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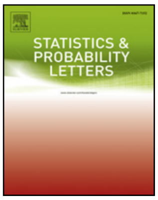

A mathematical approach to detect the Taylor property in TARCH processes

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ABSTRACT

We analyze the presence of the Taylor property in a well-known class of models for financial time series, the threshold ARCH (TARCH) model. This property is the theoretical counterpart of the stylized fact known as Taylor effect, detected in several empirical studies which have shown that the autocorrelations of the absolute returns are larger than those of the squared returns. We establish that the Taylor property is present for some parameterizations of the first order TARCH model. As this fact is strongly dependent on the distribution of the generating white noise, we analyze and compare, for several distributions of that process, the sets of parameterizations of the model presenting the Taylor property. Finally, a simulation study strongly suggests that TARCH models are considerably more likely to capture the Taylor effect than ARCH ones.

1. Introduction

In time series analysis, the selection of the class of models well-fitted to data determines the validation of the subsequent studies and conclusions. This decision is, in particular, based on our knowledge of the theoretical models' properties, as we choose the class that better captures the empirical properties detected in data.

In the case of high-frequency financial return series, several studies have shown that they share empirical regularities, usually called stylized facts (Cont, 2001). Thus, the models proposed for these series should satisfy the theoretical properties corresponding to such stylized facts. However, some of these properties are still not completely studied for many models, as is the case of the Taylor property. This property is the theoretical counterpart of the stylized fact known as Taylor effect and it states that the autocorrelations of the absolute value time series are larger than those of the squared time series. The studies on this property are generally based on simulations, due to the difficulty of handling the autocorrelations of such models.

In this work, we develop a theoretical study about the presence of the Taylor property in a well-known class of models for financial time series, the threshold ARCH (TARCH) models. More precisely, we establish that the Taylor property is present for a set of parameterizations of a TARCH model with a symmetrical generating error process. Moreover, we analyze and compare, for several distributions of that process, the sets of parameterizations of the model presenting the Taylor property. We observe that for some distributions, namely the fat tailed ones, these sets only depend on the existence of the fourth order moment of the generating white noise.

This study was motivated by the work of He and Teräsvirta (1999) on conditionally Gaussian absolute value generalized ARCH (AVGARCH) models, where, in particular, they assure the presence of the Taylor property for some of these models. We point out that in our paper, besides considering much more general models, the threshold ARCH ones, and assuring the presence of the Taylor property, we establish the explicit regions of the models parameterizations verifying that property.

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Finally, following the empirical study developed by He and Teräsvirta (1999) on conditionally Gaussian ARCH and GARCH models, we compare conditionally non-Gaussian TARCH and ARCH models in what concerns the presence of the Taylor property. He and Teräsvirta (1999) conclude, in their paper, that ARCH and GARCH models are not the best ones to capture the Taylor effect in real data. From our simulation study, as our findings for the ARCH model point out to the same conclusion drawn by He and Teräsvirta (1999), we observe that TARCH models are much more able to capture the Taylor effect than ARCH ones.

2. From the Taylor effect to the Taylor property

Taylor (1986) analyzed some features of financial returns and, to point out their non-linear structure, he studied, for 40 returns series, the autocorrelations of transformed returns. He observed that, for most of them, the sample autocorrelations of the absolute returns, $\hat{\rho}_n(1) = \widehat{corr}(|\varepsilon_t|, |\varepsilon_{t-n}|)$, were larger than those of the squared returns, $\hat{\rho}_n(2) = \widehat{corr}\left(\varepsilon_t^2, \varepsilon_{t-n}^2\right)$, for lags between 1 and 30.

Subsequent works widened the research to the autocorrelations of the powers of absolute observations, $\hat{\rho}_n(\delta) = \widehat{corr}\left(|\varepsilon_t|^\delta, |\varepsilon_{t-n}|^\delta\right)$ and, in particular, Granger and Ding (1995) called *Taylor effect* the empirical relation $\hat{\rho}_n(1) > \hat{\rho}_n(\delta)$, for any $\delta \neq 1$. However, studies referred to by Mora-Galán et al. (2004) indicate that maximum autocorrelation is not always obtained when $\delta = 1$. Nevertheless the same studies show that the empirical relation analyzed by Taylor (1986) holds.

