we obtain differential properties of the corresponding matrix biorthogonal polynomials as well as for the second kind matrix functions. It is remarkable to notice that we do not explicitly know the matrix measure, but only its differential properties.

We will consider in future work the case that the support of the measure has two finite end points, the Jacobi-type weight matrices, trying to also obtain differential properties and extensions of Painlevé discrete systems.

Author Contributions: Investigation, A.B., A.F.M., A.F. and M.M. All the authors contributed equally in the phases of this research project and contributed substantially to the work reported. All authors have read and agreed to the published version of the manuscript.

Funding: A.B. acknowledges Centro de Matemática da Universidade de Coimbra (CMUC)–UID/MAT/00324/2019, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. A.F.M. and A.F. acknowledges CIDMA Center for Research and Development in Mathematics and Applications (University of Aveiro) and the Portuguese Foundation for Science and Technology (FCT) within project UIDB/04106/2020 and UIDP/04106/2020. M.M. thanks financial support from the Spanish "Agencia Estatal de Investigación" research project [PGC2018-096504-B-C33], *Ortogonalidad y Aproximacioón: Teoría y Aplicaciones en Física Matemática*.

Institutional Review Board Statement: According to the Declaration of Helsinki no ethical approval is required.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

List of Acronyms

The following abbreviations are used in this manuscript:

MOP	matrix orthogonal polynomials
MOPRL	matrix orthogonal polynomials in the real line
OPRL	orthogonal polynomials in the real line
dPIV	discrete Painlevé IV

References

1. Krein, M.G. Infinite J-matrices and a matrix moment problem. *Dokl. Akad. Nauk SSSR* **1949**, *69*, 125–128.
2. Krein, M.G. *Fundamental Aspects of the Representation Theory of Hermitian Operators with Deficiency Index (m, m)*; Series 2; AMS Translations: Providence, RI, USA, 1971; Volume 97, pp. 75–143.
3. Berezanskii, J.M. Expansions in Eigenfunctions of Selfadjoint Operators. In *Translations of Mathematical Monographs*; 1968. Available online: https://bookstore.ams.org/mmono-17 (accessed on 24 February 2022).

