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SYMPLECTIC KEYS AND DEMAZURE ATOMS IN TYPE C

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# Symplectic keys and Demazure atoms in type $C$

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# Abstract

The type $C$ Kashiwara-Nakashima tableaux, a variation of De Concini tableaux, provide a combinatorial model for crystals associated to finite-dimensional irreducible representations of the symplectic Lie algebra. Some of these tableaux, called key tableaux, yield a tableau criterion for the Bruhat order on the hyperoctahedral group, the type $C$ Weyl group, and a tableau criterion for the Bruhat order induced on the left cosets defined by parabolic subgroups of the hyperoctahedral group. In the type $A$ crystal of semistandard Young tableaux, using the *jeu de taquin*, Lascoux-Schützenberger presented an algorithm to compute type $A$ right and left key maps, that return key tableaux, for semistandard Young tableaux. Using the Sheats-Lecouvey symplectic *jeu de taquin*, we adapt Lascoux-Schutzenberger's algorithm in order to be able to compute right and left keys for type $C$ Kashiwara-Nakashima tableaux. In fact, we can compute symplectic keys, right or left, without the use of the *jeu de taquin* and, motivated by Willis' direct way of computing right and left keys of semistandard Young tableaux, we also give a way of computing symplectic, right or left, keys without the use of *jeu de taquin*. In type $C_n$, the symplectic right and left key maps give a description of Demazure atoms and opposite Demazure atoms, respectively, and consequently of Demazure and opposite Demazure characters. The symplectic right and left key maps, and consequently Demazure atoms and opposite Demazure atoms, are related through the Lusztig involution. A type $C$ Schützenberger evacuation is defined to realize that involution.

# Resumo

Os Kashiwara-Nakashima tableaux do tipo $C$, uma variante dos tableaux de De Concini, são um modelo combinatório para os cristais associados a representações irredutíveis de dimensão finita de álgebras de Lie simplécticas. Alguns destes tableaux, chamados de key tableaux, formam critério para a ordem de Bruhat dos elementos do grupo hiperoctaedral, o grupo de Weyl do tipo $C$, e formam também um critério para a Bruhat order induzida nas suas classes laterais esquerdas relativamente aos subgrupos parabólicos. No cristal do tipo $A$ formado pelos semistandard Young tableaux, utilizando o *jeu de taquin*, Lascoux e Schützenberger apresentaram um algoritmo para calcular, no tipo $A$, duas funções que dado um semistandard Young tableau devolvem um key tableau, right key e left key. Utilizando o *jeu de taquin* simpléctico de Sheats e Lecouvey, nós adaptamos o algoritmo do Lascoux e do Schutzenberger para poder calcular right e left keys de Kashiwara-Nakashima tableaux do tipo $C$. Na realidade, conseguimos calcular keys de tableaux do tipo $C$, right ou left, sem utilizar o *jeu de taquin* e, motivados pela maneira directa de Willis' de calcular right e left keys para semistandard Young tableaux, apresentamos uma maneira de calcular keys simplécticas, right ou left, que não utiliza o *jeu de taquin*. As funções para calcular para calcular keys simplécticas, right ou left, servem como uma descrição dos Demazure atoms e dos opposite Demazure atoms, respectivamente, e consequentemente também descrevem os carácteres de Demazure e as suas versões opposite. As funções que calculam keys simplécticas, right ou left, e consequentemente os Demazure atoms e os opposite Demazure atoms, estão relacionadas pela Lusztig involution. Definimos ainda uma Schützenberger evacuation no tipo $C$ para realizar a Lusztig involution.

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# Chapter 1

# Introduction

Symplectic tableaux are a combinatorial tool to study finite-dimensional representations of the symplectic Lie algebra $sp(2n,C)$ and its Weyl group $B_n$, the *hyperoctahedral group* (or signed symmetric group). The irreducible symplectic character, or symplectic Schur function, indexed by a partition $\lambda$, can be seen as a sum on symplectic tableaux of shape $\lambda$. King has proved this using a family of tableaux [21], nowadays known as King tableaux, and later De Concini found another family of symplectic tableaux [11], known as De Concini tableaux. Although quite distinct, Sheats created a weight and shape preserving bijection between both families of tableaux [39]. Here, we will work with another family of tableaux, *Kashiwara-Nakashima tableaux* [20] (for short, KN tableaux), which is a small variation of the symplectic tableaux defined by De Concini. These KN tableaux are endowed with a type $C_n$ crystal structure that naturally contains the type $A_{n-1}$ crystal of semistandard Young tableaux (SSYT's). In fact, the SSYT's are a particular case of KN tableaux. These KN tableaux can be seen as a folding, with some additional restrictions, of the SSYT's in $A_{2n-1}$, to include negative entries. The type $C_n$ Knuth relations, or plactic relations, which include the type $A_{n-1}$ Knuth relations, allow us to define a type $C_n$ plactic monoid, studied by Lecouvey in [25, 26], compatible with an insertion algorithm, known as Baker-Lecouvey insertion [5, 25], and with sliding algorithms, known as symplectic *jeu de taquin* (SJDT), due to Sheats [39] and further developed by Lecouvey in [25]. Lecouvey have interpreted plactic relations in terms of crystal isomorphisms.

The type $C_n$ Demazure characters $\kappa_v$ are indexed by vectors $v \in \mathbb{Z}^n$ in $B_n\lambda$, the $B_n$-orbit of the partition $\lambda$, and can be seen as "partial" characters. Kashiwara [18] and Littelmann [29] have shown that they can be obtained by summing monomial weights over certain subsets $\mathfrak{B}_v$, called Demazure crystals, of the crystal $\mathfrak{B}^\lambda$, with highest weight $\lambda$. The crystal $\mathfrak{B}^\lambda$ can be partitioned into Demazure crystal atoms, $\widehat{\mathfrak{B}}_u$, where $u \in B_n\lambda$, in such a way that, for all $v \in B_n\lambda$, the Demazure crystal $\mathfrak{B}_v$ is a union of Demazure crystal atoms $\widehat{\mathfrak{B}}_u$ over the Bruhat interval $\lambda \leq u \leq v$.

Lascoux and Schützenberger, in [24], were the first ones to introduce the concept of Demazure crystal atoms, originally called standard bases, in type $A_{n-1}$. They proved that each Demazure crystal atom contains exactly one key tableau, a tableau whose columns form a nested set. Using the *jeu de taquin*, Lascoux and Schützenberger also defined a map, called *right key map*, that given a tableau returns the key tableau that identifies the atom that contains the given tableau. There is a dual definition of these Demazure atoms and Demazure crystal atoms, called opposite Demazure atoms and opposite Demazure crystal atoms. As expected, each opposite Demazure crystal atoms also contains

exactly one key tableau, and the map that, given a tableau returns the key tableau that identifies the opposite Demazure atom that contains it, is called the *left key map*. Lascoux-Schützenberger method of computing right keys via *jeu de taquin* can easily be adapted to compute left keys. More recently, Willis, in [42], found a way of computing these right and left keys without the use of the *jeu de taquin*. There are other methods and models where type $A_{n-1}$ keys are computed, such as the alcove path model [28], semi skyline augmented fillings [31], and coloured vertex models [7], for instance.

In type $C_n$, in a presentation by Azenhas in The 69th Séminaire Lotharingien de Combinatoire [4], Azenhas and Mamede identified the KN tableaux with nested columns and without symmetric entries as type $C_n$ key tableaux, and raised questions about the existence of a right key map that does the same job as Lascoux-Schützenberger's right key map, and consequently, would provide a description for the Demazure crystal that does not require to build the crystal $\mathfrak{B}^\lambda$. This was the motivation for our main theorem, Theorem 5.2.6, which provides a description of a Demazure crystal atom in type $C$ using the right key map defined in Theorem 5.1.5, via SJDT. During our work, we were informed by Jacon and Lecouvey that they have found a way to compute key maps of KN tableaux [17]. Although their approach is different from ours, their algorithm to compute the key maps is effectively the same as ours. Also in [17], Jacon and Lecouvey also suggested that Willis' direct way [42] of computing right and left should have a generalization for type $C_n$ KN tableaux. So, motivated by Willis' direct way of computing keys without the use of *jeu de taquin*, we provide algorithms, in Theorem 5.3.3 and in Theorem 5.3.9, of computing symplectic left and right keys that do not require the SJDT. Finally, we relate left and right keys through the Lusztig involution and realize it by a type $C_n$ Schützenberger evacuation. There are also keys computed in the type $C_n$ alcove path model [27, 28], which is a crystal isomorphic to the type $C_n$ model of KN tableaux, and in the coloured five vertex model [8], whose coincidence with type $C_n$ model of KN tableaux is conjectured.

This thesis is organized as follows:

- In Chapter 2 we introduce KN tableaux, as well as type $C_n$ plactic monoid, together with the Baker-Lecouvey insertion, the Sheats-Lecouvey symplectic *jeu de taquin* and the type $C_n$ Robinson-Schensted correspondence. We finish this chapter with type $C_n$ crystals and their relation with the plactic monoid in terms of crystal isomorphisms.

- In Chapter 3 we recall the type $C_n$ Weyl group, the hyperoctahedral group, its Bruhat order, and give a tableau criterion for the Bruhat order of this group, and a tableau criterion for the Bruhat order induced on the left cosets defined by the parabolic subgroups of the hyperoctahedral group. This tableau criterion uses only KN key tableaux.

- In Chapter 4 we define the Demazure crystal and the opposite Demazure crystal. Then, we embed every KN tableau in a type $A$ cocrystal, isomorphic to the crystal of SSYT's with conjugated shape. This cocrystal is motivated by Lascoux' double crystal graph in type $A$ [22], and by Heo-Kwon work in [16], where Schützenberger *jeu de taquin* slides are used as crystal operators for $\mathfrak{sl}_2$. This serves as an appetizer for the next chapter.

- In Chapter 5, we compute symplectic right and left key maps. First, we do this computation via SJDT, mimicking Lascoux-Schützenberger approach in type $A_{n-1}$, and prove, in our main theorem, that these maps describe Demazure crystal atoms and opposite Demazure crystal

atoms. Later in this chapter, motivated by Willis' algorithms for SSYT's [42], we compute right and left keys of a type $C_n$ KN tableau without using the *jeu de taquin*.

- In Chapter 6 we introduce the type $C_n$ Lusztig involution [30] and we adapt the type $A_{n-1}$ Schützenberger evacuation [38] for SSYT's to type $C_n$ KN tableaux. The Lusztig involution relates the right and left key maps, and consequently, it relates Demazure crystals and opposite Demazure crystals.

- In our final chapter, Chapter 7, we discuss some unfinished or undone work related to symplectic key tableaux, focusing mainly on a combinatorial approach for the type $C_n$ Fu-Lascoux non-symmetric Cauchy kernel. The two last sections address questions on the generalizations of our results to other combinatorial models.

Related to this thesis we published one paper, [36], and one preprint, [35], submitted to a journal. Also, for each one of these publications, an extended abstract was accepted in the proceedings of The 32nd and The 33rd Conference on Formal Power Series and Algebraic Combinatorics, [34] and [37], respectively.

# Chapter 2

# Type $C_n$ Kashiwara-Nakashima tableaux, symplectic plactic monoid and type $C_n$ crystal graphs
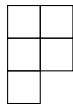
In this chapter we present type $C_n$ Kashiwara-Nakashima tableaux, as well as the Baker-Lecouvey insertion, Lecouvey-Sheats symplectic *jeu de taquin*, type $C_n$ Robinson-Schensted-Knuth correspondence and the type $C_n$ plactic monoid. We finish this chapter with Kashiwara crystals and coplactic equivalence, and their relation with the plactic monoid. Our main references for this chapter are [25], [5] and [9].

## 2.1 Kashiwara-Nakashima tableaux

The symplectic tableaux studied here were introduced by Kashiwara and Nakashima to label the vertices of the type $C_n$ crystal graphs [20].

Fix $n \in \mathbb{N}_{>0}$ and define the sets $[n] = \{1, \ldots, n\}$ and $[\pm n] = \{1, \ldots, n, \bar{n}, \ldots, \bar{1}\}$ where $\bar{i}$ is just another way of writing $-i$, hence $\bar{\bar{i}} = i$. In the second set we will consider the following order of its elements: $1 < \cdots < n < \bar{n} < \cdots < \bar{1}$ instead of the usual order.

A vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is a partition of $|\lambda| = \sum_{i=1}^{n} \lambda_i$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The *Young diagram* of shape $\lambda$, in English notation, is an array of boxes (or cells), left justified, in which the $i$-th row, from top to bottom, has $\lambda_i$ boxes. We identify a partition with its Young diagram.

For example, the Young diagram of shape $\lambda = (2, 2, 1)$ is . We define $\Delta^n = (n, n-1, \ldots, 1)$ to be the staircase partition in $\mathbb{Z}^n$.

Given $\mu$ and $\nu$ two partitions with $\nu \leq \mu$ entrywise, we write $\nu \subseteq \mu$. The Young diagram of shape $\mu/\nu$ is obtained after removing the boxes of the Young diagram of $\nu$ from the Young diagram of $\mu$.

For example, the Young diagram of shape $\mu/\nu = (2, 2, 1)/(1, 0, 0)$ is .

Let $\nu \subseteq \mu$ be two partitions and $A$ a completely ordered alphabet. A *semistandard Young tableau* (SSYT) of skew shape $\mu/\nu$, on the alphabet $A$, is a filling of the diagram $\mu/\nu$ with letters from $A$,

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such that the entries are strictly increasing, from top to bottom, in each column and weakly increasing, from left to right, in each row. When $|v| = 0$ then we obtain a semistandard Young tableau of straight shape $\mu$. Denote by $\mathcal{SSYT}(\mu/v, A)$ the set of all skew SSYT's $T$ of shape $\mu/v$, with entries in $A$. In particular, when $|v| = 0$ we write $\mathcal{SSYT}(\mu, A)$ and when $A = [n]$ we write $\mathcal{SSYT}(\mu/v, n)$.

When considering tableaux with entries in $[\pm n]$, it is usual to have some extra conditions besides being semistandard. We will use a family of tableaux known as *Kashiwara-Nakashima* tableaux. From now on we consider tableaux on the alphabet $[\pm n]$.

A *column* is a strictly increasing sequence of numbers (or letters) in $[\pm n]$ and it is usually displayed vertically. The height of a column is the number of letters in it. A column is said to be *admissible* if the following *one column condition* (1CC) holds for that column:

**Definition 2.1.1** (1CC). *Let $C$ be a column. The 1CC holds for $C$ if for all pairs $i$ and $\bar{i}$ in $C$, where $i$ is in the a-th row counting from the top of the column, and $\bar{i}$ in the b-th row counting from the bottom, we have $a + b \leq i$. Equivalently, for all pairs $i$ and $\bar{i}$ in $C$, the number $N(i)$ of letters $x$ in $C$ such that $x \leq i$ or $x \geq \bar{i}$ satisfies $N(i) \leq i$.*

If a column $C$ satisfies the 1CC then $C$ has at most $n$ letters. If 1CC does not hold for $C$ we say that $C$ *breaks the* 1CC *at* $z$, where $z$ is the minimal positive integer such that $z$ and $\bar{z}$ exist in $C$ and there are more than $z$ numbers in $C$ with absolute value less or equal than $z$.

**Example 2.1.2.** *The column* $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \bar{1} \\ \hline \end{array}$ *breaks the* 1CC *at* 1, *and* $\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \bar{3} \\ \hline \end{array}$ *is an admissible column.*

The following definition states conditions to when $C$ can be *split*:

**Definition 2.1.3.** *Let $C$ be a column and let $I = \{z_1 > \cdots > z_r\}$ be the set of unbarred letters $z$ such that the pair $(z, \bar{z})$ occurs in $C$. The column $C$ can be split when there exists a set of $r$ unbarred letters $J = \{t_1 > \cdots > t_r\} \subseteq [n]$ such that:*

1. *$t_1$ is the greatest letter of $[n]$ satisfying $t_1 < z_1$, $t_1 \notin C$, and $\overline{t_1} \notin C$,*

2. *for $i = 2, \ldots, r$, we have that $t_i$ is the greatest letter of $[n]$ satisfying $t_i < \min(t_{i-1}, z_i)$, $t_i \notin C$, and $\overline{t_i} \notin C$.*

The 1CC holds for a column $C$ (or $C$ is admissible) if and only if $C$ can be split [39, Lemma 3.1]. If $C$ can be split then we define *right column* of $C$, $rC$, and the *left column* of $C$, $\ell C$, as follows:

1. $rC$ is the column obtained by changing in $C$, $\overline{z_i}$ into $\overline{t_i}$ for each letter $z_i \in I$ and by reordering if necessary,

2. $\ell C$ is the column obtained by changing in $C$, $z_i$ into $t_i$ for each letter $z_i \in I$ and by reordering if necessary.

If $C$ is admissible then $\ell C \leq C \leq rC$ by entrywise comparison, where $\ell C$ has the same barred part as $C$ and $rC$ the same unbarred part. If $C$ does not have symmetric entries, then $C$ is admissible and $\ell C = C = rC$. In the next definition we give conditions for a column $C$ to be *coadmissible*.

**Definition 2.1.4.** *We say that a column C is coadmissible if for every pair i and $\bar{i}$ on C, where i is on the a-th row counting from the top of the column, and $\bar{i}$ on the b-th row counting from the top, then $b - a \leq n - i$. Equivalently, for every pair i and $\bar{i}$ on C, the number $N^*(i)$ of letters x in C such that $i \leq x \leq \bar{i}$ satisfies $N^*(i) \leq n - i + 1$.*

Unlike in Definition 2.1.1, in the last definition $b$ is counted from the top of the column.

**Definition 2.1.5.** *Let C be a column and let $I = \{z_1 > \cdots > z_r\}$ be the set of unbarred letters z such that the pair $(z, \bar{z})$ occurs in C. The column C is coadmissible if and only if there exists a set of r unbarred letters $H = \{h_1 > \cdots > h_r\} \subseteq [n]$ such that:*

1. *$h_r$ is the smallest letter of $[n]$ satisfying $h_r > z_r$, $h_r \notin C$, and $\overline{h_r} \notin C$,*

2. *for $i = r-1, \ldots, 1$, we have that $h_i$ is the smallest letter of $[n]$ satisfying $h_i > \max(h_{i+1}, z_i)$, $h_i \notin C$, and $\overline{h_i} \notin C$.*

Given an admissible column $C$, consider the map

$$\Phi : C \mapsto C^*$$

that sends $C$ to the column $C^*$ of the same size in which the unbarred entries are taken from $\ell C$ and the barred entries are taken from $rC$.

**Lemma 2.1.6.** *Let C be an admissible column on the alphabet $[\pm n]$, and I and J the sets in Definition 2.1.3. The entries x (barred or unbarred) of $\Phi(C)$ are such that*

1. *$x \in \Phi(C)$ and $\bar{x} \notin \Phi(C)$ if and only if $x \in C$ and $\bar{x} \notin C$.*

2. *$x, \bar{x} \in \Phi(C)$ if and only if $x \in J$ or $\bar{x} \in J$.*

*Equivalently, the set of entries in $\Phi(C)$ is $(J \cup \bar{J} \cup C) \setminus (I \cup \bar{I})$.*

Henceforth, $\Phi(C) = C$ if and only if $I = \emptyset$ (hence $J = \emptyset$), that is, $C$ does not have symmetric entries.

The *column* $\Phi(C)$ is a coadmissible column and the algorithm to form $\Phi(C)$ from $C$ is reversible [25, Section 2.2]. In particular, every column on the alphabet $[n]$ is simultaneously admissible and coadmissible. The map $\Phi$ is a bijection between admissible and coadmissible columns of the same height on the alphabet $[\pm n]$.

**Example 2.1.7.** *Let $C = \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \bar{2} \\ \hline \end{array}$ be an admissible column, so it can be split. Then $\ell C = \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \bar{2} \\ \hline \end{array}$ and $rC = \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \bar{1} \\ \hline \end{array}$. So $\Phi(C) = \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \bar{1} \\ \hline \end{array}$ is coadmissible. C is also coadmissible and $\Phi^{-1}(C) = \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \bar{3} \\ \hline \end{array}$.*

Let $T$ be a skew tableau with all of its columns admissible. The *split form* of a skew tableau $T$, $spl(T)$, is the skew tableau obtained after replacing each column $C$ of $T$ by the two columns $\ell C \, rC$. The tableau $spl(T)$ has double the amount of columns of $T$.

A semistandard skew tableau $T$ is a *Kashiwara-Nakashima (KN) skew tableau* if its split form is a semistandard skew tableau in type $A_{2n-1}$ (because it uses $2n$ letters). We define $\mathcal{KN}(\mu/\nu,n)$ to be the set of all KN tableaux of shape $\mu/\nu$ in the alphabet $[\pm n]$. When $\nu=0$, we obtain $\mathcal{KN}(\mu,n)$. The weight of $T$ is a vector whose $i$-th entry is the number of $i$'s minus the number of $\bar{i}$.

If $T$ is a skew tableau, the *column reading* of $T$, $cr(T)$, is the word read in $T$ in the Japanese way, column reading top to bottom and right to left. The *length* of a word $w$ is the total number of letters in $w$. The weight of a $w$ in the alphabet $[\pm n]$ is the vector $\operatorname{wt} w = (t_1 - t_{\bar{1}}, t_2 - t_{\bar{2}}, \ldots, t_n - t_{\bar{n}}) \in \mathbb{Z}^n$, where $t_\alpha$ is the number of $\alpha$'s in $w$, with $\alpha \in [\pm n]$. Note that the weight of a tableau and of its column reading coincide.

**Example 2.1.8.** *The split of the tableau $T = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \bar{3} \\ \cline{1-1} \end{array}$ is the tableau $spl(T) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 \\ \hline \bar{3} & \bar{1} \\ \cline{1-2} \end{array}$. Hence*
$T \in \mathcal{KN}((2,2,1),3)$, *weight* $\operatorname{wt} T = (0,2,1)$ *and* $cr(T) = 2323\bar{1}$.

If $T$ is a tableau without symmetric entries in any of its columns, i.e., for all $i \in [n]$ and for all columns $C$ in $T$, $i$ and $\bar{i}$ do not appear simultaneously in the entries of $C$, then in order to check whether $T$ is a KN tableau it is enough to check whether $T$ is semistandard in the alphabet $[\pm n]$. In particular $\mathcal{SSYT}(\mu/\nu,n) \subseteq \mathcal{KN}(\mu/\nu,n)$.

## 2.2    Symplectic *jeu de taquin*

Lecouvey-Sheats symplectic *jeu de taquin* (SJDT) [25, 39] is a procedure on KN skew tableaux, compatible with *Knuth equivalence* (or plactic equivalence on words over the alphabet $[\pm n]$) [25], that allows us to change the shape of a tableau and to rectify it. To explain how the SJDT behaves, we need to look how it works on 2-column KN skew tableaux $C_1 C_2$. A skew tableau is *punctured* if one of its box contains the symbol $*$ called the *puncture*. A punctured column is admissible if the column is admissible when ignoring the puncture. A punctured skew tableau is admissible if its columns are admissible and the rows of its split form are weakly increasing, ignoring the puncture. Let $T$ be a punctured skew tableau with two columns $C_1$ and $C_2$ with the puncture in $C_1$. In that case, the puncture splits into two punctures in $spl(T)$, and ignoring the punctures, $spl(T)$ must be semistandard. Let $\alpha$ be the entry under the puncture of $rC_1$, and $\beta$ the entry to the right of the puncture of $rC_1$.

$$spl(T) = \ell C_1 r C_1 \ell C_2 r C_2 = \begin{array}{|c|c|c|c|} \hline \cdots & \cdots & \cdots & \cdots \\ \hline * & * & \beta & \cdots \\ \hline \cdots & \alpha & \cdots & \cdots \\ \hline \cdots & \cdots \\ \cline{1-2} \end{array} ,$$

where $\alpha$ or $\beta$ may not exist. The elementary steps of SJDT are the following:

**A.** If $\alpha \leq \beta$ or $\beta$ does not exist, then the puncture of $T$ will change its position with the cell beneath it. This is a vertical slide.

**B.** If the slide is not vertical, then it is horizontal. So we have $\alpha > \beta$ or $\alpha$ does not exist. Let $C_1'$ and $C_2'$ be the columns obtained after the slide. We have two subcases, depending on the sign of $\beta$:

1. If $\beta$ is barred, we are moving a barred letter, $\beta$, from $\ell C_2$ to the punctured box of $rC_1$, and the puncture will occupy $\beta$'s place in $\ell C_2$. Note that $\ell C_2$ has the same barred part as $C_2$ and that $rC_1$ has the same barred part as $\Phi(C_1)$. Looking at $T$, we will have an horizontal slide of the puncture, getting $C_2' = C_2 \setminus \{\beta\} \sqcup \{*\}$ and $C_1' = \Phi^{-1}(\Phi(C_1) \setminus \{*\} \sqcup \{\beta\})$. In a sense, $\beta$ went from $C_2$ to $\Phi(C_1)$.

2. If $\beta$ is unbarred, we have a similar case, but this time $\beta$ will go from $\Phi(C_2)$ to $C_1$, hence $C_1' = C_1 \setminus \{*\} \cup \{\beta\}$ and $C_2' = \Phi^{-1}(\Phi(C_2) \setminus \{\beta\} \sqcup \{*\})$. Although in this case it may happen that $C_1'$ is no longer admissible. In this situation, if the 1CC breaks at $i$, we erase both $i$ and $\bar{i}$ from the column and remove a cell from the bottom and from the top column, and place all the remaining cells orderly with respect to their entries.

Applying successively elementary SJDT slides, eventually, the puncture will be a cell such that $\alpha$ and $\beta$ do not exist. In this case we redefine the shape to not include this cell and the *jeu de taquin* ends.

Given an admissible tableau $T$ of shape $\mu/\nu$, a box of the diagram of shape $\nu$ such that boxes under it and to the right are not in that shape is called an inner corner of $\mu/\nu$. An outside corner is a box of $\mu$ such that boxes under it and to the right are not in the shape $\mu$. The rectification of $T$ consists in playing the SJDT until we get a tableau of shape $\lambda$, for some partition $\lambda$. More precisely, apply successively elementary SJDT steps to $T$ until each cell of $\nu$ becomes an outside corner. At the end, we obtain a KN tableau for some shape $\lambda$. The rectification is independent of the order in which the inner corners of $\nu$ are filled [25, Corollary 6.3.9].

**Example 2.2.1.** *Consider the KN skew tableau* $T = \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & 3 \\ \hline 2 & \bar{1} \\ \hline \end{array}$ . *Let $C_1$ and $C_2$ be the first and second columns of $T$. To rectify $T$ via SJDT, one creates a puncture in the inner corner of $T$ and, by splitting, one obtains* $\begin{array}{|c|c|c|c|} \hline * & * & 2 & 2 \\ \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & \bar{1} & \bar{1} \\ \hline \end{array}$ . *So, the first two slides are vertical, obtaining* $\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 \\ \hline * & * & \bar{1} & \bar{1} \\ \hline \end{array}$ . *Finally, we do an horizontal slide, of type* **B.1***, in which we take $\bar{1}$ from $C_2$, and add it to the coadmissible column* $\Phi(C_1)$. *That is, $C_2' = (C_2 \cup \{*\}) \setminus \bar{1}$ and $C_1' = \Phi^{-1}((\Phi(C_1) \setminus \{*\}) \cup \bar{1})$, obtaining the tableau* $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \bar{3} & \\ \hline \end{array}$ .

Let $T$ be a skew tableau of shape $\mu/\nu$. Consider a punctured box that can be added to $\mu$, so that $\mu \cup \{*\}$ is a valid shape. The SJDT is reversible, meaning that we can move $*$, the empty cell outside of $\mu$, to the inner shape $\nu$ of the skew tableau $T$, simultaneously increasing both the inner and outer shapes of $T$ by one cell. The slides work similarly to the previous case: the vertical slide means that an empty cell is going up and an horizontal slide means that an entry goes from $\Phi(C_1)$ to $C_2$ or from $C_1$ to $\Phi(C_2)$, depending on whether the slid entry is barred or not, respectively. We will also call the *reverse jeu de taquin* as SJDT. In the next sections we will be mostly dealing with the *reverse jeu de taquin*. Consider the following examples, each containing a tableau and a punctured box that will be slid to its inner shape: $\begin{array}{|c|c|c|} \hline & & * \\ \hline 1 & \bar{1} & \\ \hline 2 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & \bar{1} & \\ \hline 2 & & \\ \hline \end{array}$ ; $\begin{array}{|c|c|} \hline & \\ \hline 1 & \bar{1} \\ \hline 2 & * \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline & 2 \\ \hline 2 & 2 \\ \hline \end{array}$ .

