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EXPLORING THE DYSON-SCHWINGER EQUATIONS IN QED

Dissertação no âmbito do Mestrado em Física orientada pelo Professor Doutor Orlando Olavo Aragão Aleixo e Neves de Oliveira e apresentada ao Departamento da Física da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

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Exploring the Dyson-Schwinger Equations in QED

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Resumo

Embora muito poderosa e bem-sucedida, a teoria de perturbações tem limitações e não pode ser utilizada para explicar diversos problemas em Teoria Quântica de Campos. Assim, torna-se necessário desenvolver abordagens alternativas não-perturbativas, uma das quais são as equações de Dyson-Schwinger. Estas são um conjunto de equações que relacionam as funções de Green de diferentes ordens de uma teoria entre si, formando uma torre infinita de equações integrais não lineares acopladas. Este trabalho explora a aplicação deste método à Eletrodinâmica Quântica, com a derivação de um conjunto mínimo de equações no espaço-tempo de Minkowski para uma gauge linear covariante geral que transcende os propagadores. Este conjunto de equações inclui as equações de Dyson-Schwinger para o propagador do fermião e do fotão, bem como para o vértice irredutível fotão-fermião de forma exata. A fim de construir um conjunto fechado de equações, a equação de Dyson-Schwinger para o vértice irredutível de dois-fotões-dois-fermiões também é derivada, embora em uma versão truncada para evitar a necessidade de invocar as funções de Green com um maior número de pernas externas. As identidades de Ward-Takahashi para os vértices irredutíveis fotão-fermião e dois-fotões-dois-fermiões também são derivadas. Um estudo do sistema acoplado fotão-fermião é feito utilizando o vértice de Ball-Chiu para substituir o vértice fotão-fermião. As equações são avaliadas no espaço-tempo Euclidiano e alguns resultados numéricos preliminares são discutidos.

Palavras-chave: Equações de Dyson-Schwinger; Vértice Dois-Fotões-Dois-Fermiões; Identidade de Ward-Takahashi; Teoria Quântica de Campos Não-perturbativa; Eletrodinâmica Quântica.

Abstract

Although very powerful and highly successful, perturbation theory has its limitations and cannot be used to explain several problems in Quantum Field Theories. Therefore, it becomes necessary to develop alternative non-perturbative approaches, one of which are the Dyson-Schwinger equations. These are a set of equations that relate the Green's functions of different orders of a theory to each other, forming an infinite tower of coupled nonlinear integral equations. This work explores the application of this method to Quantum Electrodynamics, with the derivation of a minimal set of equations in Minkowski spacetime for a general linear covariant gauge that goes beyond the propagators in QED. This set of equations includes the Dyson-Schwinger equations for the fermion and photon propagator, as well as for the one-particle irreducible photon-fermion vertex in exact form. In order to build a closed set of equations, the Dyson-Schwinger equation for the one-particle irreducible two-photon-two-fermion vertex is also derived, although in a truncated version to avoid the need for invoking Green's functions with a greater number of external legs. The Ward-Takahashi identities for the photon-fermion and the two-photon-two-fermion irreducible vertices are also derived. A study of the photon-fermion coupled system is made using the Ball-Chiu vertex to replace the photon-fermion vertex, where the equations are evaluated in Euclidean spacetime and some preliminary numerical results are discussed.

Keywords: Dyson-Schwinger Equations; Two-Photon-Two-Fermion Vertex; Ward-Takahashi Identity; Non-perturbative Quantum Field Theory; Quantum Electrodynamics.

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List of Acronyms

QFT	Quantum Field Theory
QED	Quantum Electrodynamics
QCD	Quantum Chromodynamics
DSE	Dyson-Schwinger Equation
1PI	One-Particle Irreducible
WTI	Ward-Takahashi Identity
BC	Ball-Chiu

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1 Introduction

The standard textbook approach used to solve interacting quantum field theories (QFTs) is perturbation theory [3, 4], in which the Green's functions are expanded in powers of the coupling constant. This is highly successful in some cases, such as Quantum Electrodynamics (QED), whose predictions are the most accurate of all fundamental theories in physics. However, the perturbative expansion breaks down if the coupling constant is not small enough and, therefore, cannot be used to explain several problems in QFT. For example, confinement in Quantum Chromodynamics (QCD) [5, 6, 7] and the dynamical mass generation in strongly coupled QED [5, 8, 9, 10, 11, 12, 13, 14] cannot be addressed in perturbation theory.

Hence, in order to study these non-perturbative problems, one needs to resort to non-perturbative methods. Although there are many possible techniques that can be used, this work will focus on the Dyson-Schwinger equations (DSEs) [15, 16, 17]. They form a set of infinite coupled nonlinear integral equations that relate the Green's functions of different orders to each other, structured in such a way that, in general, the equation for the n-point Green's function contains $(n+1)$-point Green's functions [5, 18]. The main challenge associated with this approach lies on the fact that it is not possible to solve the full set of equations simultaneously and one needs to introduce a truncation to obtain a closed set of equations. For example, in the literature, one studies the set of equations for the photon and fermion propagators in QED by replacing the photon-fermion vertex by a suitable *Ansatz* [11, 19, 20, 21, 22, 23].

Our main goal is to go a step further and solve QED via a minimal set of equations that go beyond the propagators equations. Thus, in this work, the Dyson-Schwinger equations for the photon and fermion propagators and for the one-particle irreducible (1PI) photon-fermion and two-photon-two-fermion vertices are derived. To obtain a closed set of equations, we consider the first three equations in their exact form and only the final one in a truncated version that does not contain higher-order Green's functions. Furthermore, the Ward-Takahashi identities (WTIs) for the two studied vertices are also derived. This can be found summarised in [18], along with other analyses.