This stylized fact observed by Taylor (1986) is well documented, but the analysis concerning the presence of the corresponding theoretical property in financial time series models is, as we already mentioned, very incomplete. In fact, to the best of our knowledge, until 1999 only some simulation studies had been done. In 1999, He and Teräsvirta determined the autocorrelations expressions for some models and then studied the theoretical relation for one of those models. More precisely, they concentrated their study on the autocorrelation of lag 1 and called the theoretical relation

$$\rho_1(1) > \rho_1(2) \tag{1}$$

the *Taylor property*; they explored it for the AVGARCH model with normal generating distributions, proving the presence of the Taylor property for the first order absolute value ARCH (AVARCH) model. The aim of this paper is to analyze the presence of the Taylor property for more general models, the TARCH ones with a generating symmetrical error process. These models have the advantage of taking into account in the volatility, in different ways, positive and negative reactions of the process. Moreover, the proof considered for establishing the main result allows us to obtain, explicitly, the set of parameterizations verifying the property, which is a new result, even for AVARCH models.

3. The first order TARCH model

A real stochastic process $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is said to be a first order threshold autoregressive conditional heteroscedastic, TARCH(1), model if, for any $t \in \mathbb{Z}$,

$$\begin{cases} \varepsilon_t = \sigma_t Z_t \\ \sigma_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^+ - \beta_1 \varepsilon_{t-1}^- \end{cases} \tag{2}$$

with $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $\varepsilon_t^+ = \varepsilon_t \mathrm{I}_{\{\varepsilon_t \geq 0\}}$, $\varepsilon_t^- = \varepsilon_t \mathrm{I}_{\{\varepsilon_t < 0\}}$ and where $Z = (Z_t, t \in \mathbb{Z})$ is a sequence of independent and identically distributed real random variables, with zero mean and unit variance, such that Z_t is independent of the σ-field generated by the past of ε, $\underline{\varepsilon}_{t-1} = \sigma(\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots)$.

This model (Rabemananjara and Zakoian, 1993; Zakoian, 1994) has the particularity, as one can easily observe from σ_t, that it takes into account different reactions of volatility according to the sign of past values of the process. We point out that AVARCH(1) model, studied by He and Teräsvirta (1999), is a particular case of the TARCH(1) model, obtained when $\alpha_1 = \beta_1$, that is, when that characteristic of different reactions of volatility is not taken into account.

To develop our study about the Taylor property we assume that Z_t has a symmetrical density and take into account the expressions obtained by He and Teräsvirta (1999) for the first order autocorrelation of the absolute value process, $\rho_1(1) = corr(|\varepsilon_t|, |\varepsilon_{t-1}|)$, and of the squared process, $\rho_1(2) = corr\left(\varepsilon_t^2, \varepsilon_{t-1}^2\right)$. Let us consider, under the corresponding existence conditions,

$$\gamma_i = \frac{\alpha_1^i + \beta_1^i}{2} E\left(|Z_t|^i\right), \quad \text{for } i = 1, 2, 3, 4. \tag{3}$$

If the process ε verifies the condition of strict and weak stationarity, that is $\gamma_2 < 1$, then $\rho_1(1)$ exists and is equal to

$$\rho_1(1) = \gamma_1, \tag{4}$$

and, if $\gamma_4 < 1$, then $\rho_1(2)$ exists and is equal to

$$\rho_1(2) = \gamma_2 + 2\gamma_1(1-\gamma_2)(1-\gamma_4)\left[\frac{E\left(|Z_t|^3\right)}{E(|Z_t|)}(1+2\gamma_1+2\gamma_2+\gamma_1\gamma_2)(1-\gamma_1) - (1+\gamma_1)(1-\gamma_3)\right]$$
$$\times \left[E\left(Z_t^4\right)\Delta(1-\gamma_1)(1-\gamma_2) - (1+\gamma_1)^2(1-\gamma_3)(1-\gamma_4)\right]^{-1}, \tag{5}$$

where $\Delta = 1 + 3\gamma_1 + 5\gamma_2 + 3\gamma_3 + 3\gamma_1\gamma_2 + 5\gamma_1\gamma_3 + 3\gamma_2\gamma_3 + \gamma_1\gamma_2\gamma_3$. We note that the parameter α_0 is free.

4. Main result

Let $" = ."_t; t \in Z/$ be a process following the TARCH. 1/ model defined in (2). When ${}_4 < 1$, we can consider the function F such that $F\,.\;{}_1;\;{}_1/ = {}_1\,.1/ - {}_1\,.2/$. Suppose $\frac{@}{@_1}$ satisfies the following condition:

Hypothesis H1. *The function* $\frac{@}{@_1}$ *is not null in at least one of the sets*

$$A_1 = \left(.\;{}_1;\;{}_1/ \in \;\; 0;\ \sqrt[4]{2\ E\ Z_t^4}^{\,-1\ 2} \;:\; {}_1 \le {}_1 \text{ and } F\,.\;{}_1;\;{}_1/ = 0 \right)$$

or

$$A_2 = \left(.\;{}_1;\;{}_1/ \in \;\; 0;\ \sqrt[4]{2\ E\ Z_t^4}^{\,-1\ 2} \;:\; {}_1 \ge {}_1 \text{ and } F\,.\;{}_1;\;{}_1/ = 0 \;:\right)$$

The following theorem states the existence of an open set of parameterizations of the TARCH(1) model that verifies the Taylor property. We point out that this set depends on the generator process law and strictly contains the set of parameterizations of the AVARCH model verifying the Taylor property.