If a tableau with columns $C_1$ and $C_2$ does not have symmetric entries then the SJDT applied to $C_1 C_2$ coincides with the *jeu de taquin* known for SSYT's. In sections 5.3.1 and 5.3.3, we use SJDT to swap lengths of consecutive columns in a skew tableau, to obtain skew tableaux Knuth related to a

straight tableau, which is minimal for the number of cells within its Knuth class. Recall that in the elementary step B.2 it is possible to lose cells. If we do a reverse elementary step B.2 that results in having two more cells in the skew tableau, we would have to start by adding two symmetric entries to an admissible column, making it non admissible [25, Lemma 3.2.3], and then slide an unbarred cell to the column to its right. For instance, consider the following reverse elementary step B.2 ($\equiv$ denotes type $C_n$ Knuth equivalence [25, Definition 3.2.1]):

$$
\begin{array}{cc}
\begin{array}{|c|c|}
\hline
1 & * \\
\hline
2 & \\
\cline{1-1}
\end{array}
\end{array}
\equiv
\begin{array}{|c|c|}
\hline
1 & * \\
\hline
2 & \\
\cline{1-1}
3 & \\
\cline{1-1}
\overline{3} & \\
\cline{1-1}
\end{array}
\equiv
\begin{array}{|c|c|}
\hline
* & 1 \\
\hline
2 & \\
\cline{1-1}
3 & \\
\cline{1-1}
\overline{3} & \\
\cline{1-1}
\end{array}
$$

.

The first and last skew tableaux are Knuth equivalent, but the middle tableau is not a KN skew tableau. The three semistandard tableaux are Knuth equivalent column words, via the contractor/dilator Knuth relation [25, Definition 3.2.1].

Hence, a reverse elementary step B.2 that results in having more cells in the skew tableau has to be forced, since we have to start by forcing the existence of a non admissible column. This means that if we start with a minimal skew tableau, that is, a skew-tableau with the number of cells of its rectification, we can play SJDT, or its reverse, without ever incur in a loss/gain of boxes.

## 2.3    Baker-Lecouvey insertion

Let's start by recalling the column insertion for SSYT's. Let $T \in \mathcal{SSYT}(\lambda, n)$ be a tableau with column decomposition $T = C_1 C_2 \cdots C_k$. Given a column reading word, we can recover the original tableau via *column insertion*: Let $w = w_1 \cdots w_\ell$. We start with $i := 1$, $T = \emptyset$ , the empty tableau, and $p = 1$.

1.  If $w_i$ is bigger than all entries of $C_p$, Just add a cell to the column $C_p$ with entry $w_i$. Else find $\alpha \in C_p$ the smallest entry of $C_p$ bigger or equal than $w_i$. Then replace $\alpha$ by $w_i$ in $C_p$ and redefine $w_i := \alpha$, p:=p+1 and go to (1) (this is called a *bumping*).

2.  If $i \neq \ell$, then $i := i + 1$, $p := 1$ and go to (1). Else the algorithm ends.

The Baker-Lecouvey insertion [5, 25] is a bumping algorithm that given a word in the alphabet $[\pm n]$ returns a KN tableau. Let $w$ be a word in the alphabet $[\pm n]$, we call $P(w)$ to the tableau obtained after inserting $w$. This insertion is similar to the column insertion for SSYT's. In fact both have the same behaviour unless one the following three cases happens:

Suppose that we are inserting the letter $\alpha$ in the column $C$ of the KN tableau and

(I) $\bar{y} \in C$ is the smallest letter bigger or equal then $\alpha$ and $y \in C$, for some $y \in [n]$: there is in $C$ a maximal string of consecutive decreasing integers $y, y-1, \ldots, u+1$ starting in the entry $y$ in the column $C$. Then the bump consists of replacing the entry $\bar{y}$ with $\alpha$ and subtracting 1 to the entries $y, y-1, \ldots, u+1$. The entry $\bar{u}$ is then inserted in the next column to the right. This is known as the *Type I special bump*.

(II) if $\alpha = x$ and $\bar{x} \in C$, for some $x \in [n]$: there is a maximal string of consecutive decreasing entries $\bar{x}, \overline{x+1}, \ldots, \overline{v-1}$ starting in the entry $\bar{x}$ in $C$. Let $\beta$ be the next entry above $\overline{v-1}$. Then we have two subcases:

    (a) If $v \leq \beta \leq \overline{v+1}$ then suppose $\delta$ is the smallest entry in $C$ which is bigger or equal than $v$. Then this bump consists of deleting the entry $\bar{x}$, shifting the entries $\overline{x+1}, \ldots, \overline{v-1}$ down one position, inserting $\bar{v}$ where $\overline{v-1}$ was, and replacing $\delta$ with $v$. The entry $\delta$ is then bumped into the next column. This is known as the *Type IIa special bump*.

    (b) If $\beta \leq v - 1$ or $\beta$ does not exist then there is a maximal string (possibly empty) of consecutive integers $v - 1, \ldots, u + 1$ above the entry $\overline{v-1}$. The string is not empty only when $\beta = v - 1$, or else the string is empty and $u = v - 1$. The bump consists of deleting the entry $\bar{x}$, shifting the entries $\overline{x+1}, \ldots, \overline{u+1}$ down one position, and inserting an entry $u$ where $\overline{u+1}$ (or $\overline{v-1}$, if $\beta \neq v - 1$) was. The entry $\bar{u}$ is then bumped into the next column. This is known as the *Type IIb special bump*.

(III) after adding $\alpha$ in the bottom of the column $C$, the 1CC breaks at $i$: then we will slide out the cells that contain $\bar{i}$ and $i$ via symplectic *jeu de taquin*.

In the case III of the Baker-Lecouvey insertion we will be removing a cell from the tableau instead of adding. The length of $cr(P(w))$ might be less than the length of $w$ and the weight is preserved during Baker-Lecouvey insertion, $\mathrm{wt}(w) = \mathrm{wt}(P(w))$.

**Remark 2.3.1.** *The Baker-Lecouvey insertion is different from what we would have if we use the SSYT column insertion. However, if the word $w$ does not have symmetric letters, then the insertion works just like the column insertion for SSYT's. Apart from this case, if we were to use SSYT column insertion, the final tableau may not even be a KN tableau. For instance, consider the word $w = 2\bar{1}1$. The Baker-Lecouvey insertion of $w$ creates the sequence of tableaux* $\boxed{2}$ $\begin{array}{c}\boxed{2}\\\boxed{\bar{1}}\end{array}$ $\begin{array}{cc}\boxed{2}&\boxed{2}\\\boxed{\bar{2}}\end{array} = P(2\bar{1}1)$. *The SSYT column insertion of $w$ results in the tableau* $\begin{array}{cc}\boxed{1}&\boxed{2}\\\boxed{\bar{1}}\end{array}$, *which is not a KN tableau because the first column is not admissible.*

**Example 2.3.2.** *Consider the word $w = 23\overline{23}1$. We now insert all five letters of $w$, obtaining the following sequence of tableaux:* $\boxed{2}$ $\begin{array}{c}\boxed{2}\\\boxed{3}\end{array}$ $\begin{array}{c}\boxed{2}\\\boxed{3}\\\boxed{\bar{2}}\end{array}$ $\begin{array}{cc}\boxed{1}&\boxed{\bar{1}}\\\boxed{3}\\\boxed{\bar{3}}\end{array}$ $\begin{array}{ccc}\boxed{1}&\boxed{1}&\boxed{\bar{1}}\\\boxed{3}\\\boxed{\bar{3}}\end{array} = P(w)$. *Note that the insertion of the fourth letter, $\bar{3}$, causes a type I special bump on the first column and the insertion of the fifth letter, 1, causes a type IIb special bump on the second column.*

**Proposition 2.3.3.** *[25, Corollary 6.3.9] Let $T \in \mathcal{KN}(\mu/\nu, n)$. Then the tableau obtained after rectifying $T$ via symplectic jeu de taquin coincides with $P(cr(T))$. Moreover, the insertion of $w = w_1 \ldots w_k$, $P(w)$, is the rectification of the tableau with diagonal shape $\Delta^n/\Delta^{n-1}$ and column reading $w$.*

In particular we have that if we insert $cr(T)$ we obtain $T$ again. This implies that during the insertion of $cr(T)$ the case III of the Baker-Lecouvey insertion cannot happen. In Example 2.3.2, we may conclude that $P(23\overline{23}1) = P(cr(P(23\overline{23}1))) = P(\bar{1}113\bar{3})$.

## 2.4    Robinson-Schensted type $C_n$ correspondence and plactic equivalence

Let $[\pm n]^*$ be the free monoid on the alphabet $[\pm n]$. The type $C_n$ Robinson-Schensted correspondence [25, Theorem 5.2.2] is a combinatorial bijection between words $w \in [\pm n]^*$ and pairs $(P(w), Q)$ where $P(w)$ is a KN tableau and $Q$ is an oscillating tableau, a sequence of Young diagrams that record, by order, the shapes of the tableaux obtained while inserting $w$, whose final shape is the same as $P(w)$. Every two consecutive shapes of the oscillating tableau differ in exactly one cell and its length is the same of $w$. Words with the same oscillating tableau identify the coplactic classes in the Robinson-Schensted correspondence. These words are connected by crystal operators [25, Proposition 5.2.1], that we present in the next section.

Since both the SJDT and the Baker-Lecouvey insertion are reversible [5, 25], we have that every pair $(P, Q)$, with the same final shape, is originated by exactly one word. The type $C_n$ Robinson-Schensted correspondence is the following map:

$$[\pm n]^* \to \bigsqcup_\lambda \mathcal{KN}(\lambda, n) \times \mathcal{O}(\lambda, n):$$

$$w \mapsto (P(w), Q(w))$$

where the union is over all partitions $\lambda$ with at most $n$ parts, and $\mathcal{O}(\lambda, n)$ is the set of all oscillating tableaux with final shape $\lambda$ and all shapes of the sequence have at most $n$ rows.

**Example 2.4.1.** *In Example 2.3.2, the word $w = 23\overline{23}1$ is associated to the pair*



Given $w_1, w_2 \in [\pm n]^*$, the relation $w_1 \sim w_2 \Leftrightarrow P(w_1) = P(w_2)$ defines an equivalence relation on $[\pm n]^*$ known as *Knuth equivalence* (or plactic equivalence). The type $C_n$ plactic monoid is the quotient $[\pm n]^* / \sim$ where each Knuth (plactic) class is uniquely identified with a KN tableau [23, 25]. The quotient $[\pm n]^* / \sim$ can also be described as the quotient of $[\pm n]^*$ by the *elementary Knuth relations*:

K1:  $\gamma\beta\alpha \sim \beta\gamma\alpha$, where $\gamma < \alpha \leq \beta$ and $(\beta, \gamma) \neq (\bar{x}, x)$ for all $x \in [n]$.

K2:  $\alpha\beta\gamma \sim \alpha\gamma\beta$, where $\gamma \leq \alpha < \beta$ and $(\beta, \gamma) \neq (\bar{x}, x)$ for all $x \in [n]$.

K3:  $y + 1\overline{y+1}\beta \sim \bar{y}y\beta$, where $y < \beta < \bar{y}$ and $y \in [n-1]$.

K4:  $\beta\bar{y}y \sim \beta y + 1\overline{y+1}$, where $y < \beta < \bar{y}$ and $y \in [n-1]$.

K5:  $w \sim w \setminus \{z, \bar{z}\}$, where $w \in [\pm n]^*$ and $z \in [n]$ are such that $w$ is a non-admissible column that the $1CC$ breaks at $z$, and any proper factor of $w$ is an admissible column.

**Remark 2.4.2.** *It can be proved that given a word $w \in [\pm n]^*$, any proper factor is admissible if and only if any proper prefix of $w$ is admissible. Thus, in order to be able to apply the Knuth relation $K5$ to a subword $w'$ of $w$, we only need to check that all proper prefixes of $w'$ are admissible, instead of all proper factors.*

When Knuth relations are applied to subwords of a word, the weight is preserved while the length may not. Knuth relations can be seen as *jeu de taquin* moves on words or a diagonally shaped tableau, and each SJDT slide preserves the Knuth class of the reading word of a tableau [25, Theorem 6.3.8]. In Example 2.3.2 the words $23\overline{2}31$ and $\overline{1}113\overline{3}$ are Knuth related: $\overline{1}113\overline{3} \overset{K2}{\sim} \overline{1}131\overline{3} \overset{K2}{\sim} \overline{1}133\overline{1} \overset{K3}{\sim} 2\overline{2}33\overline{1} \overset{K1}{\sim} 23\overline{2}31$.

The next proposition states that in order to be able to apply the Knuth relation $K5$ to a subcolumn $w'$ of the word $w$ we only need to check that the biggest proper prefix of $w'$ is admissible, instead of all proper factors.

**Proposition 2.4.3.** *Given a column $w$ on the alphabet $[\pm n]$, we have that any proper factor of $w$ is admissible if and only if the biggest proper prefix of $w$ is admissible.*

*Proof.* Part "only if": The biggest proper prefix is a proper factor.

Part "if": Any proper factor is contained either in the biggest proper prefix of $w$ or contains the last letter of $w$, being a proper suffix. Assuming that the statement is false, there is a proper suffix $w'$ of $w$ that breaks the $1CC$ at $y$. If $\overline{y}$ is not its last letter, if one considers the factor obtained from $w'$ after adding the last letter of $w$ not in $w'$ and removing the last of $w'$, we will have a non-admissible proper factor of $w$ contained in the biggest proper prefix of $w$, that is admissible, hence we have a contradiction. So $\overline{y}$ is the last letter of $w'$ (and $w$). In order to break the $1CC$ at $y$, since there are no letters bigger than $\overline{y}$ in $w'$, we have that $\{1, 2, \ldots, y\} \in w'$. This implies that there are no letters to the left of 1 in $w$, because this is the minimal letter of the alphabet. So $w' = w$, which is another contradiction. So $w'$ must be admissible. $\square$

## 2.5  Kashiwara crystal and $A_{n-1}$ and $C_n$ crystals

Let $V$ be an Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Fix a root system $\Phi$ with simple roots $\{\alpha_i \mid i \in I\}$ where $I$ is an indexing set and a weight lattice $\Lambda \supseteq \mathbb{Z}\text{-span}\{\alpha_i \mid i \in I\}$. A *Kashiwara crystal* of type $\Phi$ is a non-empty set $\mathfrak{B}$ together with maps [9]:

$$e_i, f_i : \mathfrak{B} \to \mathfrak{B} \sqcup \{0\} \quad \varepsilon_i, \varphi_i : \mathfrak{B} \to \mathbb{Z} \sqcup \{-\infty\} \quad \text{wt} : \mathfrak{B} \to \Lambda$$

where $i \in I$ and $0 \notin \mathfrak{B}$ is an auxiliary element, satisfying the following conditions:

1. if $a, b \in \mathfrak{B}$ then $e_i(a) = b \Leftrightarrow f_i(b) = a$. In this case, we also have $\text{wt}(b) = \text{wt}(a) + \alpha_i$, $\varepsilon_i(b) = \varepsilon_i(a) - 1$ and $\varphi_i(b) = \varphi_i(a) + 1$;

2. for all $a \in \mathfrak{B}$, we have $\varphi_i(a) = \langle \text{wt}(a), \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} \rangle + \varepsilon_i(a)$.

The crystals we deal with are seminormal [9], i.e., $\varphi_i(a) = \max\{k \in \mathbb{Z} \geq 0 \mid f_i^k(a) \neq 0\}$ and $\varepsilon_i(a) = \max\{k \in \mathbb{Z} \geq 0 \mid e_i^k(a) \neq 0\}$. An element $u \in \mathfrak{B}$ such that $e_i(u) = 0$ for all $i \in I$ is called a *highest weight element*. A *lowest weight element* is an element $u \in \mathfrak{B}$ such that $f_i(u) = 0$ for all $i \in I$. We associate with $\mathfrak{B}$ a coloured oriented graph with vertices in $\mathfrak{B}$ and edges labelled by $i \in I$: $b \overset{i}{\to} b'$ if and only if $b' = f_i(b)$, $i \in I$, $b, b' \in \mathfrak{B}$. This is the *crystal graph* of $\mathfrak{B}$.

A *morphism* $\psi : \mathfrak{B} \to \mathfrak{B}'$ of crystal graphs is a map that preserves coloured directed edges and weights. More precisely, a morphism is a map $\psi : \mathfrak{B} \to \mathfrak{B}'$ that satisfies

$$\psi(fi(b)) = fi(\psi(b))$$
$$\psi(ei(b)) = ei(\psi(b))$$
$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b).$$

where $\psi(\emptyset) = \emptyset$ by convention. Note that composing morphisms yields a morphism. An *isomorphism* of crystal graphs is a bijective morphism of crystal graphs whose inverse function is also a morphism of crystal graphs.

If $\mathfrak{B}$ and $\mathfrak{C}$ are two seminormal crystals associated to the same root system, the *tensor product* $\mathfrak{B} \otimes \mathfrak{C}$ is also a seminormal crystal. As a set, we will have the Cartesian product $\mathfrak{B} \times \mathfrak{C}$, where its elements are denoted by $b \otimes c$, $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$, with $\mathrm{wt}(b \otimes c) = \mathrm{wt}(b) + \mathrm{wt}(c)$,

$$f_i(b \otimes c) = \begin{cases} f_i(b) \otimes c \text{ if } \varphi_i(c) \leq \varepsilon_i(b) \\ b \otimes f_i(c) \text{ if } \varphi_i(c) > \varepsilon_i(b) \end{cases} , e_i(b \otimes c) = \begin{cases} e_i(b) \otimes c \text{ if } \varphi_i(c) < \varepsilon_i(b) \\ b \otimes e_i(c) \text{ if } \varphi_i(c) \geq \varepsilon_i(b) \end{cases} .$$

If $\mathfrak{B}$ and $\mathfrak{C}$ are finite, $\varphi_i(b \otimes c) = \varphi_i(b) + \max(0, \varphi_i(c) - \varepsilon_i(b))$ and $\varepsilon_i(b \otimes c) = \varepsilon_i(b) + \max(0, \varepsilon_i(b) - \varphi_i(c))$.

In type $A_{n-1}$, we consider $\{e_i\}_{i=1}^n$ the canonical basis of $\mathbb{R}^n$. The root system is $\Phi_A = \{\pm e_i \pm e_j \mid i < j\}$ and the simple roots are $\alpha_i = e_i - e_{i+1}$, for $i \in [n-1]$. In type $C_n$, we consider the same the canonical basis of $\mathbb{R}^n$. The root system is $\Phi_C = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm 2e_i\}$ and the simple roots are $\alpha_i = e_i - e_{i+1}$, if $i \in [n-1]$, $\alpha_n = 2e_n$. For both types, the weight lattice $\mathbb{Z}^n$ has dominant weights $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$.

In type $C_n$, the standard crystal is seminormal and has the following crystal graph:

$$1 \xrightarrow{1} 2 \xrightarrow{2} \ldots \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \ldots \xrightarrow{1} \bar{1}$$

with set $\mathfrak{B} = [\pm n]$, $\mathrm{wt}(\boxed{i}) = \mathbf{e_i}$, $\mathrm{wt}(\boxed{\bar{i}}) = -\mathbf{e_i}$. The highest weight element is the word 1, and the highest weight $\mathbf{e_1}$. We denote the crystal by $\mathfrak{B}^{\mathbf{e_1}}$. The type $A_{n-1}$ standard crystal uses only the first $n$ vertices of the type $C_n$ standard crystal, hence it has the same highest weight element.

The crystal $\mathfrak{B}^{\mathbf{e_1}}$ is the crystal on the words of $[\pm n]^*$ of a sole letter. The tensor product of crystals allows us to define the crystal $G_n = \bigoplus_{k \geq 0} (\mathfrak{B}^{\mathbf{e_1}})^{\otimes k}$ of all words in $[\pm n]^*$, where the vertex $w_1 \otimes \cdots \otimes w_k$ is identified with the word $w_1 \ldots w_k \in [\pm n]^*$. The action of the operators $e_i$ and $f_i$ is easily given by the signature rule [9, 20, 25]. We substitute each letter $w_j$ by $+$ if $w_j \in \{i, \overline{i+1}\}$ or by $-$ if $w_j \in \{i+1, \bar{i}\}$, and erase it in any other case. Then successively erase any pair $+-$ until all the remaining letters form a word that looks like $-^a +^b$. Then $\varphi_i(w) = b$ and $\varepsilon_i(w) = a$, $e_i$ acts on the letter associated to the rightmost unbracketed $-$ (i.e., not erased), whereas $f_i$ acts on the letter $w_j$ associated to the leftmost unbracketed $+$,

$$f_i(w_j) = \begin{cases} i+1 \text{ if } w_j = i \text{ and } i \neq n \\ \bar{i} \text{ if } w_j = \overline{i+1} \\ \bar{n} \text{ if } w_j = i \text{ and } i = n \end{cases} ,$$

and the other letters of $w$ are unchanged, and $e_i$ is the inverse map. If $b = 0$ then $f_i(w) = 0$ and if $a = 0$ then $e_i(w) = 0$. This description of the signature rule also works in type $A_{n-1}$, with the only difference being the fact that all letters of $w$ are positive.

**Example 2.5.1.** *Consider $w = \overline{2}31\overline{2}2\overline{1}$ and $i = 1$. Using the signature rule we rewrite $w$ as $+++--$.*
*Now we erase pairs $+-$ as many times as possible, obtaining only $+$, that came from the first $\overline{2}$ in $w$.*
*Given that $f_1(\overline{2}) = \overline{1}$, we have that $f_1(w) = \overline{1}31\overline{2}2\overline{1}$. Also, since there are no $-$ after eliminating all $+-$ pairs, we have that $e_1(w) = 0$.*

The crystal $G_n$, as a graph, is the union of connected components where each component has a unique highest weight word. Two connected components are isomorphic if and only if they have the same highest weight [19]. Two words in $[\pm n]^*$ are said to be crystal connected or coplactic equivalent if and only if they belong to the same connected component of $G_n$. This means that both words are obtained from the same highest weight word, through a sequence of crystal operators $f_i$, or one is obtained from another by some sequence of crystal operators $f_i$ and $e_j$, $i, j \in [n]$.

The connected components of $G_n$ are the coplactic classes in the Robinson-Schensted correspondence that identify words with the same oscillating tableau [25, Proposition 5.2.1]. Also, two words $w_1, w_2 \in [\pm n]^*$ are Knuth equivalent if and only if they occur in the same place in two isomorphic connected components of $G_n$, that is, they are obtained from two highest words with the same weight through a same sequence of crystal operators [25]. Crystal operators are coplactic and commute with the *jeu de taquin*. The next proposition identifies all highest weight words of $G_n$.

**Proposition 2.5.2.** *Let $w$ be a word in the alphabet $[\pm n]$. Then $w$ is a highest weight word if and only if the weight of all its prefixes (including itself) is a partition. In this case, one has that $P(w) = K(\lambda)$, the tableau of shape and weight $\lambda$, also known as Yamanouchi tableau.*

*Proof.* Part "if": We will prove the contraposition of the statement. There is a $i$ such that $e_i(w) \neq 0$. Let $k$ be the position of the leftmost $-$ of the signature rule of $w$, and consider the prefix $w_k$ with the first $k$ letters. Since the $k$-th letter of $w$ had an unbracketed $-$ in the signature rule then the last letter of $w_k$ will also be an unbracketed $-$. Hence there are more $-$ than $+$ in the signature rule of $w_k$. Let $t_\alpha$ be the number of $\alpha$ in $w_k$. We have that $t_i + t_{\overline{i+1}} < t_{i+1} + t_{\overline{i}} \Leftrightarrow t_i - t_{\overline{i}} < t_{i+1} - t_{\overline{i+1}}$, hence the weight of $w_k$ is not a partition.

Part "only if": We will once again prove the contrapositive of the statement. Let $w_k$ be a prefix such that its weight is not a partition. Hence there is $i \in [n]$ such that $t_i - t_{\overline{i}} < t_{i+1} - t_{\overline{i+1}} \Leftrightarrow t_i + t_{\overline{i+1}} < t_{i+1} + t_{\overline{i}}$, hence for this $i$ there will be more $-$ than $+$ in the signature rule of $w_k$. So in the first $k$ letters of $w$ there will be more $-$ than $+$, so there is an unbracketed $-$ in $w$, hence $e_i(w) \neq 0$. Note that the argument works even if $i = n$. In this case we need to assume $t_{n+1} = t_{\overline{n+i}} = 0$.

It follows from [25, Proposition 3.2.6] that the insertion of the highest word $w$ of weight $\lambda$ is $K(\lambda)$. $\qquad\square$

Choose a word $w \in [\pm n]^*$ such that the shape of $P(w)$ is $\lambda$. If we replace every word of its coplactic class with its insertion tableau we obtain the crystal of tableaux $\mathfrak{B}^\lambda$ that has all KN tableaux of shape $\lambda$ on the alphabet $[\pm n]$. The crystal $\mathfrak{B}^\lambda$ does not depend on the initial choice of word $w$, as long as $P(w)$ has shape $\lambda$. [25, Theorem 6.3.8].

**Example 2.5.3.** *Here we have the type $C_2$ crystal graph $\mathcal{KN}((2,1),2)$ containing the $A_1$ crystal $\mathcal{SSYT}((2,1),2)$:*

$\rightarrow f_1$

$\rightarrow f_2$
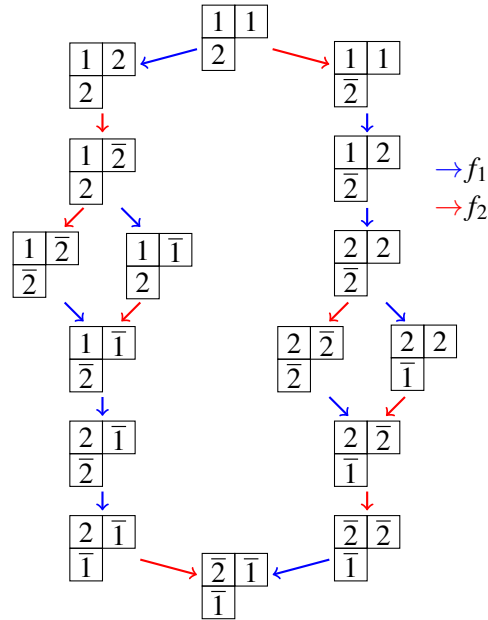
# Chapter 3

# Weyl group of type $C_n$, Bruhat order and symplectic key tableaux

In this chapter we will present the group $B_n$, known as hyperoctahedral group, which is the type $C_n$ Weyl group. The main result of this section is a tableau criterion for the Bruhat order on the elements of $B_n$ and on the coset space of $B_n$ define by a parabolic subgroup, using *symplectic key tableaux*. This shows that the combinatorics of the type $C_n$ crystal graphs are strongly connected with the Bruhat order of $B_n$.

## 3.1 Weyl group of type $C_n$

Consider the group $B_n$, with generators $s_i$, $1 \leq i \leq n$, having the following presentation, regarding the relations among the generators,

$$B_n := \langle s_1, \ldots, s_n \mid s_i^2 = 1, 1 \leq i \leq n; \ (s_i s_{i+1})^3 = 1, 1 \leq i \leq n-2; \ (s_{n-1} s_n)^4 = 1;$$
$$(s_i s_j)^2 = 1, 1 \leq i < j \leq n, |i-j| > 1 \rangle,$$

known as hyperoctahedral group or signed symmetric group. This group is a Coxeter group [6].

The elements of $B_n$ can be seen as odd bijective maps from $[\pm n]$ to itself, i.e., for all $\sigma \in B_n$ we have $\sigma(i) = \overline{\sigma(\bar{i})}$, $i \in [\pm n]$. The subgroup with the generators $s_1, \ldots, s_{n-1}$ is the symmetric group $\mathfrak{S}_n$ and its elements can be seen as bijections from $[n]$ to itself. Both groups can also be seen as groups of $n \times n$ matrices. The elements of the symmetric group can be identified with the permutation matrices, and if we allow the non-zero entries to be either 1 or $-1$, we have the elements of $B_n$. Hence $B_n$ has $2^n n!$ elements. The groups $\mathfrak{S}_n$ and $B_n$ are the Weyl groups for the root systems of types $A_{n-1}$ and $C_n$, respectively.

Let $\sigma \in B_n$. We denote $[a_1 a_2 \ldots a_n]$, where $a_i = \sigma(i)$ for $i \in [n]$, the window notation of $\sigma$, and write $\sigma = [a_1 a_2 \ldots a_n]$. The elements of $B_n$, or $\mathfrak{S}_n$, act on vectors in $\mathbb{Z}^n$ on the left. Given a vector $v \in \mathbb{Z}^n$, we have that $s_i$, with $i \in [n-1]$, acts on $v$ swapping the $i$-th and the $(i+1)$-th entries, and $s_n$ acts on $v$, $s_n v$, changing the sign of the last entry. Note that the window notation of $\sigma s_i$ is obtained after applying $s_i$ to the window notation of $\sigma$, if we see it as a vector. Ignoring signs, $\sigma v = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$, with $v = (v_1, \ldots, v_n)$. The $i$-th letter of $\sigma v$ changes its sign if and only if $\bar{i}$

appears in $\sigma$. Hence $\sigma v = (sgn(\sigma^{-1}(1))v_{|\sigma^{-1}(1)|}, \ldots, sgn(\sigma^{-1}(n))v_{|\sigma^{-1}(n)|})$, where $sgn(x) = 1$ if $x$ is positive and $-1$ if $x$ is negative, for $x \in [\pm n]$.