4. Geronimo, J.S. Scattering theory and matrix orthogonal polynomials on the real line. *Circuits Syst. Signal Process.* **1982**, *1*, 471–495. [CrossRef]
5. Aptekarev, A.I.; Nikishin, E.M. The scattering problem for a discrete Sturm-Liouville operator. *Math. USSR Sb* **1984**, *49*, 325–355. [CrossRef]
6. Durán, A.J.; Grünbaum, F.A. Orthogonal matrix polynomials, scalar-type Rodrigues' formulas and Pearson equations. *J. Approx. Theory* **2005**, *134*, 267–280. [CrossRef]
7. Durán, A.J.; Grünbaum, F.A. Structural formulas for orthogonal matrix polynomials satisfying second order differential equations, I. *Constr. Approx.* **2005**, *22*, 255–271. [CrossRef]
8. Borrego, J.; Castro, M.; Durán, A.J. Orthogonal matrix polynomials satisfying differential equations with recurrence coefficients having non-scalar limits. *Integral Transform. Spec. Funct.* **2012**, *23*, 685–700. [CrossRef]
9. Durán, A.J. Matrix inner product having a matrix symmetric second order differential operator. *Rocky Mountain J. Math.* **1997**, *27*, 585–600. [CrossRef]
10. Durán, A.J.; Grünbaum, F.A. Orthogonal matrix polynomials satisfying second order differential equations. *Int. Math. Res. Not.* **2004**, *10*, 461–484. [CrossRef]
11. Durán, A.J.; de la Iglesia, M.D. Second order differential operators having several families of orthogonal matrix polynomials as eigenfunctions. *Int. Math. Res. Not.* **2008**, *2008*, rnn084. [CrossRef]
12. Álvarez-Fernández, C.; Fidalgo, U.; Mañas, M. The multicomponent 2D Toda hierarchy: Generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems. *Inverse Probl.* **2010**, *26*, 055009. [CrossRef]
13. Álvarez-Fernández, C.; Mañas, M. On the Christoffel–Darboux formula for generalized matrix orthogonal polynomials of multigraded Hankel type. *J. Math. Anal. Appl.* **2014**, *418*, 238–247. [CrossRef]
14. Adler, M.; van Moerbeke, P. Generalized orthogonal polynomials, discrete KP and Riemann–Hilbert problems. *Commun. Math. Phys.* **1999**, *207*, 589–620. [CrossRef]
15. Adler, M.; van Moerbeke, P. Darboux transforms on band matrices, weights and associated polynomials. *Int. Math. Res. Not.* **2001**, *18*, 935–984. [CrossRef]
16. Álvarez-Fernández, C.; Ariznabarreta, G.; García-Ardila, J.C.; Mañas, M.; Marcellán, F. Christoffel transformations for matrix orthogonal polynomials in the real line and the non-Abelian 2D Toda lattice hierarchy. *Int. Math. Res. Not.* **2017**, *5*, 1285–1341. [CrossRef]
17. Ariznabarreta, G.; García-Ardila, J.C.; Mañas, M.; Marcellán, F. Non-Abelian integrable hierarchies: Matrix biorthogonal polynomials and perturbations. *J. Phys. A Math. Theor.* **2018**, *51*, 205204. [CrossRef]
18. Ariznabarreta, G.; García-Ardila, J.C.; Mañas, M.; Marcellán, F. Matrix biorthogonal polynomials on the real line: Geronimus transformation. *Bull. Math.Sci.* **2019**, *9*, 1–66. [CrossRef]
19. Fokas, A.S.; Its, A.R.; Kitaev, A.V. The isomonodromy approach to matrix models in 2D quantum gravity. *Commun. Math. Phys.* **1992**, *147*, 395–430. [CrossRef]
20. Deift, P.A. *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*; Courant Lecture Notes; American Mathematical Society: Providence, RI, USA, 2000; Volume 3.
21. Deift, P.A. Riemann–Hilbert methods in the theory of orthogonal polynomials, in Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday. In *Proceedings of Symposia in Pure Mathematics*; American Mathematical Society: Providence, RI, USA, 2007; Volume 76, pp. 715–740.
22. Deift, P.A.; Zhou, X. A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation. *Ann. Math.* **1993**, *137*, 295–368. [CrossRef]
23. Deift, P.A.; Zhou, X. Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. *Commun. Pure Appl. Math.* **2003**, *56*, 1029–1077. [CrossRef]
24. Deift, P.A.; Gioev, D. *Random Matrix Theory: Invariant Ensembles and Universality*; Courant Lecture Notes in Mathematics; American Mathematical Society: Providence, RI, USA, 2009; Volume 18.
25. Daems, E.; Kuijlaars, A.B.J. Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions. *J. Approx. Theory* **2007**, *146*, 91–114. [CrossRef]
26. Daems, E.; Kuijlaars, A.B.J.; Veys, W. Asymptotics of non-intersecting Brownian motions and a 4×4 Riemann–Hilbert problem. *J. Approx. Theory* **2008**, *153*, 225–256. [CrossRef]
27. Kuijlaars, A.B.J.; Martínez-Finkelshtein, A.; Wielonsky, F. Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weights. *Commun. Math. Phys.* **2009**, *286*, 217–275. [CrossRef]
28. Kuijlaars, A.B.J. Multiple orthogonal polynomial ensembles. *arXiv* **2010**, arXiv:0902.1058.
29. McLaughlin, K.T.-R.; Vartanian, A.H.; Zhou, X. Asymptotics of Laurent polynomials of even degree orthogonal with respect to varying exponential weights. *Int. Math. Res. Pap.* **2006**, *2006*, 62815. [CrossRef]
30. McLaughlin, K.T.-R.; Vartanian, A.H.; Zhou, X. Asymptotics of Laurent polynomials of odd degree orthogonal with respect to varying exponential weights. *Constr. Approx.* **2008**, *27*, 149–202. [CrossRef]
31. Dai, D.; Kuijlaars, A.B.J. Painlevé IV asymptotics for orthogonal polynomials with respect to a modified Laguerre weight. *Stud. Appl. Math.* **2009**, *122*, 29–83. [CrossRef]
32. Its, A.R.; Kuijlaars, A.B.J.; Östensson, J. Asymptotics for a special solution of the thirty fourth Painlevé equation. *Nonlinearity* **2009**, *22*, 1523–1558. [CrossRef]

