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Face counting on an acyclic Birkhoff polytope

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ABSTRACT

In this paper we present some algorithms allowing an exhaustive account on the number of edges and faces of the acyclic Birkhoff polytope.

1. Earlier results

In the previous paper [2], we studied some geometric properties, in terms of graph theory, of $\Omega_n(T) = \mathfrak{T}_n$, that is the set of real square matrices $[a_{ij}]$, with nonnegative entries and all rows and columns sums equal to one, such that $a_{ij} = 0$ provided ij is not an edge, for $i \neq j$, in the given tree T. The set $\mathfrak{T}_n$ is a convex polytope and is called acyclic Birkhoff polytope. In this work we are concerned with the problem of counting the faces of $\mathfrak{T}_n$. We present algorithms for counting the number of edges of $\mathfrak{T}_n$ in general, and also we find explicit expressions for this number when T has certain forms such

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as stars and spiders. Moreover, we describe algorithms for counting the number of faces of $\mathfrak{T}_n$. Some examples are provided.

In [2] we also established a bijection between the faces of any dimension of $\mathfrak{T}_n$, and the union of a finite number of bicolored subgraphs of the three following types:

Type 1. A closed vertex, $\bullet$.

To this type of subgraph we associate a one-by-one matrix [1].

Type 2. An open edge, $\circ - \circ$.

To an open edge we associate the "adjacency" matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Type 3. This type is not one of the previous two types, and is a bicolored subgraph obtained from any connected subgraph of T, with all endpoints closed.

To this type of subgraph we can also associate an "adjacency" matrix such that to closed vertices and to the edges it will correspond entries in the matrix equal to one and zero elsewhere.

In fact, given the path P_5, the path with five vertices,

$$\circ - \circ - \circ - \circ - \circ,$$

one of its bicolored subgraph of Type 3 is, for example,

$$\bullet - \bullet - \circ - \bullet.$$

Its associated "adjacency" matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The definition of *inner entry*, introduced in [12] by da Fonseca and Marques de Sá, was extended by the authors in [2], to the acyclic Birkhoff polytope:

Definition 1.1. A *T-component* is a bicolored subgraph of T of Type 3. An *inner entry* of a T-component is a non-terminal closed vertex.

Note that the concept of bicolored subgraph is also known as a 2 -stratified graph, i.e., a graph where the vertex set is partitioned into two subsets (cf. [10]).

The face of $\mathfrak{T}_n$ corresponding to a boolean sum A of $n \times n$ permutation matrices is denoted by

$$\mathscr{F}_A = \{X \in \mathfrak{T}_n | a_{ij} = 0 \Rightarrow x_{ij} = 0\}.$$

We provided in [2] a closed formula for its dimension.

Proposition 1.1. *Let t_A be the number of T-components of the bicolored subgraph of T corresponding to $\mathscr{F}_A$. Let θ_A and ι_A be, respectively, the sum of all closed endpoints and the number of inner entries in all T-components of the same bicolored subgraph of T. Then*

$$\dim \mathscr{F}_A = \theta_A + \iota_A - t_A. \tag{1.1}$$

One of the faces of $\Omega_5(P_5)$ of dimension 2 results from the union of a subgraph of Type 1 and the subgraph of Type 3 given above:

$$\bullet \quad \bullet - \bullet - \circ - \bullet$$

To the configuration of this face corresponds the "adjacency" matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

As a consequence of Proposition 1.1, we established a recurrence relation for counting the number of vertices (0-faces) of the acyclic Birkhoff polytope. A vertex of $\mathfrak{T}_n$ was identified as a bicolored subgraph of T whose diameter is at most one, it corresponds to the union of bicolored subgraphs of Types 1 and 2. Here, we consider the diameter of disconected graphs as the maximum diameter of its connected components.

If G is a graph and S is a subgraph of G, $G\backslash S$ denotes the subgraph of G deleting all the edges incident on S, without deleting the vertices. We also denote the subgraph of G induced by the subset of vertices $V' \subset V(G)$, by $G[V']$.

Note also, it follows from the bijection between the faces of any dimension of $\mathfrak{T}_n$ and the union of a finite number of bicolored subgraphs of Types 1, 2 and 3, that if T and T' are two disjoint trees with n and n' vertices, respectively, and if g_m is the number of bicolored subgraphs of a graph corresponding to faces of dimension m, then, for $m \leqslant \min\{n, n'\}$,

$$g_m(T \cup T') = \sum_{k=0}^{m} f_{m-k}(T) f_k(T').$$

Let $f_0(T)$ be the number of vertices of $\mathfrak{T}_n$ and $f_{0,ij}(T)$ the number of bicolored subgraphs of T that contains the edge ij and whose diameter is at most one. Let ij be any edge of the tree T. Then

$$f_0(T) = g_0(T\backslash ij) + f_{0,ij}(T\backslash\{i,j\} \cup ij) \tag{1.2}$$

with initial conditions $f_0(\emptyset) = f_0(v) = 1$, where v is a vertex of T.

Later on, for $p \in \{1, 2, 3\}$ we presented in [3], some explicit formulas allowing the enumeration of p-faces of Ω_n^t, the tridiagonal Birkhoff polytope. The case $p = 0$ has been considered by Dahl in [11] and it corresponds to the $(n + 1)$th Fibonacci number, denoted by f_{n+1}.

In [12], it was established a closer connection between vertex counting in Ω_n^t and Fibonacci numbers. In particular, the main results on alternating parity sequences – strictly increasing sequences of integers, with a finite numbers of entries, such that any two adjacent entries have opposite parities – are applied to determine the number of vertices of an arbitrarily given face of Ω_n^t. The authors also gave an expression for the number of edges of Ω_n^t. For $p \in \{0, \ldots, n-1\}$ we denote by $f_p(T)$, the number of p-faces of $\mathfrak{T}_n$. The case $p = 0$ has already been presented in recurrence relation (1.2) (c.f. [2,11,12]).

The summary of this paper is the following: we start Section 2 introducing an illustrative example for counting the edges of the polytope associated to a particular tree. Led by this example, we present an algorithm for counting, in the general case, the number of edges of $\mathfrak{T}_n$. In Section 3 we give another algorithm for $f_1(T)$. In Section 4 and 5 we approach the question of counting the faces of $\mathfrak{T}_n$. We also present two algorithms and, in addition, we test our techniques presenting some examples. Finally, in Sections 6 and 7, we give explicit expressions allowing us to count the number of faces of $\Omega_n(S)$, where S is a star, and the number of facets of $\mathfrak{T}_n$.

Those algorithms allow us to find the number of faces of any acyclic Birkhoff polytope. For stars and spiders we present some explicit formulas. For a general tree it seems harder to present concise formulas.

In the figures in this paper, when it is necessary, we enumerate the edges of a tree T from the left to the right and from the top to the bottom. The vertices of T are labeled in the same way.