**Example 3.1.1.** *Consider* $v = (1,2,3) \in \mathbb{Z}^3$ *and* $\sigma = [2\,\overline{3}\,1] = [s_1 s_3 s_2(1), s_1 s_3 s_2(2), s_1 s_3 s_2(3)] = s_1 s_3 s_2$ $\in B_3$. *So*

$$
\begin{aligned}
\sigma(1,2,3) &= s_1 s_3 s_2 (1,2,3) = s_1 s_3(1,3,2) = s_1(1,3,\overline{2}) = (3,1,\overline{2}) \\
&= (sgn(\sigma^{-1}(1))v_{|\sigma^{-1}(1)|}, sgn(\sigma^{-1}(2))v_{|\sigma^{-1}(2)|}, sgn(\sigma^{-1}(3))v_{|\sigma^{-1}(3)|}) \\
&= (1 \cdot 3, 1 \cdot 1, -1 \cdot 2).
\end{aligned}
$$

## 3.2   Bruhat order on $B_n$

The *length* of $\sigma \in B_n$, $\ell(\sigma)$, is the least number of generators of $B_n$ needed to go from $[1\,2\,\ldots\,n]$, the identity map, to $\sigma$. Any expression of $\sigma$ as a product of $\ell(\sigma)$ generators of $B_n$ is called reduced. We say that two letters of the window notation of $\sigma$ form an inversion if the bigger letter appears first. The next proposition gives a way to compute $\ell(\sigma)$ that only requires to look at the window notation of $\sigma$. This is a variation of the length formula presented on [6, Proposition 8.1.1], where the authors consider the usual ordering of the alphabet $[\pm n]$ and the generator that changes the sign of an entry of the window notation acts on the first entry instead of the last one.

**Proposition 3.2.1.** *Consider* $\sigma \in B_n$. *Then*

$$
\ell(\sigma) = \#\{\text{inversions of } \sigma\} + \sum_{\substack{\bar{i} \text{ appears in } \sigma}} (n+1-i).
$$

The (signed) permutation $\sigma = [2\,\overline{3}\,1]$ has two inversions: 2, 1 and $\overline{3}$, 1 and $\ell(\sigma) = 3$.

**Remark 3.2.2.**

- *If $\bar{i}$ does not appear in the window presentation of $\sigma$, for all $i \in [n]$, we may identify $\sigma$, in one-line notation, with $\sigma(1)\ldots\sigma(n) \in \mathfrak{S}_n$ and $\ell(\sigma) = \#\{\text{inversions of } \sigma\}$ [6, Proposition 1.5.2].*

- *We have that $\ell(\sigma s_i) > \ell(\sigma)$ if $i = n$ and $\sigma(n)$ is positive, or, $i \neq n$ and $\sigma(i) < \sigma(i+1)$.*

The Bruhat order on the set of the elements of $B_n$ can be defined in the following way:

**Definition 3.2.3.** *[6] Let $w = \sigma_1 \ldots \sigma_{\ell(w)}$, where $\sigma_i \in \{s_1, \ldots, s_n\}$ are generators of $B_n$, and $u$ be two elements in $B_n$. Then $u \leq w$ in the Bruhat order if*

$$
\exists 1 \leq i_1 < i_2 \cdots < i_{\ell(u)} \leq \ell(w) \text{ such that } u = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_{\ell(u)}}.
$$

By definition, if $u \leq w$ then $\ell(u) \leq \ell(w)$, but the reverse is not true. If $\sigma(n)$ is positive and $i = n$, or, $\sigma(i) < \sigma(i+1)$ and $i \neq n$, we can also say that $\sigma s_i > \sigma$.

## 3.3    Symplectic key tableaux in type $C_n$ and the Bruhat order on $B_n$

**Definition 3.3.1.** *A key tableau* of shape $\lambda$, in type $C_n$, is a KN tableau in $\mathcal{KN}(\lambda,n)$, in which the set *of elements of each column, left to right, is contained in the set of elements of the previous column, if any, and the letters $i$ and $\bar{i}$ do not appear simultaneously as entries, for any $i \in [n]$. Equivalently, the key tableaux in type $C_n$ are the KN tableaux of shape $\lambda$ whose weight is in $B_n\lambda$, the $B_n$-orbit of $\lambda$. For each element of $B_n\lambda$ there is exactly one key tableau of shape $\lambda$ with that weight (see Proposition 3.3.3).*

**Example 3.3.2.** *The KN tableau $T = $*
$$\begin{array}{|c|c|}\hline 2 & 2 \\\hline 3 & \bar{1} \\\hline \bar{1} & \\\cline{1-1}\end{array}$$
*is a key tableau.*

The set of key tableaux in type $A_{n-1}$ is the subset of the key tableaux in type $C_n$ consisting of the tableaux having only positive entries, hence they are SSYT's for the alphabet $[n]$.

Every vector $v$ of $\mathbb{Z}^n$ is in the $B_n$-orbit of exactly one partition, $\lambda_v$, which is the one obtained by sorting the absolute values of all entries of $v$. Given a partition $\lambda \in \mathbb{Z}^n$, the $B_n$-orbit of $\lambda$ is the set $B_n\lambda := \{\sigma\lambda \mid \sigma \in B_n\}$. For instance, the vector $v = (1,\bar{3},0,3,\bar{2})$ is in the $B_5$-orbit of $\lambda = (3,3,2,1,0)$.

**Proposition 3.3.3.** *Let $\lambda$ be a partition and $v \in B_n\lambda$. There is exactly one key tableau $K(v)$ whose weight is $v$. In addition, the shape of the key tableau $K(v)$ is $\lambda$. When $v = \lambda$, $K(\lambda)$ is the only KN tableau of weight and shape $\lambda$, also called Yamanouchi tableau of shape $\lambda$.*

*Proof.* <u>Existence:</u> Given $v = (v_1,\ldots,v_n) \in \mathbb{Z}^n$ there exists a key tableau $K$ of weight $v$ by putting in the first $|v_i|$ columns the letter $i$ if $v_i \geq 0$ or $\bar{i}$ if $v_i \leq 0$, and then sorting the columns properly. Clearly the columns of $K$ are nested and it is a KN tableau without symmetric entries, hence it is a key tableau. Also, its shape is $\lambda_v = \lambda$.

<u>Uniqueness:</u> Since the key tableaux do not have symmetric entries then, for all $i \in [n]$, we have that in $K$ the letter $sgn(v_i)i$ appears $|v_i|$ times in its entries. In order to the columns of $K$ be nested we have that these $|v_i|$ entries appear in the first $|v_i|$ columns, hence we have determined exactly which letters appear in each column of $K$ and now we just have to order them correctly. So the key tableau $K$ with weight $v$ is unique. When $v = \lambda$, $K(\lambda)$ has only $i$'s in the row $i$, for $i \in [n]$. $\qquad\square$

**Example 3.3.4.** *Let $v = (1,\bar{3},0,3,\bar{2})$. Then $K(v) = $*
$$\begin{array}{|c|c|c|}\hline 1 & 4 & 4 \\\hline 4 & \bar{5} & \bar{2} \\\hline \bar{5} & \bar{2} & \\\cline{1-2} \bar{2} & & \\\cline{1-1}\end{array}$$
*.*

Hence there is a bijection between vectors in $B_n\lambda$ and the key tableaux in type $C_n$ on the alphabet $[\pm n]$ with shape $\lambda$, given by the map $v \mapsto K(v)$.

**Remark 3.3.5.** *The type $C_n$ key tableaux in $\mathcal{KN}(\lambda,n)$ are characterized by their weight $\alpha\lambda$, for all $\alpha \in B_n$, and thereby denoted $K(\alpha\lambda)$. The orbit of $K(\lambda)$, the highest weight element of $\mathfrak{B}_\lambda$, under the action of the Weyl group $B_n$, is defined to be $O(\lambda) = \{K(\alpha\lambda) : \alpha \in B_n\}$. In particular, $K(w_0\lambda) = K(-\lambda)$, with $w_0$ the longest element of $B_n$, is the lowest weight element of the type $C_n$ crystal $\mathfrak{B}^\lambda$.*

If $\sigma \in B_n$ we put $K(\sigma) := K(\sigma\Delta^n)$, where $\Delta^n$ is the staircase partition. One has a natural bijection between $B_n$ and the $B_n$-orbit of $\Delta^n$.

**Proposition 3.3.6.** *If $\sigma \in B_n$ has the letter $\alpha$ in the $j$-th position then $\alpha$ appears in the first $n+1-j$ columns of the corresponding key tableau, $K(\sigma)$.*

*Proof.* Put $\Delta := \Delta^n$. Remember that, ignoring signs, $\sigma\Delta = (\Delta_{\sigma^{-1}(1)}, \ldots, \Delta_{\sigma^{-1}(n)})$. The $i$-th letter of $\sigma\Delta$ has negative sign if and only if $\bar{i}$ appears in $\sigma$. If $\alpha$ is positive, then in the position $\alpha$ of $\sigma\Delta$ will appear $\Delta_j = n+1-j$. If $\alpha$ is negative, then in the position $-\alpha$ will appear $\overline{\Delta_j} = \overline{n+1-j}$. $\qquad\square$

We now append 0 to the alphabet $[\pm n]$, obtaining $[\pm n] \cup \{0\}$, where $n < 0 < \bar{n}$, and, for all $\sigma \in B_n$, we put $\sigma(0) := 0$. Given an element $\sigma \in B_n$ consider the map

$$[\pm n] \cup \{0\} \times [\pm n] \cup \{0\} \to \mathbb{N}_0$$
$$(i,j) \mapsto |\{a \leq i : \sigma(a) \geq j\}| := \sigma[i,j].$$

This map $\sigma[\cdot,\cdot]$, originally defined in [6], produces a table which is related to key tableaux in type $C_n$. See example below:

**Example 3.3.7.** *Let $\sigma = [\bar{3}\,\bar{1}\,24]$. Then $\sigma(4,3,2,1) = (\bar{3}, 2, \bar{4}, 1)$ and $K(\sigma) =$*

| 2 | 2 | $\bar{3}$ | $\bar{3}$ |
|---|---|---|---|
| 4 | $\bar{3}$ | $\bar{1}$ | |
| 3 | $\bar{1}$ | | |
| $\bar{1}$ | | | |

*The family of numbers $\sigma[i,j]$ where $(i,j) \in [\pm n] \cup \{0\} \times [\pm n] \cup \{0\}$ originates the following table, where $i$ indexes the columns, left to right, and $j$ indexes the rows, top to bottom. We add a row of zeros at the bottom for convenience:*

|  | 1 | 2 | 3 | 4 | 0 | $\bar{4}$ | $\bar{3}$ | $\bar{2}$ | $\bar{1}$ |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **2** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| **3** | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **4** | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 6 |
| **0** | 1 | 2 | 2 | 2 | 3 | 4 | 5 | 5 | 5 |
| $\bar{4}$ | 1 | 2 | 2 | 2 | 2 | 3 | 4 | 4 | 4 |
| $\bar{3}$ | 1 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 |
| $\bar{2}$ | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| $\bar{1}$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

*To go from the table to the key tableau note that, for $i \in [n]$, the $i$-th column of the table encodes the $(n+1-i)$-th column of the tableau, in the sense that if we look to the the $i$-th column of the table, from bottom to top, if the entry of the table increases in one unity then the index of the row associated to that entry exists in the $(n+1-i)$-th column of the tableau. Knowing the entries in a column of a tableau, its ordering is unique. The columns of the tableau constructed this way are nested because the indexes in which the column $i$ increases are $\sigma(j)$, for $j \leq i$. So the tableau taken from the table is the key tableau $K(\sigma)$. It is also possible to construct the table from the key tableau and then we only need the first $n$ columns of the table.*

We then have the following result:

**Proposition 3.3.8.** *Consider $\sigma, \rho \in B_n$. $K(\sigma) \geq K(\rho)$ entrywise if and only if $\sigma[i,j] \geq \rho[i,j]$, where $i \in [n]$, and $j \in [\pm n]$.*

In [6, Theorem 8.1.8] it is proved that, for $\sigma, \rho \in B_n$, $\sigma \leq \rho$ in the Bruhat order if and only if $\sigma[i,j] \leq \rho[i,j]$ for all $i,j \in [\pm n]$. But the result in [6, Theorem 8.1.7] implies that we only need to compare $\sigma[i,j]$ and $\rho[i,j]$ for $i \in [n]$. Henceforth, we have the following criterion for the Bruhat order on $B_n$:

**Theorem 3.3.9.** *Consider $\sigma, \rho \in B_n$. $K(\sigma) \geq K(\rho)$ by entrywise comparison if and only if $\sigma \geq \rho$ in the Bruhat order.*

**Remark 3.3.10.** *In [6, Chapter 8.1] the authors use the same alphabet as here, but with the usual ordering on the integers. So, to translate the results from there to here, it is needed to apply the ordering isomorphism defined by: $i \mapsto \overline{n-i+1}$ if $i \in [n]$; $i \mapsto n+i+1$ if $i \in -[n]$; $0 \mapsto 0$. Using the usual ordering, the authors give a tableau criterion for the Bruhat order in Exercise 6, pp. 287–288, which is effectively the transpose version of the tableau criterion presented here. Also note that the generators used in [6, Chapter 8.1] are the same used here, although with different indexation. Our generator $s_i$ corresponds to the generator $s_{n-i}$ in [6, Chapter 8.1], for all $i \in [n]$.*

## 3.4 The Bruhat order on cosets of $B_n$ defined by parabolic subgroups

A *parabolic subgroup* of $B_n$ is a group generated by some of the generators $s_1, s_2, \ldots, s_n$ of $B_n$. Consider a partition $\lambda \in \mathbb{N}^n$. Let $W_\lambda = \{\rho \in B_n \mid \rho\lambda = \lambda\}$ be the stabilizer of $\lambda$, under the action of $B_n$. Since $\lambda$ is a partition, $W_\lambda$ is a *parabolic subgroup* of $B_n$, because it is generated by some of the generators of $B_n$. Also note that given a parabolic subgroup $G$ of $B_n$, there is a partition $\lambda$ such that $G = W_\lambda$. Let $J \subseteq [n]$ be the set of the indices of the generators of $W_\lambda$, i.e. $W_\lambda = \langle s_j, j \in J \rangle$, and $J^c$ the complement of this set in $[n]$. Let $B_n/W_\lambda = \{\sigma W_\lambda : \sigma \in B_n\}$ be the set of left cosets of $B_n$ determined by the subgroup $W_\lambda$. Each coset $\sigma W_\lambda$ returns a unique vector $v$ when acting on $\lambda$, and has a unique minimal length element $\sigma_v$, such that $v = \sigma_v \lambda$. Reciprocally, given a vector $v \in B_n \lambda$, there is a unique minimal length element $\sigma_v \in B_n$ such that $v = \sigma_v \lambda$. We have then a bijection between the vectors in $B_n \lambda$ and the left cosets of $B_n$, determined by the subgroup $W_\lambda$, given by $v \mapsto \sigma_v W_\lambda$. The set $J^c$ detects the minimal length coset representatives of $B_n/W_\lambda$: $\sigma$ is a minimal coset representative of $B_n/W_\lambda$ if and only if all its reduced decompositions end with a generator $s_i \in J^c$ [6]. However key tableaux, $K(v)$, $v \in B_n \lambda$, may be used to explicitly construct the minimal length coset representatives of $B_n/W_\lambda$. Given a vector $v \in B_n \lambda$, we show that there is a unique minimal length element $\sigma_v \in B_n$ such that $v = \sigma_v \lambda$ and we show how to obtain $\sigma_v$ explicitly. The next proposition is a generalization of what Lascoux does in [22] for vectors in $\mathbb{N}^n$ (hence $\sigma_v \in \mathfrak{S}_n$).

**Proposition 3.4.1.** *Let $v \in B_n \lambda$ and $T$ the tableau obtained after adding the column $C = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$ to the left of $K(v)$. The minimal length element $\sigma \in B_n$, modulo $W_\lambda$, is given by the reading word of $T$ where entries with the same absolute value are only read on their first appearance.*

*Proof.* Consider $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let $a_i$ be the multiplicity of $i$ in $\lambda$, for $0 \leq i \leq \lambda_1$. In this proof we will write $\lambda$ as $(\lambda_1^{a_{\lambda_1}}, (\lambda_1 - 1)^{a_{\lambda_1 - 1}}, \ldots, 1^{a_1}, 0^{a_0})$. Note that $\sum_{i=0}^{\lambda_1} a_i = n$.

Let $\sigma = [\alpha_1 \ldots \alpha_n] \in B_n$ read from $T$. Let's prove that $\alpha_j$ appears $\lambda_j$ times in $K(v)$: If $j = 1$ then $\alpha_1$ appears in all columns of $K(v)$, because it was the first letter read and the columns are nested. Hence it appears $\lambda_1$ times. Also, the $|\alpha_1|$-th entry of $\lambda\sigma$ is $sgn(\alpha_1)\lambda_1$ which is the weight of $|\alpha_1|$ in $K(v)$. For $j > 1$, proceeding inductively, we have that $\alpha_j$ appears in all columns of $K(v)$ not fully occupied by $\alpha_i$, with $i < j$, hence it appears $\lambda_j$ times. Also, the $|\alpha_j|$-th entry of $\lambda\sigma$ is $sgn(\alpha_j)\lambda_j$, which is the weight of $|\alpha_j|$ in $K(v)$. This makes sense even if $\lambda_j = 0$. So we have that $\sigma\lambda = v$.

We only have to see that $\sigma$ is the minimal length element of the set $\{\rho \in B_n \mid \rho\lambda = v\}$. The subset of elements $B_n$ that applied to $\lambda$ returns $v$ is the coset $\sigma W_\lambda$. Looking at $\sigma$, this allows us to swap $\alpha_i$ and $\alpha_j$ in $\sigma$ if $\lambda_i = \lambda_j$ and to change the sign of $\alpha_i$ if $\lambda_i = 0$. Since for each column the reading to obtain $\sigma$ is ordered from the least to the biggest, we have that between these elements of $B_n$, $\sigma$ has minimal number of inversions and the letter $\alpha_j$ is unbarred if $\lambda_j = 0$ because $\alpha_j$ will only be added to $\sigma$ when reading the column $C$. Hence, by Proposition 3.2.1, $\sigma$ is the minimal length element of $\sigma W_\lambda$. $\qquad\square$

Given a partition $\lambda \in \mathbb{Z}^n$ we identify each coset $\sigma W_\lambda$ with its minimal length representative $\sigma_v$, where $v = \sigma\lambda \in B_n\lambda$. Under this identification, we now induce the Bruhat order in the $B_n$-orbit of $\lambda$ and in the coset space of $B_n/W_\lambda$.

**Definition 3.4.2.** *Consider the vectors $v, w \in B_n\lambda$, where $\lambda$ is a partition. We say that $v \leq w$, in the Bruhat order, if $\sigma_v \leq \sigma_w$.*

Let $v \in B_n\lambda$. If $K := K(v)$ is the key tableau with weight $v$, consider the tableau $\widetilde{K}$ obtained from $K$ after erasing the minimal number of columns in order to have a tableau with no duplicated columns. Let $\widetilde{v}$ and $\widetilde{\lambda}$ be the weight and the shape of $\widetilde{K}$, respectively. If $K$ and $K'$ are two key tableaux with shape $\lambda$, we have that $K \geq K'$ (by entrywise comparison) if and only if $\widetilde{K} \geq \widetilde{K'}$. Note that to recover $K$ from $\widetilde{K}$ we just have to know $\lambda$, and that $\widetilde{K} = K(\widetilde{v})$.

It is also possible to obtain $\widetilde{v}$ from $v$ without having to look to key tableaux. If $i$ is positive, $i$ and $\bar{i}$ do not appear in $v$ but $i+1$ or $\overline{i+1}$ appears then change all appearances of $i+1$ and $\overline{i+1}$ to $i$ and $\bar{i}$, respectively, and repeat this as many times as possible, obtaining the vector $\widetilde{v}$. The set of the absolute values of its entries is a set of consecutive integers starting either in 0 or 1. Hence the key tableau associated to it does not have repeated columns.

Due to Proposition 3.4.1 we have that $\sigma_{\widetilde{v}} = \sigma_v$ and $\widetilde{v} = \sigma_{\widetilde{v}}\widetilde{\lambda}_v = \widetilde{\sigma_v\lambda_v}$.

**Example 3.4.3.** *Consider $v = (1, 0, \bar{3}, 3, \bar{5}) \in B_5(5, 3, 3, 1, 0)$. Hence $K(v) = $*

$$
\begin{array}{|c|c|c|c|c|}
\hline
1 & 4 & 4 & \bar{5} & \bar{5} \\
\hline
4 & \bar{5} & \bar{5} \\
\cline{1-3}
\bar{5} & \bar{3} & \bar{3} \\
\cline{1-3}
3 \\
\cline{1-1}
\end{array}
$$
*has*

*shape $\lambda = (5, 3, 3, 1, 0)$, weight $v$ and $\sigma_v = [\bar{5}\,4\,\bar{3}\,1\,2]$. Now note that $\widetilde{v} = (1, 0, \bar{2}, 2, \bar{3})$, hence $K(\widetilde{v}) = $*

$$
\begin{array}{|c|c|c|}
\hline
1 & 4 & \bar{5} \\
\hline
4 & \bar{5} \\
\cline{1-2}
\bar{5} & \bar{3} \\
\cline{1-2}
3 \\
\cline{1-1}
\end{array}
$$
*$= \widetilde{K(v)}$ has shape $(3, 2, 2, 1, 0) = \widetilde{\lambda}$ and $\sigma_{\widetilde{v}} = [\bar{5}\,4\,\bar{3}\,1\,2] = \sigma_v$.*

Recall $J$ and $J^c$ defined above. Note that the set $J$ is the same for $\lambda$ and $\widetilde{\lambda}$. If $i \in J^c$ and $i = n$ then all entries of $\lambda$ are different from 0, which implies $K(v)$ (and $\widetilde{K(v)}$) having columns of length $n$; if $i \in J^c$ and $i < n$ then $\lambda_i > \lambda_{i+1}$, hence $K(v)$ will have exactly $i$ rows with length greater then $\lambda_{i+1}$, hence $K(v)$ (and $\widetilde{K(v)}$) will have columns of length $i$. Since $\widetilde{K(v)}$ does not have repeated columns, $J^c$ have exactly the information of what column lengths exist in $\widetilde{K(v)}$. Theorem $3BC$ of Proctor's Ph.D. thesis [33] states that given a partition $\lambda$ there is a poset isomorphism between the poset formed by the key tableaux of shape $\widetilde{\lambda}$ (ordered by entrywise comparison) and the poset formed by the Bruhat order in the vectors of the orbit $B_n\widetilde{\lambda} = \{\sigma\widetilde{\lambda} : \sigma \in B_n\}$.

The following theorem gives a tableau criterion for the Bruhat order on vectors in the same $B_n$-orbit and for the corresponding $B_n$-coset space.

**Theorem 3.4.4.** *Let $v, u \in B_n\lambda$. Then $\sigma_v \leq \sigma_u$ if and only if $K(v) \leq K(u)$.*

*Proof.* We have that

$$\sigma_v \leq \sigma_u \overset{(1)}{\Leftrightarrow} v \leq u \overset{(2)}{\Leftrightarrow} \widetilde{v} \leq \widetilde{u} \overset{(3)}{\Leftrightarrow} K(\widetilde{v}) \leq K(\widetilde{u}) \Leftrightarrow \widetilde{K(v)} \leq \widetilde{K(u)} \overset{(4)}{\Leftrightarrow} K(v) \leq K(u),$$

where (1) holds by Definition 3.4.2. Note that in (2) we also need to record $\lambda$, because it is needed in (4) to recover the shape of $K(v)$ from the shape $\widetilde{K(v)}$. Finally the equivalence (3) is an application of Theorem 3BC of Proctor's Ph.D. thesis [33]. $\qquad\square$

The following example illustrates Theorem 3.4.4.

**Example 3.4.5.** *Here we have two vectors with the respective key tableaux, ordered by entrywise comparison. The corresponding minimal coset representatives, calculated using Proposition 3.4.1, preserve this order.*

$$K(3,\overline{3},0,0,\overline{2}) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \overline{5} & \overline{5} & \overline{2} \\ \hline \overline{2} & \overline{2} \\ \cline{1-2} \end{array} \leq K(\overline{3},2,0,\overline{3},0) = \begin{array}{|c|c|c|} \hline 2 & 2 & \overline{4} \\ \hline \overline{4} & \overline{4} & 1 \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array} \ and\ \sigma_v = [1\overline{2}534] \leq \sigma_u = [\overline{41}235],\ be-$$

*cause $\sigma_v$ has the reduced expression $s_5s_4s_3s_2s_3s_4s_5s_4s_3s_2$, which is a subword of the following reduced expression of $\sigma_u$: $s_4s_5s_4s_3s_2s_1s_2s_3s_4s_5s_4s_3s_2$.*

# Chapter 4

# Type $C_n$ Demazure crystals, their opposite and cocrystals

In this chapter, given a partition $\lambda$ with at most $n$ parts, the type $C_n$ crystal $\mathfrak{B}^\lambda$ of KN tableaux will be partitioned in two different ways: one into Demazure crystal atoms and the other into opposite Demazure crystal atoms. Motivated by Lascoux's double crystal graph construction in type $A$ [22], and by Heo-Kwon work in [16] where Schützenberger *jeu de taquin* slides are used as crystal operators for $\mathfrak{sl}_2$, we define the cocrystal associated to a fixed KN tableau in the type $C_n$ crystal $\mathfrak{B}^\lambda$. The main result of this section, Proposition 4.4.3, shows that given a KN tableau $T$, the cocrystal associated to $T$ is isomorphic to a type $A$ crystal of SSYT's with conjugated shape. This cocrystal is a type $A$ crystal whose elements are type $C$ objects, more precisely KN tableaux, and the crystal operators are described by SJDT slides. This cocrystal is useful in the next chapter, where SJDT is used to define right and left key maps.

## 4.1 Demazure crystal

Let $\lambda$ be a partition and $v \in B_n\lambda$. Given a subset $X$ of $\mathfrak{B}^\lambda$, consider the operator $\mathfrak{D}_i$ on $X$, with $i \in [n]$ defined by $\mathfrak{D}_i X = \{x \in \mathfrak{B}^\lambda \mid e_i^k(x) \in X \text{ for some } k \geq 0\}$ [9]. If $v = \sigma\lambda$ where $\sigma = s_{i_\ell} \cdots s_{i_1} \in B_n$ is a reduced word, we define the *Demazure crystal* $\mathfrak{B}_v$ to be

$$\mathfrak{B}_v = \mathfrak{D}_{i_\ell} \cdots \mathfrak{D}_{i_1} \{K(\lambda)\}.$$

This definition is independent of the reduced word for $\sigma$ [9, Theorem 13.5]. In particular, when $\sigma$ is the longest element fo $B_n$ we recover $B^\lambda$. Also this definition is independent of the coset representative of $\sigma W_\lambda$, that is, $\mathfrak{B}_{\sigma\lambda} = \mathfrak{B}_{\sigma_v\lambda}$. From [6, Proposition 2.4.4], $\sigma$ uniquely factorizes as $\sigma_v\sigma'$ where $\sigma' \in W_\lambda$ and $\ell(\sigma) = \ell(\sigma_v) + \ell(\sigma')$. If $\sigma' = s_{j_{\ell(\sigma')}} \cdots s_{j_1} \in W_\lambda$ is a reduced word, then $\mathfrak{B}_{\sigma'\lambda} = \mathfrak{B}_\lambda = \mathfrak{D}_{j_{\ell(\sigma')}} \cdots \mathfrak{D}_{j_1} \{K(\lambda)\} = \{K(\lambda)\}$ and we may write $\mathfrak{B}_{\sigma\lambda} = \mathfrak{B}_v$.