Theorem 1. *Let* $" = ."_t; t \in Z/$ *be the TARCH*$.1/$ *model defined in* (2) *where* Z_t *has a symmetrical density and* $E\,.|Z_t|/ > \sqrt[4]{E\ Z_t^4}^{\,-1}$. *If the Hypothesis H1 is verified, there exists an open set in* R^2, *included in* $.\;{}_1;\;{}_1/ \in R^2 : {}_4 < 1$, *of parameterizations of TARCH(1) models satisfying the Taylor property.*

Proof. We consider four parts in this proof. First, we establish that the equations ${}_4 - 1 = 0$ and ${}_1\,.1/ - {}_1\,.2/ = 0$ express ${}_1$ implicitly as a continuous function of ${}_1$ in a neighborhood of

$$B_1 = \left(.\;{}_1;\;{}_1/ \in \;\; 0;\ \sqrt[4]{2\ E\ Z_t^4}^{\,-1\ 2} \;:\; {}_1 \le {}_1 \right)$$

or of

$$B_2 = \left(.\;{}_1;\;{}_1/ \in \;\; 0;\ \sqrt[4]{2\ E\ Z_t^4}^{\,-1\ 2} \;:\; {}_1 \ge {}_1 \;:\right)$$

Then, we show that these two equations have at most six common solutions in $B_1 \cup B_2$ (they may not even have common points in that set). In the third part, we prove that, for ${}_1 = {}_1 =$, there exists a subset of $\in R^+ : {}_4 < 1$ whose elements satisfy the condition ${}_1\,.1/ > {}_1\,.2/$. Finally, we extend this conclusion to the case of ${}_1 \ne {}_1$.

(i) It is easily seen that the equation ${}_4 - 1 = 0$ expresses ${}_1$ implicitly as a continuous function of ${}_1$ in $0;\ \sqrt[4]{2\ E\ Z_t^4}^{\,-1}$.

To show that the equation ${}_1\,.1/ - {}_1\,.2/ = 0$ defines implicitly ${}_1$ as a continuous function of ${}_1$ in a neighborhood of B_1 (respectively, or in a neighborhood of B_2), we consider $A_1 = B_1 \cap .\;{}_1;\;{}_1/ \in R^2 : F\,.\;{}_1;\;{}_1/ = 0$ and $A_2 = B_2 \cap .\;{}_1;\;{}_1/ \in R^2 : F\,.\;{}_1;\;{}_1/ = 0$, where $F\,.\;{}_1;\;{}_1/ = {}_1\,.1/ - {}_1\,.2/$; it is enough to establish that $\frac{@}{@_1}$ and $\frac{@}{@_1}$ are continuous functions in a open set containing A_1 (respectively, or containing A_2) and that $\frac{@}{@_1}\ \breve{}_1;\ \breve{}_1 \ne 0$, for all $\breve{}_1;\ \breve{}_1$ in A_1 (respectively, or in A_2).

The functions $\frac{@}{@_1}$ and $\frac{@}{@_1}$ are well-defined if ${}_4 \le 1$, because ${}_1\,.2/$ exists (He and Teräsvirta, 1999). These functions are obviously continuous and the last condition mentioned, concerning the non nullity of the function $\frac{@}{@_1}$, is verified from Hypothesis H1.

(ii) Let us now determine the number of intersections between the graphics of the two functions defined by the equations ${}_4 - 1 = 0$ and ${}_1\,.1/ - {}_1\,.2/ = 0$.

Note that, in (i), we have shown that these functions are defined implicitly in a neighborhood of B_1 or of B_2. However, as $F\,.\;{}_1;\;{}_1/ = F\,.\;{}_1;\;{}_1/$ and $G\,.\;{}_1;\;{}_1/ = G\,.\;{}_1;\;{}_1/$, with $G\,.\;{}_1;\;{}_1/ = {}_4 - 1$, the curves defined by $F\,.\;{}_1;\;{}_1/ = 0$ and $G\,.\;{}_1;\;{}_1/ = 0$ are symmetrical in relation to the axis ${}_1 = {}_1$. So, we conclude that $\breve{}_1;\ \breve{}_1 \in B_1$ is one of the intersection points of interest if and only if $\breve{}_1;\ \breve{}_1 \in B_2$ is also one of those points. Therefore we will omit the set, B_1 or B_2, to which the point belongs.

