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Spontaneous magnetization in high-density quark matter

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It is shown that spontaneous magnetization occurs due to the anomalous magnetic moments of quarks in high-density quark matter under the tensor-type four-point interaction. The spin polarized condensate for each flavor of quark appears at high baryon density, which leads to the spontaneous magnetization due to the anomalous magnetic moments of quarks. The implications for the strong magnetic field in compact stars is discussed.

Subject Index D30

1. Introduction

One recent area of interest in the physics governed by quantum chromodynamics (QCD) is to clarify the phase structure of QCD with respect to the temperature, baryon chemical potential or baryon number density, external magnetic field, and so on [1]. In the region with high temperature and low density, the lattice QCD simulation gives many insights on the QCD world. However, as is well known, in the region of large quark chemical potential, there are some problems to be solved in order to calculate the physical quantities definitely. However, it is expected that there are still many interesting phenomena in the region of low temperature and large chemical potential. In particular, it is expected that there are many exotic phases such as the two-flavor color superconducting phase, the color–flavor locked phase [2–4], the quarkyonic phase [5], the phase with the inhomogeneous condensate [6], and so forth.

In heavy-ion collision experiments such as the relativistic heavy-ion collider (RHIC) experiments at Brookhaven National Laboratory, it is believed that the quark–gluon phase is realized apart from the hadronic phase. Thus, the quark matter may be created and the extreme states of QCD with finite temperature and density may be realized in the heavy-ion collision experiments. On the other hand, the high-density hadronic phase or quark phase may be realized in the inner core of compact star objects such as neutron stars, magnetars, and quark stars, if they exist. In the core of these compact stars, it is expected that the hadron or quark phase exists at low temperature and high baryon density.

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For example, quark matter is considered to exist in the core of pulsars with two solar masses in compact stars, and in the case of heavy ion collisions [7]. Therefore, the investigation of quark matter in the region of low temperature and large quark chemical potential is a very interesting and important subject in order to understand the whole world governed by QCD.

It is further known that neutron stars, especially the so-called magnetars [8–10], have a strong magnetic field. However, the origin of the strong magnetic field is not so clear. It has been pointed out that the strong magnetic field may be created if the quark liquid exists in the core of the compact stars [11]. Further, the possibility of quark spin polarization in the high-density quark matter has been investigated when the pseudovector-type interaction between quarks exists [12]. Under the pseudovector-type interaction, it was shown that the spin polarized condensate appears, which leads to ferromagnetism in quark matter [13]. However, the spin polarized condensate only appears in a narrow region of the quark chemical potential.

In addition to the pseudovector-type interaction in the extended Nambu–Jona-Lasinio (NJL) model in Eq. (1.1), it is possible to include the other four-point interactions retaining chiral symmetry, namely, tensor-type four-point interactions such as $\mathcal{L}_T$:

$$\begin{aligned}
\mathcal{L} &= i\bar{\psi}\gamma^{\mu}\partial_{\mu} + \mathcal{L}_S + \mathcal{L}_V + \mathcal{L}_T, \\
\mathcal{L}_S &= -G_S\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right], \\
\mathcal{L}_V &= -G_V\left[\left(\bar{\psi}\gamma^{\mu}\vec{\tau}\psi\right)^2 + \left(\bar{\psi}\gamma_5\gamma^{\mu}\vec{\tau}\psi\right)^2\right], \\
\mathcal{L}_T &= -G_T\left[\left(\bar{\psi}\gamma^{\mu}\gamma^{\nu}\vec{\tau}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\gamma^{\mu}\gamma^{\nu}\psi\right)^2\right].
\end{aligned} \tag{1.1}$$

As for the tensor-type four-point interaction, this interaction term was also introduced to investigate meson spectroscopy, in particular for vector and axial-vector mesons [14]. As another application, the dynamic properties of vector mesons were investigated in the extended NJL model including the tensor-type interaction [15]. As for the spin polarized condensate, it has been shown that the tensor-type interaction between quarks leads to the spin polarized phase in high-density quark matter [16–18], while the spin polarization owing to the pseudovector-type interaction disappears at high baryon density. The present authors have shown the possibility of the quark spin polarized phase in quark matter at high baryon density against the two-flavor color superconducting phase [19] and the color–flavor locked phase [20] under both the quark pairing interaction and tensor-type four-point interaction in the NJL-type model [21–23]. However, the magnetic feature has not been considered in tensor-type interactions until now.

In this paper, the magnetic features are investigated under the existence of the spin polarized condensate in the NJL model with the tensor-type four-point interactions. It will be shown that the anomalous magnetic moments of quarks play an essential role, the existence of which leads to the spontaneous magnetization of quark matter. As for the anomalous magnetic moments of quarks, there already exists much work. For example, in the massless QED, it has been shown that the magnetic catalysis of chiral symmetry breaking leads to the anomalous magnetic moment dynamically [24]. In a preceding study, it has been indicated that chiral symmetry breaking leads to the anomalous magnetic moments of massless quarks by using the NJL model [25]. In particular, in the case of the QCD-inspired nonlocal NJL model with nonlocal interaction between quarks, it has been shown that the derived anomalous magnetic moments are compatible with those derived by the constituent quark model. Further, in the NJL model with one-gluon exchange interaction, it has been shown that the values of the anomalous magnetic moments of quarks are consistent with those derived by using the

SU(6) quark model [26]. Recently, the effects of the anomalous magnetic moments of quarks in hot quark matter have also been investigated in [27], in which the inverse magnetic catalysis is reported by using the values of the anomalous magnetic moments of quarks obtained by the constituent quark model. Here, we will therefore use the values of anomalous magnetic moments of quarks described in [28], which are compatible with the values obtained by the constituent quark model. Also, the implications for the magnetic field of compact stars such as neutron stars with quark matter, namely hybrid stars, will be discussed.

This paper is organized as follows: In the next two sections, the thermodynamic potential under the external magnetic field is given with and without the spin polarized condensate. In Sect. 4, an approximate expression for the thermodynamic potential with a small magnetic field is derived. In Sect. 5, it is shown that the spontaneous magnetization does not appear in the case of no anomalous magnetic moments of quarks. In Sect. 6, the anomalous magnetic moments of quarks are introduced within the mean field approximation. As a result, spontaneous magnetization occurs in quark matter in the region of high baryon density due to both the spontaneous spin polarization and the anomalous magnetic moments of quarks. In Sect. 7, the implications for hybrid compact stars are discussed briefly and it is shown that a strong magnetic field in the surface of the compact stars may appear. The last section is devoted to a summary and concluding remarks.

2. Thermodynamic potential under external magnetic field

We consider the two-flavor case. Let us start from the following Lagrangian density with chiral symmetry:

$$\mathcal{L}_0 = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{G}{4}\Big(\left(\bar{\psi}\gamma^\mu\gamma^\nu\vec{\tau}\psi\right)\left(\bar{\psi}\gamma_\mu\gamma_\nu\vec{\tau}\psi\right) + \left(\bar{\psi}i\gamma_5\gamma^\mu\gamma^\nu\psi\right)\left(\bar{\psi}i\gamma_5\gamma_\mu\gamma_\nu\psi\right)\Big), \tag{2.1}$$

where $\vec{\tau}$ represents the flavor SU(2) generator.

In previous study, we have found that a spin polarized condensate $F_3 = -G\langle\bar{\psi}\Sigma_3\tau_3\psi\rangle$ may be realized at high baryon density, where $\Sigma_3 = -i\gamma^1\gamma^2$ is the spin operator. Thus, the Lagrangian density under the mean field approximation is obtained as

$$\begin{aligned} \mathcal{L}_{\mathrm{MF}} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - F_3\bar{\psi}\Sigma_3\tau_3\psi - \frac{F^2}{2G}, \\ F_3 &= -G\langle\bar{\psi}\Sigma_3\tau_3\psi\rangle = F\tau_f, \quad \Sigma_3 = -i\gamma^1\gamma^2 = \begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}, \end{aligned} \tag{2.2}$$

where $\tau_f = 1$ for u-quark ($f = u$) and $\tau_f = -1$ for d-quark ($f = d$) denote the eigenvalues of τ_3. Here, σ_3 is the third component of the Pauli spin matrices.