2. Counting the edges of $\mathfrak{T}_n$

In [3], we presented a formula to count the number of edges of the tridiagonal Birkhoff polytope, Ω_n^t:

$$f_1(P_n) = \sum_{p=0}^{n-2} \sum_{k=0}^{n-2-p} f_{k+1} f_{n-p-k-1}, \tag{2.1}$$

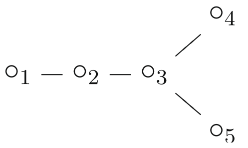
where P_n is a path with n vertices. This section is devoted to present algorithms for counting the number of edges of $\mathfrak{T}_n$.

According to [2], and due to Proposition 1.1, an edge of $\mathfrak{T}_n$ is the union of bicolored subgraphs of Type 1, Type 2, and exactly one bicolored subgraph of Type 3, without any inner entries and exactly two closed endpoints, consequently of the following type:

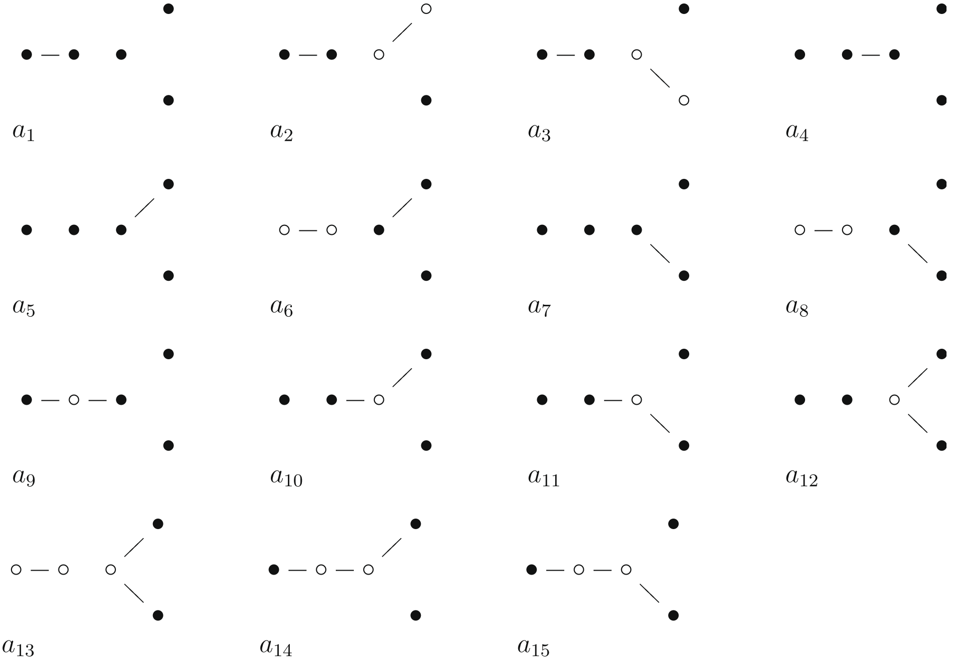
$$H_i \quad \bullet - \circ - \circ \cdots \circ - \bullet \quad H_j,$$

where H_i and H_j are the union of a finite number of bicolored subgraphs of Types 1 and 2, respectively. When $i \times j = 0$, H_0 represents the empty set and, in this case, it is conventioned that $f_0(\emptyset) = 1$.

We start presenting the edges of $\Omega_5(T_5)$, where T_5 is the tree

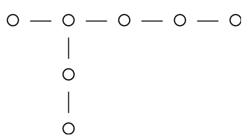


The 15 edges of $\mathfrak{T}_5$ are



If the number of vertices of the tree increases, even with a small growth, an exhaustive exhibition of all edges of $\mathfrak{T}_n$ becomes harder.

The next example provides a motivation for an algorithm to calculate the number of edges of $\Omega_n(T)$. In this example, we consider the spider $S' = S_{1,2,3}$ with three branches of lengths 1, 2, 3 presented below:



Bearing in mind Proposition 1.1, the number of edges of the polytope $\Omega_7(S')$ is equal to the number of bicolored subgraphs of S' that have one bicolored subgraph of Type 3 with two closed endpoints and without inner entries.

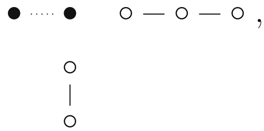
The bicolored subgraph of Type 3 (T-component) has one of the following configurations:

$$\bullet - \bullet, \bullet - \circ - \bullet, \ldots, \bullet - \circ - \circ - \circ - \circ - \bullet.$$

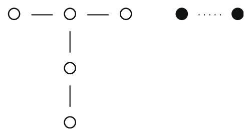
As diam$S' = 5$, there is no other possibility for the configuration of the T-component. Each of those possibilities gives rise to several distinct bicolored subgraphs that corresponds to an edge of $\Omega_7(S')$. For example, if the T-component has the first configuration, it can occupy the same position as the edges of S'. Using the procedure introduced before, we will distinguish cases to discuss.

Suppose that the T-component "occupies" the position of:

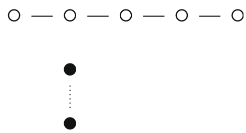
(1) the first edge of S',



(2) the fourth edge of S',



(3) the sixth edge of S',



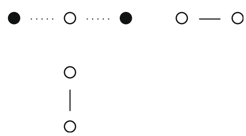
The number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given, respectively, by $f_0(P_3)f_0(P_2) = 3 \times 2 = 6, f_0(T_5) = 7$ and $f_0(P_5) = 8$.

For the remaining cases not presented in this description, the calculation of the number of edges of $\Omega_7(S')$ is determined using similar arguments. Therefore, the total number of edges obtained from this T-component is given by the expression:

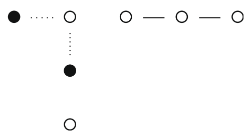
$$f_0(P_3)\, f_0(P_2) + f_0(P_1)\, f_0(P_2)\, f_0(P_2) + f_0(P_4)\, f_0(P_1) \\ + f_0(T_5) + f_0(P_1)\, f_0(P_3) f_0(P_1) + f_0(P_5) = 33.$$

If the T-component has the second configuration, it can "occupy" the same position as two consecutive edges of S', as we can see next. Assume that it "occupies" the position of:

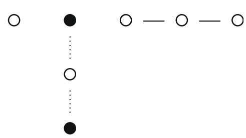
(1) the first two edges of S',



(2) the first and fifth edges of S',



(3) the last two edges of S',



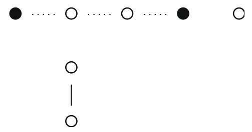
The number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given respectively by: $f_0(P_2)\, f_0(P_2) = 2 \times 2 = 4$, $f_0(P_3)\, f_0(P_1) = 3 \times 1 = 3$ and $f_0(P_1)\, f_0(P_3) = 3$.