This work is organised as follows. In Chapter 2, we go through the theoretical background needed to build the Dyson-Schwinger equations formalism in QED, which includes the definitions of the QED Lagrangian density and the various generating functionals in the path integral formulation. We also provide some useful relations used afterwards and the starting point of the DSE derivation. The Dyson-Schwinger equations for the fermion and photon propagators and for the 1PI photon-fermion and the two-photon-two-fermion vertices are derived in Chapter 3. In this chapter, we also derive the WTIs for the vertices in question and touch on the renormalisation procedure in QED. Chapter 4 regards the study of the fermion-photon coupled system. This includes solving the WTI for the photon-fermion vertex, which fixes the longitudinal part of the vertex, known as the Ball-Chiu vertex [24]. This solution is used to obtain a closed set of the propagators equations in Minkowski and in Euclidean spacetime. The latter are then solved numerically and some preliminary results are presented. Finally, Chapter 5 summarises and concludes this work, as well as mentioning some directions for future investigations. In the appendices, we provide details about some auxiliary results used throughout this work, namely the decomposition of the connected Green's functions in terms of 1PI functions and useful results involving the trace of gamma matrices.

2 Theoretical Background

This chapter reviews the theoretical foundations needed to study the Dyson-Schwinger equations in QED. To do that, we start by building the classical Lagrangian of QED using the gauge invariance principle. Afterwards, the path integral formalism and the generating functionals for the various types of Green's functions will be introduced, which allows us to derive the Dyson-Schwinger equations.

2.1 Quantum Electrodynamics

Quantum Electrodynamics is the quantum field theory of the electromagnetic force, describing the interaction between light (photons) and charged particles. To arrive at its Lagrangian density, one starts by writing the Dirac Lagrangian density, which describes the behaviour of free spin-1/2 particles, given by

$$\mathcal{L}_{Dirac} \; = \; \bar{\psi}\,(i\,\gamma^{\mu}\,\partial_{\mu} - m)\,\psi \; , \tag{2.1}$$

where ψ is a fermionic field, γ^{μ} the Dirac gamma matrices and m the fermion mass. This Lagrangian is invariant under $U(1)$ global gauge transformations

$$\psi \;\rightarrow\; \psi' \equiv e^{i\,\lambda}\,\psi \quad \text{and} \quad \bar{\psi} \;\rightarrow\; \bar{\psi}' \equiv \bar{\psi}\,e^{-i\,\lambda} \; , \tag{2.2}$$

where λ is a constant. One generalises this global symmetry to a local one by allowing the transformations to depend on the local spacetime coordinate, i.e, $\lambda \rightarrow \lambda(x)$. Then, the transformed Dirac Lagrangian density gains an extra term

$$\mathcal{L}'_{Dirac} \; = \; \mathcal{L}_{Dirac} - \bar{\psi}\,\gamma^{\mu}\,\psi\,\partial_{\mu}\,\lambda(x) \; . \tag{2.3}$$

In order to recover the gauge invariance of $\mathcal{L}_{Dirac}$, the derivative ∂_{μ} is replaced by the covariant derivative

$$\partial_{\mu} \;\rightarrow\; D_{\mu} \; = \; \partial_{\mu} + i\,g\,A_{\mu} \; , \tag{2.4}$$

where g refers to the electric charge of the spinor field and A_μ is the new vector field, which transforms as

$$A_\mu \;\rightarrow\; A'_\mu \equiv A_\mu - \frac{1}{g}\,\partial_\mu\,\lambda(x) \tag{2.5}$$

to ensure the invariance of the Lagrangian. In order to introduce dynamics to the vector field, it is necessary to include a kinetic term associated to this field to the Lagrangian density, which becomes

$$\mathcal{L} \;=\; \bar{\psi}\,(i\,\gamma^\mu\,D_\mu - m)\,\psi - \frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu}\;, \tag{2.6}$$

where the field strength tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} \;=\; \partial_\mu\,A_\nu - \partial_\nu\,A_\mu\;. \tag{2.7}$$

This Lagrangian is invariant under the following local gauge transformations:

$$\psi \;\rightarrow\; e^{i\,\lambda(x)}\,\psi\;, \tag{2.8}$$

$$\bar{\psi} \;\rightarrow\; \bar{\psi}\,e^{-i\,\lambda(x)}\;, \tag{2.9}$$

$$A_\mu \;\rightarrow\; A_\mu - \frac{1}{g}\,\partial_\mu\,\lambda(x)\;. \tag{2.10}$$

There are an infinite number of fields A_μ which are related through a gauge transformation and therefore belong to the same gauge orbit. However, since the Lagrangian density is gauge invariant, an integration over this field would be infinite. One can avoid this problem by imposing a gauge condition which is only satisfied by one field in each orbit. This can be achieved in QED by introducing a gauge fixing term to the Lagrangian density, so it becomes

$$\mathcal{L}_{QED} \;=\; \bar{\psi}\,(i\,\not{D} - m)\,\psi - \frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{2\,\xi}(\partial_\mu\,A^\mu)^2\;, \tag{2.11}$$

where $\not{D} \;=\; \gamma^\mu\,D_\mu$ and ξ is the gauge parameter.

It is worth mentioning that the calculations and results in this work assume natural units and the Minkowski metric with signature $(+\,-\,-\,-)$.

2.2 Path Integral Formalism

With the QED Lagrangian defined in Eq. (2.11), it is now possible to quantise the theory. This can be done through different methods and herein we will use the path integral approach [3, 25, 4]. This formalism is based on the superposition principle, which states that in Quantum Mechanics the probability amplitude for a certain particle to be created at the point x_1, travel through spacetime and then be detected at the point x_2 is obtained by summing over all possible paths between these points, with each path contributing with a phase given by its action. Therefore, in QFT a functional integral represents an integration over all possible field configurations and it is denoted by $\int \mathcal{D}\phi$, where ϕ is the field.