From [6, Proposition 2.5.1], if $\rho \leq \sigma$ for the Bruhat order of $B_n$, then $u = \rho\lambda \leq v$. Since $e_i^0(x) = x$, if $\rho \leq \sigma$ then $\mathfrak{B}_u \subseteq \mathfrak{B}_v$. Thus we define the *Demazure crystal atom* $\widehat{\mathfrak{B}}_v$ to be

$$\widehat{\mathfrak{B}}_v = \mathfrak{B}_v \setminus \bigcup_{u < v} \mathfrak{B}_u = \mathfrak{B}_v \setminus \bigcup_{K(u) < K(v)} \mathfrak{B}_u.$$

**Example 4.1.1.** *Recall the type $C_2$ crystal graph from Example 2.5.3, associated to the partition $\lambda = (2,1)$:*
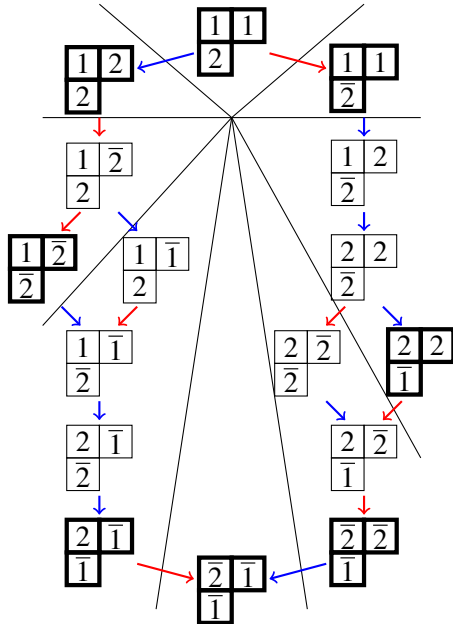


*The crystal is split into $|B_2(2,1)| = 8$ parts, the number of elements of the $B_2$-orbit of $(2,1)$. Each part is a Demazure crystal atom and contains exactly one symplectic key tableau in $O(\lambda)$, the set of key tableaux with shape $\lambda$, drawn with a thick line, so we can identify each part with the weight of that key tableau, which is a vector in the $B_2$-orbit of $(2,1)$.*

*Now we see how to compute the Demazure crystal atom $\widehat{\mathfrak{B}}_{(1,\overline{2})}$:*
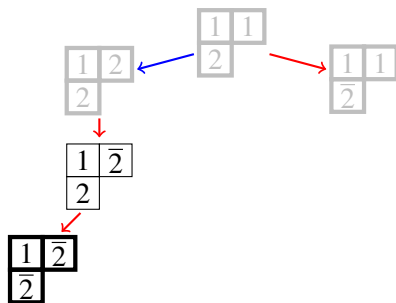


*To compute this Demazure crystal atom we start by computing the Demazure crystal $\mathfrak{B}_{(1,\overline{2})}$, which is formed by all tableaux on the left. Then, since $(2,1)$, $(1,2)$ and $(2,\overline{1})$ are smaller than $(1,\overline{2})$, we remove from the whole Demazure crystal three Demazure crystals contained in it: $\mathfrak{B}_{(2,1)}$, $\mathfrak{B}_{(1,2)}$ and $\mathfrak{B}_{(2,\overline{1})}$. The union of this sets is the greyed out section on the left.*

Like in Example 4.1.1, every Demazure crystal atom contains exactly one key tableau (see Corollary 5.2.3). Hence we can define the right key map, a map sends each tableau of $\mathfrak{B}^\lambda$ to the unique key tableau living in the Demazure crystal atom that contains the given tableau. The right key of a tableau $T$ is a key tableau of the same shape as $T$, entrywise "slightly" bigger than $T$. This is revisited in Chapter 5, where we define the right key map using the SJDT and see that both right key maps have the same output.

## 4.2  Opposite Demazure crystal

Let $\lambda$ be a partition. Analogously to the previous case, we start by creating an opposing operator $\mathfrak{D}_i^{op}$ on $X$, with $i \in [n]$ defined by $\mathfrak{D}_i^{op}X = \{x \in \mathfrak{B}^\lambda \mid f_i^k(x) \in X \text{ for some } k \geq 0\}$. If $v = \sigma\lambda$ where

$\sigma = s_{i_\ell} \cdots s_{i_1} \in B_n$ is a reduced word, we define the *opposite Demazure crystal* $\mathfrak{B}^{op}_{-v}$ to be

$$\mathfrak{B}^{op}_{-v} = \mathfrak{D}^{op}_{i_\ell} \cdots \mathfrak{D}^{op}_{i_1}\{K(-\lambda)\}.$$

We define the *opposite Demazure crystal atom* $\widehat{\mathfrak{B}}_v$ to be

$$\widehat{\mathfrak{B}}^{op}_{-v} = \mathfrak{B}^{op}_{-v} \setminus \bigcup_{-u > -v} \mathfrak{B}^{op}_{-u} = \mathfrak{B}^{op}_{-v} \setminus \bigcup_{K(-u) > K(-v)} \mathfrak{B}^{op}_{-u}.$$

The opposite Demazure crystal $\mathfrak{B}^{op}_{-v}$ is the image of $\mathfrak{B}_v$ by the Schützenberger-Lusztig involution (see Chapter 6. In particular, the tableau weights in $\mathfrak{B}_v$ and in $\mathfrak{B}^{op}_{-v}$ are symmetric.

**Example 4.2.1.** *The $C_2$ crystal graph $\mathfrak{B}^{(2,1)}$ can also be split into opposite Demazure crystal atoms:*
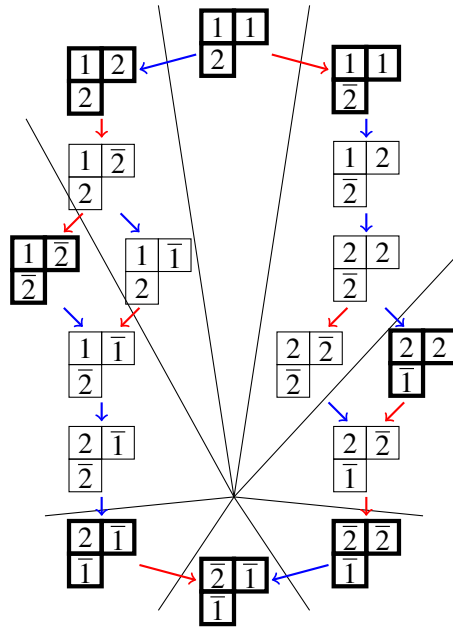


Similar to what we had in the previous section, every opposite Demazure crystal atom contains exactly one key tableau, as we shall see in the next chapter. So we can define the left key map, a map sends each tableau to the key tableau present in the opposite Demazure crystal atom that contains the given tableau. The left key of a tableau $T$ is a key tableau of the same shape as $T$, entrywise "slightly" smaller than $T$. In Chapter 5 we will compute the left key map using SJDT.

## 4.3   Demazure characters and opposite Demazure characters

The Demazure character, or key polinomial, $\kappa_v$ for $v \in B_n\lambda$, is the character of the Demazure crystal $\mathfrak{B}_v$:

$$\kappa_v(x_1, \ldots, x_n) = \sum_{T \in \mathfrak{B}_v} x^{\mathrm{wt}\,T}.$$

We also define the Demazure atom $\widehat{\kappa}_v$ as the character of the Demazure crystal atom $\widehat{\mathfrak{B}}_v$:

$$\widehat{\kappa}_v(x_1,\ldots,x_n) = \sum_{T\in\widehat{\mathfrak{B}}_v} x^{\mathrm{wt}T}.$$

Analogously, we can define opposite Demazure characters and opposite Demazure atoms:

$$\kappa^{op}_{-v}(x_1,\ldots,x_n) = \sum_{T\in\mathfrak{B}^{op}_{-v}} x^{\mathrm{wt}T}; \ \widehat{\kappa}^{op}_{-v}(x_1,\ldots,x_n) = \sum_{T\in\widehat{\mathfrak{B}}^{op}_{-v}} x^{\mathrm{wt}T}.$$

Since the tableau weights in $\mathfrak{B}_v$ and in $\mathfrak{B}^{op}_{-v}$ are symmetric, we have the following result:

**Corollary 4.3.1.**

$$\kappa_v(x_1,\ldots,x_n) = \kappa^{op}_{-v}(x_1^{-1},\ldots,x_n^{-1})$$

As a consequence, for instance, the type $C_n$ Fu-Lascoux non-symmetric Cauchy kernel, given in [12], can be written as:

$$\frac{\prod_{1\le i<j\le n}(1-x_ix_j)}{\prod_{i,j=1}^n(1-x_iy_j)\prod_{i,j=1}^n(1-x_i/y_j)} = \sum_{v\in\mathbb{N}^n}\widehat{\kappa}_v(x_1,\ldots,x_n)\kappa_{-v}(y_1,\ldots,y_n)$$

$$= \sum_{v\in\mathbb{N}^n}\widehat{\kappa}_v(x_1,\ldots,x_n)\kappa^{op}_v(y_1^{-1},\ldots,y_n^{-1})$$

**Remark 4.3.2.** *In [12], Fu-Lascoux also presented, and proved algebraically, type $A_{n-1}$ Fu-Lascoux non-symmetric Cauchy kernel. For this identity there are three combinatorial knowns proofs: one by Lascoux, in [22], a second one by Azenhas-Emami, in [3], and the most recent one, by Choi-Kwon, in [10]. In [10], Choi-Kwon, working in the Lakshmibai-Seshadri paths, started by manipulating the identity using opposite Demazure characters.*

## 4.4   Cocrystals

In this section we start working with SSYT's and type $A$ crystals, and we only address KN tableaux in the last subsection.

### 4.4.1   Dual RSK correspondence

Let T be a $T\in\mathcal{SSYT}(\lambda,n)$ with column decomposition $T = C_1C_2\cdots C_k$, and recall the column insertion for SSYT's from Section 2.3.

Given $r\ge 1$, let $E^r_n$ be the set of biwords without repeated biletters, in lexicographic order, $\begin{pmatrix}u\\v\end{pmatrix} \le \begin{pmatrix}u'\\v'\end{pmatrix}$ if $u < u'$ or if $u = u'$ and $v \le v'$, with the bottom word on the alphabet $[n]$, and the top word on the alphabet $[r]$. The set $E^r_n$ can also be thought as the set of sequences of $r$ columns, possibly some of them empty, on the alphabet $[n]$, where each pair of consecutive columns has maximum overlapping, and, in the case of two non-empty columns whose intermediate columns are empty, the top edge of the left column and the bottom edge of the right column are aligned. In particular, $E^r_n$ has

a subset identified with $\mathcal{SSYT}(\lambda,n)$, such that $\ell(\lambda') \leq r$, where $\ell(\lambda')$ is the length of $\lambda'$, the conjugate partition of $\lambda'$. Given a tableau $T \in \mathcal{SSYT}(\lambda,n)$, we create a biword, without repeated biletters, whose bottom word is $cr(T)$ and in the top word we register in which column of $T$, counted from the right, was each letter of $cr(T)$ read. Each biword will be an element of $E_n^r$, where $\ell(\lambda') \leq r$. For instance, if

$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline \end{array} \begin{array}{|c|c|} 2 & 3 \\ \hline \end{array} \begin{array}{|c|c|} 4 & 4 \\ \hline \end{array} \in \mathcal{SSYT}((3,2,2,0),4)$, the biword of $T$ is

$$w = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 2 & 2 & 3 & 4 & 1 & 2 & 4 \end{pmatrix} \in E_4^3, \text{ with } \ell(\lambda') = 3$$

.

The (type $A$) dual RSK, $RSK^*$, is a bijection [13, Section A.4.3] between $E_n^r$ and pairs of SSYT's of conjugate shapes and lengths $\leq n$ and $\leq r$, respectively:

$$RSK^* : E_n^r \to \bigsqcup_{\substack{\ell(\lambda) \leq n \\ \ell(\lambda') \leq r}} \mathcal{SSYT}(\lambda,n) \times \mathcal{SSYT}(\lambda',r) = \bigsqcup_{\substack{\ell(\lambda) \leq n \\ \ell(\lambda') \leq r \\ P \in \mathcal{SSYT}(\lambda,n)}} \{P\} \times \mathcal{SSYT}(\lambda',r)$$

$$w \mapsto (P,Q).$$

The bijection can be calculated in the following way:

Let $w = \begin{pmatrix} x_1 & x_2 & \ldots & x_m \\ y_1 & y_2 & \ldots & y_m \end{pmatrix}$. Then start with $i = 1$, $P = Q$ are empty tableaux.

1. Column insert $y_i$ into $P$.

2. Add one cell to $Q$ whose entry is $x_i$, in a position such that $P$ and $Q$, with this new cell, have conjugate shapes.

3. If $i \neq m$, then $i := i+1$ and return to (1). Else the algorithm is finished.

Given a biword $w$, the first and second components of $RSK^*(w)$ are the $P$- symbol and the $Q$-symbol of $w$.

For instance, the biword $w = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 2 & 2 & 3 & 4 & 1 & 2 & 4 \end{pmatrix}$ of $T = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline \end{array} \begin{array}{|c|c|} 2 & 3 \\ \hline \end{array} \begin{array}{|c|c|} 4 & 4 \\ \hline \end{array}$, is mapped to the pair

$$\left( T = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline \end{array} \begin{array}{|c|c|} 2 & 3 \\ \hline \end{array} \begin{array}{|c|c|} 4 & 4 \\ \hline \end{array}, K(rev((3,2,2)')) = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline \end{array} \begin{array}{|c|c|c|} 2 & 3 & 3 \\ \hline \end{array} \begin{array}{|c|} 3 \\ \hline \end{array} \right).$$

More generally, given $T \in \mathcal{SSYT}(\lambda,n)$ with $\ell(\lambda') \leq r$, the dual RSK maps its biword, $w$, to the pair $(T, K(rev(\lambda')))$, where $rev(\lambda')$ is the vector $\lambda'$ written backwards. Note that the weight of $K(\lambda')$ registers the column lengths of $T$, from right to left.

We also can compute $RSK^*$ of a biword obtained from a skew SSYT. For instance, let $\widetilde{T}$ be the

skew SSYT $\begin{array}{ccc} & & \boxed{2} \\ \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{2} & \boxed{4} \\ \boxed{4} \end{array}$ . Its biword is

$$\widetilde{w} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 2 & 3 & 2 & 4 & 1 & 2 & 4 \end{pmatrix}.$$

Finally,

$$RSK^*(\widetilde{w}) = \left( T = \text{rect}(\widetilde{T}), \widetilde{Q} = \begin{array}{ccc} \boxed{1} & \boxed{1} & \boxed{2} \\ \boxed{2} & \boxed{3} & \boxed{3} \\ \boxed{3} \end{array} \right),$$

where $\text{rect}(\widetilde{T})$ is the rectification of $\widetilde{T}$ via SJDT. The weight of $\widetilde{Q}$ records the column lengths of $\widetilde{T}$ from right to left.

### 4.4.2   Cocrystal of SSYT's

Given $T \in \mathcal{SSYT}(\lambda, n)$ with $\ell(\lambda') \leq r$, we define the *cocrystal* of $T$, $\mathfrak{CB}^{\lambda'}(T)$, to be the $\mathfrak{gl}_r$-crystal,

$$\mathfrak{CB}^{\lambda'}(T) = (RSK^*)^{-1}(\{T\} \times \mathcal{SSYT}(\lambda', r)), \tag{4.1}$$

whose crystal operators, lowering $\mathcal{F}_i$ and raising $\mathcal{E}_i$, are SJDT slides on consecutive columns $i, i+1$ of $T$, for $i = 1, \dots, r-1$. More precisely, $\mathcal{F}_i$ sends a cell from the $i$-th column to the $i+1$-th column, counting from left. The lowest weight element of $\mathfrak{CB}^{\lambda'}(T)$ is $T$, and the highest weight element is the anti rectification of $T$, that is, the rectification is performed south-eastward. The type $A_{r-1}$ crystals $\mathcal{SSYT}(\lambda', r)$ and $\mathfrak{CB}^{\lambda'}(T)$ are isomorphic. This isomorphism relies on the following proposition, a consequence of [16, Lemma 2.3, Lemma 2.4] by Heo-Kwon:

**Proposition 4.4.1.** *Let $T$ be a skew SSYT. The $Q$-symbol of $\mathcal{F}_i(T)$ is the same as $f_i$ applied to the $Q$-symbol of $T$, and the weight of the $Q$-symbol of $T$ records the column lengths of $T$ from right to left.*

**Example 4.4.2.** *Recall $T$ and $\widetilde{T}$ from the previous subsection. Note that $T = \mathcal{F}_i(\widetilde{T})$ and that the $Q$-symbols obtained from both tableaux are connected via $f_i$, that is, $\widetilde{Q} = f_1(K(rev((3,2,2)')))$.*

*This can be easily seen in the next crystal graphs. On the right, we have the cocrystal $\mathfrak{CB}^{\lambda'}(T)$, whose vertices are obtained by applying the elementary SJDT slides $\mathcal{E}_i$, for $i = 1, 2$, on $T$, the lowest weight element of the cocrystal $\mathfrak{CB}^{\lambda'}(T)$. Namely, $\mathcal{E}_1$ sends an entry from the second column to the first column, and $\mathcal{E}_2$ sends an entry from the third column to the second column, where we count columns starting from the right. $\mathcal{F}_1$ and $\mathcal{F}_2$ are the inverse operations.*

*On the left, we have the type $A_2$ crystal $\mathcal{SSYT}((3,3,1),3)$, formed by the $Q$-symbols of every skew tableau that exists in the type $A_2$ crystal $\mathfrak{CB}^{\lambda'}(T)$ on the right. The type $A_2$ crystal operators on the left are defined by the signature rule on the alphabet [3], whereas, on the right, $\mathcal{F}_1$ and $\mathcal{F}_2$ are type $A_2$ crystal operators defined by SJDT.*

Type $A_2$ crystal $\mathrm{SSYT}((3,3,1),3)$            $\mathfrak{CB}^{\lambda'}(T)$
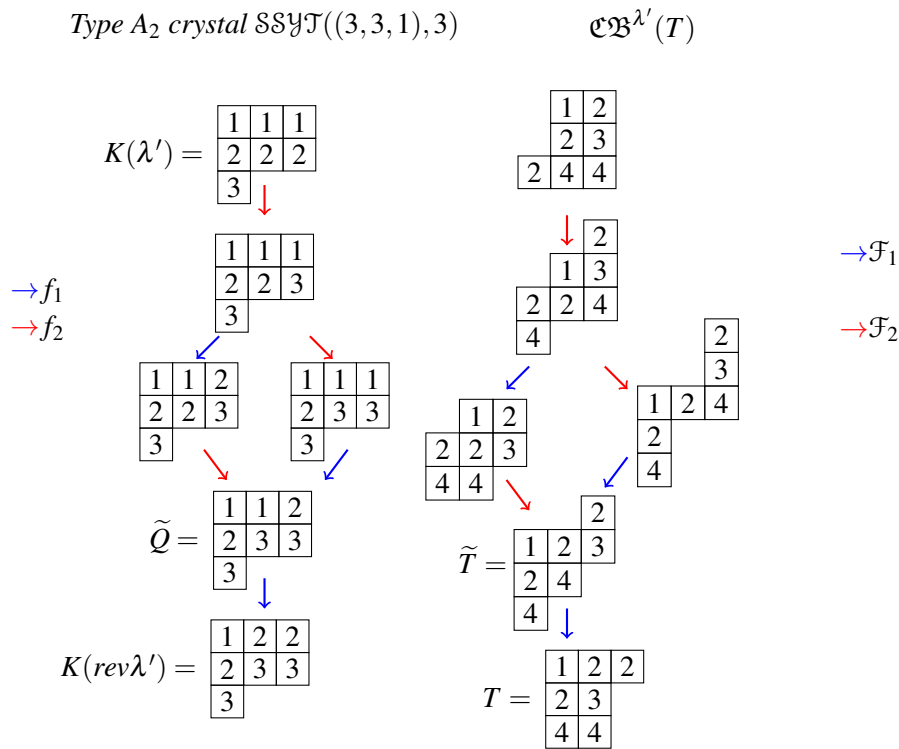


The type $A_2$ crystal operators $f_1$ and $f_2$ are given by the signature rule on the alphabet [3], whereas $\mathcal{F}_1$ and $\mathcal{F}_2$, even though they are also type $A_2$ crystal operators, are defined by SJDT.

### 4.4.3   Cocrystal of KN tableaux

Let $T \in \mathrm{SSYT}(\lambda,n)$. Note that $T$, the lowest weight of the cocrystal $\mathfrak{CB}^{\lambda'}(T)$, is also in the type $C_n$ crystal $\mathfrak{B}^\lambda$ (recall that $\mathrm{SSYT}(\lambda,n)$ is a crystal contained in $\mathfrak{B}^\lambda$). Fixed an arbitrary tableau $Y$ in the crystal $\mathfrak{B}^\lambda$, there is a sequence $S$, of type $C_n$ crystal operators of $\mathfrak{B}^\lambda$, such that $S(T) = Y$. All elements of the cocrystal $\mathfrak{CB}^{\lambda'}(T)$ are SJDT related and we can apply this sequence $S$ to all skew tableaux on the cocrystal, obtaining, for each skew tableau, a new skew tableau of the same shape. All these skew tableaux, obtained by application of the sequence $S$ to each element of $\mathfrak{CB}^{\lambda'}(T)$, will be connected via SJDT, because the SJDT and the crystal operators of $\mathfrak{B}^\lambda$ commute [25, Theorem 6.3.8], hence they are the elements of a new cocrystal $\mathfrak{CB}^{\lambda'}(S(T))$ of type $A_{r-1}$, despite the possibility that its vertices are type $C_n$ objects (i.e. KN skew tableaux). Recalling that the weight function of $\mathfrak{CB}^{\lambda'}(T)$ is given by the column lengths of each vertex, from right to left, which is preserved by any sequence $S$ of crystal operators given by the $C_n$ signature rule in $\mathfrak{B}^\lambda$, the following is a consequence of Proposition 4.4.1.

**Proposition 4.4.3.** *Given $T \in \mathcal{KN}(\lambda,n)$, with $\ell(\lambda') \geq r$, the cocrystal $\mathfrak{CB}^{\lambda'}(T)$ with lowest weight element $T$, obtained from $T$ by successive application of elementary SJDT moves, is crystal isomorphic to the $\mathfrak{gl}_r$-crystal $\mathrm{SSYT}(\lambda,r)$.*

Fulton [13] has proved the following result for type $A_{n-1}$ SSYT's.

**Proposition 4.4.4.** *[13, Proposition 7, Corollary 1, Appendix A.5] Given $T \in \mathcal{SSYT}(\lambda, n)$ and a skew shape whose column lengths are a permutation of $\lambda$, the column lengths of $T$, there is exactly one skew tableau with that shape that rectifies to $T$. Furthermore, the last and first columns only depend on their lengths.*

This means that given $T \in \mathcal{SSYT}(\lambda, n)$, the cocrystal $\mathfrak{CB}^{\lambda'}(T)$ attached to $T \in \mathcal{SSYT}(\lambda, n)$ has a distinguished set of skew tableaux whose column lengths are a permutation of $\lambda'$, the column lengths of $T$. The skew shapes of these distinguished vertices are preserved by any sequence $S$ of type $C_n$ crystal operators of the crystal $\mathfrak{B}^{\lambda}$. Thus we obtain another proof of our Proposition 40 and Corollary 41 of [36] which is an extension of Proposition 4.4.4 to KN tableaux.

**Proposition 4.4.5.** *[34, 36, Proposition 40, Corollary 41] Given $T \in \mathcal{KN}(\lambda, n)$ and a skew shape whose column lengths are a permutation of the column lengths of $T$, there is exactly one skew tableau with that shape that rectifies to $T$. Furthermore, the last and first columns only depend on their lengths.*

A key tableau in the type $A_{r-1}$ crystal $\mathcal{SSYT}(\lambda', r)$ is a tableau of shape $\lambda'$ whose weight is in $\mathfrak{S}_n \lambda'$, the $\mathfrak{S}_r$-orbit of $\lambda'$. For each element of $\mathfrak{S}_r \lambda'$ there is exactly one key tableau of shape $\lambda'$ with that weight. More precisely the key tableaux in $\mathcal{SSYT}(\lambda', r)$ are distinguished vertices and define the set $\mathfrak{S}_r K(\lambda')$ where $s_i K(\lambda') = K(s_i \lambda')$ and $s_i$, for $i = 1, \ldots, r - 1$, are the simple transpositions of $\mathfrak{S}_r$. Thereby it is natural to define keys in a cocrystal.

**Definition 4.4.6.** *Given $T \in \mathcal{KN}(\lambda, n)$, with $\ell(\lambda') \leq r$, and $X \in \mathfrak{CB}^{\lambda'}(T)$, $X$ is said to be a key (skew tableau) of $\mathfrak{CB}^{\lambda'}(T)$ if its weight as an element of the said cocrystal, the sequence column lengths of $X$, from right to left, is a permutation of the weight of $T$ as an element of the same cocrystal.*

In other words, the keys of $\mathfrak{CB}^{\lambda'}(T)$ are the image of the keys in $\mathcal{SSYT}(\lambda', r)$ via the crystal isomorphism between both crystals. We then have an action of $\mathfrak{S}_r$ on set of keys of $\mathfrak{CB}^{\lambda'}(T)$.

**Example 4.4.7.** *Recall the right hand side crystal from Example 4.4.2. $T$ is in the type $C_4$ crystal $\mathfrak{B}^{(3,2,2,0)}$. Hence we can apply to each vertex of $\mathfrak{CB}^{\lambda'}(T)$ the sequence of crystal operators $S = f_4$, obtaining a new cocrystal, on the right, whose vertices are KN skew tableaux connected via SJDT. This cocrystal $\mathfrak{CB}^{\lambda'}(f_4(T))$ is a type $A_2$ crystal.*

$$\mathfrak{CB}^{\lambda'}(T) \qquad\qquad \mathfrak{CB}^{\lambda'}(f_4(T)) = f_4\mathfrak{CB}^{\lambda'}(T)$$

The diagram displays the cocrystals $\mathfrak{CB}^{\lambda'}(T)$ and $\mathfrak{CB}^{\lambda'}(f_4(T)) = f_4\mathfrak{CB}^{\lambda'}(T)$ as connected arrangements of Kashiwara–Nakashima (skew) tableaux, with arrows labelled $\to\mathcal{F}_1$ and $\to\mathcal{F}_2$. The left cocrystal contains

$$\mathcal{E}_1 K(\lambda) \to \cdots \to \widetilde{T} = \begin{smallmatrix}1&2&3\\2&4\\4\end{smallmatrix} \xrightarrow{\ f_4\ } f_4(\widetilde{T}) = \begin{smallmatrix}1&2&3\\2&\bar{4}\\4\end{smallmatrix} \to \cdots \to \mathcal{E}_1 K(-\lambda)$$

$$K(\lambda) \to \cdots \to T = \begin{smallmatrix}1&2&2\\2&3\\4&4\end{smallmatrix} \xrightarrow{\ f_4\ } f_4(T) = \begin{smallmatrix}1&2&2\\2&3\\4&\bar{4}\end{smallmatrix} \to \cdots \to K(-\lambda)$$
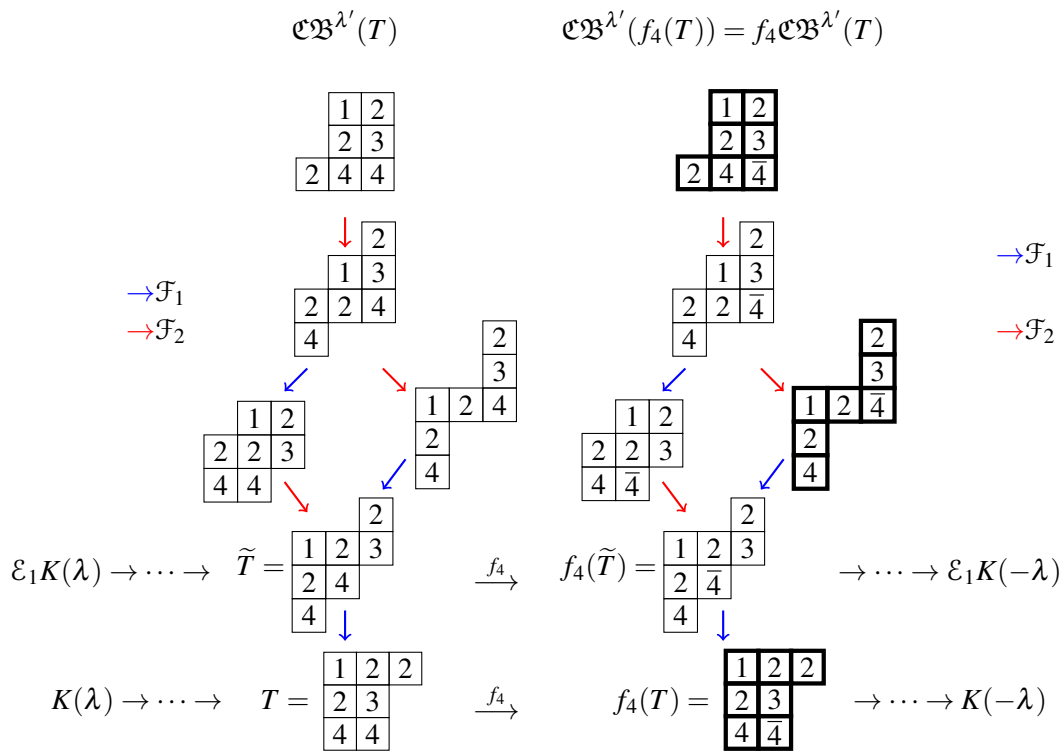
*The bold skew tableaux in the cocrystal on the right are its key (skew) tableaux. The KN tableaux $T$ and $f_4(T)$ are contained in a type $C_4$ crystal with highest weight element $K(\lambda)$ and lowest weight element $K(-\lambda)$. The KN skew tableaux in a same position of the cocrystal define a type $C_4$ crystal isomorphic to the crystal $\mathfrak{B}^\lambda$. In fact, their highest weight are the Littlewood-Richardson tableaux [13] of weight $\lambda$, defining the cocrystal attached to $K(\lambda)$, the Yamanouchi tableau of weight and shape $\lambda$. For instance, the type $C_4$ crystal containing $\widetilde{T}$ and $f_4(\widetilde{T})$ has highest weight element the*

*Littlewood-Richardson tableau $\mathcal{E}_1(K(\lambda)) = \begin{smallmatrix} & &1\\1&1&2\\2&3\\3\end{smallmatrix}$ and lowest weight element its symmetric, in the*

*sense of Lusztig involution (see Chapter 6), $\mathcal{E}_1(K(-\lambda)) = \begin{smallmatrix} & &\bar{3}\\\bar{3}&\bar{2}&\bar{1}\\\bar{2}&\bar{1}\\\bar{1}\end{smallmatrix}$.*

In the next chapter, we compute the right and left keys of a tableau using the SJDT. The cocrystal $\mathfrak{CB}^{\lambda'}(T)$ contains all the needed information to compute both keys.