Hereafter, let us consider a system that is subject to an external magnetic field B along the z-axis. The Lagrangian density is recast into

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - F_3\bar{\psi}\Sigma_3\tau_3\psi - \frac{F^2}{2G}, \tag{2.3}$$

where D_μ represents the covariant derivative:

$$D_\mu = \partial_\mu + iQA_\mu, \quad A_\mu = \left(0,\ \frac{By}{2},\ -\frac{Bx}{2},\ 0\right) = (0,\ -\boldsymbol{A}). \tag{2.4}$$

Here, for up (down) quark, $Q = 2e/3$ $(-e/3)$ where e is the elementary charge. Because we investigate quark matter at finite density, the Hamiltonian density with quark chemical potential μ is obtained as

$$\begin{aligned}\mathcal{H}-\mu\mathcal{N}&=\bar{\psi}\left(-i\boldsymbol{\gamma}\cdot(\nabla-iQ\boldsymbol{A})-\mu\gamma^0\right)\psi+F_3\bar{\psi}\Sigma_3\tau_3\psi+\frac{F^2}{2G}\\&=\psi^\dagger(h-\mu)\psi+\frac{F^2}{2G},\end{aligned}\tag{2.5}$$

where $\mathcal{N}$ represents the quark-number operator. Here, h can be expressed as

$$\begin{aligned}h&=\boldsymbol{\alpha}\cdot\left(\hat{\boldsymbol{p}}-Q\boldsymbol{A}\right)+F_3\tau_3\beta\Sigma_3\\&=\alpha_z\hat{p}_z+\alpha_x\hat{P}_x+\alpha_y\hat{P}_y+F_3\tau_3\beta\Sigma_3,\end{aligned}\tag{2.6}$$

where, by using the Dirac representation of the Dirac gamma matrices,

$$\begin{aligned}&\alpha_i=\gamma^0\gamma^i=\begin{pmatrix}0&\sigma_i\\\sigma_i&0\end{pmatrix},\quad\beta=\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\\&\hat{p}_i=-i\frac{\partial}{\partial x^i},\quad\hat{P}_x=\hat{p}_x+\frac{QB}{2}y,\quad\hat{P}_y=\hat{p}_y-\frac{QB}{2}x.\end{aligned}\tag{2.7}$$

Hereafter, let us consider two cases, namely $F>0$ and $F=0$, respectively.

2.1. $F>0$ case

The Dirac equation is written by $i\partial\psi/\partial t=h\psi=E\psi$, namely

$$\begin{pmatrix}F_3\tau_3\sigma_z-E&\hat{p}_z\sigma_z+\hat{P}_x\sigma_x+\hat{P}_y\sigma_y\\\hat{p}_z\sigma_z+\hat{P}_x\sigma_x+\hat{P}_y\sigma_y&-F_3\tau_3\sigma_z-E\end{pmatrix}\begin{pmatrix}\phi\\\varphi\end{pmatrix}=0,\tag{2.8}$$

where ϕ and φ are two-component spinors. Eliminating φ and noting that $[\hat{P}_x,\hat{P}_y]=iQB$, we obtain the following equation:

$$\begin{aligned}&\Bigg[\left\{-E+\frac{E}{E^2-F^2}\left(\hat{p}_z^2+\hat{P}_x^2+\hat{P}_y^2\right)-\frac{F}{E^2-F^2}QB\right\}\\&\quad+\sigma_z\left\{F-\frac{F}{E^2-F^2}\left(\hat{p}_z^2-\hat{P}_x^2-\hat{P}_y^2\right)-\frac{E}{E^2-F^2}QB\right\}\\&\quad-\frac{2F}{E^2-F^2}\hat{p}_z\left(\sigma_x\hat{P}_x+\sigma_y\hat{P}_y\right)\Bigg]\phi=0,\\&F_3\tau_3=F\tau_f\tau_3=\begin{pmatrix}F&0\\0&F\end{pmatrix}=F\mathbf{1},\quad Q=\begin{cases}\frac{2}{3}e&\text{for up quark}\\-\frac{1}{3}e&\text{for down quark}\end{cases},\end{aligned}\tag{2.9}$$

where $\mathbf{1}$ is the identity matrix in the isospin space. Let us consider the $Q>0$ case, that is, the case of the up quark. We introduce new operators instead of $\hat{P}_x$ and $\hat{P}_y$ as

$$a=\frac{1}{\sqrt{2QB}}\left(\hat{P}_x+i\hat{P}_y\right),\quad a^\dagger=\frac{1}{\sqrt{2QB}}\left(\hat{P}_x-i\hat{P}_y\right),\tag{2.10}$$

where the commutation relation $[a,\ a^\dagger,] = 1$ is satisfied. Then, Eq. (2.9) is recast into

$$\left[\left\{-E + \frac{E}{E^2 - F^2}\left(\hat{p}_z^2 + QB\left(aa^\dagger + a^\dagger a\right) - \frac{F}{E^2 - F^2}QB\right\}\right.\right.$$
$$+ \sigma_z\left\{F - \frac{F}{E^2 - F^2}\left(\hat{p}_z^2 - QB\left(aa^\dagger + a^\dagger a\right) - \frac{E}{E^2 - F^2}QB\right\}$$
$$\left. - \frac{2F}{E^2 - F^2}\hat{p}_z\sqrt{\frac{QB}{2}}\left(\sigma_x\left(a + a^\dagger\right) - i\sigma_y\left(a - a^\dagger\right)\right)\right]\phi = 0. \tag{2.11}$$

Here, expressing the two-component spinor ϕ as $\phi = {}^t(\phi_1, \phi_2)$, the above equation can be expressed as

$$\left[X_0 + Y_0 + (X_1 + Y_1)\, a^\dagger a\right]\phi_1 - Ca^\dagger\phi_2 = 0,$$
$$-Ca\phi_1 + \left[X_0 - Y_0 + (X_1 - Y_1)\, a^\dagger a\right]\phi_2 = 0, \tag{2.12}$$

and, eliminating ϕ_2, the following equation can be derived from Eq. (2.11):

$$\left[\left[X_0 + Y_0 + (X_1 + Y_1)\, a^\dagger a\right] - \left[X_0 - Y_0 + (X_1 - Y_1)\left(a^\dagger a - 1\right)\right]^{-1} C^2 a^\dagger a\right]\phi_1 = 0, \tag{2.13}$$

where we define the following quantity for simplicity:

$$X_0 = -E + \frac{E\hat{p}_z^2}{E^2 - F^2} + \frac{E - F}{E^2 - F^2}QB, \quad Y_0 = F - \frac{F\hat{p}_z^2}{E^2 - F^2} - \frac{E - F}{E^2 - F^2}QB,$$
$$X_1 = \frac{2EQB}{E^2 - F^2}, \quad Y_1 = \frac{2FQB}{E^2 - F^2}, \quad C = \frac{2F\hat{p}_z\sqrt{2QB}}{E^2 - F^2}. \tag{2.14}$$

Here, $\hat{p}_z$ and $a^\dagger a$ should be replaced by their eigenvalues p_z and ν $(= 0, 1, 2, \ldots)$. Further, since the case $Q > 0$ is treated, namely the case of the up quark, so $F_3 = F$. Then, from the coefficient of ϕ_1 being 0, we can get the eigenvalue of the Dirac equation, E, which is expressed as $\epsilon_{p_z,\nu}^{\text{flavor}}$, as

$$\epsilon_{p_z,\nu}^{\text{up}} = \sqrt{\left(F \pm \sqrt{2QB\nu}\right)^2 + p_z^2} \quad (\text{for}\ \ \nu = 1,\ 2, \ldots) \tag{2.15}$$

with $Q = 2e/3$. It should be noted here that it is necessary to pay special attention to the case $\nu = 0$ because $a\phi(\nu = 0) = a|\nu = 0\rangle = 0$ is satisfied. For $\nu = 0$, from $\phi_1 = |\nu = 0\rangle$, Eq. (2.11) is reduced to

$$X_0 + Y_0 = 0, \quad \phi_2 = 0 \tag{2.16}$$

because $a\phi_1 = a|\nu = 0\rangle = 0$. From $X_0 + Y_0 = 0$, which leads to a quadratic equation with respect to E, we have

$$\epsilon_{p_z,\nu=0}^{\text{up}} = \sqrt{F^2 + p_z^2} \tag{2.17}$$

for the positive energy solution. Thus, for $\nu = 0$, the solution only appears once, and the energy corresponding to $\nu = 0$ is not degenerate.