A QFT is completely characterised by its Green's functions [25], which can be defined using the path integral formalism. Using the scalar theory as an example, the vacuum expectation value of a time-ordered product of n field operators, i.e., the n-point Green's function can be written as:

$$
\begin{aligned}
G^{(n)}(x_1,\ldots,x_n) &\equiv \langle 0|\, T[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)]\,|0\rangle \\
&= \frac{\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, e^{i\,S}}{\int \mathcal{D}\phi\, e^{i\,S}} \,, \qquad (2.12)
\end{aligned}
$$

where $\phi(x)$ is the scalar field at a spacetime point x, T indicates the time-ordering of the operators and S is the action, given by $S = \int d^4x \mathcal{L}(x)$. However, it is possible to write this in a simpler way by defining the generating functional for the Green's functions $Z[J]$ as

$$
Z[J] = \frac{1}{Z[0]} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[\mathcal{L}(x) + J(x)\phi(x) \right] \right\} , \qquad (2.13)
$$

where an external source $J(x)$ was introduced. Then, the n-point Green's function can be generated by

$$
G^{(n)}(x_1,\ldots,x_n) = (-i)^n \left. \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)} \right|_{J\,=\,0} . \qquad (2.14)
$$

Here we have also introduced a new concept: functional differentiation [4], i.e., the derivative of a functional with respect to a function, defined as

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon\to 0} \frac{F[f(x)+\epsilon\delta(x-y)] - F[f(x)]}{\epsilon} , \tag{2.15}$$

where $\delta(x-y)$ is the usual Dirac function.

When doing calculations using the generating functional, it is very often useful to replace the field inside the path integral by a functional derivative with respect to its source acting upon the generating functional, since:

$$\frac{\delta}{i\,\delta J(x)} \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[\mathcal{L} + J\phi \right] \right\} = \int \mathcal{D}\phi \, \phi(x) \, \exp\left\{ i \int d^4x \left[\mathcal{L} + J\phi \right] \right\} . \tag{2.16}$$

In the case of QED, the expression for the generating functional is given by

$$\begin{aligned} Z[J,\bar{\eta},\eta] = \frac{1}{Z[0,0,0]} \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi \, \exp\Big\{ i \int d^4x \Big[\mathcal{L}_{QED}(x) \\ + J^\mu(x)\,A_\mu(x) + \bar{\eta}(x)\,\psi(x) + \bar{\psi}(x)\,\eta(x) \Big] \Big\} , \end{aligned} \tag{2.17}$$

where $\mathcal{L}_{QED}$ is defined in Eq. (2.11), J^μ is the external source associated with the gauge field A_μ and the sources $\bar{\eta}$ and η are associated with the fermion fields ψ and $\bar{\psi}$, respectively. It is worth noting that the fermionic fields and sources are Grassmann variables.

Just like it was done in the scalar theory example in Eq. (2.16), in the functional integration over the fields A_μ, $\bar{\psi}$ and ψ, these fields can be substituted by a functional derivative with respect to their respective source. Thus, we can do the following replacements, where in the case of differentiation with respect with Grassmann numbers we consider left differentiation:

$$\psi(x) \leftrightarrow \frac{\delta}{i\,\delta\bar{\eta}(x)} , \quad \bar{\psi}(x) \leftrightarrow \frac{\delta}{-i\,\delta\eta(x)} \quad \text{and} \quad A^\mu(x) \leftrightarrow \frac{\delta}{i\,\delta J_\mu(x)} . \tag{2.18}$$

Furthermore, while Z generates both disconnected and connected diagrams, it is useful to define a generating functional only for the connected Green's functions, denoted by W. These two types of generating functionals are related by

$$Z[J,\bar{\eta},\eta] = e^{i\,W[J,\bar{\eta},\eta]} \tag{2.19}$$

and from this, one defines the classical fields:

$$A_{cl,\,\mu}(x) \;=\; \frac{\delta W[J,\bar{\eta},\eta]}{\delta J^{\mu}(x)} \;, \tag{2.20}$$

$$\bar{\psi}_{cl}(x) \;=\; -\frac{\delta W[J,\bar{\eta},\eta]}{\delta \eta(x)} \;, \tag{2.21}$$

$$\psi_{cl}(x) \;=\; \frac{\delta W[J,\bar{\eta},\eta]}{\delta \bar{\eta}(x)} \;. \tag{2.22}$$

These, in turn, can be used to obtain the generating functional for one-particle irreducible (1PI) diagrams, i.e., diagrams that cannot be separated into two disconnected pieces by cutting any internal line. This generating functional is denoted by Γ and it is defined via the following Legendre transformation:

$$\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}] \;=\; W[J,\bar{\eta},\eta] - (J,A_{cl}) - (\bar{\eta},\psi_{cl}) - (\bar{\psi}_{cl},\eta) \;. \tag{2.23}$$

Here we have introduced the notation

$$(J,A_{cl}) \;=\; \int d^4x\; J^{\mu}(x)A_{cl,\,\mu}(x) \;, \tag{2.24}$$

$$(\bar{\eta},\psi_{cl}) \;=\; \int d^4x\; \bar{\eta}_{\alpha}(x)\,\psi_{cl,\,\alpha}(x) \;, \tag{2.25}$$

$$(\bar{\psi}_{cl},\eta) \;=\; \int d^4x\; \bar{\psi}_{cl,\,\alpha}(x)\,\eta_{\alpha}(x) \tag{2.26}$$

where α is the index associated with the Dirac spinor. It follows that

$$\frac{\delta\,\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(x)} \;=\; -J^{\mu}(x) \tag{2.27}$$

$$\frac{\delta\,\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta \psi_{cl,\,\alpha}(x)} \;=\; \bar{\eta}_{\alpha}(x) \;, \tag{2.28}$$

$$\frac{\delta\,\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta \bar{\psi}_{cl,\,\alpha}(x)} \;=\; -\eta_{\alpha}(x) \;. \tag{2.29}$$

Having stablished the various types of generating functionals, Eqs. (2.17), (2.19) and (2.23), one defines the following Green's functions which will be used in the study of the Dyson-Schwinger equations in QED. The two-point Green's functions, also known as the photon and fermion propagators, are given, respectively, by

$$i\,D_{\mu\nu}(x-y) \;=\; -i\,\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^\mu(x)\,\delta J^\nu(y)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0}\;, \tag{2.30}$$

$$i\,S_{\alpha\beta}(x-y) \;=\; i\,\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x)\,\delta\eta_\beta(y)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} \;=\; -i\,\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_\alpha(x)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0}\;. \tag{2.31}$$