# Chapter 5

# Right and left keys and Demazure atoms in type $C_n$

Frank words were introduced in type $A$ by Lascoux and Schützenberger in [24]. In this chapter, we define type $C_n$ frank words on the alphabet $[\pm n]$ and use them to create the right and left key maps, that send KN tableaux to key tableaux in type $C_n$. Lascoux and Schützenberger constructed a right key map, via *jeu de taquin*, that provided a description for type $A_{n-1}$ Demazure crystal atoms [24, Theorem 3.8]. The main result of this section, Theorem 5.2.6, shows that the right key map defined in this chapter, via SJDT, provides a description for type $C_n$ Demazure crystal atoms. We finish this chapter with a way of computing right and left keys that does not require the use of SJDT.

## 5.1 Frank words in type $C_n$

We start by defining frank words in the alphabet $[\pm n]$. Given a ordered alphabet and a word on that alphabet, a column of the word is a maximal factor whose letters are strictly increasing. Hence, we can decompose a word into columns, and such decomposition is unique.

**Definition 5.1.1.** *Let $w$ be word on the alphabet $[\pm n]$. We say that $w$ is a type $C_n$ frank word if the lengths of its columns form a multiset equal to the multiset formed by the lengths of the columns of the tableau $P(w)$, the Baker-Lecouvey insertion of $w$.*

**Example 5.1.2.** *In Example 2.3.2 we have that $P(23\overline{2}31) = P(\overline{1}113\overline{3}) = $*
$\begin{array}{|c|c|c|}
\hline
1 & 1 & \overline{1} \\
\hline
3 \\
\cline{1-1}
\overline{3} \\
\cline{1-1}
\end{array}$
*. Since $23\overline{2}31$ and $\overline{1}113\overline{3}$ have one column of length $3$ and two columns of length $1$, they are frank words.*

Given a frank word $w$, the number of letters of $w$ is the same as the number of cells of $P(w)$, hence the case 3 of the Baker-Lecouvey insertion does not happen.

**Proposition 5.1.3.** *Let $w$ be frank word on the alphabet $[\pm n]$. All columns of $w$ are admissible.*

*Proof.* Suppose that the statement is false. So there is a factor of $w$ that is a non-admissible column with all of its proper factors admissible. Hence we can apply the Knuth relation $K5$, meaning that

$w$ is Knuth related to a smaller word $w'$. But in this case, the number of letters of $w'$ is less then the number of cells of $P(w) = P(w')$, which is a contradiction. $\qquad\square$

Fixed a KN tableau $T$, consider the set of all possible last columns taken from skew tableaux SJDT connected to $T$ and with same number of columns of each length as $T$. Proposition 4.4.5 implies that for each permutation of the columns lengths of $T$, there is a skew tableau with that sequence of column lengths that rectifies to $T$. In particular, it implies that for every column length of $T$, there is a skew tableau SJDT connected to $T$ with last column having that length. Proposition 4.4.5 also implies that this set of last columns has exactly one element for each distinct column length of $T$. For every column $C$ in this set, consider the columns $rC$, its right column. The next proposition implies that this set of right columns is nested, if we see each column as the set formed by its elements.

**Proposition 5.1.4.** *Consider $T$ a two-column KN skew tableau $C_1 C_2$ with a puncture in the first column. Slide that cell once via SJDT, obtaining a punctured two-column KN skew tableau $C_1' C_2'$. Then $rC_2' \subseteq rC_2$.*

*Proof.* If the sliding was vertical then $C_2' = C_2$, hence $rC_2' = rC_2$. If the sliding was horizontal, Let $\beta$ be the number on the cell right of the puncture on $spl(T)$. Recall $\Phi$, the function that takes an admissible column to the associated coadmissible column.

If $\beta = b$ is unbarred then $C_2' = \Phi^{-1}(\Phi(C_2) \setminus \{b\} \sqcup \{*\})$. In this case $\Phi(C_2') = \Phi(C_2) \setminus \{b\} \sqcup \{*\}$, hence $rC_2$ and $rC_2'$ have the same barred part. Consider $z_1 < \cdots < z_\ell$ the unbarred letters that appear on $C_2$ and not on $\Phi(C_2)$. When we take $b$ from $\Phi(C_2)$, if $\overline{b} \in \Phi(C_2)$ our set of letters $z_1 < \cdots < z_\ell$ will lose an element, giving the inclusion of the unbarred part of $C_2'$ in $C_2$; if $\overline{b} \notin \Phi(C_2)$, then $b \in C_2$ and in $C_2'$ the least $z_i > b$ may reduce to $b$, and subsequent $z_j$ may reduce to $z_{j-1}$. Hence we have the inclusion of the unbarred part of $C_2'$ in $C_2$.

If $\beta = \overline{b}$ is barred then $C_2' = C_2 \setminus \{\overline{b}\} \sqcup \{*\}$. In this case $rC_2$ and $rC_2'$ have the same unbarred part. Consider $\overline{t_1} > \cdots > \overline{t_\ell}$ the barred letters that appear on $\Phi(C_2)$ and not on $C_2$. When we take $\overline{b}$ from $C_2$, if $b \in C_2$ our set of $\overline{t_1} > \cdots > \overline{t_\ell}$ letters will lose an element, giving the inclusion of the barred part of $rC_2'$ in $rC_2$; if $b \notin C_2$, then $\overline{b} \in \Phi(C_2)$ and in $C_2'$ the least $\overline{z_i} > \overline{b}$ may reduce to $\overline{b}$, and subsequent bigger $\overline{z_j}$'s may reduce to $\overline{z_{j+1}}$. Hence we have the inclusion of the barred part of $\Phi(C_2')$ in $\Phi(C_2)$. $\qquad\square$

This proposition allows us to define a map that sends a KN tableau to a key tableau in type $C_n$, called the (symplectic) right key map of a given KN tableau. We shall see in the next chapter that this right key map has the same output as the right key map defined in the previous chapter.

**Theorem 5.1.5** (Right key map)**.** *Given a KN tableau $T$, we can replace each column with a column of the same size taken from the right columns of the last columns of all skew tableaux associated to it. The tableau obtained is a key tableau. We call this tableau the right key tableau of $T$ and denote it by $K_+(T)$.*

*Proof.* The previous proposition implies that the columns of $K_+(T)$ are nested and do not have symmetric entries. So, it is indeed a KN key tableau. $\qquad\square$

**Remark 5.1.6.** *Recall the set up of Proposition 4.4.5. If the shape of $S$, $\mu/\nu$, is such that every two consecutive columns have at least one cell in the same row, then each column of $S$ is a column of the*

word $cr(S)$, hence $cr(S)$ is a frank word. Moreover, the columns of $S$ appear in reverse order in $cr(S)$. Therefore, given a KN tableau $T$, the columns of $K_+(T)$ can be also found as the right columns of the first columns of frank words associated to $T$.

If $T$ is a SSYT then this right key map coincides with the one defined by Lascoux and Schützenberger in [24].

**Example 5.1.7.** The tableau $T = \begin{array}{|c|c|c|} \hline 1 & 3 & \bar{1} \\ \hline 3 & 3 \\ \hline \bar{3} \\ \hline \end{array}$ gives rise to the cocrystal $\mathfrak{CB}^{\lambda'}(T)$, with $\lambda = (3,2,1)$.

The following are the vertices of $\mathfrak{CB}^{\lambda'}(T)$ consisting of the six KN skew tableaux with the same number of columns of each length as $T$, each one corresponding to a permutation of its column lengths.
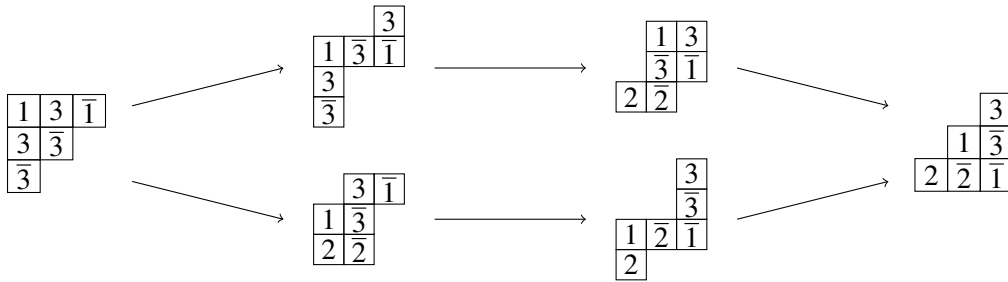


The right key tableau associated to $T$ has as columns $r\begin{array}{|c|}\hline 3 \\ \hline 3 \\ \hline \bar{1} \\ \hline\end{array}$, $r\begin{array}{|c|}\hline 3 \\ \hline \bar{1} \\ \hline\end{array}$ and $r\begin{array}{|c|}\hline \bar{1} \\ \hline\end{array}$. Hence $K_+(T) = $

$\begin{array}{|c|c|c|} \hline 3 & 3 & \bar{1} \\ \hline \bar{2} & \bar{1} \\ \hline \bar{1} \\ \hline \end{array}$ .

**Remark 5.1.8.** *Proposition 4.4.5 shows that the action defined by the SJDT on two consecutive columns of a straight KN tableau $T$ of shape $\lambda$ gives rise to a permutohedron where the vertices are all the KN skew tableaux in the Knuth class of $T$ whose column length sequence is a permutation of the column length sequence of $T$ [24]. For instance, in Example 5.1.7 we have a permutohedron (hexagon) for $\mathfrak{S}_3$. In fact, these vertices are the key (skew) tableaux of the cocrystal $\mathfrak{CB}^{(3,2,1)'}(T)$. Hence the cocrystal $\mathfrak{CB}^{(3,2,1)'}(T)$ contains all the needed information to compute the right key of $T$.*

In the same spirit of the right key, we define the left key of a KN tableau. Just like in Proposition 5.1.4, we can prove that the slides of the SJDT are effectively adding an entry to $\ell C_1$, i.e. $\ell C_1 \subseteq \ell C_1'$, hence the left columns of the first columns of all skew tableaux with the same number of columns of each length as $T$ will be nested.

So, if we replace each column of $T$ with a column of the same size taken from the left columns of the first columns of all skew tableaux associated to it we obtain the left key $K_-(T)$.

**Example 5.1.9.** *In Example 5.1.7 we have that the left key of $T = \begin{array}{|c|c|c|} \hline 1 & 3 & \bar{1} \\ \hline 3 & 3 \\ \hline \bar{3} \\ \hline \end{array}$ has as columns*

$\ell\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \bar{3} \\ \hline\end{array}$, $\ell\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ and $\ell\begin{array}{|c|}\hline 2 \\ \hline\end{array}$. *Hence $K_-(T) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \hline \bar{3} \\ \hline \end{array}$ . So, the cocrystal $\mathfrak{CB}^{(3,2,1)'}(T)$ also contains all the needed information to compute the left key of $T$.*

## 5.2   Demazure crystals and right key tableaux

Let $\lambda \in \mathbb{Z}^n$ be a partition and $v \in B_n\lambda$. We define

$$\mathfrak{U}(v) = \{T \in \mathcal{KN}(\lambda, n) \mid K_+(T) = K(v)\}$$

the set of KN tableaux of $B^\lambda$ with right key $K(v)$.

In the following lemma we identify how does the crystal operators affect the weight of a column:

**Lemma 5.2.1.** *Let $\sigma = s_i$ be a generator of $B_n$ and $C$ an admissible column such that $f_i(C) \neq 0$. Then* $\mathrm{wt}(rC) = \mathrm{wt}(r(f_i(C)))$ *or* $\mathrm{wt}(rC) = \sigma(\mathrm{wt}(r(f_i(C))))$.

*Proof.* Let $i = n$. We can apply $f_i$ to $C$ if and only $n \in C$ and $\bar{n} \notin C$. In this case $n \in rC$ and after applying $f_i$ we have $n \notin C$ and $\bar{n} \in C$, hence $\bar{n} \in rC$. So $\mathrm{wt}(rC) = s_n(\mathrm{wt}(r(f_n(C))))$.

Let $i < n$. We can apply $f_i$ to $C$, so we have 6 cases to study:

1. $i \in C, i+1, \overline{i+1}, \bar{i} \notin C$: In this case we have that $i+1 \in f_i(C), i, \overline{i+1}, \bar{i} \notin f_i(C)$. Note that $\bar{i} \notin rC$ and $\overline{i+1} \notin r(f_i(C))$. If $\overline{i+1} \notin rC$ then $\bar{i} \notin r(f_i(C))$, hence $f_i$ swaps the weight of $i$ and $i+1$ from $(1,0)$ to $(0,1)$, respectively. If $\overline{i+1} \in rC$ then $\bar{i} \in r(f_i(C))$, hence $f_i$ swaps the weight of $i$ and $i+1$ from $(1,-1)$ to $(-1,1)$.

2. $i, \overline{i+1} \in C$, $i+1, \bar{i} \notin C$: In this case we have that $i+1, \overline{i+1} \in f_i(C), i, \bar{i} \notin f_i(C)$. Note that $i, \overline{i+1} \in rC, i+1, , \bar{i} \notin rC$ and that $i+1, \bar{i} \in r(f_i(C)), i, \overline{i+1} \notin r(f_i(C))$, and all other appearances in $rC$ are intact. Hence $f_i$ swaps the weight of $i$ and $i+1$ from $(1,-1)$ to $(-1,1)$.

3. $i+1, \overline{i+1} \in C$, $i, \bar{i} \notin C$: In this case we have that $i+1, \bar{i} \in f_i(C), i, \overline{i+1} \notin f_i(C)$. Note that $i+1, \bar{i} \in rC, i, \overline{i+1} \notin rC$ and that $i+1, \bar{i} \in r(f_i(C)), i, \overline{i+1} \notin r(f_i(C))$, and all other appearances in $rC$ are intact. Hence $f_i$ did nothing to weight of $rC$.

4. $i, i+1, \overline{i+1} \in C$, $\bar{i} \notin C$: In this case we have that $i, i+1, \bar{i} \in f_i(C), \overline{i+1} \notin f_i(C)$. Note that $i, i+1 \in rC, \overline{i+1}, \bar{i} \notin rC$ and that $i, i+1 \in r(f_i(C)), \overline{i+1}, \bar{i} \notin r(f_i(C))$, and all other appearances in $rC$ are intact. Hence $f_i$ did nothing to weight of $rC$.

5. $i, \overline{i+1}, \bar{i} \in C$, $i+1 \notin C$: In this case we have that $i+1, \overline{i+1}, \bar{i} \in f_i(C), i \notin f_i(C)$. Note that $i, \overline{i+1} \in rC, i+1, \bar{i} \notin rC$ and that $i+1, \bar{i} \in r(f_i(C)), i, \overline{i+1} \notin r(f_i(C))$, and all other appearances in $rC$ are intact. Hence $f_i$ swaps the weight of $i$ and $i+1$ from $(1,-1)$ to $(-1,1)$.

6. $\overline{i+1} \in C, i, i+1, \bar{i} \notin C$: In this case we have that $\bar{i} \in f_i(C), i, i+1, \overline{i+1} \notin f_i(C)$. Note that $i, i+1 \notin rC$ and $\overline{i+1} \in rC$. If $\bar{i} \in rC$ then we have $i, i+1 \notin r(f_i(C))$ and $\overline{i+1}, \bar{i} \in r(f_i(C))$, so $f_i$ did nothing to weight of $rC$. If $\bar{i} \notin rC$ then $\overline{i+1} \notin r(f_i(C))$ and $\bar{i} \in r(f_i(C))$, hence $f_i$ swaps the weight of $i$ and $i+1$ from $(0,-1)$ to $(-1,0)$.

$\square$

**Remark 5.2.2.** *All the cases where the weight is preserved happen to have equal weight for $i$ or $i+1$ in $rC$ or we are in a column $C$ in which we can also apply $e_i$. If the weights for $i$ and $i+1$ in $rC$ swap, then if $rC$ the weight of $i$ is bigger (in the usual ordering) then the weight of $i+1$.*

The following corollary shows that every Demazure crystal atom defined in the previous chapter has exactly one key tableau:

**Corollary 5.2.3.** *Let $T$ be a KN tableau and $i \in [n]$. If $K_+(T) = K(v)$, for some $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, then $K_+(f_i(T)) = K(v)$ or $K_+(f_i(T)) = K(s_i v)$. Moreover, $K_+(f_i(T)) = K(s_1 v)$ only if $v_i > v_{i+1}$ (in the usual ordering of real numbers) and $1 \leq i < n$, or, $v_i > 0$ and $i = n$.*

*Proof.* Consider a multiset of frank words $F$ such that the multiset of length of their first columns is the same of the multiset of lengths of columns of $T$.

If $K_+(f_i(T)) = K_+(T)$ then we are done. Else there are two cases: $1 \leq i < n$ and $i = n$.

Consider $1 \leq i < n$. Since there is a change in the weight of the key tableau, we have that in at least one first column of words in $F$ weight of $i$ is bigger or equal than the weight of $i+1$. These first columns form a nested set without symmetric entries, hence in all first column of words in $F$ weight of $i$ is bigger or equal than the weight of $i+1$.

Let $A$ be the subset of $F$ such that the weight of $i$ and $i+1$ in the right column of its first column is different and does not swap when we apply $f_i$ to the frank word. Consider $(a, b)$ the sum of weights of $i$ and $i+1$, respectively, of all right columns of first columns of words in $A$, and $(c, d)$ defined analogously to $F \setminus A$.

The weights of $i$ and $i+1$ in $K_+(T)$ is $(a, b) + (c, d) = (a+c, b+d)$ and the weights of $i$ and $i+1$ in $K_+(f_i(T))$ is $(a, b) + (d, c) = (a+d, b+c)$, and note that $(a+c, b+d) \in B_2(a+d, b+c)$, because $f_i$ doesn't change any other weight (Lemma 5.2.1).

Since in all first columns of $F$ weight of $i$ is bigger or equal than the weight of $i+1$, $a \geq 0$ and $b \leq 0$, and they are equal when $A = \emptyset$, so $(a+c, b+d) = s_1(a+d, b+c)$, hence $\mathrm{wt}(K_+(f_i(T))) = s_i v$. Hence we assume $a \neq b$. If $c = d$ we have $\mathrm{wt}(K_+(f_i(T))) = v$, hence $K_+(f_i(T)) = K(v) = K_+(T)$, which is a contradiction.

This implies that $(a+c, b+d) = \sigma(a+d, b+c)$ where $\sigma = \overline{12}$ or $\sigma = \overline{21}$. The first case implies that $a = \frac{-c-d}{2} = b$ and the second case implies $c = \frac{-a-b}{2} = d$, hence there are not more possibilities for the weight of $K_+(f_i(T))$.

The case $i = n$ is a simpler version of this one. $\qquad\square$

Analogously, one can also prove that every opposite Demazure crystal atom has exactly one key tableau, using the following corollary of Lemma 5.2.1:

**Corollary 5.2.4.** *Let $\sigma = s_i$ be a generator of $B_n$ and $C$ an admissible column. Then $\mathrm{wt}(rC) = \mathrm{wt}(r(e_i(C)))$ or $\mathrm{wt}(rC) = \sigma(\mathrm{wt}(r(e_i(C))))$.*

*Proof.* Let $C' = e_i(C)$. By Lemma 5.2.1 we have that $\mathrm{wt}(rC') = \mathrm{wt}(r(f_i(C')))$ or $\mathrm{wt}(C') = \sigma(\mathrm{wt}(r(f_i(C'))))$. Hence $\mathrm{wt}(r(e_i(C))) = \mathrm{wt}(rC)$ or $\mathrm{wt}(e_i(C)) = \sigma(\mathrm{wt}(rC)) \Leftrightarrow \sigma(\mathrm{wt}(e_i(C))) = \mathrm{wt}(rC)$. $\qquad\square$

The following lemma identifies when we can apply $e_i$ to a column without taking it to 0.

**Lemma 5.2.5.** *Let $i \in [n]$ and $C$ be an admissible column such that one of the following happens*

*1. $i < n$ and the weight of $i$ in $rC$ is less than the weight of $i+1$ in $rC$;*

*2. $i = n$ and weight of $i$ is negative in $rC$,*

*then we can apply $e_i$ to $C$ (in the sense $e_i(C) \neq 0$).*

*Proof.* If $i = n$ then $-n$ appears on $rC$ and $n$ does not. Since $n$ is the biggest unbarred letter of the alphabet we have that $-n$ also appears in $C$ and $n$ does not. Hence we can apply $e_n$ to $C$.

If $i < n$ and the weight of $i$ in $rC$ is less than the weight of $i+1$ in $rC$ then the weight of both can be one of the following three options: $(0,1)$, $(-1,1)$, $(-1,0)$. Note that $rC$ does not have symmetric entries. So in the first two cases we have that $i+1$ exists in $rC$ and $i$ does not, hence $i+1$ exists in $C$ and $i$ does not, so we can apply $e_i$ to $C$. In the last case, we have that $\bar{i}$ exists in $rC$ and $i+1$ and $\overline{i+1}$ does not. Hence we have that $\bar{i}$ exists in $C$ and $i$ or $\overline{i+1}$ does not, so we can apply $e_i$ to $C$.            $\square$

The next theorem gives a description of a Demazure crystal atom in type $C$ using the right key map Theorem 5.1.5. Lascoux and Schützenberger, in [24, Theorem 3.8], proved the type $A$ version of this theorem, which consists in considering the case when $v \in \mathbb{N}^n$ and, consequently, $\sigma_v \in \mathfrak{S}_n$. For inductive reasoning, used in what follows, we recall the chain property on the set of minimal length coset representatives modulo $W_\lambda$ [6, Theorem 2.5.5].

**Theorem 5.2.6** (Main theorem). *Let $v \in B_n\lambda$. Then $\mathfrak{U}(v) = \widehat{\mathfrak{B}}_v$.*

*Proof.* Let $\rho$ be a minimal length coset representative modulo $W_\lambda$ such that $v = \rho\lambda$. We will proceed by induction on $\ell(\rho)$. If $\ell(\rho) = 0$ then $\rho = id$ and $v = \lambda$. In this case we have that $\widehat{\mathfrak{B}}_\lambda = \{K(\lambda)\} = \mathfrak{U}(\lambda)$.

Let $\rho \geq 0$. Consider $\sigma = s_i$ a generator of $B_n$ such that $\sigma\rho > \rho$ and $\sigma\rho\lambda \neq \rho\lambda = v$, i.e., $\rho^{-1}\sigma\rho \notin W_\lambda$. Recall $e_i$, $\varepsilon_i$, $f_i$ and $\phi_i$ from the definition of the crystal $\mathfrak{B}^\lambda$. If $T \in \widehat{\mathfrak{B}}_{\sigma\rho\lambda}$ then $T$ is obtained after applying $f_i$ (maybe more than once) to a tableau in $\widehat{\mathfrak{B}}_{\rho\lambda}$, which by inductive hypothesis exists in $\mathfrak{U}(v)$. By Corollary 5.2.3, if $f_i(T) \notin \mathfrak{U}(v)$ then $f_i(T) \in \mathfrak{U}(\sigma v)$. So it is enough to prove that given a tableau $T \in \mathfrak{U}(v) \cup \mathfrak{U}(\sigma v)$ then $e_i^{\varepsilon_i(T)}(T) \in \mathfrak{U}(v)$.

We have two different cases to consider: $i = n$ and $i < n$.

If $T \in \mathfrak{U}(\sigma v)$ then, if $i < n$, there exists a frank word of $T$ such that, if $V_1$ is its first column then $rV_1$ has less weight for $i$ than for $i+1$ (less in the usual ordering of real numbers); if $i = n$, there exists a frank word of $T$ such that, if $V_1$ is its first column then $rV_1$ has negative weight for $i$. Since we are in the column $rV_1$, if $i < n$, $i$ and $i+1$ can have weights $(0,1)$, $(-1,1)$ or $(-1,0)$ and if $i = n$ then $i$ has weight $-1$. Note that these are the exact conditions of Lemma 5.2.5. In either case, due to Lemma 5.2.5, we can applying $e_i$ enough times to the frank word associated until this no longer happens. This is true because we only need to look to $V_1$ to see if it changes after applying $e_i$ enough times to the frank word. In the signature rule we have that successive applications of $e_i$ changes the letters of a word from the end to the beginning, so, from the remark after Lemma 5.2.1, the number of times that we need to apply $e_i$, in order to conditions of Lemma 5.2.5 do not hold for the first column, is $\varepsilon_i(T)$. So $K_+\left(e_i^{\varepsilon_i(T)}(T)\right) \neq K(\sigma v)$, hence, from Corollary 5.2.4, we have that $e_i^{\varepsilon_i(T)}(T) \in \mathfrak{U}(v)$.

If $T \in \mathfrak{U}(v)$ then $e_i^{\varepsilon_i(T)}(T) \in \mathfrak{U}(v)$ because if not, $e_i^{\varepsilon_i(T)}(T)$ will be in a Demazure crystal associated to $\rho' \in B_n$, with $\rho' < \rho$ such that $\sigma\rho' = \rho$. This cannot happen because in this case $\rho' = \sigma\rho < \rho$, which is a contradiction.            $\square$

### 5.2.1  Combinatorial description of type $C_n$ Demazure characters and atoms

Recall the Demazure characters and Demazure atoms defined in 4.3. Theorem 5.2.6 detects the KN tableaux in $\mathfrak{B}^\lambda$ contributing to the Demazure atom $\widehat{\kappa}_v$, $\widehat{\kappa}_v = \sum\limits_{\substack{K_+(T)=K(v) \\ T\in\mathfrak{B}^\lambda}} x^{\mathrm{wt}\,T}$.

**Proposition 5.2.7.** *Given $v \in B_n\lambda$, one has $\kappa_v = \sum\limits_{u\leq v} \widehat{\kappa}_u$.*

*Proof.* It is enough to prove that $\mathfrak{B}_v = \bigcup\limits_{u\leq v} \widehat{\mathfrak{B}}_u$, because $\kappa_v$ and $\widehat{\kappa}_u$ are the generating functions of the tableau weights in $\mathfrak{B}_v$ and $\widehat{\mathfrak{B}}_u$, respectively. Since $v = \sigma\lambda$, where $\sigma := \sigma_v$, we can rewrite the identity as $\mathfrak{B}_{\sigma\lambda} = \bigcup\limits_{\rho\leq\sigma} \widehat{\mathfrak{B}}_{\rho\lambda}$.

We will proceed by induction on $\ell(\sigma)$. If $\ell(\sigma) = 0$ then the result follows because $\mathfrak{B}_\lambda = \widehat{\mathfrak{B}}_\lambda = \{K(\lambda)\}$. From the definition of Demazure crystal atom, we have $\widehat{\mathfrak{B}}_{\sigma\lambda} = \mathfrak{B}_{\sigma\lambda} \setminus \bigcup\limits_{\rho<\sigma} \mathfrak{B}_{\rho\lambda}$, and by inductive hypothesis, we have that $\mathfrak{B}_{\rho\lambda} = \bigcup\limits_{\rho'\leq\rho} \widehat{\mathfrak{B}}_{\rho'\lambda}$. Hence:

$$\widehat{\mathfrak{B}}_{\sigma\lambda} = \mathfrak{B}_{\sigma\lambda} \setminus \bigcup\limits_{\rho<\sigma} \mathfrak{B}_{\rho\lambda} = \mathfrak{B}_{\sigma\lambda} \setminus \bigcup\limits_{\rho<\sigma}\bigcup\limits_{\rho'\leq\rho} \widehat{\mathfrak{B}}_{\rho'\lambda} = \mathfrak{B}_{\sigma\lambda} \setminus \bigcup\limits_{\rho'<\sigma} \widehat{\mathfrak{B}}_{\rho'\lambda}$$

$\square$

Proposition 5.2.7, the equivalence $u \leq v \Leftrightarrow K(u) \leq K(v)$, and Theorem 5.2.6, allow us to detect the KN tableaux contributing to a key polynomial in type $C$:

$$\kappa_v = \sum_{u\leq v} \widehat{\kappa}_u = \sum_{\substack{u\leq v \\ T\in\mathfrak{U}(u)}} x^{\mathrm{wt}\,T} = \sum_{\substack{K(u)\leq K(v) \\ T\in\mathfrak{U}(u)}} x^{\mathrm{wt}\,T} = \sum_{K_+(T)\leq K(v)} x^{\mathrm{wt}\,T}.$$

**Example 5.2.8.** *Recall the type $C_2$ crystal $\mathfrak{B}^{(2,1)}$, partitioned into Demazure crystal atoms.*
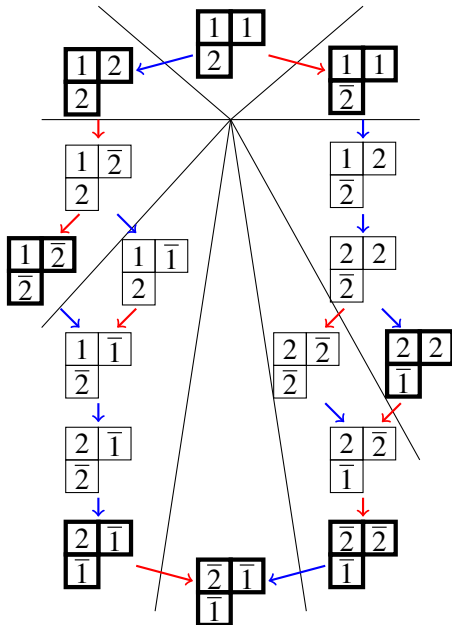


*One can check that, for example*
$$\mathfrak{U}((1,\overline{2})) = \left\{ \begin{array}{|c|c|} \hline 1 & \overline{2} \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & \overline{2} \\ \hline \overline{2} \\ \cline{1-1} \end{array} \right\} = \widehat{\mathfrak{B}}_{\lambda s_1 s_2}.$$
*Also,*
$$\mathfrak{B}_{(1,\overline{2})} = \{T \in \mathfrak{B}^\lambda \mid K_+(T) \leq K((1,\overline{2}))\} = \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \overline{2} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|} \hline 1 & \overline{2} \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|} \hline 1 & \overline{2} \\ \hline \overline{2} \\ \cline{1-1}\end{array} \right\}.$$

## 5.3   Symplectic right and left keys - direct way

In this section, we start by by introducing two maps, $K_+^1$ and $K_-^1$, that, given a tableau, return the leftmost column of the right key and the rightmost column of the left key, respectively. Given a tableau $T$, we can express its right key $K_+(T)$ in terms of $K_+^1$ applied to $T$ and to some subtableaux of $T$, and analogously for left side. For each side, we start by introducing an algorithm, based on SJDT, to compute these maps $K_+^1$ and $K_-^1$. Motived by Willis' direct way of computing right and left keys of SSYT's [42], we introduce a way of computing these maps $K_+^1$ and $K_-^1$, and consequently symplectic right and left keys, without the use of SJDT. We end this section with an example of these direct algorithms.