We intend to solve the system

$$\begin{array}{l} {}_4 = 1 \\ {}_1\,.1/ = {}_1\,.2/ \end{array}$$

which is equivalent to

$$\begin{cases} 1-\frac{E\ Z_t^4}{2}\quad {}_1^4+{}_1^4=0 \\ E\,.|Z_t|/.\ {}_1+{}_1/-{}_1^2+{}_1^2=0: \end{cases}$$

From the second equation, we obtain

$$_1=\frac{E\,.|Z_t|/}{2}\pm\sqrt{-{}_1^2+E\,.|Z_t|/\ {}_1+\left(\frac{E\,.|Z_t|/}{2}\right)^2}$$

and, so,

$$\begin{aligned} P\,.\ {}_1/ &= -16\ {}_1^8+32E\,.|Z_t|/\ {}_1^7-16\,.E\,.|Z_t|//^2\ {}_1^6 \\ &\quad -8\left(.E\,.|Z_t|//^4-\frac{4}{E\ Z_t^4}\right){}_1^4-\frac{32E\,.|Z_t|/}{E\ Z_t^4}\ {}_1^3 \\ &\quad -\frac{16\,.E\,.|Z_t|//^2}{E\ Z_t^4}\ {}_1^2+\frac{32\,.E\,.|Z_t|//^3}{E\ Z_t^4}\ {}_1 \\ &\quad +\frac{8}{E\ Z_t^4}\left(.E\,.|Z_t|//^4-\frac{2}{E\ Z_t^4}\right)=0: \end{aligned}$$

By the Descartes' sign rule, the maximum number of positive roots of this equation is equal to the number of sign changes of the coefficients of the polynomial P, as we proceed from the highest to the lowest power. Since the sign of $.E\,.|Z_t|//^4-\frac{2}{E\ Z_t^4}$ and that of $.E\,.|Z_t|//^4-\frac{4}{E\ Z_t^4}$ depend on the Z law, we can only state that the number of sign changes varies between 3 and 6. Therefore, the maximum number of positive roots of this equation is 6 and, consequently, the maximum number of intersections between the graphics of the two functions defined by the equations $_4-1=0$ and $_1\,.1/-{}_1\,.2/=0$ is also 6.

(iii) Let us now consider the points such that $_1={}_1$. We denote $_1$ and $_1$ by and, to stress the fact that the autocorrelations depend of , $_1\,.1/$ by $_1\,.1;\ /$ and $_1\,.2/$ by $_1\,.2;\ /$. As >0 and $_4<1$, then >0 and $^4E\ Z_t^4<1$, therefore $0<\ <\sqrt[4]{\frac{1}{E\ Z_t^4}}$. Moreover, we have $_1\,.1;\ /={}_1=\ E\,.|Z_t|/$ and $_1\,.2;\ /={}_2+.1-{}_4/\times\ =\ ^2+.1-{}_4/\times$, where $=\ .\ /$ is obtained from (5). Note that $_1\,.1;\ /$ and $_1\,.2;\ /$ are both continuous functions of . Moreover,

$$_1\,.1;\ /=\ E\,.|Z_t|/\quad\text{converges to}\ \sqrt[4]{\frac{1}{E\ Z_t^4}}E\,.|Z_t|/$$

and

$$_1\,.2;\ /={}^2+.1-{}_4/\times\qquad\text{converges to}\ \left(\sqrt[4]{\frac{1}{E\ Z_t^4}}\right)^2;$$

when converges, by smaller values, to $\left(\sqrt[4]{\frac{1}{E\ Z_t^4}}\right)$. In addition, as we suppose that $E\,.|Z_t|/>\sqrt[4]{\frac{1}{E\ Z_t^4}}$, there exists a left neighborhood of $\sqrt[4]{\frac{1}{E\ Z_t^4}}$ such that $_1\,.1;\ />{}_1\,.2;\ /$.

(iv) Considering now the general form of $_1\,.1/$ and of $_1\,.2/$ (as functions of $_1$ and $_1$, not necessarily equal), we can finally state, by the continuity of these functions, that, in a neighborhood of the point $\left(\sqrt[4]{\frac{1}{E\ Z_t^4}};\sqrt[4]{\frac{1}{E\ Z_t^4}}\right)$ intersected with the set $.\ {}_1;\ {}_1/\in\mathrm{R}^2:\ {}_4<1$, we have $_1\,.1/>{}_1\,.2/$. Thus, there exists a region of the plane or, eventually, a reunion of two or three regions of the plane (depending on the number of intersections), which is a subset of $.\ {}_1;\ {}_1/\in\mathrm{R}^+\times\mathrm{R}^+:\ {}_4<1$, whose elements satisfy the condition $_1\,.1/>{}_1\,.2/$. ■