Again, for the remaining cases the calculation uses similar arguments. Therefore, the number of edges obtained from this T-component is given by the following expression:

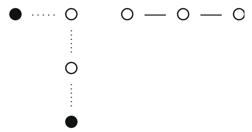
$$f_0(P_2)\, f_0(P_2) + f_0(P_1)\, f_0(P_2)\, f_0(P_1) + f_0(P_4) + f_0(P_3)\, f_0(P_1) \\ + f_0(P_1)\, f_0(P_2)\, f_0(P_1) + f_0(P_1)\, f_0(P_3) = 19.$$

If the T-component has the third configuration, it can "occupy" the same position as three consecutive edges of S'. Suppose that it "occupies" the position of:

(1) the three first consecutive edges of S',



(2) the first, fifth and sixth edges of S',

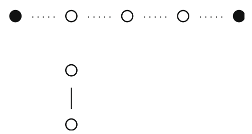


The number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given respectively by: $f_0(P_1)\, f_0(P_2) = 2$ and $f_0(P_3) = 3$. For the remaining cases the calculation is similar. Therefore, the total number of edges that is obtained from this T-component is given by the following expression:

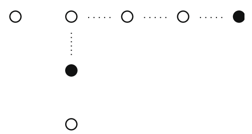
$$f_0(P_1)\, f_0(P_2) + f_0(P_1)\, f_0(P_2) + f_0(P_1)\, f_0(P_2) + f_0(P_1)\, f_0(P_1)\, f_0(P_1) = 10.$$

If the T-component has the fourth configuration, it can "occupy" the same position as four consecutive edges of S'. Consider that it "occupies" the position of:

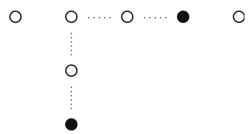
(1) the first four edges of S',



(2) the second, third, fourth and fifth edges of S',



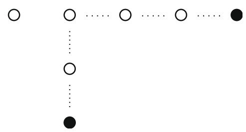
(3) the second, third, fifth and sixth edges of S',



Similarly, the number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given respectively by: $f_0(P_2) = 2$, $f_0(P_1)\, f_0(P_1) = 1$ and $f_0(P_1)\, f_0(P_1) = 1$.

If the T-component has the last configuration, it can "occupy" the same position as five consecutive edges of S'. Consider that it "occupies" the position of:

(1) the second, third, fourth, fifth and sixth edges of S',



The number of edges of the polytope $\Omega_7(S')$ having the previous T-component is given by $f_0(P_1) = 1$.

Therefore the total number of edges of $\Omega_7(S')$ is the sum of all the previous values, i.e, $f_1(S') = 67$.

The illustration presented above gives rise to the first algorithm that allow us to count the number of edges of $\Omega_n(T)$, for a given tree T.

An edge is a face of dimension 1. From Proposition 1.1, as $1 = \theta_A + \iota_A - t_A, \theta_A \geqslant 2, t_A \geqslant 1$ and $\iota_A \geqslant 0$, we obtain the unique solution $\theta_A = 2$, $t_A = 1$ and $\iota_A = 0$. Therefore, the bicolored subgraphs that represent an edge of $\mathfrak{T}_n$, have one T-component with two endpoints and without inner entries.

In fact, each edge considered individually, is a path of length 1 and can be regarded as a T-component in the previous conditions but without open circles. Each of these paths gives origin to $g_0(T \backslash P_2)$ different bicolored subgraphs.

Each pair of consecutive edges, consider individually, is a path of length 2 and can be regarded as a T-component with two endpoints and without inner entries but with an open circle. Each path of length 2 gives origin to $g_0(T \backslash P_3)$ different bicolored subgraphs. We continue in a constructive process until the number of consecutive adjacent edges reaches the diameter of the tree.

We proceed as follows:

Algorithm 1

The input is a tree T such that $\operatorname{diam} T = p$.

Step 1 ⧫ For $i \in \{1, \ldots, p\}$ consider each path P of T, with i edges, and calculate the value $g_0(T \backslash P)$;

Final step ⧫ Sum all the values obtained in the previous steps and exit.

The sum obtained in the final step is the number of edges of $\mathfrak{T}_n$.

The application of this algorithm provides a closed formula for the number of edges of the polytope $\Omega_n(S)$, where S is a star with n vertices.

Proposition 2.1. *Let S be a star with n vertices, then*

$$f_1(S) = \frac{n(n-1)}{2}.$$

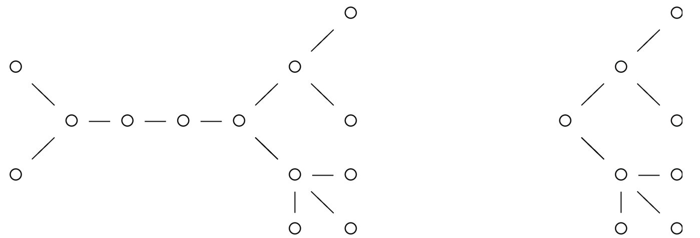
Proof. In fact, S has $n-1$ edges and each of them gives rise to an edge of the acyclic Birkhoff polytope $\Omega_n(S)$; the graph S has C_2^{n-1} pairs of consecutive edges and each pair gives rise again to an edge of $\Omega_n(S)$. Since $\operatorname{diam} S = 2$, there is only one possibility to count the edges of $\Omega_n(S)$ and therefore,

$$f_1(S) = (n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}. \qquad \square$$

3. An alternative algorithm for $f_1(T)$

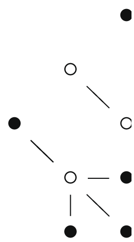
Note that if T' and T are two trees such that T' is a subgraph of T and if T' has a p-face, $p \geqslant 0$, with a specific configuration, then T has $g_0(T \backslash T')$ different p-faces that "contain" the configuration of the referred p-face.

In order to illustrate this property let us consider the trees T and T' presented below:

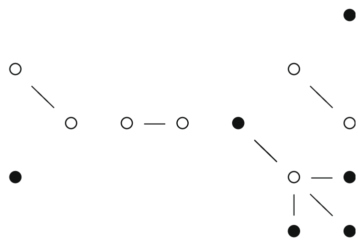


T $\qquad$ T'

One 3-face of $\Omega_8(T')$ has the following configuration:



The polytope $\Omega_{13}(T)$ has $g_0(T\backslash T') = g_0(T_5) = 7$ faces of dimension 3 that contains the referred configuration. The next example is one of these 7 faces:



Expressions giving the number of edges of the tridiagonal Birkhoff polytope are known, [3]. Therefore, the next algorithm for calculating $f_1(T)$ has the underlying idea to consider all different paths obtained using two terminal vertices of the original tree. We compute the number of the edges of polytopes associated to them. Some of the configurations of the edges of $\mathfrak{T}_n$ are going to be repeated and must be removed.