In the same way, we can get the eigenvalue E in the case of the down quark where $Q < 0$. Instead of (2.10), we define new operators as

$$a = \frac{1}{\sqrt{-2QB}}\left(\hat{P}_x - i\hat{P}_y\right), \quad a^\dagger = \frac{1}{\sqrt{-2QB}}\left(\hat{P}_x + i\hat{P}_y\right), \tag{2.18}$$

where $[a,\ a^\dagger] = 1$ is satisfied. Then, we obtain the equation instead of (2.11) as

$$\begin{aligned}&\left[\left\{-E + \frac{E}{E^2-F^2}\left(\hat{p}_z^2 - QB\left(aa^\dagger + a^\dagger a\right) - \frac{F}{E^2-F^2}QB\right\}\right.\right.\\ &\quad + \sigma_z\left\{F - \frac{F}{E^2-F^2}\left(\hat{p}_z^2 + QB\left(aa^\dagger + a^\dagger a\right) - \frac{E}{E^2-F^2}QB\right\}\right.\\ &\quad \left. - \frac{2F}{E^2-F^2}\hat{p}_z\sqrt{-\frac{QB}{2}}\left(\sigma_x\left(a+a^\dagger\right) + i\sigma_y\left(a-a^\dagger\right)\right)\right]\phi = 0.\end{aligned} \tag{2.19}$$

Eliminating ϕ_1, which is the upper component of the two-component spinor ϕ, we obtain the following equation:

$$\left[\left[\tilde{X}_0 + \tilde{Y}_0 - (\tilde{X}_1 + \tilde{Y}_1)a^\dagger a\right] - \left[\tilde{X}_0 - \tilde{Y}_0 - \left(\tilde{X}_1 - \tilde{Y}_1\right)\left(a^\dagger a - 1\right)\right]^{-1}\tilde{C}^2 a^\dagger a\right]\phi_2 = 0, \tag{2.20}$$

where

$$\begin{aligned}&\tilde{X}_0 = -E + \frac{E\hat{p}_z^2}{E^2-F^2} - \frac{E+F}{E^2-F^2}QB, \quad \tilde{Y}_0 = -F + \frac{F\hat{p}_z^2}{E^2-F^2} + \frac{E+F}{E^2-F^2}QB,\\ &\tilde{X}_1 = \frac{2EQB}{E^2-F^2}, \quad \tilde{Y}_1 = -\frac{2FQB}{E^2-F^2}, \quad \tilde{C} = -\frac{2F\hat{p}_z\sqrt{-2QB}}{E^2-F^2}.\end{aligned} \tag{2.21}$$

Thus, we obtain the eigenvalue for down quark, $\epsilon^{\mathrm{down}}_{p_z,\nu}$, by replacing $\hat{p}_z$ and $a^\dagger a$ into their eigenvalues p_z and ν $(= 0, 1, 2, \ldots)$:

$$\begin{aligned}&\epsilon^{\mathrm{down}}_{p_z,\nu} = \sqrt{\left(F \pm \sqrt{-2QB\nu}\right)^2 + p_z^2} \qquad (\text{for}\ \ \nu = 1,\ 2, \ldots)\\ &\epsilon^{\mathrm{down}}_{p_z,\nu=0} = \sqrt{F^2 + p_z^2} \qquad (\text{for}\ \ \nu = 0)\end{aligned} \tag{2.22}$$

with $F_3 = -F$ and $Q = -e/3$.

Thus, the single-particle energy of quarks with flavor $f = u$ or d for up and down quarks can be expressed as

$$\epsilon^f_{p_z,\nu,\eta} = \sqrt{\left(F + \eta\sqrt{2|Q_f|B\nu}\right)^2 + p_z^2} \quad \begin{cases}\nu = 0,\ 1,\ 2, \ldots & \text{for}\ \ \eta = 1,\\ \nu = 1,\ 2, \ldots & \text{for}\ \ \eta = -1,\end{cases} \tag{2.23}$$

with $Q_u = 2e/3$, $Q_d = -e/3$, and $\eta = \pm 1$.

The thermodynamic potential Φ can be expressed in terms of the vacuum expectation values of the Hamiltonian $\hat{H}$ and the particle number operator $\hat{N}$, in which the quarks occupy the energy levels from that with the lowest energy to that with the Fermi energy. Thus, we obtain Φ as

$$\Phi = \frac{1}{V}\langle \hat{H} - \mu\hat{N}\rangle = 3\cdot\frac{1}{V}\sum_{p_z,\nu\ \left(\epsilon^f_{p_z,\nu,+}\le\mu\right)}\left[\left(\epsilon^u_{p_z,\nu,+}-\mu\right)+\left(\epsilon^d_{p_z,\nu,+}-\mu\right)\right]$$
$$+3\cdot\frac{1}{V}\sum_{p_z,\nu\ \left(\epsilon^f_{p_z,\nu,-}\le\mu\right)}\left[\left(\epsilon^u_{p_z,\nu,-}-\mu\right)+\left(\epsilon^d_{p_z,\nu,-}-\mu\right)\right]+\frac{F^2}{2G}, \tag{2.24}$$

where V represents the volume under consideration and the factor 3 represents the color degree of freedom. Here, it should be noted that the sum with respect to p_z and ν can be regarded as the following:[1]

$$\frac{1}{V}\sum_{p_z,\nu} = \int\frac{dp_z}{2\pi}\cdot\frac{|Q_f|B}{2\pi}\sum_{\nu=\nu^{f(\eta)}_{\min}}^{\nu^{f(\eta)}_{\max}}. \tag{2.25}$$

Thus, the thermodynamic potential can be obtained finally as

$$\Phi = 3\int_{-p_F}^{p_F}\frac{dp_z}{2\pi}\sum_{f=u,d,\eta=\pm}\frac{|Q_f|B}{2\pi}\sum_{\nu=\nu^{f(\eta)}_{\min}}^{\nu^{f(\eta)}_{\max}}\left[\sqrt{\left(F+\eta\sqrt{2|Q_f|B\nu}\right)^2+p_z^2}-\mu\right]+\frac{F^2}{2G}, \tag{2.26}$$

where p_F, $\nu^{f(\eta)}_{\min}$, and $\nu^{f(\eta)}_{\max}$ are determined by the condition $\epsilon^f_{p_z,\nu,\eta}\le\mu$.

2.2. $F = 0$ case

Next, let us consider the $F = 0$ case. For $Q > 0$ and $Q < 0$, Eqs. (2.11) and (2.19) are valid with $F = 0$. Thus, the following equation for the two-component spinor ϕ is obtained:

$$\left[\left\{-E+\frac{1}{E}\left(\hat{p}_z^2+|Q|B\left(2a^\dagger a+1\right)\right)\right\}\mp\sigma_z\frac{1}{E}|Q|B\right]\phi = 0, \tag{2.27}$$

where the upper (lower) sign in front of σ_z corresponds to the case $Q > 0$ ($Q < 0$). Replacing $\hat{p}_z$ and $a^\dagger a$ by their eigenvalues and $\phi = {}^t(\phi_1, \phi_2)$, the above equation is written as

$$\begin{aligned}\left(-E^2+p_z^2+2|Q|B\nu\right)\phi_{\substack{1\\2}} &= 0,\\ \left(-E^2+p_z^2+2|Q|B\,(\nu+1)\right)\phi_{\substack{2\\1}} &= 0.\end{aligned} \tag{2.28}$$

Thus, the single-particle energy E is obtained as

$$E = \begin{cases}\sqrt{p_z^2+2|Q|B\nu}\\ \sqrt{p_z^2+2|Q|B\,(\nu+1)}\end{cases} \tag{2.29}$$

[1] Usually, $\frac{1}{V}\sum_{\boldsymbol{p}}$ is replaced with $\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}$. Here, $p_x^2+p_y^2$ can be regarded as $2|Q|B\nu$ $(= p_\perp^2)$. Thus, the following correspondence may be understood: $\iint dp_x dp_y = \int 2\pi p_\perp dp_\perp = 2\pi\int\sqrt{2|Q|B\nu}\frac{\sqrt{2|Q|B}}{2\sqrt{\nu}}d\nu = 2\pi|Q|B\int d\nu = 2\pi|Q|B\sum_\nu$.