5. Applications

In this section we illustrate in a constructive way the region of parameterizations with the Taylor property for some TARCH(1) models. The aim of this section is also to analyze how this set behaves when we change the distribution of the

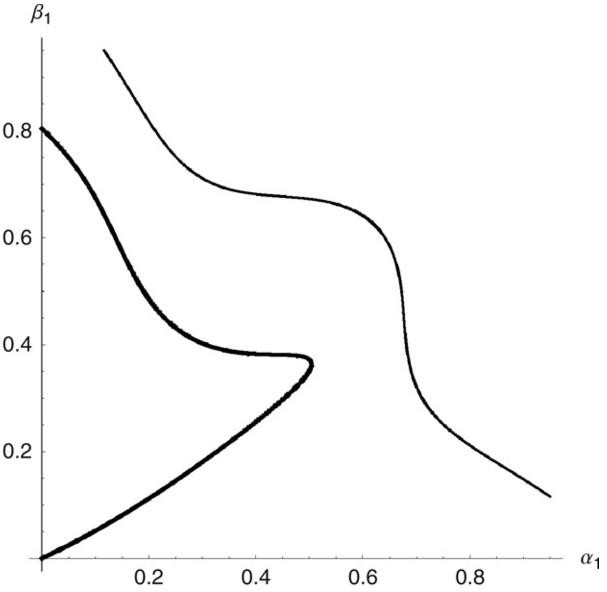

Fig. 1. Curves defined by $F(\ _1;\ _1) = 0$ (thinner line) and by $\frac{\partial}{\partial\ _1}(\ _1;\ _1) = 0$ (thicker line), when Z_t follows the standard normal distribution.

generator process Z. To achieve this, we use Theorem 1 and then determine such a set for the particular case of the absolute value ARCH(1) model considering several distributions for the generator noise. The distributions considered for Z_t must be symmetrical, of unit variance and the moments must verify the condition $E(|Z_t|) > \sqrt[4]{E(Z_t^4)}^{\,-1}$. Some examples of these distributions are the standard normal distribution, the unit variance distribution based on the Student's t-distribution with n degrees of freedom $(n > 4)$ and the symmetrical and unit variance triangular distribution. In what concerns the Hypothesis H1, we establish its validity graphically. Namely, we verify if the curves defined by $F(\ _1;\ _1) = 0$ and by $\frac{\partial}{\partial\ _1}(\ _1;\ _1) = 0$ do not intersect or, when they intersect, if the intersections points are only above or only below the line $\ _1 = \ _1$.

5.1. Normal distribution

Let us first consider the case where Z_t follows the standard normal distribution, that is, where " follows a conditionally Gaussian TARCH(1) model. In this case $\ _4 < 1 \Leftrightarrow 0 < \ _1 < \sqrt[4]{\frac{2}{3} - \ _1^4}$, which defines the region of existence of the autocorrelations. So, from the analysis of Fig. 1, we can state that the Hypothesis H1 is verified whenever this condition is fulfilled. Hence the Taylor property is present for a subset of parameterizations of this model, some of them shown in Fig. 2 and obtained by an exploratory study.

For the absolute value ARCH(1) model, this conclusion coincides with the one drawn by He and Teräsvirta (1999). Moreover, we are able to identify completely, in this case, the subset of parameterizations of the absolute value ARCH(1) model where the Taylor property is present. In fact,

- The case of $\ _1 = \ _1 = \ $ was considered in the third part of the proof of Theorem 1, where we established that the Taylor property is present in a left neighborhood of $\sqrt[4]{\frac{1}{E(Z_t^4)}}$. Note that $\ _4 < 1 \Leftrightarrow \ < \sqrt[4]{\frac{1}{E(Z_t^4)}}$ is the condition for the existence of the fourth order moment of the process following the absolute value ARCH(1) model. Therefore we have the upper limit of the interval, which is $\sqrt[4]{\frac{1}{3}} \simeq 0.7598$.
- To find the lower limit of the interval, we need to find the solutions of $\ _1(1) - \ _1(2) = 0$ in $\left]0; \sqrt[4]{\frac{1}{3}}\right[$, where

$$
\begin{aligned}
\ _1(1) - \ _1(2) = {}& \ ^2[-2\ \ + \ ^2 + (2 - \ )\sqrt{2\ } \ + (6\ - 20)\ \ ^2 \\
&+ (3\ - 8)\sqrt{2\ }\ ^3 + (56 - 2\ - 6\ ^2)\ ^4 + (8 - 3\ )\sqrt{2\ }\ ^5 + (18\ - 48)\ ^6] \\
&\times [-\ ^2 - 2\ \sqrt{2\ }\ + (10 - 6\ )\ \ ^2 - 4\ \sqrt{2\ }\ ^3 + (6\ - 16)\ \ ^4 \\
&+ 2(14 - 3\ )\sqrt{2\ }\ ^5 + 12\ \ ^6 + 12(-2 + \ )\sqrt{2\ }\ ^7]^{-1}.
\end{aligned}
$$