Algorithm 2

The input is a tree T with n vertices.

Step 1 ⧫ for each pair of different terminal vertices of the tree T consider the path P between them and compute $f_1(P)\, g_0(T\backslash P)$;
Step 2 ⧫ sum all the numbers obtained in step 1;
Step 3 ⧫ for each pair of different paths considered at step 1, let P' be the path from their intersection. Compute $f_1(P')\, g_0(T\backslash P')$;
Step 4 ⧫ sum all the numbers obtained in step 3;
Final step ⧫ calculate the difference between the numbers obtained in steps 2 and 4.

It seems that, restricting the complexity of the algorithms only to the number of necessary iterations to get results, maintaining the other operational parameters constant, the Algorithm 1 needs

δ^2 iterations while Algorithm 2 needs $\delta + \delta^2$, where δ is the total number of paths obtained from T considering all possible lengths. However, it depends on the implementations that can be done. The computational implementation of the algorithms is not already done but we present below a different form to write Algorithms 1 and 2 in such a way that we can study its complexity and compare them.

Consider the following data structure: V is the set of all vertices of the tree T and *Paths* is the set of all different paths in the tree considering all possible lengths. All paths saved in this data structure have an unique index.

Consider now the following methods:

- diam(T) gives the length of one of the largest path of T;
- LengthPath (P) gives 0 if the path does not exist and gives l_P if l_P is the length of the path P;
- $\delta =$ LengthPaths (T) gives the total number of paths existing in the tree and saved in the data structure *Paths*;
- PathTerminal (P) gives *true* if the path P is formed with terminal vertices of the tree, *false* otherwise;
- IntersectPath (P_a, P_b) gives 0 if the path P_a does not intersect the path P_b and gives the path P_i if $P_a \cap P_b = P_i$.

As we said we can rewrite the previous algorithms in the following form:

Algorithm 1

```
Let Sum = 0
   For i = 1 to diam(T)
      For p = 1 to LengthPaths (T)
         If LengthPath (P) = i then let Sum = Sum + g0(T\P)
      Next p
   Next i
```

To get the final result, after running Algorithm 1, the total number of iterations is δ^2. Note that, to determine diam(T) we need to run all the paths of the tree and therefore we need δ iterations.

We present Algorithm 2 in two stages. The Stage *I* finds all the paths P of the tree with terminal vertices and, for each one, calculate $f_1(P)g_0(T\backslash P)$.

The Stage II finds all the intersections P' between two different paths and, for each one, calculate $f_1(P')g_0(T\backslash P')$.

We present the details below:

Algorithm 2

```
Stage I
   Let Sum1 = 0
   For p = 1 to LengthPaths (T)
         If PathTerminal (P) = true then let Sum1 = Sum1 + f1(P)g0(T\P)
   Next p
```

To obtain a result after running this stage, we need δ iterations.

The Stage II can be implemented in two forms:

```
   Stage II
   (Implementation 1)
   Let Sum2 = 0
   For Pa = 1 to LengthPaths (T)
      For Pb = 1 to LengthPaths (T)
         If Pa ≠ Pb then
            P' = IntersectPath (Pa, Pb)
```

If $P' \neq 0$ then let $Sum2 = Sum2 + f_1(P')g_0(T\backslash P')$
end If
Next P_b
Next P_a.

To obtain a result after running Stage II with this implementation, we need δ^2 iterations.

Stage II
(Implementation 2)
Let $Sum2 = 0$
For $P_a = 1$ to LengthPaths $(T) - 1$
For $P_b = P_a + 1$ to LengthPaths (T)
$P' =$ IntersectPath (P_a, P_b)
If $P' \neq 0$ then let $Sum2 = Sum2 + f_1(P')g_0(T\backslash P')$
Next P_b
Next P_a.

We need $1 + \sum_{j=3}^{\delta}(j-1) = \frac{\delta^2-\delta}{2}$ iterations to obtain a result after running Stage II with this implementation. This expression can be obtained by induction.

Now, we can compare the different implementations of Algorithm 2:

(a) Algorithm 2 with Stage I and Stage II (implementation 1) needs $\delta + \delta^2$ iterations to obtain the final result.

(b) Algorithm 2 with Stage I and Stage II (implementation 2) needs $\delta + 1 + \sum_{j=3}^{\delta}(j-1) = \frac{\delta^2+\delta}{2}$ iterations to obtain the final result.

It seems that (a) is not more efficient than (b).

Now, we can compare Algorithms 1 and 2.

Algorithm 1 is more efficient than Algorithm 2 (with implementation 1) but, Algorithm 2 (with implementation 2) is more efficient than Algorithm 1.

The following proposition follows from Algorithm 2:

Proposition 3.1. *Let $S' = S_{p_1,p_2,\ldots,p_n}$ be a spider with n branches of lengths $p_1, \ldots, p_n$ and $N = p_1 + p_2 + \cdots + p_n + 1$ vertices. The number of edges of $\Omega_N(S')$ is given by*

$$f_1(S') = \sum_{1 \leqslant i < j \leqslant n} \left[f_1(P_{p_i+p_j+1}) \prod_{k \neq i,j} f_0(P_{p_k}) \right] - (n-2) \sum_{i=1}^{n} \left[f_1(P_{p_{i+1}}) \prod_{k \neq i} f_0(P_{p_k}) \right].$$

Proof. For each pair of different branches of S' of lengths p_i and p_j, consider the path $P_{p_i+p_j+1}$. Without loss of generality, for each $i \in \{1, \ldots, n-1\}$, j will run over all values from $i+1$ to n. From this, the next sum represents the number of the configurations of all edges of polytopes associated to the mentioned paths

$$\sum_{1 \leqslant i < j \leqslant n} \left[f_1(P_{p_i+p_j+1}) \prod_{k \neq i,j} f_0(P_{p_k}) \right]. \tag{3.1}$$

The previous number includes edges of $\Omega_N(S')$ that appear repeatedly and must be removed. They correspond to edges of polytopes associated to paths resulting from the intersection of two different paths that have a common part and that number is given by the following expression:

$$(n-2) \sum_{i=1}^{n} \left[f_1(P_{p_{i+1}}) \prod_{k \neq i} f_0(P_{p_k}) \right]. \tag{3.2}$$

The final number of the edges of $\Omega_N(S')$ is given by the difference between the expressions (3.1) and (3.2). □

Recall that the formulas (3.1) and (3.2) involve only the expressions of $f_0(P_n)$ and $f_1(P_n)$ for any n, that can easily be determined (c.f. [2,3]).