### 5.3.1   The right key of a tableau - *Jeu de taquin* approach

Let $T = C_1 C_2 \cdots C_k$ be a straight KN tableau with columns $C_1, C_2, \ldots, C_k$. Note that, to compute which entries appear in the $i$-th column of $K_+(T)$ we do not need to look to the first $i-1$ columns of $T$. We only need the last column of a skew tableau obtained by applying the SJDT to the columns $C_i \cdots C_k$ of $T$, so that the last column has the length of $C_i$, because, by Proposition 4.4.5, all last columns of skew tableaux associated to $T$ with the same length are equal. Let $K_+^1(T)$ be the map that given a tableau returns the first column of $K_+(T)$. This is noticeable in Example 5.1.7 where $K_+(T) = K_+^1(C1C2C3)K_+^1(C2C3)K_+^1(C3)$. In general, $K_+(T) = K_+^1(C_1 \cdots C_k)K_+^1(C_2 \cdots C_k) \cdots K_+^1(C_k)$. Based on this observation and Proposition 4.4.5, next algorithm summarizes our way to compute $K_1^+(T)$ using SJDT:

**Algorithm 5.3.1.** *Let $T$ be a straight KN tableau:*

1. *Let $i = 2$.*

2. *If $T$ has exactly one column, return the right column of $T$. Otherwise, let $T_i = T_2$ be the tableau formed by the first two columns of $T$.*

3. *If the length of the two columns of $T_i$ is the same, put $T_i' := T_i$. Else, play the SJDT on $T_i$ until both column lengths are swapped, obtaining $T_i'$.*

4. *If $T$ has more than $i$ columns, redefine $i := i+1$, and define $T_i$ to be the two-columned tableau formed by the rightmost column of $T_{i-1}'$ and the $i$-th column of $T$, and go back to 3.. Else, return the right column of the rightmost column of $T_i'$.*

This algorithm is illustrated on the bottom path of Example 5.1.7.

**Corollary 5.3.2.** *If $T$ is a rectangular tableau, $K_+(T) = rC_k rC_k \cdots rC_k$ (k times).*

Next, we present a way of computing $K_+^1(T)$ that does not require the SJDT. Willis has done this when $T$ is a SSYT [42]. It is a simplified version of the algorithm presented here.

### 5.3.2   Right key - a direct way

Let $T = C_1C_2$ be a straight KN two column tableau and $spl(T) = \ell C_1 rC_1 \ell C_2 rC_2$ a straight semistandard tableau. In particular, $rC_1 \ell C_2$ is a semistandard tableau. The *matching between $rC_1$ and $\ell C_2$* is defined as follows:

- Let $\beta_1 < \cdots < \beta_{m'}$ be the elements of $\ell C_2$. Let $i$ go from $m'$ to 1, match $\beta_i$ with the biggest, not yet matched, element of $rC_1$ smaller or equal than $\beta_i$.

**Theorem 5.3.3** (The direct way algorithm for the right key). *Let $T$ be a straight KN tableau with columns $C_1, C_2, \ldots, C_k$, and consider its split form $spl(T)$. For every right column $rC_2, \ldots, rC_k$, add empty cells to the bottom in order to have all columns with the same length as $rC_1$. We will fill all of these empty cells recursively, proceeding from left to right. The extra numbers that are written in the column $rC_2$ are found in the following way:*

- *match $rC_1$ and $\ell C_2$.*

- *Let $\alpha_1 < \cdots < \alpha_m$ be the elements of $rC_1$. Let $i$ go from 1 to $m$. If $\alpha_i$ is not matched with any entry of $\ell C_2$, write in the new empty cells of $rC_2$ the smallest element bigger or equal than $\alpha_i$ such that neither it or its symmetric exist in $rC_2$ or in its new cells. Let $C'_2$ be the column defined by $rC_2$ together with the filled extra cells, after ordering.*

*To compute the filling of the extra cells of $rC_3$, we do the same thing, with $C'_2$ and $C_3$. If we do this for all pairs of consecutive columns, we eventually obtain a column $C'_k$, consisting of $rC_k$ together with extra cells, with the same length as $rC_1$. We claim that $C'_k = K^1_+(T)$.*

**Example 5.3.4.** *Let $T = C_1C_2C_3 = \begin{array}{|c|c|c|} \hline 1 & 3 & \bar{1} \\ \hline 3 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$ , with split form $spl(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & \bar{1} & \bar{1} \\ \hline 2 & 3 & 3 & 2 \\ \cline{1-4} 3 & 2 \\ \cline{1-2} \end{array}$ . We match*

*$rC_1$ and $\ell C_2$, as indicated by the letters a and b: $\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1^a & 2^a & 3 & \bar{1} & \bar{1} \\ \hline 2 & 3^b & \bar{3}^b & \bar{2} \\ \cline{1-4} 3 & \bar{2} \\ \cline{1-2} \end{array}$ . Hence $\bar{2}$ creates a $\bar{1}$ in $rC_2$,*

*completing the right column $rC_2$: $\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & \bar{1}^a & \bar{1} \\ \hline 2 & 3 & \bar{3} & \bar{2} \\ \cline{1-4} 3 & \bar{2} & & \bar{1}^a \\ \cline{1-2}\cline{4-4} \end{array}$ . Now we match $C'_2$ and $\ell C_3$, which is already*

*done, and see what new cells 3 and $\bar{2}$ create in $rC_3$, obtaining $\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & \bar{1} & \bar{1} \\ \hline 2 & 3 & \bar{3} & \bar{2} & & 3 \\ \cline{1-4}\cline{6-6} 3 & \bar{2} & & \bar{1} & & \bar{2} \\ \cline{1-2}\cline{4-4}\cline{6-6} \end{array}$ . Hence $K^1_+(T) = \begin{array}{|c|} \hline 3 \\ \hline \bar{2} \\ \hline \bar{1} \\ \hline \end{array}$*

*is obtained from $C'_3$ after reordering its entries.*

**The proof of Theorem 5.3.3**

It is enough to prove that by the end of this algorithm, the entries in $C'_k$ are the entries on the right column of the rightmost column of $T'_k$ from Algorithm 5.3.1. In fact, it is enough to do this for $k = 2$. For bigger $k$ note that the entries that are "slid" into $C_k$ come from $rC_{k-1}$, so, to go to the next step on the SJDT algorithm we only need to know the previous right column, which is exactly what we claim to compute this way. The next lemma determines which number is added to $rC_2$ given that we know $\alpha$, the entry that is horizontally slid:

**Lemma 5.3.5.** *Suppose that $T = C_1C_2$ is a non-rectangular two-column tableau (if the tableau is rectangular then we have nothing to do). Play the SJDT on this tableau which ends up moving one cell from the first column to the second (some entries may change their values). Then,*

    • *Immediately before the horizontal slide of the SJDT, the entry $\alpha$, on the left of the puncture, is an unmatched cell of $rC_1$.*

    • *Call $C_1'$ and $C_2'$ to both columns after the horizontal slide on $T$. The new entry in $rC_2'$, compared to $rC_2$, is the smallest element bigger or equal than $\alpha$ such that neither it or its symmetric exist in $rC_2$.*

**Example 5.3.6.** *Let $T = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 5 & \bar{5} \\ \hline \bar{5} & \\ \cline{1-1} \bar{2} & \\ \cline{1-1} \end{array}$. After splitting, and just before the first horizontal slide, we have*

$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 3 & 3 & 4 & 4 \\ \hline 4 & 5 & \bar{5} & \bar{5} \\ \hline \bar{5} & \bar{4} & * & * \\ \hline \bar{2} & \bar{1} & & \\ \cline{1-2} \end{array}$ *. The new entry in $rC_2$ is $\bar{2}$, as predicted by the lemma:* $\begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 3 & 4 \\ \hline 4 & 4 & \bar{5} & \bar{5} \\ \hline * & * & \bar{4} & \bar{2} \\ \hline \bar{2} & \bar{1} & & \\ \cline{1-2} \end{array}$ .

*Proof.* **Case 1:** $\alpha$ is barred. Then $C_2' = C_2 \cup \{\alpha\}$. If $\overline{\alpha}$ does not exist neither in $C_2$ nor in $\Phi(C_2)$, then $\alpha$ will exist in both $C_2'$ and $\Phi(C_2')$. If $\overline{\alpha}$ does exist in $C_2$, and consequently in $\Phi(C_2)$ (but $\alpha \notin \Phi(C_2)$), then $\alpha$ and $\overline{\alpha}$ will both exist in $C_2'$. Hence, in the construction of the barred part of $\Phi(C_2')$, compared to $\Phi(C_2)$, there will be a new barred number which is the smallest number bigger (or equal, but the equality can not happen) than $\alpha$ such that neither it nor its symmetric exist in the barred part of $\Phi(C_2)$ or the unbarred part of $C_2$ (i.e., $rC_2$). If $\alpha$ existed in $\Phi(C_2)$, then $\overline{\alpha}$ existed in $\Phi(C_2)$. That means that whatever number got sent to $\alpha$ in the construction of $\Phi(C_2)$ will be sent to the next available number, meaning that in $rC_2$ will appear a new number, the smallest number bigger (or equal, but the equality can not happen because $\alpha$ is already there) than $\alpha$ such that neither it nor its symmetric exist in $rC_2$.

    **Case 2:** $\alpha$ is unbarred. Then $C_2' = \Phi^{-1}(\Phi(C_2) \cup \{\alpha\})$. If $\overline{\alpha}$ does not exist in $C_2$ nor in $\Phi(C_2)$, then $\alpha$ will exist in both $C_2'$ and $\Phi(C_2')$. If $\overline{\alpha}$ existed in $\Phi(C_2)$, and consequently in $C_2$, then both $\alpha$ and $\overline{\alpha}$ will exist in $\Phi(C_2')$, hence, if we start in the coadmissible column, in the construction of the unbarred part of $C_2'$, compared to $C_2$, there will be a new unbarred number which is the smallest number bigger than $\alpha$ such that neither it nor its symmetric exist in $rC_2$. Finally, if $\alpha$ existed in $C_2$, then $\overline{\alpha}$ also existed in $C_2$. That means that whatever number got sent to $\alpha$ in the construction of $C_2$, from $\Phi(C_2)$, will be sent to the next available number, meaning that in $rC_2$ will appear a new number, the smallest number bigger than $\alpha$ such that neither it nor its symmetric exist in $rC_2$. $\qquad\square$

*Proof of Theorem 5.3.3:* Each SJDT in $T$, a two-column skew tableau, moves a cell from the first to the second column. We will prove that if we apply the direct way algorithm after each SJDT, the output $C_2'$ does not change. The cells on $\ell C_2$ without cells to its left do not get to be matched. When we slide horizontally, the columns $rC_1$ and $\ell C_2$ may change more than the adding/removal of $\alpha$, the horizontally slid entry. Since the horizontal slides happen from top to bottom, we only need to see what changes happen to bigger entries than the one slid. All entries above $\alpha$ are matched to the entry in the same row in $\ell C_2$.

If $\alpha$ is barred then, the remaining barred entries of $rC_1$ and $\ell C_2$ remain unchanged, and since all entries above $\alpha$, including the unbarred ones, are matched to the entry directly on their right, there is no noteworthy change and everything runs as expected.

If $\alpha$ is unbarred then, the remaining unbarred entries of $rC_1$ and $\ell C_2$ remain unchanged. In the barred part of $rC_1$ either nothing happens, or there is an entry bigger than $\overline{\alpha}$, $\overline{x}$, that gets replaced by $\overline{\alpha}$. Note that $\overline{x}$ must be such that for every number between $\overline{x}$ and $\overline{\alpha}$, either it or its symmetric existed in $rC_1$. In the barred part of $\ell C_2$, if $\overline{\alpha} \in \ell C_2$, then $\overline{\alpha}$ gets replaced by $\overline{y}$, smaller than $\overline{\alpha}$, such that for every number between $\overline{y}$ and $\overline{\alpha}$, either it or its symmetric existed in $\ell C_2$, and both $y$ and $\overline{y}$ do not exist in $\ell C_2$.

Let's look to $\ell C_2$. Let $\alpha < p_1 < p_2 < \cdots < p_m = y$ be the numbers between $\alpha$ and $y$ that does not exist in $\ell C_2$, right before the horizontal slide. Then, their symmetric exist in $\ell C_2$. For all numbers in $rC_2$ between $\alpha$ and $y$, there exists, in the same row in $rC_1$, a number between $\alpha$ and $y$. Let $\alpha < p'_1 < p'_2 < \cdots < p'_m = y$ be the missing numbers between $\alpha$ and $y$ in $rC_1$, then $p_i \le p'_i$. Note that $\overline{p_1} > \overline{p_2} > \cdots > \overline{p_m} = \overline{y}$ exist in $\ell C_2$ after the horizontal slide and that the biggest numbers between $\overline{\alpha}$ and $\overline{y}$ (not including $\overline{\alpha}$) that can exist in $rC_1$ are $\overline{p'_1} > \overline{p'_2} > \cdots > \overline{p'_m}$, and since $\overline{p_i} \ge \overline{p'_i}$, the matching holds for this interval after swapping $\overline{\alpha}$ by $\overline{y}$ in $\ell C_2$.

Now let's look to $rC_1$. Before the slide, call $\overline{x'}$ to the biggest unmatched number of $rC_1$ smaller or equal than $\overline{x}$ and bigger than $\overline{\alpha}$. If there is no such $\overline{x'}$, then everything in $rC_1$ between $\overline{\alpha}$ and $\overline{x}$ is matched, hence swapping $\overline{x}$ by $\overline{\alpha}$ will keep all of them matched, meaning that the algorithm works in this scenario. Let $x' < q_1 < q_2 < \cdots < q_m < \alpha$ be the numbers between $x'$ and $\alpha$ that does not exist in $rC_1$, right before the horizontal slide. Then, their symmetric exist in $rC_1$. For all numbers in $rC_1$ between $x'$ and $\alpha$, there exists, in the same row in $\ell C_2$, a number between $x'$ and $\alpha$, because $\alpha$ is unmatched. Let $x' < q'_1 < q'_2 < \cdots < q'_m < \alpha$ be the missing numbers between $x'$ and $\alpha$ in $\ell C_2$, then $q_i \ge q'_i$. Note that $\overline{q_1} > \overline{q_2} > \cdots > \overline{q_m} > \overline{\alpha}$ exist in $rC_1$ after the horizontal slide and the numbers between $\overline{x'}$ and $\overline{\alpha}$ that can exist in $\ell C_2$ are $\overline{q'_1} > \overline{q'_2} > \cdots > \overline{q'_m}$, and since $\overline{q_i} \le \overline{q'_i}$, these numbers are matching a number bigger or equal then $\overline{q_i}$ in $rC_1$, meaning that $\overline{\alpha}$ is unmatched in $rC_1$ after the slide. Ignoring signs, the numbers that appear in either $rC_2$ or $\ell C_2$ are the same. So before playing the SJDT, applying the direct way algorithm we have that the unmatched numbers in $rC_1$ are sent to the not used numbers of $\overline{q'_1} > \overline{q'_2} > \cdots > \overline{q'_m}$ in $\ell C_2$ (this is a bijection), and $\overline{x'}$ is sent to the smallest available number, bigger or equal than $\overline{x'}$. Now consider $rC_1$ and $\ell C_2$ after the slide. In $rC_1$ we replace $\overline{x'}$ by $\overline{\alpha}$ and remove $\alpha$ and in $\ell C_2$ there is $\alpha$ and not $\overline{\alpha}$. In the direct algorithm, all unmatched numbers of $\overline{q_1} > \overline{q_2} > \cdots > \overline{q_m} > \overline{\alpha}$ are sent to the not used numbers of $\overline{q'_1} > \overline{q'_2} > \cdots > \overline{q'_m}$ in $\ell C_2$, but now we have more numbers in the first set than in the second, meaning that $\overline{\alpha}$ will bump the image of the smallest unmatched number, which will bump the image of the second smallest unmatched number, and so on, meaning that the image of biggest unmatched will be out of this set. This image will be the smallest number available, which was the image of $\overline{x'}$ before the horizontal slide.

Hence, the outcome of the direct way does not change due to the changes to the columns when we play the SJDT, meaning that the outcome is what we intend. $\qquad\square$

### 5.3.3 The left key of a tableau - *Jeu de taquin* approach

Now we present a way to compute the leftmost column of the left key of a tableau, via SJDT, and in Section 5.3.4 we present a way of doing that does not require the use of SJDT.

Analogously to the symplectic right key map, let $T = C_1 C_2 \cdots C_k$ be a KN tableau and note that, to compute which entries appear in the $i$-th column of $K_-(T)$, we only need to look to the first $i$ columns of $T$. We need the first column of a skew tableau obtained by applying the SJDT to the columns $C_1 \cdots C_i$ of $T$, so that the first column has the length of $C_i$. Let $K_-^1(T)$ be the map that given a tableau returns the last column of $K_-(T)$. Then, $K_-(T) = K_-^1(C_1) \cdots K_-^1(C_1 \cdots C_{k-1}) K_-^1(C_1 \cdots C_k)$. In Example 5.1.9 we have $K_-(T) = K_-^1(C1) K_-^1(C1C2) K_-^1(C1C2C3)$.

Next we present how we compute $K_-^1(T)$ using SJDT:

**Algorithm 5.3.7.** *Let $k$ be the number of columns of $T$ and $i = k - 1$.*

1. *If $T$ has exactly one column, return the left column of $T$. Otherwise, let $T_i := T_{k-1}$ be the tableau formed by the last two columns of $T$.*

2. *If the length of the two columns of $T_i$ is the same, put $T_i' := T_i$. Else, play the SJDT on $T_i$ until both column lengths are swapped, obtaining $T_i'$.*

3. *If $i \neq 1$, redefine $i := i - 1$, and define $T_i$ as the two-columned tableau formed with the leftmost column of $T_{i+1}'$ and the $i$-th column of $T$, and go back to (1). Else, return the left column of the leftmost column of $T_i'$.*

This algorithm is exemplified on the top path of Example 5.1.7.

**Corollary 5.3.8.** *If $T$ is a rectangular tableau, $K_-(T) = \ell C_1 \ell C_1 \cdots \ell C_1$ (k times).*

Next, we present a way of computing $K_-^1(T)$ that does not require the use of SJDT. In [42], this is done when $T$ is a SSYT. It is simplified version of the algorithm presented here.

### 5.3.4   Left key - a direct way

**Theorem 5.3.9.** *Let $T$ be a KN tableau with columns $C_1, C_2, \ldots, C_k$, and consider its split form $spl(T)$.*

*We will now delete entries from the left columns, proceeding from right to left, in such a way that in the end every left column has as many entries as $C_k$. The entries deleted from $\ell C_{k-1}$ are found in the following way:*

*We start by creating a matching between $r C_{k-1}$ and $\ell C_k$. Let $\beta_1 < \cdots < \beta_m$ be the unmatched elements of $r C_{k-1}$. For $i$ between 1 and $m$, let $\alpha_i$ be the entry on $\ell C_{k-1}$ next to $\beta_i$. Let $i$ go from 1 to $m$. Starting at $\alpha_i$ and going up, delete the first entry of $\ell C_{k-1}$ bigger than the entry directly Northeast of it. If there is no entry in this conditions, delete the top entry of $\ell C_{k-1}$. Also delete $b_i$ from $r C_{k-1}$. By the end of this procedure we obtain $\ell C_{k-1}'$ with the same number of cells as $C_k$.*

*To continue the algorithm, we do the same thing with $C_{k-2}$ and $\ell C_{k-1}'$. If we do this for all pairs of consecutive columns, we eventually obtain a column $\ell C_1'$, consisting of $\ell C_1$ with some entries deleted, with the same length as $C_k$. We claim that $\ell C_1' = K_-^1(T)$.*

**Example 5.3.10.** *Consider* $T = \begin{array}{|c|c|c|}\hline 2 & 3 & \bar{3} \\\hline 3 & 3 \\\cline{1-2} 3 \\\cline{1-1}\end{array}$ *, whose split form is* $spl(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{3} & \bar{3} \\\hline 2 & 3 & \bar{3} & \bar{2} \\\cline{1-4} 3 & \bar{1} \\\cline{1-2}\end{array}$ *. We*

*match* $rC_2$ *and* $\ell C_1$*, obtaining:* $\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3^a & \bar{3}^a & \bar{3} \\\hline 2 & 3 & \bar{3} & \bar{2} \\\cline{1-4} 3 & \bar{1} \\\cline{1-2}\end{array}$ *. Hence* $\bar{2}$ *is unmatched in* $rC_2$*. So it will get deleted,*

*alongside the* $\bar{3}$ *in* $\ell C_2$*. Thus we have* $\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2^a & 2^a & 3 & \bar{3} & \bar{3} \\\hline 2 & 3 & \bar{3} & \bar{2} \\\cline{1-4} 3 & \bar{1} \\\cline{1-2}\end{array}$ *(the deleted entries are greyed out).*

Now we have to create the match between $\ell C_2'$ and $rC_1$, which is already done. The entries $3$ and
$\bar{1}$ are unmatched in $rC_1$, hence they will be removed alongside the entries $1$ and $\bar{3}$ in $\ell C_1$, obtaining

$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{3} & \bar{3} \\\hline 2 & 3 & \bar{3} & \bar{2} \\\cline{1-4} \bar{3} & \bar{1} \\\cline{1-2}\end{array}$ *. Hence* $K_-^1(T) = \boxed{2}$ *.*

**Proof of Theorem 5.3.9**

It is enough to prove that by the end of this algorithm, the entries in $\ell C_j'$ are the entries on the left
column of the leftmost column of $T_j'$ from Algorithm 5.3.7. Just like in the right key case, it is enough
to do this for $j = k-1$. For smaller $j$ note that we only need to know what remains in the left column
$\ell C_j'$, which is exactly what we claim to compute this way.

So only need to prove this when $T$ is a two-column tableaux.

**Lemma 5.3.11.** *Suppose that $T$ is a non-rectangular two-column tableau (if the tableau is rectangular
then we have nothing to do). Play the SJDT on this tableau, which ends up moving one cell from the
first column to the second (some entries may change its value). Immediately before the horizontal
slide of the SJDT, the entry $\beta$, on the left of the puncture, is an unmatched cell of $rC_1$. Call $C_1'$ and $C_2'$
to both columns after the slide.*

*Then $\ell C_1'$ will lose an entry, compared to $\ell C_1$, which is the biggest entry of $\ell C_1$, in a row not under
the row that contains $\beta$, bigger than the entry directly Northeast of it.*

**Example 5.3.12.** *Consider the tableau* $T = \begin{array}{|c|c|}\hline 2 & 3 \\\hline 4 & 4 \\\hline 5 & \bar{2} \\\hline 5 \\\cline{1-1} 2 \\\cline{1-1}\end{array}$ *. After split, and just before the horizontal slide,*

*we have* $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\\hline 3 & 4 & 4 & 4 \\\hline 4 & 5 & * & * \\\hline 5 & 3 & \bar{2} & \bar{2} \\\hline 2 & \bar{1} \\\cline{1-2}\end{array}$ *. So 5 slides from $rC_1$ to $\ell C_2$, obtaining the tableau* $\begin{array}{|c|c|}\hline 2 & 3 \\\hline 4 & 4 \\\hline * & 5 \\\hline 5 & \bar{2} \\\hline 2 \\\cline{1-1}\end{array}$ *, whose split is*

$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\\hline 4 & 4 & 4 & 4 \\\hline * & * & 5 & 5 \\\hline \bar{5} & \bar{5} & \bar{2} & \bar{2} \\\hline 2 & \bar{1} \\\cline{1-2}\end{array}$ *. The entry removed from $\ell C_1$ is 3, as predicted by the lemma.*

*Proof.* If $\beta$ is unbarred then look at all numbers $\beta \le i \le n$, and count, in $C_1$, count how many of
them exist together with its symmetric and it is not matched to a number with bigger than $\beta$ in the

coadmissible column. Let $k$ be that count. Now let $i$ go from $\beta - 1$ to 1. If $i$ and $\bar{i}$ exist in $C_1$ then $k := k + 1$, and if neither exist then $k := k - 1$. Since $C_1$ is admissible, eventually $k = 0$ and this is the $i$ removed from $\ell C_1$. So, the columns $\ell C_1$ and $rC_1$ have same number of entries with absolute value bigger or equal than $i$, hence the entry $i$ of $\ell C_1$ is bigger than the entry directly Northeast of it.

If $\beta$ is barred then look at all numbers $\beta \leq i \leq \bar{1}$, and count, in $C_1$, count how many of them exist together with its symmetric and it is not matched to a number bigger than $\beta$ in the coadmissible column. Let $k$ be that count. Now let $i$ go from $\beta - 1$ to $\bar{n}$. If $i$ and $\bar{i}$ exist in $C_1$ then $k := k + 1$, and if neither exist then $k := k - 1$. Since $\Phi(C_1)$ is coadmissible, eventually $k = 0$ and this is the $i$ removed from $\ell C_1$. The columns $\ell C_1$ and $rC_1$ have same number of entries with absolute value smaller or equal than $\bar{i}$, hence the entry $i$ of $\ell C_1$ is bigger than the entry directly Northeast of it (remember that $i$ is negative). $\hfill\square$

*Proof of Theorem 5.3.9:* Hence we have determined which entry is removed from $\ell C_1$ given that we know $\beta$, the entry of the cell that is horizontally slid. The SJDT on $T$ may change the entries or the matching in $rC_1$. We need to prove that, even with these eventual changes, the entries removed from $\ell C_1$ are the ones that we calculated in the beginning, before doing any SJDT slide.

If $\beta$ is barred, since we run the unmatched entries of $rC_1$ from smallest to biggest, when removing $\beta$ from $rC_1$ the unbarred part of $rC_1$ remains the same, hence, the remaining entries and matched entries do not change, hence the outcome will be the one predicted.

If $\beta$ is unbarred then the remaining unbarred entries of $rC_1$ remain unchanged. In the barred part of $rC_1$ either nothing happens, or there is an entry bigger than $\overline{\beta}$, $\bar{x}$, that gets replaced by $\overline{\beta}$. Note that $\bar{x}$ must be such that for every number between $\bar{x}$ and $\overline{\beta}$, either it or its symmetric existed in $rC_1$. This can only happen if $k$, from the proof of Lemma 5.3.11 starts being bigger than 0.

Since for all numbers between $\bar{x}$ and $\overline{\beta}$ either it or its symmetric exist in $rC_1$, all unmatched entries here will remove from $\ell C_1$ an entry smaller or equal than $\bar{x}$. In fact, the way of constructing $\bar{x}$ and $i$, from the proof of Lemma 5.3.11, is effectively the same. Since, after the slide of $\beta$, we may have different matches in the numbers between $\bar{x}$ and $\overline{\beta}$, and the number of unmatched entries remains the same after the slide. Since all unmatched entries in here will remove something smaller or equal than $\overline{\beta}$ from $\ell C_1$, the outcome of the algorithm is the same as if we apply it to $\ell C_1$, $rC_1$ before or after the horizontal slide. Hence we do not need to do any SJDT in order to know the entries of $\ell C_1$ after the SJDT. $\hfill\square$

### 5.3.5 Example

In this section we present a KN tableau and compute its right and left keys via SJDT and using the direct way.