This result is included in Eq. (2.23). Thus, the thermodynamic potential Φ_0 can be calculated as

$$\Phi_0 = 3\sum_{f=u,d}\int_{-p_0}^{p_0}\frac{dp_z}{2\pi}\cdot\frac{|Q_f|B}{2\pi}\left[\sum_{\nu=0}^{\nu_{\rm M}^f}\left(\sqrt{p_z^2+2|Q_f|B\nu}-\mu\right)+\sum_{\nu=0}^{\nu_{\rm M}^f-1}\left(\sqrt{p_z^2+2|Q_f|B(\nu+1)}-\mu\right)\right]$$
$$= 3\sum_{f=u,d}\int_{-p_0}^{p_0}\frac{dp_z}{(2\pi)^2}|Q_f|B\,(|p_z|-\mu)+3\sum_{f=u,d}\int_{-p_0}^{p_0}\frac{dp_z}{(2\pi)^2}2|Q_f|B\sum_{\nu=1}^{\nu_{\rm M}^f}\left[\sqrt{p_z^2+2|Q_f|B\nu}-\mu\right], \tag{2.30}$$

where p_0 and $\nu_{\rm M}^f$ should be determined later.

3. The thermodynamic potential in three cases: $F > \mu$, $0 < F < \mu$, and $F = 0$

First, let us integrate out with respect to p_z in the thermodynamic potential (2.26) and (2.30). The condition $(F+\eta\sqrt{2|Q_f|B\nu})^2+p_z^2\le\mu^2$ gives the range of integration with respect to p_z. Namely,

$$-\sqrt{\mu^2-\left(F+\eta\sqrt{2|Q_f|B\nu}\right)^2}\le p_z\le\sqrt{\mu^2-\left(F+\eta\sqrt{2|Q_f|B\nu}\right)^2}\ (\equiv p_F)\,. \tag{3.1}$$

By using the following integration formula,

$$\int dp_z\sqrt{(F+X)^2+p_z^2}=\frac{p_z}{2}\sqrt{(F+X)^2+p_z^2}+\frac{(F+X)^2}{2}\ln\left(p_z+\sqrt{(F+X)^2+p_z^2}\right), \tag{3.2}$$

we can carry out the integration with respect to p_z in Eq. (2.26), which leads to

$$\Phi=\frac{3}{2\pi}\sum_{f=u,d;\eta=\pm}\frac{|Q_f|B}{2\pi}\sum_{\nu=\nu_{f,m}^{(\eta)}}^{\nu_{f,M}^{(\eta)}}\left[-\mu\sqrt{\mu^2-\left(F+\eta\sqrt{2|Q_f|B\nu}\right)^2}+\left(F+\eta\sqrt{2|Q_f|B\nu}\right)^2\ln\frac{\mu+\sqrt{\mu^2-\left(F+\eta\sqrt{2|Q_f|B\nu}\right)^2}}{F+\eta\sqrt{2|Q_f|B\nu}}\right]+\frac{F^2}{2G}$$
$$=\frac{F^2}{2G}+\frac{3}{8\pi^2}\sum_{f=u,d;\eta=\pm}2|Q_f|B\sum_{\nu=\nu_{f,m}^{(\eta)}}^{\nu_{f,M}^{(\eta)}}g_\eta\left(2|Q_f|B\nu\right), \tag{3.3}$$

where $\nu_{f,m}^{(\eta)}$ and $\nu_{f,M}^{(\eta)}$ are determined by the condition

$$\mu^2-\left(F+\eta\sqrt{2|Q_f|B\nu}\right)^2\ge 0, \tag{3.4}$$

which guarantees that p_z is real. Here, $g_\eta(x)$ is defined by

$$g_\eta(x) = -\mu\sqrt{\mu^2 - \left(F + \eta\sqrt{x}\right)^2} + \left(F + \eta\sqrt{x}\right)^2 \ln \frac{\mu + \sqrt{\mu^2 - \left(F + \eta\sqrt{x}\right)^2}}{F + \eta\sqrt{x}}. \tag{3.5}$$

3.1. $F \geq \mu$ case

For $\eta = 1$, the condition (3.4) is not satisfied for any F. On the other hand, for $\eta = -1$, if $F < \sqrt{2|Q_f|B\nu}$, the condition (3.4) gives $\sqrt{2|Q_f|B\nu} \leq \mu + F$. If $F > \sqrt{2|Q_f|B\nu}$, the condition (3.4) gives $\sqrt{2|Q_f|B\nu} \geq F - \mu$. Thus, we summarize the condition for ν as follows:

$$0 \neq \nu_{f,m}^{(-)} \equiv \left[\frac{(F-\mu)^2}{2|Q_f|B}\right] + 1 \leq \nu \leq \left[\frac{(F+\mu)^2}{2|Q_f|B}\right] \equiv \nu_{f,M}^{(-)}, \tag{3.6}$$

where $[\cdots]$ represents the Gauss symbol. Thus, the thermodynamic potential can be expressed as

$$\begin{aligned}\Phi &= \frac{3}{2\pi}\sum_{f=u,d}\frac{|Q_f|B}{2\pi}\sum_{\nu=\nu_{f,m}^{(-)}}^{\nu_{f,M}^{(-)}}\left[-\mu\sqrt{\mu^2 - \left(F - \sqrt{2|Q_f|B\nu}\right)^2}\right. \\ &\quad \left. + \left(F - \sqrt{2|Q_f|B\nu}\right)^2 \ln\frac{\mu + \sqrt{\mu^2 - \left(F - \sqrt{2|Q_f|B\nu}\right)^2}}{F - \sqrt{2|Q_f|B\nu}}\right] + \frac{F^2}{2G} \\ &= \frac{3}{8\pi^2}\sum_{f=u,d}2|Q_f|B\sum_{\nu=\nu_{f,m}^{(-)}}^{\nu_{f,M}^{(-)}} g_-\left(2|Q_f|B\nu\right) + \frac{F^2}{2G}\end{aligned} \tag{3.7}$$

with (3.6).

3.2. $F < \mu$ case

For $\eta = 1$, the condition (3.4) gives the condition $0 \leq \sqrt{2|Q_f|B\nu} \leq \mu - F$. On the other hand, for $\eta = -1$, if $F < \sqrt{2|Q_f|B\nu}$, the condition (3.4) gives $F \leq \sqrt{2|Q_f|B\nu} \leq \mu + F$. If $F > \sqrt{2|Q_f|B\nu}$, the condition (3.4) gives $0 \leq \sqrt{2|Q_f|B\nu} \leq F$. Thus, we summarize the condition for ν as follows:

$$\begin{aligned}\nu_{f,m}^{(+)} &\equiv 0 \leq \nu \leq \left[\frac{(\mu-F)^2}{2|Q_f|B}\right] \equiv \nu_{f,M}^{(+)} \quad \text{for} \quad \eta = 1, \\ \nu_{f,m}^{(-)} &\equiv 1 \leq \nu \leq \left[\frac{(\mu+F)^2}{2|Q_f|B}\right] \equiv \nu_{f,M}^{(-)} \quad \text{for} \quad \eta = -1.\end{aligned} \tag{3.8}$$