33. Van Assche, W. Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials. In *Proceedings of the International Conference on Difference Equations, Special Functions and Orthogonal Polynomials*; World Scientific: Singapore, 2007; pp. 687–725.
34. Van Assche, W. *Orthogonal Polynomials and Painlevé Equations*; Australian Mathematical Society Lecture Series; Cambridge University Press: Cambridge, UK, 2017; Volume 27.
35. Freud, G. On the coefficients in the recursion formulae of orthogonal polynomials. *Proc. R. Ir. Acad. Sect. A* **1976**, *76*, 1–6.
36. Magnus, A.P. Freud's equations for orthogonal polynomials as discrete Painlevé equations. In *Symmetries and Integrability of Difference Equations*; London Mathematical Society Lecture Note Series; Cambridge University Press: Cambridge, UK, 1999; Volume 255, pp. 228–243.
37. Periwal, V.; Shevitz, D. Unitary-Matrix Models as Exactly Solvable String Theories. *Phys. Rev. Lett.* **1990**, *64*, 1326–1329. [CrossRef] [PubMed]
38. Periwal, V.; Shevitz, D. Exactly solvable unitary matrix models: Multicritical potentials and correlations. *Nucl. Phys.* **1990**, *B344*, 731–746. [CrossRef]
39. Hisakado, M. Unitary matrix models and Painlevé III. *Mod. Phys. Lett.* **1996**, *A11*, 3001–3010. [CrossRef]
40. Baik, J. Riemann–Hilbert problems for last passage percolation in Recent Developments in Integrable Systems and Riemann–Hilbert Problems. *Contemp. Math.* **2003**, *326*, 1–21.
41. Baik, J.; Deift, P.; Johansson, K. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **1999**, *12*, 1119–1178. [CrossRef]
42. Tracy, C.A.; Widom, H. Random unitary matrices, permutations and Painlevé. *Commun. Math. Phys.* **1999**, *207*, 665–685. [CrossRef]
43. Creswell, C.; Joshi, N. The discrete first, second and thirty-fourth Painlevé hierarchies. *J. Phys. A Math. Gen.* **1999**, *32*, 655–669. [CrossRef]
44. Clarkson, P.A. Painlevé equations–Nonlinear Special Functions. In *Orthogonal Polynomials and Special Functions*; Marcellán, F., Van Assche, W., Eds.; Lecture Notes in Mathematics; Springer: New York, NY, USA, 2011; Volume 1883, pp. 331–400.
45. Clarkson, P.A.; Jordaan, K. The Relationship Between Semiclassical Laguerre Polynomials and the Fourth Painlevé Equation, *Constr. Approx.* **2014**, *29*, 223–254. [CrossRef]
46. Clarkson, P.A.; Loureiro, A.F.; Van Assche, W. Unique positive solution for an alternative discrete Painlevé I equation. *J. Differ. Equ. Appl.* **2016**, *22*, 656–675. [CrossRef]
47. Zhou, X. The Riemann–Hilbert problem and inverse scattering. *SIAM J. Math. Anal.* **1989**, *20*, 966–986. [CrossRef]
48. Cassatella-Contra, G.A.; Mañas, M. Riemann–Hilbert Problems, Matrix Orthogonal Polynomials and Discrete Matrix Equations with Singularity Confinement. *Stud. Appl. Math.* **2011**, *128*, 252–274. [CrossRef]
49. Cassatella-Contra, G.A.; Mañas, M.; Tempesta, P. Singularity confinement for matrix discrete Painlevé equations. *Nonlinearity* **2014**, *27*, 2321–2335. [CrossRef]
50. Grünbaum, F.A.; de la Iglesia, M.D.; Martínez-Finkelshtein, A. Properties of matrix orthogonal polynomials via their Riemann–Hilbert characterization. *SIGMA* **2011**, *7*, 098. [CrossRef]
51. Cassatella-Contra, G.A.; Mañas, M. Riemann–Hilbert problem and matrix discrete Painlevé II system. *Stud. Appl. Math.* **2019**, *143*, 272–314. [CrossRef]
52. Branquinho, A.; Foulquié Moreno, A.; Mañas, M. Matrix biorthogonal polynomials: Eigenvalue problems and non-abelian discrete Painlevé equations: A Riemann–Hilbert problem perspective. *J. Math. Anal. Appl.* **2021**, *494*, 124605. [CrossRef]
53. Gel'fand, I.M.; Gel'fand, S.; Retakh, V.S.; Wilson, R. Quasideterminants. *Adv. Math.* **2005**, *193*, 56–141. [CrossRef]
54. Branquinho, A.; Foulquié Moreno, A.; Mañas, M. Riemann–Hilbert problem and Matrix Biorthogonal Polynomials. In *Orthogonal Polynomials: Current Trends and Applications: Proceedings of the 7th EIBPOA Conference*; Springer International Publishing: Berlin/Heidelberg, Germany, 2021; pp. 1–15.
55. Kuijlaars, A.B.J. Riemann–Hilbert analysis for orthogonal polynomials, In *Orthogonal Polynomials and Special Functions*; Springer: Berlin/Heidelberg, Germany, 2003; pp. 167–210.
56. Gakhov, F.D. *Boundary Value Problems*; Dover Publications, Inc.: New York, NY, USA, 1990.
57. Wasow, W. *Asymptotic Expansions for oRdinary Differential Equations*; Pure and Applied Mathematics; Interscience Publishers: New York, NY, USA, 1965; Volume XIV.
58. Muskhelishvili, N.I. *Singular Integral Equations*; Dover Publications, Inc.: New York, NY, USA, 1992.
59. Cantero, M.J.; José, M.; Moral, L.; Velázquez, L. Matrix orthogonal polynomials whose derivatives are also orthogonal. *J. Approx. Theory* **2007**, *146*, 174–211. [CrossRef]
60. Boelen, L.; Van Assche, W. Discrete Painlevé equations for recurrence coefficients of semiclassical Laguerre polynomials. *Proc. Am. Math. Soc.* **2010**, *138*, 1317–1331. [CrossRef]
61. Grammaticos, B.; Hietarinta, J.; Ramani, A. Discrete versions of the Painlevé equations. *Phys. Rev. Lett.* **1991**, *67*, 1829–1832.