Let $T$ be the KN tableau
$$\begin{array}{|c|c|c|c|}\hline 2 & 3 & 3 & 4 \\\hline 4 & \bar{4} & \bar{4} & \bar{4} \\\hline \bar{5} & \bar{3} & \bar{2} \\\cline{1-3} \bar{4} \\\cline{1-1} \bar{3} \\\cline{1-1}\end{array}$$
with split form
$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\\hline 3 & 4 & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{3} \\\hline \bar{5} & \bar{5} & \bar{3} & \bar{2} & \bar{2} & \bar{2} \\\cline{1-6} \bar{4} & \bar{3} \\\cline{1-2} \bar{3} & \bar{1} \\\cline{1-2}\end{array}.$$

In order to find the right (resp. left) key of $T$, we play the SJDT to swap heights of consecutive columns, and find skew tableaux, Knuth related to $T$, such that for every column height there is a skew tableau whose last column (resp. first) has that height.

$$
\begin{array}{l}
\begin{array}{|c|c|c|c|}\hline 2&3&3&4\\\hline 4&\bar4&\bar4&\bar4\\\hline \bar5&\bar3&\bar2\\\cline{1-3} \bar4\\\cline{1-1}\bar3\\\cline{1-1}\end{array}
\quad
\begin{array}{|c|c|c|}\hline 3&3&4\\\hline \bar5&\bar4&\bar4\\\hline 2&\bar5&\bar2\\\hline \bar5&\bar4\\\cline{1-2}\bar3&\bar3\\\cline{1-2}\end{array}
\quad
\begin{array}{l}
\begin{array}{|c|}\hline 4\\\hline\end{array}\\
\begin{array}{|c|c|c|}\hline 3&3&\bar4\\\hline \bar5&\bar4&\bar2\\\hline 2&\bar5\\\cline{1-2}\bar5&\bar4\\\cline{1-2}\bar3&\bar3\\\cline{1-2}\end{array}
\end{array}
\quad
\begin{array}{l}
\begin{array}{|c|c|}\hline 3&4\\\hline \bar5&\bar4\\\hline \bar5&\bar2\\\hline\end{array}\\
\begin{array}{|c|c|c|}\hline 2&2&\bar4\\\hline \bar5&\bar4&\bar2\\\hline \bar3\\\cline{1-1}\end{array}
\end{array}
\quad
\begin{array}{l}
\begin{array}{|c|c|}\hline 3&4\\\hline \bar5&\bar4\\\hline 2&\bar5&\bar2\\\hline 2&\bar4&\bar4\\\hline \bar5&\bar3&\bar2\\\hline\end{array}
\end{array}
\quad
\begin{array}{l}
\begin{array}{|c|}\hline 4\\\hline \bar5\\\hline\end{array}\\
\begin{array}{|c|c|c|}\hline 2&3&\bar5\\\hline 2&\bar4&\bar4&\bar4\\\hline \bar5&\bar3&\bar2&\bar2\\\hline\end{array}
\end{array}
\end{array}
$$
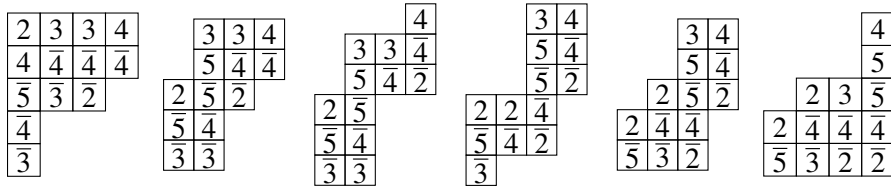
Each tableau is obtained from the previous after playing SJDT in two consecutive columns, swapping their heights.

If we compute the right (resp. left) columns of all last (resp. first) columns of these tableaux, we find the columns of the right (resp. left) key associated to $T$:

$$
K_+(T) = \begin{array}{|c|c|c|c|}\hline 4&4&4&4\\\hline \bar5&\bar3&\bar3&\bar3\\\hline \bar3&\bar2&\bar2\\\cline{1-3}\bar2\\\cline{1-1}\bar1\\\cline{1-1}\end{array}
\quad\text{and}\quad
K_-(T) = \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline 2&\bar5&\bar5&\bar5\\\hline \bar5&\bar3&\bar3\\\cline{1-3}\bar4\\\cline{1-1}\bar3\\\cline{1-1}\end{array}\quad.
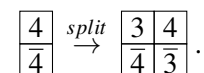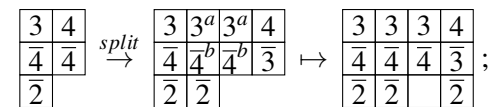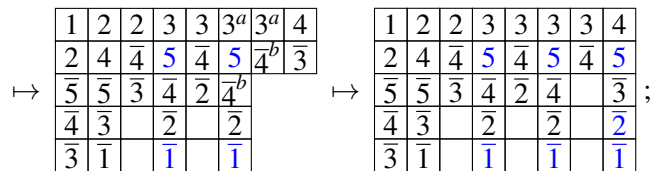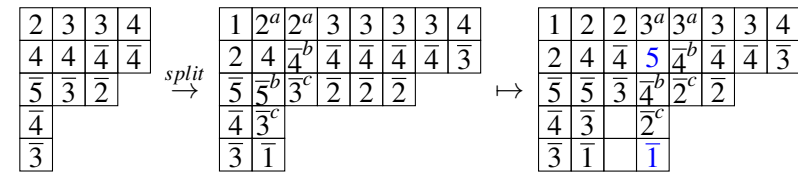$$

Note that we have 9 horizontal slides in our sequence of tableaux, and for each horizontal slide we have to apply the map $\Phi$, or its inverse, two times. This means that we are effectively computing the split form of 9 skew tableaux, even though we only need 3 tableaux (the first, the third and the last one) to have all column heights in each end of the tableau.

Now we compute both keys using the direct way. In here we only need to compute one split form, and make some calculations on it, and on subtableaux of the split form.

To compute the right key, via direct way, we need to compute the columns $K_+^1\left(\begin{array}{|c|c|c|c|}\hline 2&3&3&4\\\hline 4&\bar4&\bar4&\bar4\\\hline \bar5&\bar3&\bar2\\\cline{1-3}\bar4\\\cline{1-1}\bar3\\\cline{1-1}\end{array}\right),$

$$
K_+^1\left(\begin{array}{|c|c|c|}\hline 3&3&4\\\hline \bar4&\bar4&\bar4\\\hline \bar3&\bar2\\\cline{1-2}\end{array}\right) = K_+^1\left(\begin{array}{|c|c|}\hline 3&4\\\hline \bar4&\bar4\\\hline \bar2\\\cline{1-1}\end{array}\right)\quad\text{and}\quad K_+^1\left(\begin{array}{|c|}\hline 4\\\hline \bar4\\\hline\end{array}\right).
$$

We start by splitting and matching, and every $\mapsto$ marks when new entries, written in blue, are added to a right column, and we do these until there are no columns left.

$$
\begin{array}{|c|c|c|c|}\hline 2&3&3&4\\\hline 4&\bar4&\bar4&\bar4\\\hline \bar5&\bar3&\bar2\\\cline{1-3}\bar4\\\cline{1-1}\bar3\\\cline{1-1}\end{array}
\;\xrightarrow{split}\;
\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2^a&2^a&3&3&3&3&4\\\hline 2&4&\bar4^b&\bar4&\bar4&\bar4&\bar4&3\\\hline \bar5&\bar5^b&\bar3^c&\bar2&\bar2&\bar2\\\cline{1-6}\bar4&\bar3^c\\\cline{1-2}\bar3&\bar1\\\cline{1-2}\end{array}
\;\mapsto\;
\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&3^a&3^a&3&3&4\\\hline 2&4&\bar4&5&\bar4^b&\bar4&\bar4&3\\\hline \bar5&\bar5&\bar3&\bar4^b&\bar2^c&\bar2\\\cline{1-6}\bar4&\bar3&&\bar2^c\\\cline{1-4}\bar3&\bar1&&\bar1\\\cline{1-4}\end{array}
$$

$$
\mapsto\;
\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&3&3&3^a&3^a&4\\\hline 2&4&\bar4&5&\bar4&5&\bar4^b&3\\\hline \bar5&\bar5&\bar3&\bar4&\bar2&\bar4^b\\\cline{1-6}\bar4&\bar3&&\bar2&&\bar2\\\cline{1-6}\bar3&\bar1&&\bar1&&\bar1\\\cline{1-6}\end{array}
\;\mapsto\;
\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&2&2&3&3&3&3&4\\\hline 2&4&\bar4&5&\bar4&5&\bar4&5\\\hline \bar5&\bar5&\bar3&\bar4&\bar2&\bar4&&\bar3\\\cline{1-7}\bar4&\bar3&&\bar2&&\bar2&&\bar2\\\cline{1-8}\bar3&\bar1&&\bar1&&\bar1&&\bar1\\\cline{1-8}\end{array}\;;
$$

$$
\begin{array}{|c|c|}\hline 3&4\\\hline \bar4&\bar4\\\hline \bar2\\\cline{1-1}\end{array}
\;\xrightarrow{split}\;
\begin{array}{|c|c|c|c|}\hline 3&3^a&3^a&4\\\hline \bar4&\bar4^b&\bar4^b&3\\\hline \bar2&\bar2\\\cline{1-2}\end{array}
\;\mapsto\;
\begin{array}{|c|c|c|c|}\hline 3&3&3&4\\\hline \bar4&\bar4&\bar4&3\\\hline \bar2&\bar2&&\bar2\\\cline{1-4}\end{array}\;;
$$

$$
\begin{array}{|c|}\hline 4\\\hline \bar4\\\hline\end{array}
\;\xrightarrow{split}\;
\begin{array}{|c|c|}\hline 3&4\\\hline \bar4&\bar3\\\hline\end{array}\quad.
$$

To compute the left key, via direct way, we need to compute the columns $K_-^1 \left( \begin{array}{cccc} 2 & 3 & 3 & 4 \\ 4 & \bar 4 & \bar 4 & \bar 4 \\ \bar 5 & \bar 3 & \bar 2 \\ \bar 4 \\ \bar 3 \end{array} \right)$,

$K_-^1 \left( \begin{array}{cc} 2 & 3 \\ 4 & \bar 4 \\ \bar 5 & \bar 3 \\ \bar 4 \\ \bar 3 \end{array} \right) = K_-^1 \left( \begin{array}{ccc} 2 & 3 & 3 \\ 4 & \bar 4 & \bar 4 \\ \bar 5 & \bar 3 & \bar 2 \\ \bar 4 \\ \bar 3 \end{array} \right)$ and $K_-^1 \left( \begin{array}{c} 2 \\ 4 \\ \bar 5 \\ \bar 4 \\ \bar 3 \end{array} \right)$. We start by splitting and matching, and every
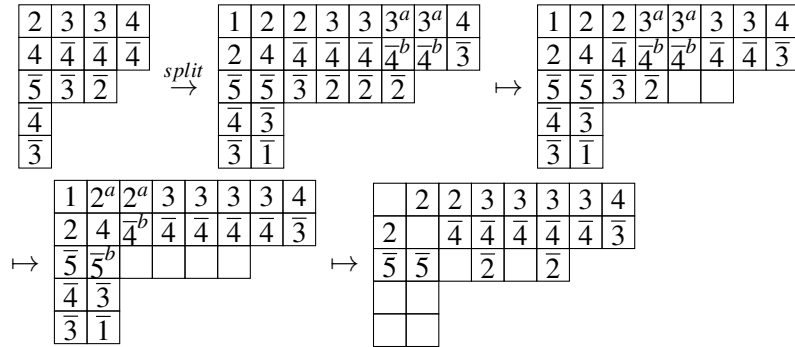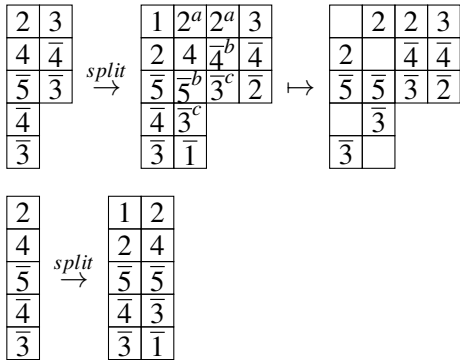
$\mapsto$ marks when entries are removed from a left column, and we do these until there are no columns left. Recall that this algorithm goes from right to left.

$$
\begin{array}{cccc} 2 & 3 & 3 & 4 \\ 4 & \bar 4 & \bar 4 & \bar 4 \\ \bar 5 & \bar 3 & \bar 2 \\ \bar 4 \\ \bar 3 \end{array}
\xrightarrow{split}
\begin{array}{ccccccc} 1 & 2 & 2 & 3 & 3 & 3^a & 3^a & 4 \\ 2 & 4 & \bar 4 & \bar 4 & \bar 4 & \bar 4^b & \bar 4^b & \bar 3 \\ \bar 5 & \bar 5 & \bar 3 & \bar 2 & \bar 2 & \bar 2 \\ \bar 4 & \bar 3 \\ \bar 3 & \bar 1 \end{array}
\mapsto
\begin{array}{cccccccc} 1 & 2 & 2 & 3^a & 3^a & 3 & 3 & 4 \\ 2 & 4 & \bar 4 & \bar 4^b & \bar 4^b & \bar 4 & \bar 4 & \bar 3 \\ \bar 5 & \bar 5 & \bar 3 & \bar 2 \\ \bar 4 & \bar 3 \\ \bar 3 & \bar 1 \end{array}
$$

$$
\mapsto
\begin{array}{ccccccc} 1 & 2^a & 2^a & 3 & 3 & 3 & 3 & 4 \\ 2 & 4 & \bar 4^b & \bar 4 & \bar 4 & \bar 4 & \bar 4 & \bar 3 \\ \bar 5 & \bar 5^b \\ \bar 4 & \bar 3 \\ \bar 3 & \bar 1 \end{array}
\mapsto
\begin{array}{ccccccc} & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\ 2 & & \bar 4 & \bar 4 & \bar 4 & \bar 4 & \bar 4 & \bar 3 \\ \bar 5 & \bar 5 & & \bar 2 & & \bar 2 \\ & \\ & \end{array}
$$

In the final step, we are removing $\bar 3$ from $\ell C_1$, because the entry directly Northeast of it is $\bar 5$, because the $\bar 3$ of $r C_1$ has already been slid out.

$$
\begin{array}{cc} 2 & 3 \\ 4 & \bar 4 \\ \bar 5 & \bar 3 \\ \bar 4 \\ \bar 3 \end{array}
\xrightarrow{split}
\begin{array}{cccc} 1 & 2^a & 2^a & 3 \\ 2 & 4 & \bar 4^b & \bar 4 \\ \bar 5 & \bar 5^b & \bar 3^c & \bar 2 \\ \bar 4 & \bar 3^c \\ \bar 3 & \bar 1 \end{array}
\mapsto
\begin{array}{cccc} & 2 & 2 & 3 \\ 2 & & \bar 4 & \bar 4 \\ \bar 5 & \bar 5 & \bar 3 & \bar 2 \\ & \bar 3 \\ \bar 3 \end{array}
$$

$$
\begin{array}{c} 2 \\ 4 \\ \bar 5 \\ \bar 4 \\ \bar 3 \end{array}
\xrightarrow{split}
\begin{array}{cc} 1 & 2 \\ 2 & 4 \\ \bar 5 & \bar 5 \\ \bar 4 & \bar 3 \\ \bar 3 & \bar 1 \end{array}
$$

# Chapter 6

# Realization of the Lusztig involution in types $A_{n-1}$ and $C_n$

In this chapter we present an involution on a crystal, known as Lusztig involution [30]. For the type $A_{n-1}$ crystal of SSYT's this involution is usually known as Schützenberger involution or evacuation, and can be realized via *jeu da taquin* or, equivalently, via column insertion. Here we adapt the type $A_{n-1}$ Schützenberger evacuation to type $C_n$ KN tableaux, via SJDT or Baker-Lecouvey insertion. We relate the right and left key maps of a tableau via Lusztig involution.

## 6.1 Lusztig involution and evacuation algorithms

In type $A_{n-1}$, the Lusztig involution [30] on the crystal with set $\mathcal{SSYT}(\lambda, n)$ coincides with the Schützenberger involution or evacuation [13, 38, 40], *Ev*, and takes $T \in \mathcal{SSYT}(\lambda, n)$ to $T^{Ev} \in \mathcal{SSYT}(\lambda, n)$, whose weight is $\omega_0(\text{wt}\,T)$, where $\omega_0$ is the longest permutation of $\mathfrak{S}_n$, in the Bruhat order. Note that $\omega_0(\text{wt}\,T)$ is the vector $\text{wt}\,T$ in reverse order, i.e., $\omega_0(v_1, \ldots, v_n) = (v_n, \ldots, v_1)$. In type $C_n$ we will work with KN tableaux instead of SSYT's. Consider $T \in \mathcal{KN}(\lambda, n)$. In this case, $T^{Ev} \in \mathcal{KN}(\lambda, n)$ and $\text{wt}\,T = -\text{wt}\,T^{Ev} = \omega_0^C(\text{wt}\,T^{Ev})$, where $\omega_0^C$ is the longest permutation of $B_n$.

**Definition 6.1.1.** *The Lusztig involution $L : \mathfrak{B}^\lambda \to \mathfrak{B}^\lambda$ is the only set involution such that for all $i \in I$ ($I = [n-1]$ in type $A_{n-1}$ and $I = [n]$ in type $C_n$):*

1.  *$\text{wt}(L(x)) = \omega_0(\text{wt}(x))$, where $\omega_0$ is the longest element of the Weyl group;*

2.  *$e_i(Lx) = L(f_{i'}(x))$ and $f_i(Lx) = L(e_{i'}(x))$ where $i'$ is such that $\omega_0(\alpha_i) = -\alpha_{i'}$ and $\alpha_i$ is the $i$-th simple root;*

3.  *$\varepsilon_i(Lx) = \varphi_{i'}(x)$ and $\varphi_i(Lx) = \varepsilon_{i'}(x)$.*

**Remark 6.1.2.** *One can prove that the map defined by this three conditions is, in fact, unique [9, Chapter 5].*

For type $A_{n-1}$ we have that $\omega_0$ is the reverse permutation and $i' = n - i$, and for type $C_n$ we have $\omega_0 = -\text{Id}$ and $i' = i$, where Id is the identity map. In type $C_n$ the involution can be seen as flipping the crystal upside down.

The type $C_n$ Lusztig involution can be seen as a realization of the dual crystal:

**Definition 6.1.3.** *[9] Let $\mathfrak{C}$ be a connected component in the type $C_n$ crystal $G_n$. The dual crystal $\mathfrak{C}^\vee$ is the crystal obtained from $\mathfrak{C}$ after reversing the direction of all arrows. Also, the if $x \in \mathfrak{C}$, then for its correspondent in $\mathfrak{C}^\vee$, $x^\vee$, we have $wt(x) = -wt(x^\vee)$.*

In type $C_n$, since $i' = i$ and $\omega_0 = -\text{Id}$, it follows from the definition of Lusztig involution that $\mathfrak{C}$ and $\mathfrak{C}^\vee$, as crystals in $G_n$, have the same highest weight. Therefore, they are isomorphic. Hence the crystal $\mathfrak{B}^\lambda$ with set $\mathcal{KN}(\lambda, n)$ is self-dual. We shall see other realizations of the dual.

The complement of a tableau or a word in types $A_{n-1}$ or $C_n$ consists in applying $\omega_0$ or $\omega_0^C$, respectively, to all of its entries. In type $A_{n-1}$, it sends $i$ to $n+1-i$ for all $i \in [n]$, i.e., $\omega_0(i) = n+1-i$ and in type $C_n$ we have $\omega_0(i) = -i$. Given a SSYT, there are several algorithms, due to Schützenberger, to obtain a SSYT with the same shape whose weight is its reverse. We recall some versions of them for which one is able to find analogues for KN tableaux.

**Algorithm 6.1.4.**

1. *Define $w = cr(T)$.*

2. *Define $w^\star$ the word obtained by complementing its letters and writing it backwards.*

3. *$T^{Ev} := P(w^\star)$.*

**Example 6.1.5.** *In type A, the tableau $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$ has reading $w = 32313124$. Then $w^\star =$*

*13424232, and the column insertion of this word is $T^{Ev} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$ .*

*In type C, consider the KN tableau $T = \begin{array}{|c|c|c|} \hline 1 & 3 & \bar{1} \\ \hline 3 & \bar{3} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array}$ . Then, $w = cr(T) = \bar{1}3\bar{3}13\bar{3}$ and $w^\star = 3\bar{3}1\bar{3}3\bar{3}1$.*

*So now we insert $w^\star$, obtaining the following sequence of tableaux:*

$\boxed{3}$   $\begin{array}{|c|} \hline 3 \\ \hline \bar{3} \\ \hline \end{array}$   $\begin{array}{|c|} \hline 3 \\ \hline \bar{3} \\ \hline \bar{1} \\ \hline \end{array}$   $\begin{array}{|c|c|} \hline 2 & \bar{2} \\ \hline 3 \\ \cline{1-1} \bar{1} \\ \cline{1-1} \end{array}$   $\begin{array}{|c|c|} \hline 2 & \bar{2} \\ \hline 3 & \bar{1} \\ \hline \bar{3} \\ \cline{1-1} \end{array}$   $\begin{array}{|c|c|c|} \hline 1 & 2 & \bar{2} \\ \hline 3 & \bar{1} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \end{array}$ $= P(w^\star)$.

**Algorithm 6.1.6.**

1. *Define $T^0 := complement(\pi\text{-}rotate(T))$.*

2. *$T^{Ev} := rectification$ of $T^0$.*

**Example 6.1.7.** *In type A, consider the tableau $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$ . After $\pi$-rotation and complement we*

*have the skew tableau $T^0 = \begin{array}{ccc|c|} \cline{4-4} & & & 1 \\ \cline{2-4} & 2 & 2 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \hline \end{array}$ which, after rectification, gives the tableau $T^{Ev} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$ .*

*In type C, consider the KN tableau $T = $* 
$\begin{array}{|c|c|c|}\hline 1 & 3 & \bar{1} \\\hline \end{array}$
$\begin{array}{|c|c|}\hline 3 & 3 \\\hline \end{array}$
$\begin{array}{|c|}\hline \bar{3} \\\hline \end{array}$ 
. *Then, $T_0 = $* 
$\begin{array}{|c|}\hline 3 \\\hline \end{array}$
$\begin{array}{|c|c|}\hline 3 & \bar{3} \\\hline \end{array}$
$\begin{array}{|c|c|c|}\hline 1 & 3 & \bar{1} \\\hline \end{array}$
. *So now we have to*

*rectify this skew tableau obtaining $T^{Ev} = $* 
$\begin{array}{|c|c|c|}\hline 1 & 2 & \bar{2} \\\hline \end{array}$
$\begin{array}{|c|c|}\hline 3 & \bar{1} \\\hline \end{array}$
$\begin{array}{|c|}\hline \bar{3} \\\hline \end{array}$ .

Given a KN tableau (resp. SSYT) $T$, the algorithm characterize $T^{Ev}$ as the unique KN tableau (resp. SSYT) Knuth equivalent to $\text{wt}(T)^\star$ and coplactic equivalent to $T$.

In both Cartan types we have that algorithms 6.1.4 and 6.1.6 produce the same tableau since the column reading of $T^0$ is $w^\star$, $P(w^\star) = rect(T^0) = rect(w^\star)$, assuming that, in type $C_n$, $T^0$ is admissible. This can be concluded using the following lemma.

**Lemma 6.1.8.** *For type $C_n$, the split of a column $C$, $(\ell C, rC)$ is the rotation and complement of the split of the column $C^0 = complement(\pi\text{-rotate}(C))$, $(\ell C^0, rC^0)$.*

*Proof.* Let's say that $(\ell C, rC) = $ $\begin{array}{|c|c|}\hline A' & A \\\hline B & B' \\\hline \end{array}$ where $C = $ $\begin{array}{|c|}\hline A \\\hline B \\\hline \end{array}$, $\ell C = $ $\begin{array}{|c|}\hline A' \\\hline B \\\hline \end{array}$ and $rC = $ $\begin{array}{|c|}\hline A \\\hline B' \\\hline \end{array}$ , where $A$ and $A'$ are the unbarred letters of the columns $C$ and $\ell C$, respectively, and $B$ and $rB$ are the barred letters of $C$ and $rC$, respectively. Note that $\ell C$ and $C$ share the barred part and $C$ and $rC$ share the unbarred part.

We have that $C^0 = $ $\begin{array}{|c|}\hline B^0 \\\hline A^0 \\\hline \end{array}$ and its split $(\ell C^0, rC^0) = $ $\begin{array}{|c|c|}\hline B^{0'} & B^0 \\\hline A^0 & A^{0'} \\\hline \end{array}$ . Ignoring bars and counting multiplicities, the letters that appear in $C$ and $C^0$ are the same. Hence $B^{0'}$ has the same letters as $B'$, but they appear unbarred, hence $B^{0'} = B'^0$. The same happens with $A^{0'}$ and $A'^0$. Now it is easy to see that $(\ell C^0, rC^0)$ is obtained from $(\ell C, rC)$ rotating and complementing. In particular $(rC)^0 = \ell C^0$ and $(\ell C)^0 = rC^0$. $\square$

We now set the Cartan type to be $C_n$. Given a word $w \in [\pm n]^*$, we define the $w^\star$ like in the Algorithm 6.1.4 and show that the map $^\star$ preserves Knuth equivalence.

**Theorem 6.1.9.** *Let $v, w \in [\pm n]^*$. Then $v \sim w$ if and only if $v^\star \sim w^\star$.*

*Proof.* It is enough to consider $v$ and $w$ only one Knuth relation apart, because all other cases are obtained by composing multiple Knuth relations. It is enough to consider each transformation applied in one direction, since the other direction is the same case, after swapping the roles of $v$ and $w$.

K1 Consider $v = v_p \gamma \beta \alpha v_s$, with $\gamma < \alpha \leq \beta$ and $(\beta, \gamma) \neq (\bar{x}, x)$, where $v_p$ is a prefix of $v$, $v_s$ is a suffix of $v$, and $\gamma \beta \alpha$ are three consecutive letters of $v$. Then, $v \overset{K1}{\sim} w = v_p \beta \gamma \alpha v_s$. Note that $v^\star = v_s^\star \overline{\alpha} \overline{\beta} \overline{\gamma} v_p^\star$ and $w^\star = v_s^\star \overline{\alpha} \overline{\gamma} \overline{\beta} v_p^\star$, with $(\overline{\gamma}, \overline{\beta}) \neq (\bar{x}, x)$ and $\overline{\beta} \leq \overline{\alpha} < \overline{\gamma}$. Hence $v^\star \overset{K2}{\sim} w^\star$, so they are Knuth related.

K2 Consider $v = v_p \alpha \beta \gamma v_s$, with $\gamma \leq \alpha < \beta$ and $(\beta, \gamma) \neq (\bar{x}, x)$, where $v_p$ is a prefix of $v$, $v_s$ is a suffix of $v$, and $\alpha \beta \gamma$ are three consecutive letters of $v$. Then, $v \overset{K2}{\sim} w = v_p \alpha \gamma \beta v_s$. Note that $v^\star = v_s^\star \overline{\gamma} \overline{\beta} \overline{\alpha} v_p^\star$ and $w^\star = v_s^\star \overline{\beta} \overline{\gamma} \overline{\alpha} v_p^\star$, with $(\overline{\gamma}, \overline{\beta}) \neq (\bar{x}, x)$ and $\overline{\beta} < \overline{\alpha} \leq \overline{\gamma}$. Hence $v^\star \overset{K1}{\sim} w^\star$, so they are Knuth related.

K3   Consider $v = v_p(y+1)\overline{y+1}\beta v_s$, with $y < \beta < \bar{y}$, where $v_p$ is a prefix of $v$, $v_s$ is a suffix of $v$, and $(y+1)\overline{y+1}\beta$ are three consecutive letters of $v$. Then, $v \overset{K3}{\sim} w = v_p\bar{y}y\beta v_s$. Note that $v^\star = v_s^\star\bar{\beta}(y+1)\overline{y+1}v_p^\star$ and $w^\star = v_s^\star\bar{\beta}\bar{y}yv_p^\star$, with $y < \beta < \bar{y}$. Hence $v^\star \overset{K4}{\sim} w^\star$, so they are Knuth related.

K4   Consider $v = v_p\alpha\bar{x}xv_s$, with $x < \alpha < \bar{x}$, where $v_p$ is a prefix of $v$, $v_s$ is a suffix of $v$, and $\alpha\bar{x}x$ are three consecutive letters of $v$. Then, $v \overset{K4}{\sim} w = v_p\alpha(x+1)\overline{x+1}v_s$. Note that $v^\star = v_s^\star\bar{x}x\bar{\alpha}v_p^\star$ and $w^\star = v_s^\star(x+1)\overline{x+1}\bar{\alpha}v_p^\star$, with $x < \alpha < \bar{x}$. Hence $v^\star \overset{K3}{\sim} w^\star$, so they are Knuth related.