Thus, the thermodynamic potential is (3.3) with (3.8) for $\eta = \pm$. Namely,

$$\Phi = \frac{F^2}{2G} + \frac{3}{8\pi^2}\sum_{f=u,d;\eta=\pm}2|Q_f|B\sum_{\nu=\nu_{f,m}^{(\eta)}}^{\nu_{f,M}^{(\eta)}} g_\eta\left(2|Q_f|B\nu\right). \tag{3.9}$$

3.3. $F = 0$ case

Next, let us consider the $F = 0$ case. Thus, the thermodynamic potential Φ_0 can be given in (2.30), which we show again:

$$\begin{aligned}
\Phi_0 &= 3\sum_{f=u,d}\int_{-\mu}^{\mu}\frac{dp_z}{(2\pi)^2}|Q_f|B\left(|p_z|-\mu\right)\\
&\quad+3\sum_{f=u,d}\int_{-\sqrt{\mu^2-2|Q_f|B\nu}}^{\sqrt{\mu^2-2|Q_f|B\nu}}\frac{dp_z}{(2\pi)^2}2|Q_f|B\sum_{\nu=1}^{\nu_{\rm M}^f}\left[\sqrt{p_z^2+2|Q_f|B\nu}-\mu\right]\\
&=-\frac{3}{4\pi^2}\sum_{f=u,d}|Q_f|B\mu^2\\
&\quad+\frac{3}{4\pi^2}\sum_{f=u,d}2|Q_f|B\sum_{\nu=1}^{\nu_{\rm M}^f}\left[-\mu\sqrt{\mu^2-2|Q_f|B\nu}+2|Q_f|B\nu\ln\frac{\mu+\sqrt{\mu^2-2|Q_f|B\nu}}{\sqrt{2|Q_f|B\nu}}\right]\\
&=-\frac{3}{4\pi^2}\sum_{f=u,d}|Q_f|B\mu^2+\frac{3}{4\pi^2}\sum_{f=u,d}2|Q_f|B\sum_{\nu=1}^{\nu_{\rm M}^f}g_0\left(2|Q_f|B\nu\right),\\
g_0(x) &= -\mu\sqrt{\mu^2-x}+x\ln\frac{\mu+\sqrt{\mu^2-x}}{\sqrt{x}},
\end{aligned}\tag{3.10}$$

with

$$\nu_{\rm M}^f=\left[\frac{\mu^2}{2|Q_f|B}\right].\tag{3.11}$$

4. Approximation of the thermodynamic potential by replacing summation by integration with respect to the Landau level

In the thermodynamic potential (2.26) and (2.30), the quantum number ν, which labels the Landau level, has to be summed up. However, since it is interesting to consider the spontaneous magnetization, it may be assumed that the external magnetic field B is small and that finally B becomes 0. Therefore, let us replace the sum with respect to ν by an approximate integral [16].

In general, let us consider a function $f(x)$. Here, we introduce a small quantity a and let us consider the Tailor expansion around $x = a\nu$ as follows:

$$\begin{aligned}
\int_{a(\nu-1)}^{a\nu}dxf(x) &= \int_{a(\nu-1)}^{a\nu}dx\left[f(a\nu)+\left.\frac{df}{dx}\right|_{x=a\nu}(x-a\nu)+\frac{1}{2}\left.\frac{d^2f}{dx^2}\right|_{x=a\nu}(x-a\nu)^2+\cdots\right]\\
&= af(a\nu)-\frac{1}{2}a^2f'(a\nu)+\frac{1}{6}a^3f''(a\nu)+\cdots.
\end{aligned}\tag{4.1}$$

Thus, the following relation is obtained:

$$\begin{aligned}
\sum_{\nu=\nu_m+1}^{\nu_M}\int_{a(\nu-1)}^{a\nu}dxf(x) &\equiv \int_{a\nu_m}^{a\nu_M}dxf(x)\\
&= a\sum_{\nu=\nu_m+1}^{\nu_M}f(a\nu)-\frac{a^2}{2}\sum_{\nu=\nu_m+1}^{\nu_M}f'(a\nu)+\frac{a^3}{6}\sum_{\nu=\nu_m+1}^{\nu_M}f''(a\nu)+\cdots.
\end{aligned}\tag{4.2}$$

Here, it should be noted that the definition of integral can be used when a is infinitesimally small, namely,

$$a\sum_{\nu=\nu_m+1}^{\nu_M} f'(a\nu) = \int_{a\nu_m}^{a\nu_M} dx f'(x) = f(a\nu_M) - f(a\nu_m), \tag{4.3}$$

and so on. Thus, adding $af(a\nu_m)$ on both sides of Eq. (4.2), a useful approximate formula is obtained as follows:

$$a\sum_{\nu=\nu_m}^{\nu_M} f(a\nu) = \int_{a\nu_m}^{a\nu_M} dx f(x) + \frac{a}{2}\left[f(a\nu_M) + f(a\nu_m)\right] - \frac{a^2}{6}\left[f'(a\nu_M) - f'(a\nu_m)\right] + \cdots . \tag{4.4}$$

From this, let us approximate the thermodynamic potential in the case of small B.

For $F > \mu$, Eq. (3.7) is approximated by using the formula (4.4) as follows:

$$\begin{aligned}\Phi = \frac{F^2}{2G} + \frac{3}{8\pi^2}\sum_{f=u,d}\Bigg[2\int_{p^f_{\min}}^{p^f_{\max}} dp_\perp p_\perp g_-\left(p_\perp^2\right) + |Q_f|B\left(g_-\left(2|Q_f|B\nu^{(-)}_{f,m}\right)\right.\\
\left. + g_-\left(2|Q_f|B\nu^{(-)}_{f,M}\right)\right) - \frac{2|Q_F|^2B^2}{3}\left(g'_-\left(2|Q_f|B\nu^{(-)}_{f,M}\right) - g'_-\left(2|Q_f|B\nu^{(-)}_{f,m}\right)\right) + \cdots\Bigg]\end{aligned} \tag{4.5}$$

with (3.5). Here, $p^f_{\min} = \sqrt{2|Q_f|B\nu^{(-)}_{f,m}}$ and $p^f_{\max} = \sqrt{2|Q_f|B\nu^{(-)}_{f,M}}$, respectively.

For $F < \mu$, the thermodynamic potential (3.9) is approximated by using (4.2) directly as

$$\begin{aligned}\Phi &= \frac{F^2}{2G} + \frac{3}{8\pi^2}\sum_{f=u,d} 2|Q_f|B\left[g_+(0) + \sum_{\nu=1}^{\nu^{(+)}_{f,M}} g_+\left(2|Q_f|B\nu\right) + \sum_{\nu=1}^{\nu^{(-)}_{f,M}} g_-\left(2|Q_f|B\nu\right)\right]\\
&= \frac{F^2}{2G} + \frac{3}{8\pi^2}\sum_{f=u,d} 2|Q_f|B g_+(0) + \frac{3}{8\pi^2}\sum_{f=u,d;\eta=\pm}\Bigg[2\int_0^{p^{f(\eta)}_{\max}} dp_\perp p_\perp g_\pm\left(p_\perp^2\right)\\
&\quad + |Q_f|B\left(g_\pm\left(2|Q_f|B\nu^{(\eta)}_{f,M}\right) - g_\eta(0)\right) - \frac{2|Q_f|^2B^2}{3}\left(g'_\eta\left(2|Q_f|B\nu^{(\eta)}_{f,M}\right) - g'_\eta(0)\right) + \cdots\Bigg],\end{aligned} \tag{4.6}$$

where $\nu^{(\eta)}_{f,M} = \left[(\mu \mp F)^2/(2|Q_f|B)\right]$ and $p^{f(\eta)}_{\max} = \sqrt{2|Q_f|B\nu^{(\eta)}_{f,M}}$, respectively.