Example 3.1. For $S' = S_{1,2,3}$ presented in the first part of this section, we count $f_1(S')$. Here attending to the last algorithm we have

$$\begin{aligned} f_1(S') &= f_1(P_5)\,f_0(P_2) + f_1(P_4)\,f_0(P_3) + f_1(P_6)\,f_0(P_1) \\ &\quad - [f_1(P_2)\,f_0(P_2)\,f_0(P_3) + f_1(P_3)\,f_0(P_1)\,f_0(P_3) + f_1(P_4)\,f_0(P_1)\,f_0(P_2)] \\ &= 18 \times 2 + 8 \times 3 + 38 \times 1 - [1 \times 2 \times 3 + 3 \times 1 \times 3 + 8 \times 1 \times 2] = 67. \end{aligned}$$

4. Counting the 2-faces of $\mathfrak{T}_n$

In [3], we presented a formula to count the number of 2-faces of Ω_n^t:

$$f_2(P_n) = \sum_{p=1}^{n-2} p \sum_{k=0}^{n-2-p} f_{k+1}\, f_{n-p-k-1} + \sum_{p=1}^{n-3} p \sum_{j=0}^{n-3-p} \sum_{k=0}^{n-3-p-j} f_{k+1}\, f_{j+1}\, f_{n-p-j-k-2}. \tag{4.1}$$

Motivated by [2], we start enumerating the faces of $\Omega_n(S)$, where S is a star with n vertices.

Proposition 4.1. *Let S be a star with n vertices. The number of 2-faces of $\Omega_n(S)$ is*

$$f_2(S) = \frac{n(n-1)(n-2)}{6}.$$

Proof. According to Proposition 1.1, the number of faces of $\Omega_n(S)$ is equal to the sum of the number of all bicolored subgraphs with one T-component, with two closed endpoints and one inner entry, with the number of all bicolored subgraphs with one T-component with three closed endpoints and without inner entries.

The T-component can have one of the configurations presented below:

- ⊛ two closed endpoints and one inner entry:

$$\bullet - \bullet - \bullet$$

- ⊛ three closed endpoints without inner entries:

$$\begin{array}{ccccc} \bullet & - & \circ & - & \bullet \\ & & | & & \\ & & \bullet & & \end{array}$$

Recall that S has $n-1$ edges. Therefore, due to the configurations of the T-component it follows that we have C_2^{n-1}, in the first case, and C_3^{n-1}, in the second case, different bicolored subgraphs of S. The sum of these two values gives $f_2(S)$. □

As in Section 2, we use again Proposition 1.1, to obtain the composition (in number and structure) of the T-components presented in the configuration of a bicolored subgraph that represents a 2-face.

As $\dim \mathscr{F}_A = 2$, from the relation $2 = \theta_A + \iota_A - t_A$ where $\theta_A \geqslant 2$, $\iota_A \geqslant 0$ and $t_A \geqslant 1$, we only have three possibilities.

(1) $\theta_A = 2, \iota_A = 1$ and $t_A = 1$;
(2) $\theta_A = 3, \iota_A = 0$ and $t_A = 1$;
(3) $\theta_A = 4, \iota_A = 0$ and $t_A = 2$.

Each one leads to a different stage of the next algorithm that will allowing an exhaustive account for the number of the 2-faces of $\mathfrak{T}_n$.

Algorithm 3

The input is a tree with vertex set V and diam$T = q$.

Stage I Computation of the number of bicolored subgraphs with a T-component with two endpoints and one inner entry:

Step 1 ▲ For $i \in \{2, \ldots, q\}$ consider each path P of T with i edges calculate $g_0(T\backslash P)$, compute $(i-1)g_0(T\backslash P)$, and sum all the values obtained;

Final step ▲ Sum all the values obtained at step 1 and exit.

Stage II Computation of the number of bicolored subgraphs with a T-component with three endpoints and without inner entries:

Step 1 ▲ for each vertex v of T whose degree is greater than 2, we consider each of the triplets of incident edges on v, i.e., stars with three branches and central vertex v;

Step 2 ▲ for each of these stars, S, we consider all spiders, S' with central vertex v containing S as a subgraph. For each S' we calculate $g_0(T\backslash S')$;

Step 3 ▲ sum all the values obtained in step 2;

Step 4 ▲ consider the spiders with origin in the same star and with a common vertex $i \neq v$ whose degree in T is greater than 2, for each pair of these spiders consider their intersection $S^{\star}$ and calculate $g_0(T\backslash S^{\star})$;

Step 5 ▲ sum all the values obtained in step 4;

Final step ▲ calculate the difference between the values obtained in steps 3 and 5, respectively.

Stage III Computation of the number of bicolored subgraphs with two T-components each one with two endpoints and without inner entries:
We start fixing a T-component (called first T-component) and we vary the another one in configuration and in position. Then, for each $p \in \{1, \ldots, \text{diam}T\}$ the first T-component can occupy the position of a path in T, $P_{e_i,p}$, with length p, where e_i is its initial edge.

Step 1 ▲ If the first T-component occupies the position of a path in T with initial edge e_1 and length p, $P_{e_1,p}$, compute $f_0(\emptyset)g_1(T[V\backslash V(P_{e_1,p})]$;

Step 2 ▲ If the first T-component occupies the position of a path in T with initial edge e_2 and length p, $P_{e_2,p}$ compute $f_0(P_1)g_1(T[V\backslash(V(e_1) \cup V(P_{e_2,p}))]$;

Step i ▲ If the first T-component occupies the position of a path in T with initial edge e_i and length p, $P_{e_i,p}$ compute $g_0(H)g_1(T[V\backslash(V(H) \cup V(P_{e_i,p}))]$.
Here, H is the subgraph of T with edge set

$$E(H) = \{e_1, e_2, \ldots, e_{i-1}\}\backslash\{e : e \text{ is incident on some vertex of } P_{e_i,p}\},$$

and vertex set:

$$V(H) = V(e_1 \cup e_2 \cup \cdots \cup e_{i-1})\backslash V(P_{e_i,p}).$$

Final step ▲ Repeat the previous step until the diameter of each connected component of the induced subgraph $T[V\backslash(V(H) \cup V(P_{e_i,p}))]$ is equal to 0 and sum all values obtained at this stage.

Stage IV Sum all values computed at Stages I, II and III.

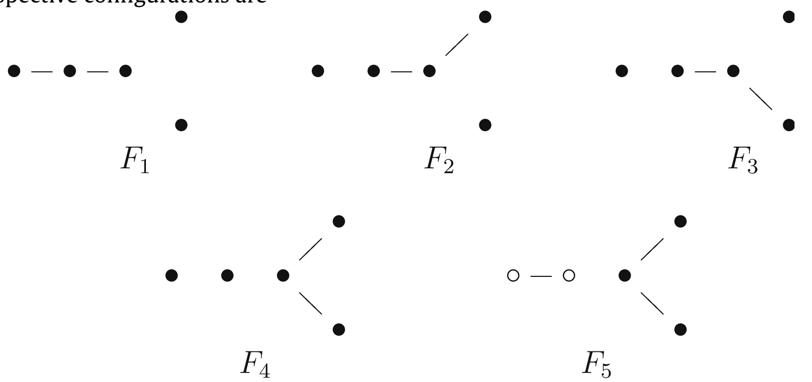
In order to illustrate this algorithm we present a description of the way we can count the 2-faces of $\mathfrak{T}_5$.