While the photon-fermion vertex, or the 1PI three-point Green's function, is defined as

$$-i\,g\,\Gamma^\mu_{\alpha\beta}(x,y;z) \;=\; i\,\left(\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta}(y)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} \tag{2.32}$$

and the two-photon-two-fermion vertex, or the 1PI four-point Green's function, as

$$-i\,g^2\,\Gamma^{\mu\nu}_{\alpha\beta}(x,y;z,w) \;=\; i\,\left(\frac{\delta^4\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta A_{cl,\,\nu}(w)\,\delta\psi_{cl,\,\beta}(y)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0}\;. \tag{2.33}$$

These functions in momentum space [25] are defined as

$$G^{(n)}(x_1,\ldots,x_n) \;=\; \int \frac{d^4p_1}{(2\,\pi)^4}\,\cdots\,\frac{d^4p_n}{(2\,\pi)^4}\;e^{-i\,(p_1\,x_1\;+\;\cdots\;+\;p_n\,x_n)}$$
$$\delta(p_1\;+\;\cdots\;+\;p_n)\,G^{(n)}(p_1,\ldots,p_n)\;, \tag{2.34}$$

where all momenta are incoming. Fig. 2.1 shows the diagrammatic representation of the 1PI two- and three-point Green's function, where the straight lines with arrows refers to fermion legs and the wavy lines to photon legs.

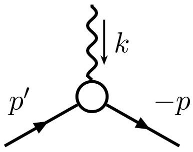
Finally, the following identities relate the second derivatives of Γ with the inverse of the propagators of QED and will be useful in the derivation of the Dyson-Schwinger equations

$$\int d^4z\,\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^\mu(x)\,\delta J^\zeta(z)}\right)\left(\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\zeta}(z)\,\delta A^{\,\nu}_{cl}(y)}\right) \;=\; -g_{\mu\nu}\,\delta(x-y)\;, \tag{2.35}$$

$$\int d^4z\,\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\iota(z)\,\delta\bar{\eta}_\alpha(x)}\right)\left(\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta\psi_{cl,\,\beta}(y)\,\delta\bar{\psi}_{cl,\,\iota}(z)}\right) \;=\; -\delta_{\alpha\beta}\,\delta(x-y)\;, \tag{2.36}$$

$$\int d^4z\,\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\bar{\eta}_\iota(z)\,\delta\eta_\alpha(x)}\right)\left(\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta\bar{\psi}_{cl,\,\beta}(y)\,\delta\psi_{cl,\,\iota}(z)}\right) \;=\; -\delta_{\alpha\beta}\,\delta(x-y)\;. \tag{2.37}$$

$$-i\,g\,\Gamma^{\mu}(p', p; k) \;=$$



$$-i\,g^2\,\Gamma^{\mu\nu}(p', p; k, q) \;=$$

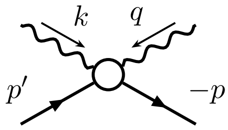


Figure 2.1: Diagrammatic representation of the photon-fermion (top) and the two-photon-two-fermion (bottom) one-particle irreducible vertices in momentum space. The straight lines with arrows refers to fermion legs and the wavy lines to photon legs. All Feynman diagrams were built using TikZ-FeynHand [1, 2].

2.3 Derivation of the Dyson-Schwinger Equations

The Dyson-Schwinger equations [3, 26], also known as the field equations of motion for a QFT, follow from the vanishing of the functional integral of a functional derivative with respect to a field

$$\int \mathcal{D}\phi \, \frac{\delta}{\delta\,\phi(x)} \equiv 0 \; . \tag{2.38}$$

Applying this to the scalar theory as an example, one writes:

$$\begin{aligned} 0 = & \int \mathcal{D}\phi \, \frac{\delta}{\delta\,\phi(x)} \exp\left\{ i\Big(S(\phi) + \int d^4x \; J(x)\,\phi(x) \Big) \right\} \\ = & \int \mathcal{D}\phi \; i \left[\frac{\delta\,S}{\delta\,\phi} + J \right] \exp\left\{ i\Big(S(\phi) + \int d^4x \; J(x)\,\phi(x) \Big) \right\} \; . \end{aligned} \tag{2.39}$$

And, by using the replacement explained in Eq. (2.16), this can be written as a differential equation in terms of the generating functional:

$$\left[\frac{\delta\,S}{\delta\,\phi} \left(\frac{\delta}{i\,\delta J} \right) + J \right] Z[J] \;=\; 0 \; . \tag{2.40}$$

Naturally, these are just the first few steps of the full derivation and one would need to do other derivatives, in accordance with the Green's function in question, as well as setting the sources to zero. In chapter 3 the Dyson-Schwinger equations for the fermion and photon propagator, as well as for the photon-fermion vertex for the two-photon-two-fermion vertex in QED will be derived.

3 Dyson-Schwinger equations for QED

The Dyson-Schwinger equations are a set of equations that relate all the Green's functions of the theory. In this chapter, we will use the definitions given in Sec. 2.2 and follow the procedure that was introduced in Sec. 2.3 applied to QED to derive the DSE for the fermion and photon propagator, the photon-fermion vertex and the two-photon-two-fermion vertex. For the latter, we will present an approximate equation in order to build a closed set of equations. Furthermore, we will derive the Ward-Takahashi identities for the two studied vertices and discuss the renormalisation of QED.