For $F = 0$, the thermodynamic potential (3.10) is approximated as

$$\begin{aligned}\Phi_0 = -\frac{3}{4\pi^2}\sum_{f=u,d}|Q_f|B\mu^2 + \frac{3}{4\pi^2}\sum_{f=u,d}\Bigg[\int_0^{a_f\nu^f_M} g_0(x)\,dx + |Q_f|B\left(g_0\left(a_f\nu^f_M\right) - g_0(0)\right)\\
- \frac{2|Q_f|^2B^2}{3}\left(g'_0\left(2|Q_f|B\nu^f_M\right) - g'_0(0)\right) + \cdots\Bigg].\end{aligned} \tag{4.7}$$

Here, since B is small, ν may be regarded as a continuum variable. Remembering the condition which determines $\nu_{f,m}^{(\eta)}$ and $\nu_{f,M}^{(\eta)}$, namely, $g_{\pm}(2|Q_f|B\nu_{f,M}^{(\eta)})=0$, and $g_{\pm}(0)\neq 0$ with (3.5), then Eqs. (4.5), (4.6), and (4.7) are simply calculated by performing the integration as

$$\begin{aligned}
\Phi &= \Phi_{>} = -\frac{F\mu^3}{2\pi}+\frac{F^2}{2G}+O\big(B^2\big), \quad \text{for} \quad F>\mu, \\
\Phi &= \Phi_{<} = \frac{3}{\pi^2}\left[\sqrt{\mu^2-F^2}\left(\frac{\mu^3}{6}+\frac{F^2\mu}{4}\right)-\frac{F\mu^3}{3}\arctan\frac{F}{\sqrt{\mu^2-F^2}}\right. \\
&\quad \left.+\frac{F^4}{12}\ln\frac{\mu+\sqrt{\mu^2-F^2}}{F}\right]+\frac{F^2}{2G}+O\big(B^2\big), \qquad \text{for} \quad F<\mu, \\
\Phi &= \Phi_0 = -\frac{\mu^4}{2\pi^2}+O\left(B^2\right), \qquad \text{for} \quad F=0,
\end{aligned} \tag{4.8}$$

where we used the integration formulae:

$$\begin{aligned}
\int dyy\sqrt{\mu^2-(F+\eta y)^2} &= \sqrt{\mu^2-(F+\eta y)^2}\left[-\frac{\mu^2}{3}-\frac{F}{2}(F+\eta y)+\frac{1}{3}(F+\eta y)^2\right] \\
&\quad -\frac{F\mu^2}{2}\arctan\left[\frac{(F+\eta y)^2\sqrt{\mu^2-(F+\eta y)^2}}{\mu^2-(F+\eta y)^2}\right], \\
\int dyy\,(F+\eta y)^2\ln\frac{\mu+\sqrt{\mu^2-(F+\eta y)^2}}{F+\eta y} \\
&= \sqrt{\mu^2-(F+\eta y)^2}\left[-\frac{\mu^3}{6}+\frac{F\mu}{6}(F+\eta y)-\frac{\mu}{12}(F+\eta y)^2\right] \\
&\quad +\frac{F\mu^3}{6}\arctan\left[\frac{(F+\eta y)\sqrt{\mu^2-(F+\eta y)^2}}{-\mu^2+(F+\eta y)^2}\right] \\
&\quad +\frac{1}{12}(F+\eta y)^3\left[-4F+3(F+\eta y)\right]\ln\frac{\mu+\sqrt{\mu^2-(F+\eta y)^2}}{F+\eta y}.
\end{aligned} \tag{4.9}$$

5. Spontaneous magnetization for quark matter under tensor-type interaction between quarks

First, let us calculate the spin polarization F with $B=0$ as a function of the chemical potential μ. This task has been done in our previous papers, [18] or [19]. The thermodynamic potential can be expressed as

$$\begin{aligned}
\Phi_{B=0} &= 6\cdot\frac{1}{V}\sum_{\boldsymbol{p}\left(\epsilon_{\boldsymbol{p}^{(+)}}\leq\mu\right)}\left(\epsilon_{\boldsymbol{p}}^{(+)}-\mu\right)+6\cdot\frac{1}{V}\sum_{\boldsymbol{p}\left(\epsilon_{\boldsymbol{p}^{(-)}}\leq\mu\right)}\left(\epsilon_{\boldsymbol{p}}^{(-)}-\mu\right)+\frac{F^2}{2G}, \\
\epsilon_{\boldsymbol{p}}^{(\pm)} &= \sqrt{p_3^2+\left(F+\eta\sqrt{p_1^2+p_2^2}\right)^2},
\end{aligned} \tag{5.1}$$

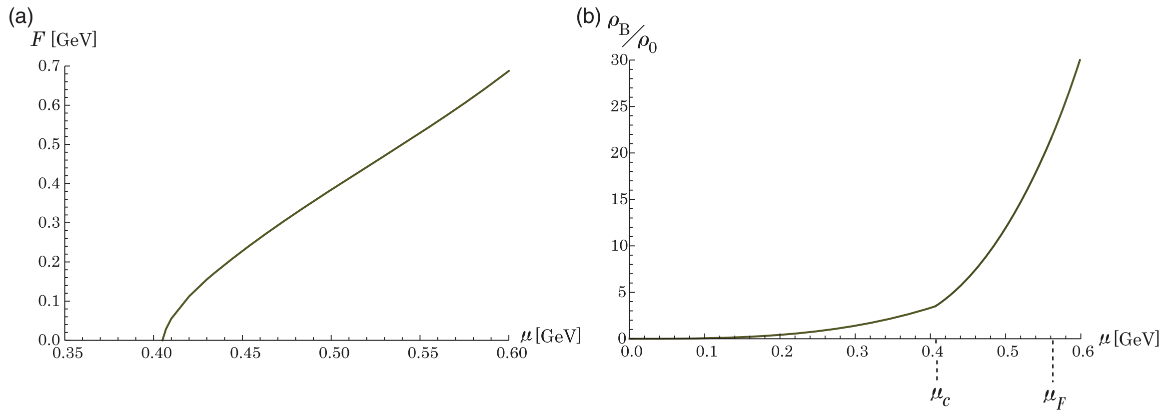

Fig. 1. (a) The spin polarization condensate F is depicted as a function of the quark chemical potential μ. (b) The baryon number density of quark matter divided by the normal nuclear density $\rho_0 = 0.17\,\text{fm}^{-3}$ is shown as a function of the quark chemical potential. Here, $\mu_c \approx 0.407\,\text{GeV}$ is the value at which the spin polarization occurs. Also, in the region of $\mu \geq \mu_F \approx 0.5605\,\text{GeV}$, the spin polarization condensate is greater than the chemical potential, $F \geq \mu$.

where the factor 6 represents the color and flavor degrees of freedom and $p_1 = p_x$, $p_2 = p_y$, and $p_3 = p_z$. Here, $(1/V) \cdot \sum_{\boldsymbol{p}}$ can be replaced to the integration $\int d^3\boldsymbol{p}/(2\pi)^3$. Integrating the three-momentum, we have obtained the thermodynamic potential as

$$\Phi_{B=0} = \begin{cases} \dfrac{F^2}{2G} - \dfrac{1}{\pi^2}\left[\dfrac{\sqrt{\mu^2 - F^2}}{4}\left(3F^2\mu + 2\mu^3\right) + F\mu^3 \arctan\dfrac{F}{\sqrt{\mu^2 - F^2}} \right. & \\ \qquad\qquad \left. -\dfrac{F^4}{4}\ln\dfrac{\mu + \sqrt{\mu^2 - F^2}}{F}\right] & \text{for} \quad F < \mu \\ \dfrac{F^2}{2G} - \dfrac{F\mu^3}{2\pi} & \text{for} \quad F > \mu. \end{cases} \tag{5.2}$$

It should be here noted that Eq. (5.2) is identical to (4.8) with $B = 0$. The gap equation is derived by $\partial\Phi_{B=0}/\partial F = 0$. The quark number density ρ_q is also derived through the thermodynamic relation $\rho_q = -\partial\Phi/\partial\mu$ [18] for the solution of the gap equation, $F = F_{\min}$:

$$\rho_q = \begin{cases} \dfrac{1}{\pi^2}\left[\left(F_{\min}^2 + 2\mu^2\right)\sqrt{\mu^2 - F_{\min}^2} + 3F_{\min}\mu^2 \arctan\dfrac{F_{\min}}{\sqrt{\mu^2 - F_{\min}^2}}\right] & \text{for} \quad F < \mu \\ \dfrac{3G}{4\pi^2}\mu^5 & \text{for} \quad F > \mu. \end{cases} \tag{5.3}$$