So, we need to determine the zeros of sixth order polynomial of the numerator. Since we have a sixth degree equation, we have to use numerical analysis techniques, namely Sturm theorem, to assess the number of real roots and find them. That

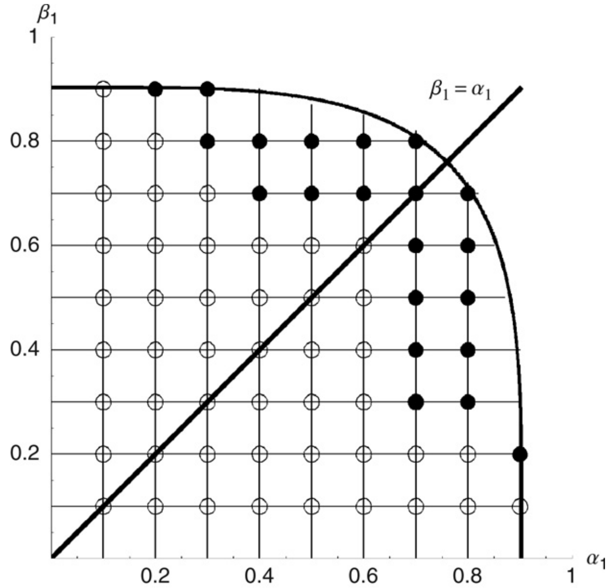

Fig. 2. The curve is the boundary of existence of the autocorrelations, that is, ${}_4 = 1$. The circles (respectively, the circumferences) mark a discretization of the set of parameterizations of the TARCH(1) model which verify (respectively, do not verify) the Taylor property.

Table 1
$S_i . \ /$, for $i = 0; : : : ; 6$ and $\in \{0; 0{:}6227; 0{:}6228; 0{:}76\}$.

	$S_0 . \ /$	$S_1 . \ /$	$S_2 . \ /$	$S_3 . \ /$	$S_4 . \ /$	$S_5 . \ /$	$S_6 . \ /$	$N . \ /$
0	3.59	−2.86	−3.55	0.14	3.55	−12.05	831.17	4
0.6227	$9{:}61 \times 10^{-4}$	−10.27	−0.95	0.57	0.94	−11.43	831.17	4
0.6228	$-6{:}61 \times 10^{-5}$	−10.27	−0.95	0.58	0.94	−11.43	831.17	3
0.76	−1.57	−11.80	0.18	14.00	−0.20	−11.30	831.17	3

theorem states that the number of real roots of an algebraic equation with real coefficients in an interval, whose limits are not roots, is equal to the difference between the number of sign changes of the Sturm chain in both interval limits (Durand, 1960). Let us define $S_0 . \ /$ as the the sixth order polynomial of the numerator, $S_1 . \ /$ the first derivative of $S_0 . \ /$ and $S_i . \ /$ the symmetrical of the polynomial remainder resulting of the division of $S_{i-2} . \ /$ by $S_{i-1} . \ /$, for $i = 2; : : : ; 6$. Therefore $S_0 . \ / ; S_1 . \ / ; : : : ; S_6 . \ /$ form the Sturm chain in this case. Let $N . \ /$ be the number of sign changes of the Sturm chain calculated at .

From Table 1, we can conclude that the polynomial has only one real zero in $\mathrm{i}\ 0; \sqrt[4]{\frac{1}{3}}\ \mathrm{h}$, which we denote by $\breve{}$ (with an approximation error inferior to 5×10^{-5}, $\breve{} = 0{:}62275$).

- Thus, the Taylor property is present in the absolute value ARCH(1) model, when $\in \mathrm{i}\ \breve{};\ \sqrt[4]{\frac{1}{3}}\ \mathrm{h}$.

5.2. Student's t-distribution

Let us now suppose that Z_t follows the unit variance distribution based on the Student's t-distribution with 6 degrees of freedom, whose density function is

$$f\,.x/ = \frac{15}{2} \quad 1 + \frac{x^2}{4} \quad^{-\frac{7}{2}} ; \qquad x \in \mathrm{R}:$$

This is a leptokurtic distribution, since its kurtosis equals 6. We derive from Fig. 3 that the Hypothesis H1 is satisfied when $0 < {}_1 < \sqrt[4]{\frac{1}{3} - {}^{4}_{1}}$, hence there is a subset of parameterizations of this model where Taylor property is verified.