Stage I. If we have exactly one T-component, then one of the following configurations may occur:
⊛ the T-component has two endpoints and one inner entry:

$$\bullet - \bullet - \bullet$$

From the first step of the algorithm we have: $g_0(T\backslash\{1,2,3\}) = g_0(T\backslash\{2,3,5\}) = g_0(T\backslash\{2,3,4\}) = f_0(P_1) = 1$ and $g_0(T\backslash\{3,4,5\}) = f_0(P_2) = 2$. We sum all of these numbers. Therefore we have five 2-faces of $\mathfrak{T}_5$

and the respective configurations are



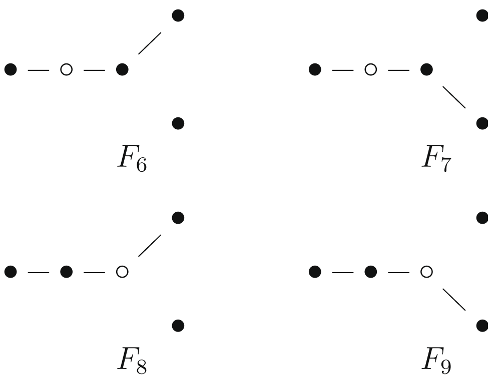
⊛ the T-component has one open circle and its configuration can be

$\bullet - \circ - \bullet - \bullet$

From the application of the step 2 we have

$$g_0(T\backslash\{1,2,3,4\}) = 1 \quad \text{and} \quad g_0(T\backslash\{1,2,3,5\}) = 1.$$

We compute $2g_0(T\backslash\{1,2,3,4\}) + 2g_0(T\backslash\{1,2,3,5\}) = 4$. The corresponding configurations of the four 2-faces of $\mathfrak{T}_5$ are

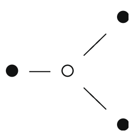


Note that the two last one are due from the fact that the configuration of the T-component can also be

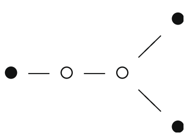
$\bullet - \bullet - \circ - \bullet$

As $\mathrm{diam}(T_5) = 3$ we do not have other possibility, we finish this stage adding the values obtained: $5 + 4 = 9$.

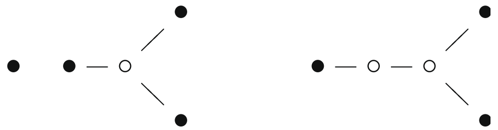
Stage II. If the T-component has three endpoints without inner entries it can have the following configuration:



or



As in T_5, we only have one vertex with degree greater than 2 and we only have a triplet of incident edges in this vertex, which corresponds to the 2-faces, respectively,



F_{10} F_{11}

Note that from the application of step 2 we only have two spiders.

As result of this stage we have two faces of $\mathfrak{T}_5$ and this number of faces results from the sum of $g_0(T\backslash\{2,3,4,5\}) = f_0(v) = 1$ with $g_0(T\backslash\{1,2,3,4,5\}) = f_0(\emptyset) = 1$.

Stage III. If the bicolored subgraph has two T-components with two endpoints and without inner entries.

We start fixing the first T-component without open circles. The second one can have or not open circles. As $\operatorname{diam}(T_5) = 3$, the number of open circles must be necessarily one. Therefore we have two possibilities:

$\bullet - \bullet \quad \bullet - \bullet$

and

$\bullet - \bullet \quad \bullet - \circ - \bullet$

Attending to step1, we calculate

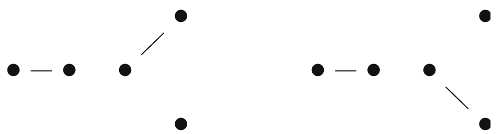
$$f_0(\emptyset) f_1(P_3) = 3.$$

Suppose now that the first T-component will occupy the position of e_2, as

$$\operatorname{diam} T[V\backslash V(e_1 \cup e_2)] = 0$$

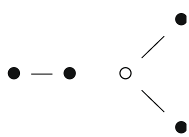
we must stop the process.

So, the configurations of the three faces obtained are



F_{12} F_{13}

and



F_{14}

Stage IV. In this way we obtained the number of all the 2-faces of $\mathfrak{T}_5$

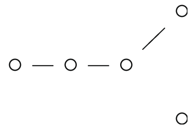
$$5 + 2 \times 2 + 2 + 3 = 14.$$

5. Counting the faces of $\mathfrak{T}_n$ revisited

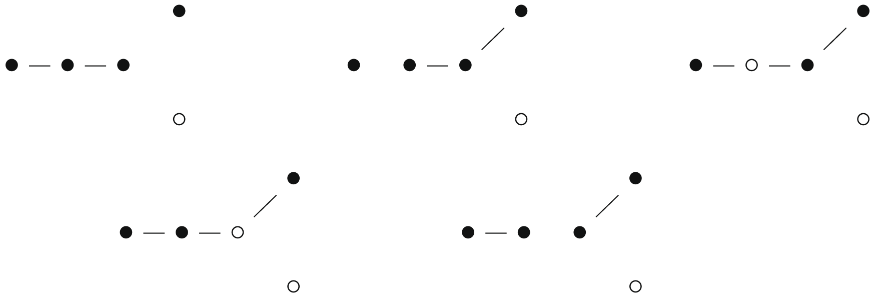
In this section, as we did before for the counting of the edges of $\mathfrak{T}_n$, we will consider paths between terminal vertices of the tree. We count the faces associated to them.

So, in order to count the faces of $\mathfrak{T}_5$ we start with maximal paths between terminal vertices in T_5,that is whose diameter is equal to diam(T_5). We have two possibilities:

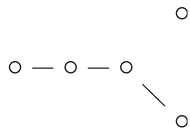
⊛ let P_4 the path constituted by the vertices $1, 2, 3$ and 4



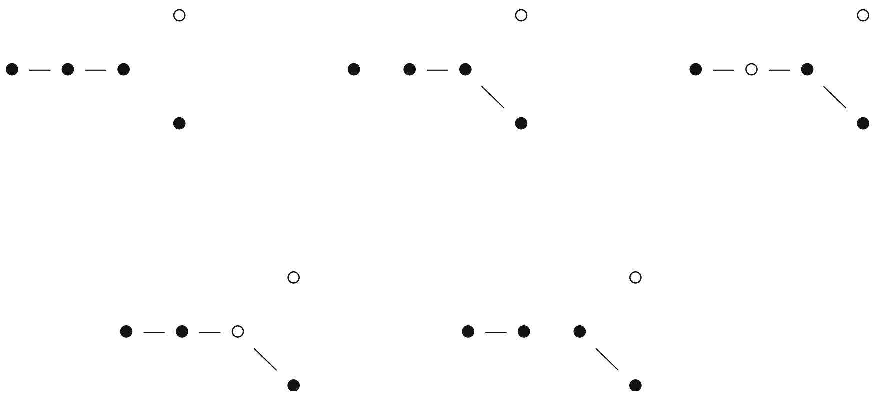
The faces of the tridiagonal Birkhoff polytope associated to the previous path corresponds to the following representations:



⊛ let P_4' be the path formed by vertices $1, 2, 3$ and 5



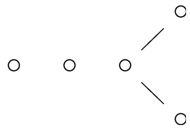
The faces of the respective tridiagonal Birkhoff polytope corresponds to:



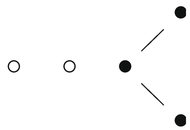
The previous representations will correspond to the 2-faces F_1, F_2, F_3, F_6, F_7, F_8, F_9, F_{12} and F_{13} of $\mathfrak{T}_5$. Observe that the representation of F_1 appears in both cases. This is due to the fact that the vertices 1,2 and 3 are common to both paths, $P_4 \cap P_4' = P_3$, here is the path with vertices 1, 2 and 3. So in the end we must remove the faces of $\mathfrak{T}_5$ that appear repeated. Therefore, so far we have nine different faces.

Now we consider all paths between terminal vertices of T_5 whose diameter is equal to diam$(T_5) - 1$.

⊛ Let P_3' be the path constituted by the vertices 3, 4 and 5 of T_5



The face of the respective tridiagonal Birkhoff polytope associated to this path is obtained from:

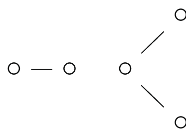


The representation above corresponds the faces F_4 and F_5 of $\mathfrak{T}_5$.
Until here we got 12 2-faces but one of them is repeated. Therefore we have 11 different faces.

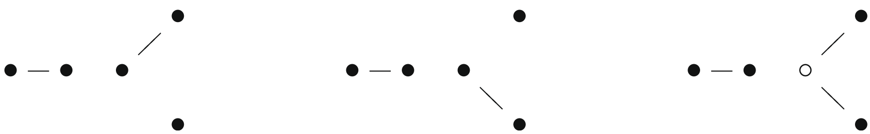
There is no possibility to obtain from the original tree, more paths with terminal vertices of T_5.

As in the configuration of a face we can have two T-components, we must consider the situation that involves two disjoint paths. Each T-component belongs to one of the paths, corresponding to an edge of the corresponding tridiagonal Birkhoff polytope.

The faces of $\mathfrak{T}_5$ obtained from an 1-face of $\Omega_2(P_2)$, where P_2 has vertices 1 and 2 of T_5, and from a 1-face of $\Omega_3(P_3')$, that is,



are



In this case the two first 2-faces correspond to F_{12} and F_{13} and they have already emerged before from the polytopes associated to P_4 and P_4', and the third 2-face corresponds to the face F_{14} of $\mathfrak{T}_5$.

Due to the nature of the initial graph the former bicolored subgraphs were the only possibility.

Finally, we are going to analyze the faces which bicolored subgraphs have a T-component with three endpoints and without inner entries. As this T-component needs at least four vertices, three endpoints and one open vertex in its interior, it remains only a free vertex. From this, we obtain two configurations corresponding to the faces F_{10} and F_{11} of $\mathfrak{T}_5$.

Therefore, it is possible to express $f_2(T_5)$ from the number of faces, edges and vertices of polytopes corresponding to paths and from the number of bicolored subgraphs which have a T-component with three endpoints and without inner entries.

This illustration leads to the following algorithm for counting the faces of $\mathfrak{T}_n$.

Algorithm 4

The input is a tree T with vertex set V.

Stage I Computation of the number of configuration of all faces of polytopes associated to different paths.

Step 1 ⧫ for each pair of different terminal vertices of the tree T consider the path P between them and compute $f_2(P) \times g_0(T\backslash P)$;
Step 2 ⧫ sum all the numbers obtained in step 1;
Step 3 ⧫ for each pair of different paths of step1 let P' be its intersection. Compute $f_2(P') \times g_0(T\backslash P')$;

Step 4 ⧫ sum all the numbers obtained in step 3;
Step 5 ⧫ calculate the difference between the numbers obtained in steps 2 and 4;
Step 6 ⧫ for each path P considered in step 1 compute $f_1(P)$;
Step 7 ⧫ delete all edges incident on any vertex of P and call to the remaining graph G;
Step 8 ⧫ let $\widetilde{G}$ be any connected component of G. For each pair of terminal vertices of $\widetilde{G}$ consider the path $\widetilde{P}$ between them and compute $f_1(\widetilde{P})$;
Step 9 ⧫ compute $f_1(P) \times f_1(\widetilde{P})$;
Step 10 ⧫ consider all paths of T formed with all terminal vertices of P and $\widetilde{P}$ and distinct from them;
Step 11 ⧫ in each paths formed in step 10, delete all edges that join P and $\widetilde{P}$ and let M and M' be the two subgraphs obtained;
Step 12 ⧫ compute $g_1(M) \times g_1(M')$;
Step 13 ⧫ sum all the products obtained in step 12;
Step 14 ⧫ subtract the value obtained in step 13 to the value obtained in step 9;
Step 15 ⧫ calculate $g_0(T\backslash(P \cup \widetilde{G}))$;
Step 16 ⧫ multiply the values obtained in steps 14 and 15;
Final step ⧫ sum the values obtained in steps 5 and 16.
Stage II Computation of the number of bicolored subgraphs corresponding to the T-component which has 3 endpoints and no inner entries.
Step 1 ⧫ for each vertex v of T, whose degree is greater than 2, we consider each of the triplets of incident edges on v, i.e, spiders with branches with maximum length.
Step 2 ⧫ for each $S_{p_1,p_2,p_3} = S'$ let $i = 0, 1 \ldots, p_1 - 1$, compute

$$f_0(P_{p_1-i})f_0(P_{p_2-j})f_0(P_{p_3-k})g_0(T\backslash S'), \text{forall} j = 0, 1, \ldots, p_2 - 1, k = 0, 1, \ldots, p_3 - 1;$$

Final step ⧫ sum all the values determined in previous step.

Stage III Sum the values obtained in final steps of stages I and II.

Recall that, as we have seen, the maximum number of closed endpoints of a T-component is at most three.