Figure 1 shows (a) the spin polarization condensate F and (b) the baryon density of quark matter divided by the normal nuclear density, $\rho_0 = 0.17\,\text{fm}^{-3}$, as a function of the quark chemical potential μ. Here, we used a parameter $G = 20\,\text{GeV}^{-2}$ which has been used[2] in our previous papers [18,19]. About $\mu = \mu_c \approx 0.407\,\text{GeV}$, which corresponds to the baryon density being $3.53\rho_0$, the spin

[2] If the vacuum polarization is taken into account, the parameter G with a rather small value, such as $G = 11.1\,\text{GeV}^{-2}$, gives the same results quantitatively under the standard three-momentum cutoff $\Lambda = 0.631\,\text{GeV}$, although we do not consider the vacuum polarization in this paper.

polarization appears against the free quark phase. Of course, if the two-flavor color superconductivity is considered, it has been shown that the spin polarized phase appears about $\mu = 0.442$ GeV ($\rho_B \approx 5.85\rho_0$) [19].

We derive the spontaneous magnetization through the thermodynamic relation. The spontaneous magnetization per unit volume, $\mathcal{M}$, is defined by

$$\mathcal{M} = -\left.\frac{\partial \Phi}{\partial B}\right|_{B=0}. \tag{5.4}$$

In the right-hand side, $B = 0$ is adopted. However, from (4.8), the thermodynamic potential does not depend on B linearly. Thus, the spontaneous magnetization is equal to zero under this consideration, namely

$$\mathcal{M} = -\left.\frac{\partial \Phi}{\partial B}\right|_{B=0} = 0, \tag{5.5}$$

even if quark-spin polarization occurs, while the magnetic polarization which is proportional to B may appear.

6. Spontaneous magnetization originated from the anomalous magnetic moments of quarks

In the previous section it has been found that no spontaneous magnetization of polarized high density quark matter is predicted by the normal coupling to an external magnetic field. However, the spin polarization F should be observed in some way. We know that the quark has an anomalous magnetic moment. In this section, let us investigate the effect of the anomalous magnetic moment of quarks.

The effect of the anomalous magnetic moment μ_A is introduced at the level of the mean field approximation in this paper. Here, it is known that the anomalous magnetic moment is expressed as [28]

$$\mu_A \mathbf{1} = \begin{pmatrix} \mu_u & 0 \\ 0 & \mu_d \end{pmatrix}, \quad \mu_u = 1.85\mu_N, \quad \mu_d = -0.97\mu_N,$$
$$\mu_N = \frac{e}{2m_p} = 3.15 \times 10^{-17}\,\mathrm{GeV\,T^{-1}}. \tag{6.1}$$

We introduce the effects of the anomalous magnetic moment in the Lagrangian density within the mean field approximation as

$$\mathcal{L}_A = \mathcal{L} - \frac{i}{2}\bar{\psi}\mu_A\gamma^\mu\gamma^\nu F_{\mu\nu}\psi, \tag{6.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F_{12} \equiv -B_z = -B$. Here, $\mathcal{L}$ is nothing other than Eq. (2.3). We only take μ and ν as $\mu = 1,\ \nu = 2$ and $\mu = 2,\ \nu = 1$ because the magnetic field only has a z-component. Then, in the mean field approximation, Eq. (2.3) is recast into

$$\mathcal{L}_A = i\bar{\psi}\gamma^\mu D_\mu\psi - \bar{\psi}\left(F_3\tau_3 + \mu_A B\mathbf{1}\right)\Sigma_3\psi. \tag{6.3}$$

Here, $F_3\tau_3 = F\mathbf{1}$ where $\mathbf{1}$ is the 2×2 identity matrix for the isospin space, as denoted in Eqs. (2.2) and (2.9). Thus, we introduce the flavor-dependent variables $\widetilde{F}_f$ as

$$\widetilde{F}_f = F + \mu_f B, \quad \text{namely} \quad \widetilde{F}_u = F + \mu_u B, \qquad \widetilde{F}_d = F + \mu_d B. \tag{6.4}$$

Therefore, the Lagrangian density can be expressed as

$$\mathcal{L}_A = i\bar{\psi}\gamma^\mu D_\mu \psi - \bar{\psi}\left(F + \mu_A B\right)\Sigma_3\psi - \frac{F^2}{2G}$$
$$= i\bar{\psi}\gamma^\mu D_\mu \psi - \bar{\psi}\widetilde{F}\Sigma_3\psi - \frac{F^2}{2G}. \tag{6.5}$$

Thus, we learn that the variable F should be replaced by $\widetilde{F}_f = F + \mu_f B$ for each flavor except for the last term in (6.5). In this replacement, we can derive the thermodynamic potential in the same way developed in Sect. 4. Namely, the results including the effects of the anomalous magnetic moment of quarks should be obtained by replacing F by $\widetilde{F}_f$ for each flavor in the previous calculation developed in Sect. 4, except for the term originated from the mean field approximation, $F^2/2G$. The results are summarized as follows:

$$\Phi = \Phi_> = -\frac{1}{2}\sum_{f=u,d}\frac{\widetilde{F}_f\mu^3}{2\pi} + \frac{F^2}{2G} + O\left(B^2\right), \quad \text{for} \quad F > \mu,$$
$$\Phi = \Phi_< = \frac{1}{2}\sum_{f=u,d}\frac{3}{\pi^2}\left[-\sqrt{\mu^2 - \widetilde{F}_f^2}\left(\frac{\mu^3}{6} + \frac{\widetilde{F}_f^2\mu}{4}\right) - \frac{\widetilde{F}_f\mu^3}{3}\arctan\frac{\widetilde{F}_f}{\sqrt{\mu^2 - \widetilde{F}_f^2}}\right.$$
$$\left. + \frac{\widetilde{F}_f^4}{12}\ln\frac{\mu + \sqrt{\mu^2 - \widetilde{F}_f^2}}{\widetilde{F}_f}\right] + \frac{F^2}{2G}, \quad \text{for} \quad \mu \geq F > 0. \tag{6.6}$$

Thus, the spontaneous magnetization per unit volume, $\mathcal{M}$, can be derived through Eq. (5.4), that is,

$$\mathcal{M} = -\frac{\partial \Phi}{\partial B}\bigg|_{B=0} = -\sum_{f=u,d}\frac{\partial \Phi}{\partial \widetilde{F}_f}\bigg|_{B=0} \cdot \frac{\partial \widetilde{F}_f}{\partial B}. \tag{6.7}$$

From (6.6), we can derive the spontaneous magnetization originated from the anomalous magnetic moment and the spin polarization:

$$\mathcal{M} = -\frac{\partial \Phi_>}{\partial B}\bigg|_{B=0} = \frac{\mu^3}{4\pi}\left(\mu_u + \mu_d\right), \tag{6.8a}$$

$$\mathcal{M} = -\sum_{f=u,d}\frac{\partial \Phi_<}{\partial \widetilde{F}_f}\bigg|_{B=0} \cdot \frac{\partial \widetilde{F}_f}{\partial B}$$
$$= \frac{1}{2\pi^2}\left[2F\mu\sqrt{\mu^2 - F^2} + \mu^3 \arctan\left[\frac{F}{\sqrt{\mu^2 - F^2}}\right] - F^3 \ln\frac{\mu + \sqrt{\mu^2 - F^2}}{F}\right]\left(\mu_u + \mu_d\right)$$
$$= \frac{F}{2G}\left(\mu_u + \mu_d\right), \tag{6.8b}$$

where the gap equation $\partial\Phi/\partial F = \sum_{f=u,d}\partial\Phi/\partial\widetilde{F}_f + \partial(F^2/(2G))/\partial F = 0$ is used from the second line to the third line in (6.8b). It should be noted that if $F = 0$, the spontaneous magnetization disappears because $F = 0$ in Eq. (6.8b).