Considering the absolute value ARCH(1) model and following the same procedure as in example 1, we determine that the interval of values of where the model satisfies the Taylor property is $\mathrm{i}\ 0; \sqrt[4]{\frac{1}{6}}\ \mathrm{h}$. We emphasize that this set only depends on the existence of the fourth order moment of the process.

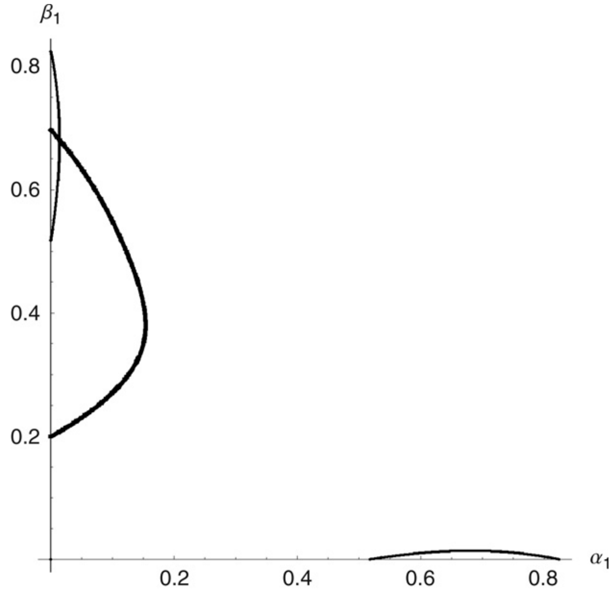

Fig. 3. Curves defined by $F(\alpha_1,\beta_1)=0$ (thinner line) and by $\frac{\partial F}{\partial \alpha_1}(\alpha_1,\beta_1)=0$ (thicker line), when Z_t follows the unit variance distribution based on the Student's t-distribution with 6 degrees of freedom.

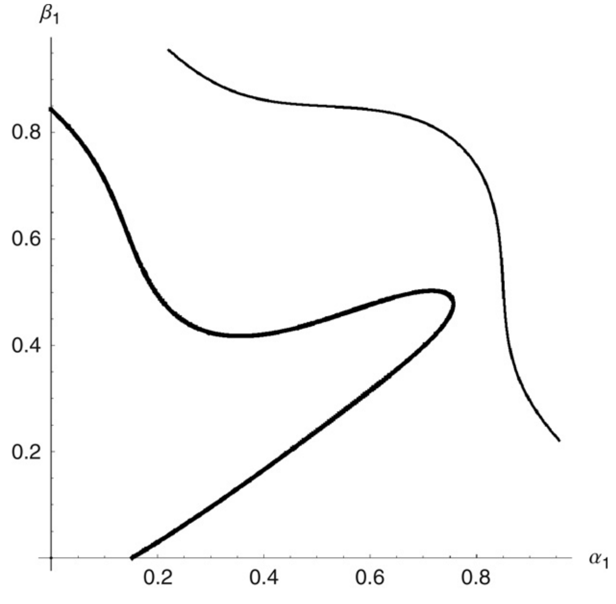

Fig. 4. Curves defined by $F(\alpha_1,\beta_1)=0$ (thinner line) and by $\frac{\partial F}{\partial \alpha_1}(\alpha_1,\beta_1)=0$ (thicker line), when Z_t follows the triangular distribution.

5.3. *Triangular distribution*

To finalize, we consider Z_t following the symmetrical and unit variance triangular distribution, whose density function is

$$f(x)=\frac{\sqrt{6}-|x|}{6}\mathbb{I}_{\left]-\sqrt{6},\sqrt{6}\right[}(x),\quad x\in\mathbb{R}.$$

We now have a platykurtic distribution, as its kurtosis is equal to $12/5=2.4$. In this case, as we can derive from the analysis of Fig. 4, the Hypothesis H1 is verified when $0<\alpha_1<\sqrt[4]{\frac{5}{6}-\beta_1^4}$, and so the Taylor property is present for a subset of parameterizations of this model.

Table 2
The first and second columns have the values considered for parameters α_1 and β_1, respectively, of the TARCH process ε, the third has the kurtosis of ε and the fourth and fifth ones have the lower and upper bound, respectively, of the 95% confidence interval for the proportion of verifications of the Taylor effect.