3.1 Fermion Dyson-Schwinger Equation

To get the DSE for the fermion propagator, also known as the fermion gap equation, we follow the procedure introduced in Sec. 2.3, taking in this case a derivative with respect to $\bar{\psi}_\alpha(x)$, which reads

$$\begin{aligned} 0 \;&=\; \int \mathcal{D}A\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \frac{\delta}{\delta\bar{\psi}_\alpha(x)} \exp\Bigg\{ i\, S_{QED}(A, \bar{\psi}, \psi) \\ &\qquad\qquad + \; i \int d^4 z \, [J^\mu(z)\, A_\mu(z) \;+\; \bar{\eta}(z)\, \psi(z) \;+\; \bar{\psi}(z)\, \eta(z)] \Bigg\} \\ &=\; \left[\frac{\delta\, S_{QED}}{\delta\bar{\psi}_\alpha(x)} \left(\frac{\delta}{i\,\delta J}, \frac{\delta}{-i\,\delta\eta}, \frac{\delta}{i\,\delta\bar{\eta}} \right) \;+\; \eta_\alpha(x) \right] Z[J, \bar{\eta}, \eta] \; . \end{aligned} \tag{3.1}$$

Using the Lagrangian density for QED, defined in Eq. (2.11), we can write

$$\begin{aligned} \frac{\delta\, S_{QED}}{\delta\bar{\psi}_\alpha(x)} \;&=\; \frac{\delta}{\delta\bar{\psi}_\alpha(x)} \int d^4 z \, \mathcal{L}_{QED}(z) \\ &=\; (i\, \partial\!\!\!/_x \;-\; m)_{\alpha\beta}\, \psi_\beta(x) \;-\; g\, (\gamma^\mu)_{\alpha\beta}\, A_\mu(x)\, \psi_\beta(x) \; , \end{aligned} \tag{3.2}$$

so that Eq. (3.1) becomes

$$\left\{[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta}\left(\frac{\delta}{i\,\delta\bar{\eta}_\beta(x)}\right) \;-\; g\,(\gamma^\mu)_{\alpha\beta}\left(\frac{\delta}{i\,\delta\bar{\eta}_\beta(x)}\right)\left(\frac{\delta}{i\,\delta J^\mu(x)}\right) \;+\; \eta_\alpha(x)\right\} Z[J,\bar{\eta},\eta] \;=\; 0\;, \quad (3.3)$$

where the index in $\partial\!\!\!/_x$ indicates that the partial derivative is taken with respect to x. Since we are interested in obtaining the equation for the fermion 2-point Green's function, we still need to take one additional functional derivative. Thus, deriving Eq. (3.2) with respect to $i\,\delta/\delta\eta_\theta(y)$, it becomes

$$[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta}\left(\frac{\delta^2 Z[J,\bar{\eta},\eta]}{\delta\eta_\theta(y)\,\delta\bar{\eta}_\beta(x)}\right) \;+\; g\,(\gamma^\mu)_{\alpha\beta}\,i\left(\frac{\delta^3 Z[J,\bar{\eta},\eta]}{\delta\eta_\theta(y)\,\delta\bar{\eta}_\beta(x)\,\delta J^\mu(x)}\right)$$
$$+\; i\,\delta_{\alpha\theta}\,\delta(x-y)Z[J,\bar{\eta},\eta] \;+\; \eta_\alpha(x)\,i\,\frac{\delta Z[J,\bar{\eta},\eta]}{\delta\eta_\theta(y)} \;=\; 0\;. \quad (3.4)$$

Next, using the definition of the generating functional of the connected Green's functions, Eq. (2.19), and the chain rule, as well as setting the sources to zero, we obtain

$$\left\{i\,[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta}\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\theta(y)\,\delta\bar{\eta}_\beta(x)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} \;-\; g\,(\gamma^\mu)_{\alpha\beta}\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta\eta_\theta(y)\,\delta\bar{\eta}_\beta(x)\,\delta J^\mu(x)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0}\right.$$
$$\left.+\; i\,\delta_{\alpha\theta}\,\delta(x-y)\right\} Z[0,0,0] \;=\; 0\;. \quad (3.5)$$

The normalisation of the generating functional, see Eq. (2.17), sets $Z[0,0,0] \;=\; 1$. We can also substitute the fermion propagator, defined in Eq. (2.31), so that

$$-\,i\,[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta}\,S_{\beta\theta}(x-y) \;-\; g\,(\gamma^\mu)_{\alpha\beta}\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta\eta_\theta(y)\,\delta\bar{\eta}_\beta(x)\,\delta J^\mu(x)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0}$$
$$+\; i\,\delta_{\alpha\theta}\,\delta(x-y) \;=\; 0\;. \quad (3.6)$$

By multiplying this whole equation by the inverse of the fermion propagator $S^{-1}_{\theta\theta'}(y-y')$, integrating it with respect to y and relabelling, we find

$$S^{-1}_{\alpha\beta}(x-y) \;=\; [i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta}\,\delta(x-y)$$
$$-\; i\,g\,(\gamma^\mu)_{\alpha\alpha'}\int d^4z\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta\eta_{\beta'}(z)\,\delta\bar{\eta}_{\alpha'}(x)\,\delta J^\mu(x)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} S^{-1}_{\beta'\beta}(z-y)\;. \quad (3.7)$$

Furthermore, we can use the results obtained in Appendix A, more specifically Eq. (A.5), to rewrite the connected three-point Green's function in terms of one-particle irreducible functions. Then, doing the integrations with respect to z and u_3, we obtain the fermion gap equation in coordinate space:

$$
\begin{aligned}
S^{-1}(x-y) \;=\; & [i\,\not{\partial}_x \;-\; m]\,\delta(x-y) \\
& -\; i\,g^2 \int d^4u_1\, d^4u_2\, D_{\mu\mu'}(x-u_1)\Big[\gamma^\mu S(x-u_2)\,\Gamma^{\mu'}(u_2,y;u_1)\Big]\;, \qquad (3.8)
\end{aligned}
$$

where the vertex Γ^μ is defined in Eq. (2.32). Using the Fourier transform defined in Eq. (2.34), we obtain the fermion gap equation in momentum space:

$$
S^{-1}(p) \;=\; (\not{p} \;-\; m) \;-\; i\,g^2 \int \frac{d^4k}{(2\,\pi)^4}\, D_{\mu\nu}(k)\,\gamma^\mu\, S(p-k)\,\Gamma^\nu(p-k,-p;k)\;, \qquad (3.9)
$$

where the vertex in defined with all momenta incoming, see Fig. 2.1. In this equation, we can identify $-i\,(\not{p}-m)$ as the inverse of the tree-level fermion propagator and $-i\,g\,\gamma^\mu$ as the tree-level vertex. Eq. (3.9) is represented diagrammatically in Fig. 3.1, where the propagators with solid blobs are full quantities and the vertex with an empty blob is 1PI.. The vertex with a dot refers to the tree-level vertex.