K5   Consider $w$ and $\{z, \bar{z}\} \in w$ such that $w \overset{K5}{\sim} w \setminus \{z, \bar{z}\}$. It is clear to see that a word $v$ breaks the $1CC$ at $z$ if and only if $v^\star$ breaks the $1CC$ at $z$. So, if $w$ is non admissible and all its factors are admissible then the same will happen to $w^\star$, because all of its factors are obtained after applying $^\star$ to a factor of $w$. So we have that $w^\star \overset{K5}{\sim} w^\star \setminus \{z, \bar{z}\}$.

Hence the word operator $^\star$ preserves Knuth equivalence.      □

Consider a KN tableau $T$ with column reading $w$. The column reading of the tableau obtained after applying Algorithm 6.1.4 to $T$ is Knuth-related to $w^\star$, because both give the same tableau if inserted. Since $^\star$ is an involution $((w^\star)^\star = w)$, if we apply the algorithm again we will get a tableau whose column reading, by the last theorem, is Knuth equivalent to $(w^\star)^\star = w$, hence we will have $T$ again. So Algorithm 6.1.4 is an involution. Next we conclude that algorithms 6.1.4 and 6.1.6 is a realization of the Lusztig involution for type $C_n$.

**Theorem 6.1.10.** *Let $w \in [\pm n]^\star$. The connected component of the crystal $G_n$ that contains the word $w$ is isomorphic to the one that contains the word $w^\star$. Therefore $P(w)$ and $P(w^\star)$ have the same shape and weights of opposite sign. Moreover, the two crystals are dual of each other and the $^\star$ map is a realization of the dual crystal.*

*Proof.* Remember the crystal operators $e_i$ and $f_i$ from the definition of crystal. Note that $(f_i(w))^\star = e_i(w^\star)$, because in the signature rule applied to $w$ and $w^\star$, the distance of the leftmost unbracketed $+$ of $w$ to the beginning of the word is equal to the distance of the rightmost unbracketed $-$ of $w^\star$ to the end of this word. Hence, the letter that changes when applying $f_i$ to $w$ is the complement of the letter that changes when applying $e_i$ to $w^\star$, and the letter obtained on their position after applying the crystal operators are also complement of each other. Hence the crystal that contains the word $w^\star$ is the dual to the one that contains $w$. But the crystal that contains $w$ is self-dual, hence the crystals that contains any of the words are isomorphic. From [25, Theorem 3.2.8] $P(w)$ and $P(w^\star)$ have the same shape.      □

## 6.2   Right and left keys and Lusztig involution

The next result shows that the right and left key maps defined for KN tableaux anticommutes with the Lusztig involution. The evacuation of the right key of a tableau is the left key of the evacuation of the same tableau.

**Proposition 6.2.1.** *Let T be a KN tableau and Ev the type $C_n$ Lusztig involution. Then*

$$K_+(T)^{Ev} = K_-(T^{Ev}).$$

*Proof.* Since the tableaux $K_+(T)$ and $K_-(T^{Ev})$ are key tableaux, they are completely determined by their weights. Then we just need to prove that their weights are symmetric.

Fix a column $C$ of $K_+(T)$. There is a frank word $w$, Knuth related to $cr(T)$, such that $C$ is the right column of the first column of $w$. Let's say the $w_k$ is the first column of $w$. From Proposition 6.1.9, $w^\star$ is Knuth related to $cr(T)^\star$, hence $P(w^\star) = T^{Ev}$. Also note that the $w^\star$ has the same number of columns of each length as $w$, hence it is a frank word, and its last column is $w_k^\star$. Note that Lemma 6.1.8 implies that if $v$ is an admissible column, then $l(v^\star) = (rv)^\star$. So we have that $l(w_k^\star) = (rw_k)^\star$ is a column of $K_-(T^{Ev})$. Therefore, for each column $C$ of $K_+(T)$ there is a column of $K_-(T^{Ev})$ whose weight is $\omega_0(\mathrm{wt}(C))$, hence $K_+(T)$ and $K_-(T^{Ev})$ have symmetric weights.            $\square$

**Remark 6.2.2.** *Using Proposition 6.2.1 and the definition of the Lusztig involution, it is now clear that the tableau weights in $\mathfrak{B}_v$ and in $\mathfrak{B}_{-v}^{op}$ are symmetric.*

# Chapter 7

# Final remarks and open questions

In this chapter we discuss some unfinished work and open problems related to the topics presented in this thesis.

## 7.1 Type $C_n$ Fu-Lascoux non-symmetric Cauchy kernel

Let's start by recalling the type $C_n$ Fu-Lascoux non-symmetric Cauchy kernel:

$$\frac{\prod\limits_{1 \le i < j \le n} (1 - x_i x_j)}{\prod\limits_{i,j=1}^{n} (1 - x_i y_j) \prod\limits_{i,j=1}^{n} (1 - x_i / y_j)} = \sum_{v \in \mathbb{N}^n} \widehat{\kappa}_v(x_1, \ldots, x_n) \kappa_{-v}(y_1, \ldots, y_n)$$

$$\Leftrightarrow \prod_{1 \le i,j \le n} (1 - x_i y_j)^{-1} \prod_{1 \le i \le j \le n} (1 - x_i y_j^{-1})^{-1} = \prod_{1 \le i < j \le n} (1 - x_i x_j)^{-1} \sum_{v \in \mathbb{N}^n} \widehat{\kappa}_v(x) \kappa_{-v}(y).$$

In this section we present a combinatorial interpretation for each side of the last identity, and propose an algorithm that relates both combinatorial interpretations.

### 7.1.1 Warm up for the combinatorial interpretations

Let $\lambda_v$ be the only partition on the $\mathfrak{S}_n$-orbit of $v$, for $v \in \mathbb{N}^n$. The next lemma is a reformulation of the tableau criterion for the Bruhat order on $\mathfrak{S}_n$.

**Lemma 7.1.1.** *Let $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathfrak{S}_n \lambda$.*

$$u \le v \text{ if and only if } \forall i \in [n], \lambda_{(u_1, \ldots, u_i)} \supseteq \lambda_{(v_1, \ldots, v_i)}.$$

*Proof.* From Theorem 3.4.4, restricted to the symmetric group, we have that $u \le v$ if and only if $K(u) \le K(v)$. For all $i \in [n]$, if we know the first $i$ entries of $u$ then we know all entries of $K(u)$ less or equal than $i$. Also, the shape occupied by this entries is exactly $\lambda_{(u_1, \ldots, u_i)}$. The shape occupied by the entries less or equal than $i$ in $K(u)$ contains the shape occupied by the entries less or equal than $i$ in $K(v)$. Hence $K(u) \le K(v)$ if and only if $\lambda_{(u_1, \ldots, u_i)} \supseteq \lambda_{(v_1, \ldots, v_i)}, \forall i \in [n]$. $\qquad\square$

Due to the natural embedding of $B_n$ into $\mathfrak{S}_{2n}$, this can be extended to $B_n$. Given a $v \in \mathbb{Z}_n$, its correspondent in $\mathbb{N}_{2n}$ is the vector $v^{\#}$, with $2n$ entries indexed by $[\pm n]$, where $v_i^{\#} = \begin{cases} 0 \text{ if } i \times v_{|i|} \leq 0 \\ |v_{|i|}| \text{ otherwise} \end{cases}$.

For instance, if $u = (4, 1, \overline{3}, \overline{2}, 3)$. Then $u^{\#} = (4, 1, 0, 0, 3, 0, 2, 3, 0, 0)$.

Embedding $B_n$ into $\mathfrak{S}_{2n}$ and applying the previous lemma, we have the following:

**Corollary 7.1.2.** *Let $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathfrak{B}_n \lambda$.*

$$u \leq v \text{ if and only if } \forall i \in [\pm n], \lambda_{(u_1^{\#}, \ldots, u_i^{\#})} \supseteq \lambda_{(v_1^{\#}, \ldots, v_i^{\#})}.$$

The next proposition is the conjugate version of [13, Proposition 7, Appendix A].

**Proposition 7.1.3.** *Let $T$ be a SSYT tableau of shape $\lambda$. Let $\mu/\nu$ be a skew diagram with same number of rows of each length as $T$. Then there is a unique KN skew tableau $S$ with shape $\mu/\nu$ that rectifies to $T$.*

*Proof.* The number of skew tableaux of shape $\mu/\nu$ that rectify to $T$ of shape $\lambda$ is given by the Littlewood-Richardson coefficient $c_{\lambda\,\nu}^{\mu}$.

Remember the identity $c_{\lambda\,\nu}^{\mu} = c_{\lambda'\,\nu'}^{\mu'}$ [1, 2, 15], where $\mu'$, $\nu'$ and $\lambda'$ are the conjugate diagrams of $\mu$, $\nu$ and $\lambda$, respectively. Given $\lambda$, $\mu$ and $\nu$ like in the statement we have that $\lambda'$ and $\mu'/\nu'$ have the same number of columns of each length, hence from [13, Proposition 7, Appendix A], we have that $c_{\lambda'\,\nu'}^{\mu'} = 1$. So $c_{\lambda\,\nu}^{\mu} = 1$. $\square$

We can generalize the last proposition to KN tableaux, obtaining the following:

**Proposition 7.1.4.** *Let $T$ be a KN tableau of shape $\lambda$. Let $\mu/\nu$ be a skew diagram with same number of rows of each length as $T$. Then there is a unique KN skew tableau $S$ with shape $\mu/\nu$ that rectifies to $T$.*

*Proof.* If $T$ is the Yamanouchi tableau $K(\lambda)$ and $S \in \mathcal{KN}(\mu/\nu, n)$ rectifies to $K(\lambda)$, then, since $S$ and $K(\lambda)$ have the same number of cells, all entries of $S$ are unbarred, hence $S$ is a semistandard skew tableau. So, it follows from 7.1.3 that $S$ exists and is unique. If $T$ is not the Yamanouchi tableau, note that $T$ is crystal connected to $K(\lambda)$ and from [25, Theorem 6.3.8] we have that the SJDT slides commute with the action of the crystal operators. Consider $Y_\lambda$ the only tableau on the skew shape $\mu/\nu$ that rectifies to $K(\lambda)$, which exists due to 7.1.3. Since $S$ rectifies to $T$, which is crystal connected to $K(\lambda)$, and $Y_\lambda$ rectifies to $K(\lambda)$, then $S$ is crystal connected to $Y_\lambda$ and the path has same sequence of colours as the one from $T$ to $K(\lambda)$. Hence $S$ exists and is uniquely defined. $\square$

### 7.1.2 Combinatorial interpretation of the left hand side of the identity

Let's look at the left hand side of type $C_n$ Fu-Lascoux non-symmetric Cauchy kernel:

$$\prod_{1 \leq i,j \leq n} (1 - x_i y_j)^{-1} \prod_{1 \leq i \leq j \leq n} (1 - x_i y_j^{-1})^{-1}.$$

In the same spirit of the combinatorial proof of the Cauchy identity via RSK correspondence [13, Section 4], we can see that this is the generating function of all biwords in which the top column

only have positive letters and if $i$ is the top letter of a column then the bottom letter is less or equal than $\bar{i}$. If a word satisfies this, we say it satisfies the *LHS (left hand side) condition.* In same spirit of the biwords from $E_n^r$, from Subsection 4.4.1, we will consider our biletters ordered lexicographically, obtaining:

$$\begin{pmatrix} 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots & n \\ x_1 & \leq & x_i & x_{i+1} & \leq & x_{i'} & \ldots & x_{i''} \end{pmatrix}$$

Note that the top row separates the bottom row into weakly increasing sequence of words.

### 7.1.3 Combinatorial interpretation of the right hand side of the identity

Now, recall the right hand side of type $C_n$ Fu-Lascoux non-symmetric Cauchy kernel:

$$\prod_{1 \leq i < j \leq n} (1 - x_i x_j)^{-1} \sum_{v \in \mathbb{N}^n} \hat{\kappa}_v(x) \kappa_{-v}(y).$$

Our combinatorial interpretation of this side will be a tuple consisting of three entries: a biword, a reverse SSYT and a KN tableau. Our biword will have biletters ordered lexicographically, and all entries in $[n]$, and with the top entry bigger than the respective bottom entry. Before translating the key polynomial and the Demazure atom as generating functions of some sets of tableaux, we will start by doing an algebraic manipulation, and with that in mind we will need the following lemma:

**Lemma 7.1.5.** *[6, Proposition 2.3.4] Let $u, v \in B_n\lambda$. If $u \leq v$ then $-v \leq -u$.*

Hence:

$$\begin{aligned} \sum_{v \in \mathbb{N}^n} \hat{\kappa}_v(x) \kappa_{-v}(y) &= \sum_{v \in \mathbb{N}^n} \hat{\kappa}_v(x) \sum_{u \leq -v} \hat{\kappa}_u(y) \\ &= \sum_{v \in \mathbb{N}^n} \sum_{u \leq -v} \hat{\kappa}_v(x) \hat{\kappa}_u(y) \\ &= \sum_{u \in \mathbb{Z}^n} \sum_{\substack{v \in \mathbb{N}^n \\ u \leq -v}} \hat{\kappa}_v(x) \hat{\kappa}_u(y) \\ &= \sum_{u \in \mathbb{Z}^n} \sum_{\substack{v \in \mathbb{N}^n \\ v \leq -u}} \hat{\kappa}_v(x) \hat{\kappa}_u(y) \end{aligned}$$

Using the tableau criterion for the Bruhat order on Lemma 7.1.1, we can find a maximal $u'$ such that $u' \in \mathbb{N}^n \subset \mathbb{Z}^n$ and $u' \leq -u$ in the following way:

We will determine the entries of $u'$ recursively. The first entry of $u'$ is $-u_1$ if $-u_1$ is positive and $\lambda_n$ otherwise. For $i$ going from 2 to $n$, $u'_i$ is the minimal not yet used entry of $\lambda$ such that $\lambda_{(-u_1^\#, \ldots, -u_i^\#)} \supseteq \lambda_{(u'_1, \ldots, u'_i)}$. Hence we find a vector $u'$ such that

$$\sum_{\substack{v \in \mathbb{N}^n \\ v \leq -u}} \hat{\kappa}_v(x) = \kappa_{u'}(x).$$

**Example 7.1.6.** *Consider $-u = (1, -4, 4, 0, -3)$. Then $u' = (1, 0, 4, 3, 4)$.*

### 7.1.4   Reverse SSYT's

Let $\lambda$ be a partition and consider the type $A_{n-1}$ crystal $\mathbb{SSYT}(\lambda, n)$. For every SSYT in $\mathbb{SSYT}(\lambda, n)$, $\pi$-rotate it, obtaining a reverse skew SSYT. Now rectify each tableau, using the *jeu de taquin* for reverse SSYT, obtaining a straight reverse SSYT's. Now we have a crystal of reverse SSYT's. We call this crystal $\mathbb{RSSYT}(\lambda, n)$. Their reading word also follows the signature rule if applied backwards (swapping left and right).

**Example 7.1.7.** *The type $A_2$ crystal of reverse SSYT's $\mathfrak{B}^{(2,1,0)}$ is represented by the graph, where red represents $f_1$ and blue represents $f_2$.*
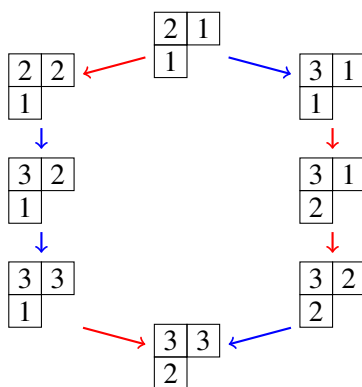


### 7.1.5   Another cocrystal of KN tableaux

Inspired by Lascoux' double crystal graph, we create a cocrystal, similar to the one in Section 4.4. Although, the cocrystal here is created based on RSK correspondence, instead of the dual RSK, meaning that the crystal operators on the cocrystal are *jeu de taquin* moves between consecutive rows.

Fix a partition $\lambda$ and a KN tableau $T \in \mathbb{KN}(\lambda, n)$. Starting from the bottom, label $n$ rows with the numbers 1 to $n$. In these rows we will place some cells (possibly 0) that form a skew tableau.

Now we create a map from $\mathbb{RSSYT}(\lambda, n)$ to cocrystal consisting of skew KN tableaux connected to $T$ via SJDT. We will also force all skew tableau to have columns of length 1, by moving all rows horizontally via SJDT. So all of skew tableau in the cocrystal will only have columns of length 1 and are SJDT connected to $T$. This forces the reverse of the row reading word (the word obtain after concatenating each row, from bottom to top) of our skew tableaux to be Knuth equivalent to $T$. This happens because the reverse of the row reading word of the skew tableaux will be its column reading word, and since the skew tableau rectifies to $T$, the column reading word of the skew tableau is Knuth related to the column reading word of $T$.

The highest weight of the cocrystal is a KN skew tableau Knuth equivalent to $T$, with exactly one cell per column, and with $\lambda_i$ cells in the $i$-th row. This tableau is unique thanks to Proposition 7.1.4. It is obtained from $T$ by doing some horizontal SJDT slides.

Given a tableau $Q$ in $\mathbb{RSSYT}(\lambda, n)$, if we can apply $f_i$ to it (i.e. $f_i(Q) \neq 0$) then the correspondent of $f_i(Q)$ in the cocrystal is obtained from the correspondent of $Q$ by taking a cell from the $i$-th row to the $i+1$-th row via SJDT. An arrow of colour $i$ (just like in the the crystal) connects both skew tableaux. Thus, we create a graph isomorphic (as a coloured graph) to $\mathbb{SSYT}(\lambda, n)$, in which each vertex is

Knuth equivalent to $T$ and the number of cells in the row $i$ is the weight of $i$ in the corresponding vertex on the crystal graph. This is done by Lascoux in [22] when $T$ is a SSYT. The extension of the cocrystal to type $C_n$ objects (even though it is still a type $A$ crystal) follows the same idea presented in Section 4.4.

Hence, given a KN tableau and a RSSYT with same shape, we have a unique skew tableau without two cells in the same column. Note that in Section 4.4 we had something similar to this, where given a skew SSYT $T$ we could get a biword, and $RSK^*$ sent that word to a pair consisting of two conjugated tableaux, one identifying the Knuth class of $T$ and the other identifying its skew shape.

**Example 7.1.8.** *Let $T = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array}$. Here we have, side by side, the crystal $\mathcal{RSSYT}(\lambda,n)$ and the cocrystal associated to it with respect to T, where the crystal operators are SJDT moves on consecutive rows. To the left of each tableau in the cocrystal we have a numbering of the rows, in order to be easier to keep track of its row lengths. They are isomorphic type $A_2$ crystal.*
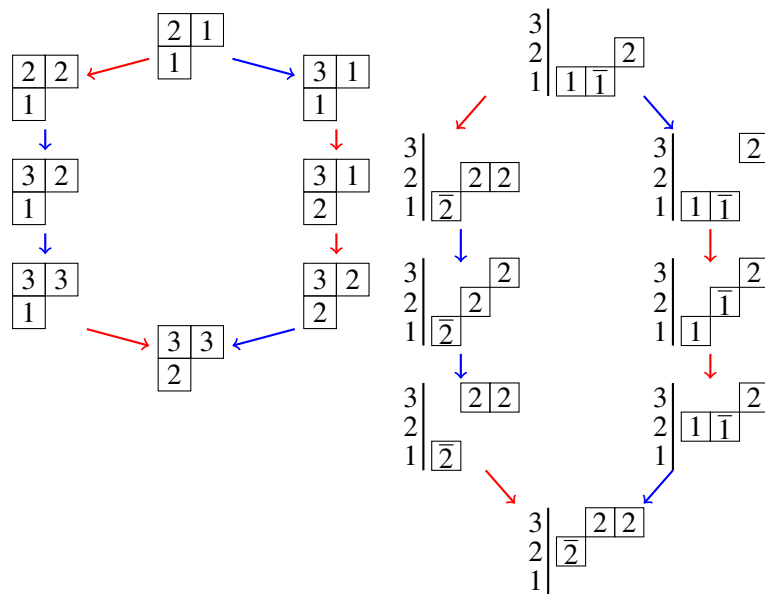


### 7.1.6 The algorithm

In this subsection we present an algorithm that given $L$, a biword with positive entries, where the top letter is strictly bigger then the bottom letter, a KN tableau $T$ with right key $K(u)$ and a reverse SSYT $Q$ in the Demazure crystal $\mathfrak{B}_{u'}$, returns a biword that satisfies the LHS condition. The creation of the biword is somewhat close to Sundaram's combinatorial bijection for the type $C_n$ symmetric Cauchy kernel in [41].

The algorithm relies heavily on the inverse of the Baker-Lecouvey insertion [5, 25]. The inverse algorithm accepts two possible inputs, each one having its output:

- Case 1: The input is a KN tableau $T$ and one of its corners. The algorithm returns a tableau $\widetilde{T}$ without that corner and a number $y \in [\pm n]$, such that if we insert $y$ in $\widetilde{T}$ we recover $T$.

- Case 2: The input is a KN tableau $T$ and a cell outside of $T$ such that the shape of $T$ together with that cell is a Ferrers diagram of shape $\lambda$. The inverse Baker-Lecouvey insertion returns a

tableau $\widetilde{T}$ of shape $\lambda$ and a number $y \in [\pm n]$, such that if we insert $y$ in $\widetilde{T}$ we recover $T$. Note that, in this case, Baker-Lecouvey insertion of $y$ in $\widetilde{T}$ incurs in a loss of cells.

The algorithm starts by looking to the smallest number in $L$ or $Q$, $x$. In each step of the algorithm there are two different cases, with the second case having two subcases:

- Case 1: If $x$ is in $Q$, delete the rightmost $x$ in $Q$, and, via the inverse of the Baker-Lecouvey insertion, we remove the entry in $T$ in the same position, obtaining a $y$, and we write $\genfrac{}{}{0pt}{}{x}{y}$ in the biword.

- Case 2: if $x$ is in $L$ and not in $Q$, we row insert in $Q$ the biggest $x'$ paired with $x$ in $L$, obtaining a tableau one cell bigger, and let $y$ be the letter that causes that change on the shape. Note that $y$ is uniquely determined by the reverse Baker-Lecouvey insertion the case where the insertion makes the tableau lose cells, hence its reverse makes the tableau gain cells, which is this case. So

  - Case 2a: if $-y \geq x$, we add $\genfrac{}{}{0pt}{}{x}{y}$ in the biword.

  - Case 2b: if $-y < x$ we add $\genfrac{}{}{0pt}{}{x}{\bar{x}} \genfrac{}{}{0pt}{}{x'}{x}$ to the biword (the second biletter may be misplaced, and needs to be corrected later). $Q, T$ returns to what they were in the beginning of this case, and $L$ loses the biletter $\genfrac{}{}{0pt}{}{x'}{x}$.

**Example 7.1.9.** *Let $T = \begin{array}{|c|c|}\hline 2 & 2 \\\hline 2 \\\cline{1-1}\end{array}$. Then $u = wt(K_+(T)) = (-1, 2, 0)$. Then $u' = (1, 0, 2)$. Let $Q = \begin{array}{|c|c|}\hline 3 & 1 \\\hline 2 \\\cline{1-1}\end{array} \in \mathfrak{B}_{u'}$ and let $L = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$.*
*We will now apply the algorithm:*

| Case applied | T | Q | L | biword |
|---|---|---|---|---|
| | $\begin{array}{|c|c|}\hline 2 & 2 \\\hline 2 \\\cline{1-1}\end{array}$ | $\begin{array}{|c|c|}\hline 3 & 1 \\\hline 2 \\\cline{1-1}\end{array}$ | $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ | $\emptyset$ |
| Case 1 | $\begin{array}{|c|}\hline 2 \\\hline \bar{1} \\\hline\end{array}$ | $\begin{array}{|c|}\hline 3 \\\hline 2 \\\hline\end{array}$ | $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ | $\begin{array}{c} 1 \\ 1 \end{array}$ |
| Case 2a, $y = \bar{1}$ | $\begin{array}{|c|c|}\hline 2 & 2 \\\hline 2 \\\cline{1-1}\end{array}$ | $\begin{array}{|c|c|}\hline 3 & 3 \\\hline 2 \\\cline{1-1}\end{array}$ | $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $\begin{array}{cc} 1 & 1 \\ 1 & \bar{1} \end{array}$ |
| Case 2a, $y = \bar{1}$ | $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\\hline 2 \\\cline{1-1}\end{array}$ | $\begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\\hline 2 \\\cline{1-1}\end{array}$ | $\emptyset$ | $\begin{array}{ccc} 1 & 1 & 1 \\ 1 & \bar{1} & \bar{1} \end{array}$ |
| Case 1, four times | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & \bar{1} & \bar{1} & 1 & \bar{2} & 2 & 2 \end{array}$ |

And now we shall see a small example in which we have the case 2*b*.

**Example 7.1.10.** *Let $T = \emptyset$, the empty tableau. Then $u = wt(K_+(T)) = (0) = u'$ and $Q = \emptyset$ and let $L = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.*

| Case applied | T | Q | L | biword |
|---|---|---|---|---|
| | $\emptyset$ | $\emptyset$ | $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ | $\emptyset$ |
| Case 2b, $y = \bar{1}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\begin{matrix} 2 & 3 \\ 2 & 2 \end{matrix}$ |

**Conjecture 7.1.11.** *The algorithm is a bijection between all tuples consisting of a biword with positive entries where the top letter is strictly bigger then the bottom letter, a KN tableau in $\hat{\mathfrak{B}}_u$ and a reverse SSYT in $\mathfrak{B}_{u'}$, and biwords that satisfies the LHS condition.*

If this conjecture is true, then we have a combinatorial proof of the type $C_n$ Fu-Lascoux non-symmetric Cauchy kernel:

$$\prod_{1 \leq i,j \leq n} (1 - x_i y_j)^{-1} \prod_{1 \leq i \leq j \leq n} (1 - x_i y_j^{-1})^{-1} = \prod_{1 \leq i < j \leq n} (1 - x_i x_j)^{-1} \sum_{v \in \mathbb{N}^n} \hat{\kappa}_v(x) \kappa_{-v}(y).$$

In fact, it is not even clear that the output of the algorithm always satisfies the LHS condition. Here we will not prove the conjecture, and we will not even prove that its output satisfies the LHS condition. However, when $L$ is empty, then the output of the algorithm satisfies the LHS condition:

**Lemma 7.1.12.** *Let $L$ be empty and $u \in \mathbb{Z}^n$. If $T \in \hat{\mathfrak{B}}_u$ and $Q \in \mathfrak{B}_{u'}$ LHS condition holds for the biword produced by the algorithm.*

*Proof.* Remember the double cocrystal in Subsection 7.1.5.

Given $T$ and $Q$, consider the skew tableau obtained from $T$ and $Q$. The biword obtained from $T$ and $Q$ is found in this skew tableau, because each biletter represents a row and an entry of each cell of the skew tableau. In particular, if $T = K(u)$ and $Q = K(u')$, the LHS condition holds because of the way $u'$ was constructed.

If $T \leq K(u)$ by entrywise comparison, the skew tableau associated to $T$ with $Q$-symbol $K(u')$ is, entrywise, smaller or equal than the skew tableau associated to $K(u)$ with $Q$-symbol $K(u')$. Hence, the LHS condition holds.

Now we assume $T = K(u)$. Fix an element $Q$ in $\mathfrak{B}_{u'}$. Since $Q \leq K(u')$, the skew tableau associated to $Q$ is obtained from the skew tableau with $Q$-symbol $K(\lambda)$ by pushing less cells to higher rows than the skew tableau with $Q$-symbol $K(u')$, which satisfies. Hence, the LHS condition holds fro the biword obtained from $K(u)$ and $Q \in \mathfrak{B}_{u'}$.

So, LHS condition holds for the biword obtained from $T \in \hat{\mathfrak{B}}_u$ and $Q \in \mathfrak{B}_{u'}$.

$\square$

## 7.2 Further questions

### 7.2.1 Types $B$ and $D$ Kashiwara-Nakashima tableaux

In [20], Kashiwara-Nakashima also presented two other families of tableaux, compatible with a type $B$ and type $D$ crystal structure. Lecouvey, in [26], explored the crystal of these families of tableaux

and endowed them with a plactic monoid compatible with a RSK correspondence and an insertion algorithm. In fact, for type *B* he even introduced a *jeu de taquin*. Hence, a natural question is to ask whether the algorithms developed in this thesis, to compute right or left keys with or without *jeu de taquin*, are applicable to these types.

### 7.2.2 Type $C_n$ semi skyline augmented filling

In [32], Mason showed that Demazure atoms are specializations of non-symmetric Macdonald polynomials of type $A_{n-1}$ with $q = t = 0$. This allowed us to use the shapes of semi-skyline augmented fillings, in the combinatorial formula of non-symmetric Macdonald polynomials [14], which are in weight preserving bijection with semi standard Young tableaux, to detect the right keys. It would be interesting to obtain a similar object for a KN tableau in type $C_n$. For example, semi-skyline augmented fillings have been instrumental to obtain a RSK type bijective proof [3] for the Lascoux non-symmetric Cauchy identity in type $A_{n-1}$ [22]. Such a generalization of skyline fillings for type *C* could also lead to a combinatorial formula for some specialization of nonsymmetric Macdonald polynomials in type $C_n$.

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