α_1	β_1	k_ε	Lower bound	Upper bound
0.2	0.75	357.48	0.80	0.90
0.58	0.65	112.20	0.84	0.93
0.3	0.3	7.60	0.97	1.00
0.2	0.1	6.40	0.96	1.00

Table 3
The first column has the values considered for parameter α_1 of the ARCH process ε, the second has the kurtosis of ε and the third and fourth ones have the lower and upper bound, respectively, of the 95% confidence interval for the proportion of verifications of the Taylor effect.

α_1	k_ε	Lower bound	Upper bound
0.405	316.46	0.22	0.34
0.4	126.00	0.24	0.36
0.2	7.58	0.11	0.21
0.1	6.32	0.20	0.32

Following the prior examples, for the absolute value ARCH(1) model the interval of values of the parameter α for which the Taylor property is satisfied is $\left]\breve{\alpha};\sqrt[4]{\frac{1}{2.4}}\right[$, with $\breve{\alpha}$ equal to 0.77255, with an approximation error inferior to 5×10^{-5}.

6. Empirical studies

In this work, our main concern was the study of the presence of the Taylor property in the TARCH(1) models. With the theoretical study presented here, we obtain a subclass of those models verifying such property. Since the Taylor effect is common in financial time series, this study reinforces the importance of the TARCH models.

Due to the weight of ARCH models in financial time series analysis, a comparison between the ARCH(1) and the TARCH(1) models is unavoidable. These models differ in the expression of the conditional variance of the process, which is defined, in the ARCH(1) model, as $\sigma_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2$, $\alpha_0 > 0$, $\alpha_1 \geq 0$. A theoretical comparison study in what concerns the Taylor property is, for the moment, impossible to be done as the expression of autocorrelations of the absolute value process of the ARCH model is not available. So, in order to do this comparison, a simulation study is presented here.

As mentioned in the introduction, He and Teräsvirta (1999) have already considered conditionally Gaussian ARCH(1) and GARCH(1,1) models, concluding that these models do not appear capable of generating series with the Taylor effect. In our simulation, we decide to consider ARCH and TARCH models with the same non-Gaussian leptokurtic distribution for the generating white noise, namely the Student's t-distribution with 6 degrees of freedom.

He and Teräsvirta (1999) also mention that the kurtosis of the process ε seems to play a role in the presence of the Taylor property. So, to better compare ARCH(1) and TARCH(1) models, we choose four parameterizations for each model in order to have series with similar values of kurtosis. We fix $\alpha_0 = 1$ for both models, since this parameter does not appear in the known theoretical expressions of the kurtosis and of the autocorrelations of the models. Based on 200 simulated series of 100 000 observations (after deleting 50 to discard the initial effects), we register the number of times the Taylor effect is present (that is, $\hat{\rho}_1(1) > \hat{\rho}_n(2)$) and determine the 95% confidence interval for its proportion. The results are presented for the TARCH(1) model in Table 2 and for the ARCH(1) model in Table 3.

Comparing the results presented in these two tables, we see that the Taylor effect occurs a much more significant number of times in the TARCH(1) model, even almost always for some of these parameterizations. This allows us to conclude that the Taylor effect is more likely to be captured by the TARCH(1) model than by the ARCH(1) model. Comparing our results with those of He and Teräsvirta (1999), we observe that the conditionally Gaussian ARCH(1) model that they consider is even less likely to capture the Taylor effect than the non-Gaussian leptokurtic one studied here. This, along with our findings of the previous section, support our suspicion that high values of kurtosis of the generating white noise favor the appearance of the Taylor property.

studies will be done in this direction, namely, analyzing the influence of the distribution tail in the presence of the Taylor property.

The simulations studies developed to compare the ARCH and TARCH models, strongly suggest that TARCH models are considerably more likely to capture the Taylor effect than ARCH ones.

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References

Cont, R., 2001. Empirical properties of asset returns: Stylized facts and statistical issues. Quant. Finance 1, 223–236.
Durand, E., 1960. Solutions Numériques des Équations Algébriques, Tome I. Masson, Paris.
Granger, C.W.J., Ding, Z., 1995. Some properties of absolute return: An alternative measure of risk. Ann. Économ. Statist. 40, 67–95.
He, C., Teräsvirta, T., 1999. Properties of moments of a family of GARCH processes. J. Econom. 92, 173–192.
Mora-Galán, A., Pérez, A., Ruiz, E., 2004. Stochastic volatility models and the Taylor effect. Statistics and Econometrics Working Papers ws046315, Universidad Carlos III.
Rabemananjara, R., Zakoian, J.M., 1993. Threshold ARCH models and asymmetries in volatility. J. Appl. Econom. 8, 31–49.
Taylor, S., 1986. Modelling Financial Time Series. Wiley.
Zakoian, J.M., 1994. Threshold Heteroskedastic Models. J. Econom. Dynam. Control 18, 931–955.