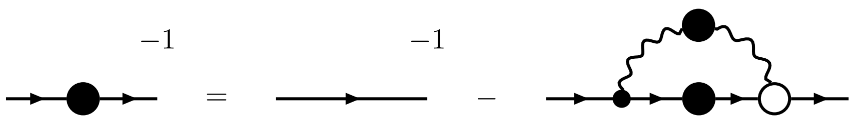


Figure 3.1: Diagrammatic representation of the fermion gap equation in momentum space. The solid blobs indicate full quantities, while the empty blobs indicate one-particle irreducible quantities. The vertex with a dot refers to the tree-level vertex. This notation is used in all diagrams.

3.1.1 The Fermion Propagator

In general, we can decompose the Dirac structure of the fermion propagator and write it in terms of two scalar functions as

$$i\,S(p) \;=\; i\,\frac{\mathcal{F}(p^2)}{\not p \;-\; \mathcal{M}(p^2) \;+\; i\epsilon}\;, \tag{3.10}$$

where $\mathcal{F}(p^2)$ is the fermion wave-function renormalisation and $\mathcal{M}(p^2)$ the dynamical fermion mass. Or, equivalently,

$$i\,S^{-1}(p) \;=\; i\left[\mathcal{A}(p^2)\,\not p \;-\; \mathcal{B}(p^2) \;+\; i\epsilon\right] \tag{3.11}$$

where $\mathcal{F}(p^2) \;=\; 1/\mathcal{A}(p^2)$ and $\mathcal{M}(p^2) \;=\; \mathcal{B}(p^2)/\mathcal{A}(p^2)$.

It follows that, by considering the parametrisation given in Eq. (3.11), taking the trace of the fermion gap equation, Eq. (3.9), and dividing the resulting equation by (-4), we find an equation for $\mathcal{B}(p^2)$:

$$\mathcal{B}(p^2) \;=\; m \;+\; \frac{1}{4}\,\mathrm{Tr}\Big[\Sigma(p)\Big]\;, \tag{3.12}$$

where the fermion self-energy $\Sigma(p)$ is defined as

$$\Sigma(p) \;=\; i\,g^2 \int \frac{d^4k}{(2\,\pi)^4}\, D_{\mu\nu}(k)\,\gamma^{\mu}\,S(p-k)\,\Gamma^{\nu}(p-k,-p;k)\;. \tag{3.13}$$

Just as we did with the fermion propagator, we can also decompose the fermion self-energy in terms of a scalar and a vectorial functions, $\Sigma_s(p^2)$ and $\Sigma_s(p^2)$ respectively,

$$\Sigma(p) = \Sigma_v(p^2)\,\not p \;+\; \Sigma_s(p^2)\;, \tag{3.14}$$

so Eq. (3.12) becomes

$$\mathcal{B}(p^2) \;=\; m \;+\; \Sigma_s(p^2)\;. \tag{3.15}$$

On the other hand, if we first multiply Eq. (3.9) by $\not p$ and then take the trace, followed by dividing the resulting equation by $(4\,p^2)$, we obtain the equation for $\mathcal{A}(p^2)$:

$$\mathcal{A}(p^2) \;=\; 1 \;-\; \frac{1}{4\,p^2}\,\mathrm{Tr}\Big[\not{p}\,\Sigma(p)\Big]$$
$$=\; 1 \;-\; \Sigma_v(p^2)\;. \tag{3.16}$$

3.2 Photon Dyson-Schwinger Equation

To derive the Dyson-Schwinger equation for the photon propagator, we follow a very similar procedure to the one employed in the case of the fermion gap equation and start by taking a derivative with respect to the field $A_\mu(x)$:

$$0 \;=\; \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\;\frac{\delta}{\delta A_\mu(x)}\,\exp\Bigg\{ i\,S_{QED}(A,\bar{\psi},\psi) \;+\; i\int d^4z\,[J^\lambda(z)\,A_\lambda(z) \;+\; \bar{\eta}(z)\,\psi(z) \;+\; \bar{\psi}(z)\,\eta(z)]\Bigg\}$$
$$=\; \left[\frac{\delta\,S_{QED}}{\delta A_\mu(x)}\left(\frac{\delta}{i\,\delta J},\frac{\delta}{-i\,\delta\eta},\frac{\delta}{i\,\delta\bar{\eta}}\right) \;+\; J^\mu(x)\right] Z[J,\bar{\eta},\eta]\;. \tag{3.17}$$

Performing the derivative of the action, we find:

$$\Bigg\{\left[\Box_x\,g^{\mu\mu'} \;-\; \left(1 \;-\; \frac{1}{\xi}\right)\partial^\mu_x\,\partial^{\mu'}_x\right]\left(\frac{\delta}{i\,\delta J^{\mu'}(x)}\right) \;-\; g\,(\gamma^\mu)_{\alpha\beta}\left(-\,\frac{\delta}{i\,\delta\eta_\alpha(x)}\right)\left(\frac{\delta}{i\,\delta\bar{\eta}_\beta(x)}\right) \;+\; J^\mu(x)\Bigg\}\,Z[J,\bar{\eta},\eta] = 0\;, \tag{3.18}$$

where $\Box_x \;=\; \partial^\lambda_x\,\partial_{\lambda,x}$ is the d'Alembert operator with respect to x. Taking a further derivative with respect to $J^\nu(y)$ and writing the equation in terms of the generating functional of the connected Green's functions, using for that the definition given in Eq. (2.19) and the chain rule, this becomes

$$\Bigg\{\left[\Box_x\,g^{\mu\mu'} \;-\; \left(1 \;-\; \frac{1}{\xi}\right)\partial^\mu_x\,\partial^{\mu'}_x\right]\left(\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^\nu(y)\,\delta J^{\mu'}(x)}\right) \;-\; i\,g\,(\gamma^\mu)_{\alpha\beta}\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^\nu(y)\,\delta\eta_\alpha(x)\,\delta\bar{\eta}_\beta(x)}\right) \;+\; g^\mu_{\;\;\nu}\,\delta(x-y) + \cdots\Bigg\}\,Z[J,\bar{\eta},\eta] = 0\;, \tag{3.19}$$

where $\cdots$ represents the terms that vanish when the sources are set to zero. Using the definition of the photon propagator, given in Eq. (2.30), after setting the sources to zero, Eq. (3.19) becomes

$$\left[\Box_x\, g^{\mu\mu'} \;-\; \left(1 \;-\; \frac{1}{\xi}\right) \partial_x^{\mu}\, \partial_x^{\mu'}\right] D_{\mu'\nu}(x-y)$$
$$+\; i\, g\, (\gamma^{\mu})_{\alpha\beta} \left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^{\nu}(y)\, \delta\eta_{\alpha}(x)\, \delta\bar{\eta}_{\beta}(x)} \right) \;-\; g^{\mu}{}_{\nu}\, \delta(x-y) = 0 \; . \quad (3.20)$$