Proposition 5.1. *Let $S' = S_{p_1,p_2,\ldots,p_n}$ be a spider with n branches of lengths $p_1, \ldots, p_n$ and $N = p_1 + p_2 + \cdots + p_n + 1$ vertices. The number of faces of $\Omega_N(S')$ is given by*

$$\begin{aligned} f_2(T) = {} & \sum_{1 \leqslant i<j \leqslant n} \left[f_2(P_{p_i+p_{j+1}}) \prod_{k \neq i,j} f_0(P_{p_k}) \right] - (n-2) \sum_{i=1}^{n} f_2(P_{p_i+1}) \prod_{k \neq i} f_0(P_{p_k}) \\ & + \left[\sum_{\substack{1 \leqslant i<j \leqslant n \\ j \neq k \neq i}} (f_1(P_{p_i+p_j+1}) - f_1(P_{p_i+1}) - f_1(P_{p_j+1}))f_1(P_{p_k}) \right] \prod_{\ell \neq i,j,k} f_0(P_{p_\ell}) \\ & + \sum_{\substack{j \neq k \neq i \\ i \neq j}} \left(\sum_{\substack{0 \leqslant r \leqslant p_i - 1 \\ 0 \leqslant s \leqslant p_j - 1 \\ 0 \leqslant t \leqslant p_k - 1}} f_0(P_{p_i-r})f_0(P_{p_j-s})f_0(P_{p_k-t}) \prod_{\ell \neq i,j,k} f_0(P_{p_l}) \right). \end{aligned}$$

Proof. For each pair of different branches of S' p_i and p_j, consider the path $P_{p_i+p_j+1}$. Without loss of generality, for each i, $i = 1, \ldots, n-1$, j will run all values from $i+1$ to n.

The next sum represents the number of the configurations of all faces of polytopes associated to the referred paths

$$\sum_{1\leqslant i<j\leqslant n}\left[f_2(P_{p_i+p_{j+1}})\prod_{i\neq k\neq j}f_0(P_{p_k})\right].$$

When we consider the faces of the polytopes corresponding to those paths,

$$(n-2)\sum_{i=1}^{n}f_2(P_{p_i+1})\prod_{k\neq i}f_0(P_{p_k})$$

is the number of faces that are going to appear repeated. Therefore that number has to be excluded from the previous expression.

Now, we must consider the 2-faces resulting from two T-components, where one of them is in a branch and the other T-component has its two endpoints in two different branches. It results the following number:

$$\left[\sum_{\substack{1\leqslant i<j\leqslant n\\ j\neq k\neq i}}(f_1(P_{p_i+p_j+1})\ -f_1(P_{p_i+1})-f_1(P_{p_j+1}))f_1(P_{p_k})\right]\prod_{\ell\neq i,j,k}f_0(P_{p_\ell}).$$

Finally, the number of bicolored subgraphs corresponding to the T-components which have three endpoints and no inner entries is

$$\sum_{\substack{j\neq k\neq i\\ i\neq j}}\left(\sum_{\substack{0\leqslant r\leqslant p_i-1\\ 0\leqslant s\leqslant p_j-1\\ 0\leqslant t\leqslant p_k-1}}f_0(P_{p_i-r})f_0(P_{p_j-s})f_0(P_{p_k-t})\prod_{\ell\neq i,j,k}f_0(P_{p_\ell})\right).$$

Here, if $k\leqslant n$, then $f_n(P_k)=0$.

From the previous considerations we get the desire result. $\square$

6. Counting faces of any dimension of $\Omega_n(S)$

For a given star with n vertices, we have already seen that

$$f_1(S)=(n-1)+C_2^{n-1}=\frac{n(n-1)}{2}$$

and

$$f_2(S)=C_2^{n-1}+C_3^{n-1}=\frac{n(n-1)(n-2)}{6}.$$

In this section our aim is to obtain an expression for the number of p-faces of the acyclic Birkhoff polytope associated to S, for $3\leqslant p\leqslant n-1$.

This number is equal to the number of bicolored subgraphs with one T-component and whose sum of closed endpoints and inner entries equals to $p+1$.

Bearing in mind that $\operatorname{diam}S=2$, the T-components that we can consider to characterize a p-face, with $3\leqslant p\leqslant n-1$, are only of the two different types:

(i) with p closed endpoints and one inner entry; and

(ii) with $p+1$ closed endpoints and without inner entries.

The number of p-faces of $\Omega_n(S)$, whose bicolored subgraphs have, respectively, the first and second T-component is given by

$$C_p^{n-1}\quad\text{and}\quad C_{p+1}^{n-1}.$$

Therefore, the number of p-faces of $\Omega_n(S)$ is given by

$$f_p(S)=C_p^{n-1}+C_{p+1}^{n-1}=\frac{n!}{(p+1)!(n-p-1)!}.$$

7. Counting facets of $\mathfrak{T}_n$

Finally, we present an expression for the number of facets of $\mathfrak{T}_n$. Taking into account Proposition 1.1, the next proposition allows us to determine the number of facets of any acyclic Birkhoff polytope $\mathfrak{T}_n$. Here, an *end-edge* is an edge of the tree that is terminal, i.e., one of its vertices is an endpoint of the graph and an *inner-edge* is an edge of the tree that is not an end-edge, i.e., whose both vertices are not endpoints of the tree.

Proposition 7.1. *Let T be a tree with n ($n \geqslant 2$) vertices, where p of them are endpoints. The number of facets of the polytope $\Omega_n(T)$ is $2n - p - 1$.*

Proof. Considering Proposition 1.1, we are looking for all different bicolored subgraphs of T verifying

$$n - 2 = \theta_A + \iota_A - t_A.$$

We know that in any bicolored subgraph $\theta_A + \iota_A \leqslant n$. Therefore the bicolored subgraphs that we are searching for have at most two T-components. If the bicolored subgraph has only one T-component, we can have two different cases:

Case 1. All the endpoints of T are endpoints of the T-component as well. By (1.1), the T-component has $n - p - 1$ inner entries. Therefore, the T-component must have one (and only one) vertex of T that is not an inner entry. As this vertex can occupy $n - p$ different positions in T, we have $n - p$ different bicolored subgraphs and each of them represents a facet.

Case 2. The T-component has $p - 1$ endpoints that are also endpoints of the graph T and one endpoint that does is not an endpoint of the graph T, i.e, one of the end-edges of the graph T is not an edge of the T-component.

The number of different end-edges of the graph T is p. Therefore we have p different possibilities to get the bicolored subgraph and we have p different bicolored subgraphs whose corresponding face has dimension $n - 2$, and each of them represents a facet of $\mathfrak{T}_n$.

If the bicolored subgraph has two T-components, all the vertices of the bicolored subgraph must be closed. In this particular situation, the T-component must be “apart” by one inner edge. Since the number of edges of the tree T is $n - 1$, we have $n - 1 - p$ inner edges and consequently we have $n - 1 - p$ bicolored subgraphs of the graph T and each of them represents a facet of $\mathfrak{T}_n$. From the previous calculations we obtain the number of facets of the polytope $\mathfrak{T}_n$:

$$(n - p) + p + (n - 1 - p) = 2n - p - 1. \quad \square$$

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