In Fig. 2, the spontaneous magnetization is depicted as a function of the quark chemical potential. The spontaneous magnetization is shown in SI units. For $\mu < \mu_c \approx 0.407$ GeV, the magnetization does not occur because the spin polarization does not appear, $F = 0$. At $\mu = \mu_c$, the spontaneous magnetization suddenly appears.

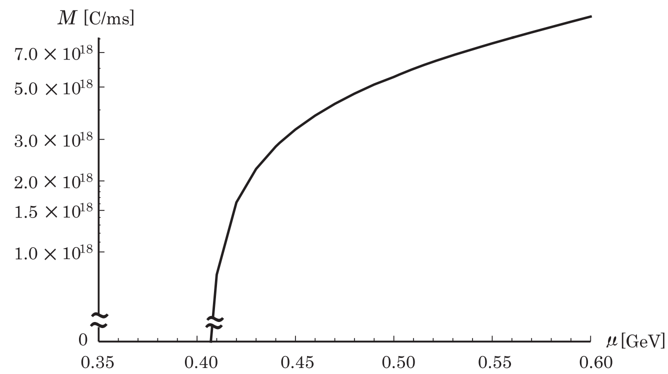

Fig. 2. The spontaneous magnetization per unit volume $\mathcal{M}$ is depicted as a function of the quark chemical potential μ. The vertical axis represents the magnetization $\mathcal{M}$ in a logarithmic scale. For $\mu < 0.407$ GeV, the magnetization does not occur because the spin polarization does not appear, $F = 0$.

7. Magnetic field of hybrid compact star

As seen in the previous section, for $\mu > \mu_c$, the spontaneous magnetization in the quark matter appears. Here, the spontaneous magnetization per unit volume $\mathcal{M}$ may be regarded as the magnetic dipole moment. Under this identification, we can calculate the strength of the magnetic field yielded by the spontaneous magnetization. As is well known in classical electromagnetism, the magnetic field at position $\boldsymbol{r}$, namely magnetic flux density $\boldsymbol{B}(\boldsymbol{r})$, created by the magnetic dipole moment $\boldsymbol{m}$ can be expressed as

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\left[-\frac{\boldsymbol{m}}{r^3} + \frac{3\boldsymbol{r}\,(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5}\right], \tag{7.1}$$

where μ_0 represents the vacuum permeability. In our case, $\boldsymbol{m} = (0,\ 0,\ \mathcal{M}\times V)$, where V represents a volume, because $\mathcal{M}$ is nothing but the magnetization per unit volume.

Here, let us consider a hybrid star with quark matter in the core of a neutron star. Let us assume that the hybrid (neutron) star has radius $R = 10$ km. If there exists quark matter in the inner core of the star from the center to r_q km, the strength of the magnetic flux density on the surface at the north or south pole of the hybrid star is roughly estimated as

$$B_z = \frac{\mu_0}{4\pi}\left(-\frac{\mathcal{M}}{R^3} + \frac{3z^2\mathcal{M}}{R^5}\right)\times\frac{4}{3}\pi r_q^3 = \mu_0\frac{2\mathcal{M}r_q^3}{3R^3}\ [T]. \tag{7.2}$$

Figure 3 shows the magnetic flux density as a function of the quark chemical potential μ in the cases (a) $r_q = 1$ km and (b) $r_q = 2.15$ km where quark matter occupies (a) 0.1% and (b) 1% of the total volume of the star $[(4\pi r_q^3/3)/(4\pi R^3/3) =$ (a) 0.001 and (b) 0.01], respectively. It should be noted that the SI unit, tesla, is converted to gauss, namely, 1 T (tesla) $= 10^4$ G (gauss). In a magnetar, the strength of the magnetic field at the surface of star is near 10^{15} G. Thus, in our calculation, if quark matter exists and spin polarization occurs, a strong magnetic field of about 10^{13} or 10^{14} gauss may be created.

8. Summary and concluding remarks

It has been shown that spontaneous magnetization occurs due to the anomalous magnetic moments of quarks in high-density quark matter under the tensor-type four-point interaction. In the Nambu–Jona-Lasinio model as an effective model of QCD, the tensor-type four-point interaction has been introduced. Owing to this interaction, the spin polarized condensate appears for each quark flavor in

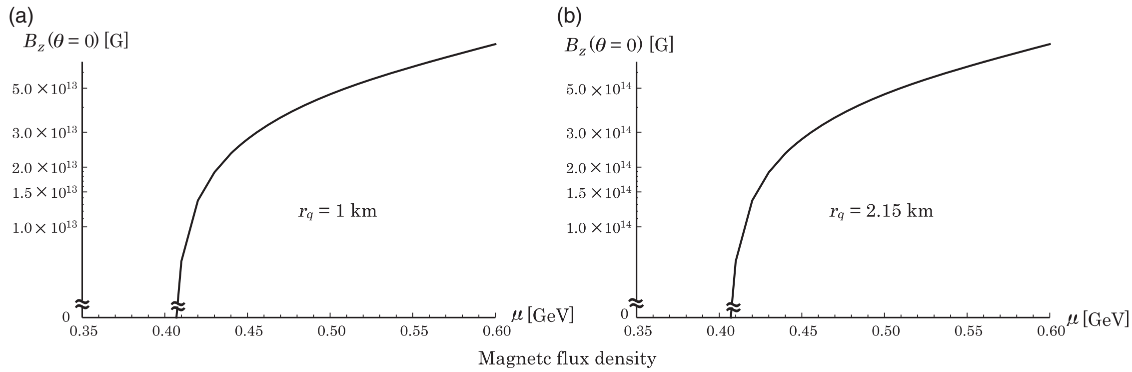

Fig. 3. The magnetic flux density is shown in a logarithmic scale as a function of the quark chemical potential μ in the case (a) $r_q = 1$ km where the quark matter occupies 0.1% of the total volume of the star $\left[\left(4\pi r_q^3/3\right)/\left(4\pi R^3/3\right) = 0.001\right]$ and (b) $r_q = 2.15$ km where the quark matter occupies 1% of the total volume of the star $\left[\left(4\pi r_q^3/3\right)/\left(4\pi R^3/3\right) = 0.01\right]$, respectively. It should be noted that the SI unit, tesla, is converted to gauss, namely, 1 T (tesla) = 10^4 G (gauss).

the region of large quark chemical potential. It has been shown that the spin polarized condensate leads to spontaneous magnetization of quark matter due to the anomalous magnetic moments of quarks. Also, it has been pointed out that spontaneous magnetization does not occur if no anomalous magnetic moments of quarks exist.

In this paper, furthermore, the implications for the strong magnetic field in compact stars such as hybrid stars has been discussed. If there exists quark matter in the core of neutron stars and the quark number density is rather high, spontaneous magnetization may occur. If quark matter occupies a volume of 1% of the neutron star, the strength of the magnetic field at the surface of the neutron star is of the order of 10^{14} gauss, which is comparable to the strength of the magnetic field in the so-called magnetar.

As indicated in this paper, the spin polarized condensate appears in the region with a large quark chemical potential. Thus, the chiral symmetry is broken in this model because the starting Lagrangian density is constructed with chiral symmetry. Under the magnetic field, the chiral symmetry is broken and it is shown that the chiral condensate grows at most linearly as a function of the magnetic field B [29]. Further, in Ref. [30], the tensor-type four-point interaction was introduced in the NJL model in the case of one quark flavor. In that paper, both the chiral condensate and spin polarized condensate are treated equally in a finite temperature system. Thus, it may be an interesting problem that the coexistence of the chiral condensate and the spin polarized condensate is also considered in a finite baryon density system.

As for the implications for compact stars such as neutron stars, it is important to impose the charge neutrality condition including electrons in addition to up and down quarks. Then, the chemical equilibrium condition is also necessary. This is an interesting and important task for future studies to investigate. Further, it has been shown that there is the possibility of the existence of massive hybrid quark stars with two solar mass under the strong magnetic field [31]. Thus, the investigation of the equation of state of quark matter in the spin polarized phase revealing spontaneous magnetization may be one of the interesting future problems.

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