Considering a linear covariant gauge with $\xi \neq 0$, we can multiply Eq. (3.20) by the inverse of the photon propagator $[D^{\nu\nu'}]^{-1}(y-z)$, integrate with respect to y and relabel. This leaves us with

$$[D^{\mu\nu}(x-y)]^{-1} \;=\; \left[\Box_x\, g^{\mu\nu} \;-\; \left(1 \;-\; \frac{1}{\xi}\right) \partial_x^{\mu}\, \partial_x^{\nu}\right] \delta(x-y)$$
$$+\; i\, g\, (\gamma^{\mu})_{\alpha\beta} \int d^4z \left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^{\sigma}(z)\, \delta\eta_{\alpha}(x)\, \delta\bar{\eta}_{\beta}(x)} \right) [D^{\sigma\nu}(z-y)]^{-1} \; . \quad (3.21)$$

We can use once again the decomposition of the three-point connected Green's function given in Eq. (A.5). Substituting this in Eq. (3.21) and doing the integrations with respect to z and u_1, we obtain the Dyson-Schwinger equation for the photon propagator in coordinate space:

$$[D^{\mu\nu}(x-y)]^{-1} \;=\; \left[\Box_x\, g^{\mu\nu} \;-\; \left(1 \;-\; \frac{1}{\xi}\right) \partial_x^{\mu}\, \partial_x^{\nu}\right] \delta(x-y)$$
$$+\; i\, g^2\, N_f \int d^4u_2\, d^4u_3\; \mathrm{Tr}[\gamma^{\mu}\, S(x-u_2)\, \Gamma^{\nu}(u_2,u_3;y)\, S(u_3-y)] \; , \quad (3.22)$$

where N_f stands for the number of fermion flavours, assuming that all of them couple to the electromagnetic field with the same strength g. This accounts for the number of distinct flavour loops which can occur.

Using the Fourier transformation defined in Eq. (2.34) we obtain the photon DSE in momentum space, given by

$$[D^{\mu\nu}(k)]^{-1} \;=\; k^2 \left[-\, g^{\mu\nu} \;+\; \left(1 \;-\; \frac{1}{\xi}\right) \frac{k^\mu\, k^\nu}{k^2} \right] \\ +\; i\, g^2\, N_f \int \frac{d^4p}{(2\,\pi)^4}\, \mathrm{Tr}[\gamma^\mu\, S(p)\, \Gamma^\nu(p, -p+k; -k)\, S(p-k)] \;, \quad (3.23)$$

where the first term in the right-hand side of the equation can be identified as the tree-level photon propagator. The diagrammatic representation of Eq. (3.23) is shown in Fig. 3.2.

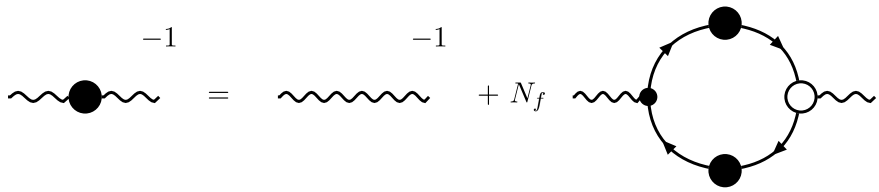


Figure 3.2: Diagrammatic representation of the Dyson-Schwinger equation for the photon propagator in momentum space.

It is worth mentioning that the results for the Landau gauge must be obtained from the ones derived for a general linear covariant gauge by setting the gauge parameter to zero. Since there is no inverse for the photon propagator in the Landau gauge, some of the formal manipulations done here would be meaningless if we considered $\xi \;=\; 0$ from the very beginning.

3.2.1 The Photon Propagator

Similarly to what we did in Sec. 3.1.1, the photon propagator can be written as

$$i\, D_{\mu\nu}(k) \;=\; i \left[-\, P^T_{\mu\nu}(k)\, \frac{\mathcal{G}(k^2)}{k^2} \;-\; \frac{\xi}{k^2}\, P^L_{\mu\nu}(k) \right] \,, \qquad (3.24)$$

where $\mathcal{G}(k^2)$ is a scalar function called the photon wave-function renormalisation and $P^T_{\mu\nu}(k)$ and $P^L_{\mu\nu}(k)$ are, respectively, the transverse and longitudinal projection operators with respect to the photon momentum, defined as

$$P^T_{\mu\nu}(k) \;=\; g_{\mu\nu} - \frac{k_\mu\, k_\nu}{k^2} \;, \qquad (3.25)$$

$$P^L_{\mu\nu}(k) \;=\; \frac{k_\mu\, k_\nu}{k^2} \;. \qquad (3.26)$$

Substituting the expression for the photon propagator in momentum space given in Eq. (3.24) into the DSE given in Eq. (3.23), it becomes

$$\frac{k^2\,P^T_{\mu\nu}(k)}{\mathcal{G}(k^2)} \;=\; k^2\,P^T_{\mu\nu}(k) \;-\; i\,g^2 \int \frac{d^4p}{(2\,\pi)^4}\,\mathrm{Tr}[\gamma^\mu\,S(p)\,\Gamma^\nu(p,-p+k;-k)\,S(p-k)]\;. \tag{3.27}$$

And by contracting the Lorentz indices, we obtain an equation for the inverse of $\mathcal{G}(k^2)$:

$$\frac{1}{\mathcal{G}(k^2)} \;=\; 1 \;+\; \Pi(k^2)\;, \tag{3.28}$$

where the scalar photon polarisation $\Pi(k^2)$ is given by

$$\Pi(k^2) \;=\; -i\,N_f\,\frac{g^2}{3}\,\frac{1}{k^2}\int\frac{d^4p}{(2\,\pi)^4}\,\mathrm{Tr}\Big[\gamma_\mu\,S(p)\,\Gamma^\mu(p,-p+k;-k)\,S(p-k)\Big]\;. \tag{3.29}$$

3.3 Photon-Fermion Vertex Dyson-Schwinger Equation

To obtain the Dyson-Schwinger equation for the photon-fermion vertex, we start by taking an additional derivative with respect to $\delta/i\delta J^\nu(w)$ of Eq. (3.4), which reads

$$-\,i\,[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta}\left(\frac{\delta^3 Z[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\theta(y)\,\delta\bar\eta_\beta(x)}\right) \;+\; g\,(\gamma^\mu)_{\alpha\beta}\left(\frac{\delta^4 Z[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^\mu(x)\,\delta\eta_\theta(y)\,\delta\bar\eta_\beta(x)}\right)$$
$$+\;\delta_{\alpha\theta}\,\delta(x-y)\,\frac{Z[J,\bar\eta,\eta]}{\delta\,J^\nu(w)} \;+\; \eta_\alpha(x)\left(\frac{\delta^2 Z[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\theta(y)}\right) \;=\; 0\;. \tag{3.30}$$

It follows from Eq. (2.19) and the chain rule that, after relabelling,

$$[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta'}\left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)}\right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0}$$
$$+\;i\,g\,(\gamma^\mu)_{\alpha\beta'}\left(\frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^\mu(x)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)}\right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0}$$
$$-\;g\,(\gamma^\mu)_{\alpha\beta'}\,S_{\beta'\beta}(x-y)\,D_{\mu\nu}(x-w) \;=\; 0\;, \tag{3.31}$$

where the sources are set to zero and the photon and fermion propagators are defined in Eqs. (2.30) and (2.31). Following the same procedure used before, we can decompose the connected Green's functions in terms of 1PI functions. The decomposition of the three- and four-point connected Green's functions can be found in Eqs. (A.5) and (A.10) of Appendix A, respectively. Thus, in momentum space we obtain

$$
\begin{aligned}
&\Big[\gamma^\sigma\,(p_\sigma \;-\; k_\sigma) - m\Big]_{\alpha\beta'} D_{\nu\nu'}(k)\,\Big[S(p-k)\,\Gamma^{\nu'}(p-k,-p;k)\,S(p)\Big]_{\beta'\beta} \\
&-\; i\,g^2 \int \frac{d^4q}{(2\,\pi)^4}\, D_{\nu\nu'}(k)\, D_{\mu\mu'} \\
&\qquad \Big\{\gamma^\mu\, S(p-k-q)\,\Big[\Gamma^{\nu'\mu'}(p-k-q,-p;k,q) \\
&\qquad\qquad +\; \Gamma^{\mu'}(p-k-q,-p+k;q)\,S(p-k)\,\Gamma^{\nu'}(p-k,-p;k) \\
&\qquad\qquad +\; \Gamma^{\nu'}(p-k-q,-p+q;k)\,S(p-q)\,\Gamma^{\mu'}(p-q,-p;q)\Big]\,S(p)\Big\}_{\alpha\beta} \\
&-\; D_{\nu\nu'}\,\Big[\gamma^{\nu'}\,S(p)\Big]_{\alpha\beta} \;=\; 0\;. \qquad (3.32)
\end{aligned}
$$

We can simplify this further by using the fermion gap equation, Eq. (3.9), in the first term. This will result in two terms, one of which cancels exactly with the second term inside the integral. Then, after multiplying the equation by the inverse of the photon and fermion propagators and relabelling, the DSE for the photon-fermion vertex in momentum space becomes

$$
\begin{aligned}
\Gamma^\mu(p,-p-k;k) \;&=\; \gamma^\mu \;+\; i\,g^2 \int \frac{d^4q}{(2\,\pi)^4} D_{\nu\nu'}(q)\,\Big[\gamma^\nu\,S(p-q)\,\Gamma^{\mu\nu'}(p-q,-p-k;k,q) \\
&+\; \gamma^\nu\,S(p-q)\,\Gamma^\mu(p-q,-p-k+q;k)\,S(p+k-q)\,\Gamma^{\nu'}(p+k-q,-p-k;q)\Big] \qquad (3.33)
\end{aligned}
$$

and it is diagrammatically represented in Fig. 3.3.

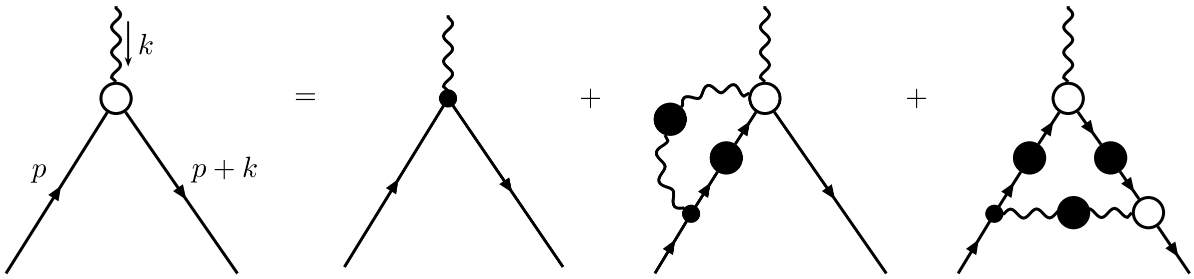


Figure 3.3: Diagrammatic representation of the Dyson-Schwinger equation for the photon-fermion vertex in momentum space.

3.4 Two-Photon-Two-Fermion Vertex Dyson-Schwinger Equation

Finally, by differentiating once more Eq. (3.30) with respect to $\delta/i\delta J^{\rho}(z)$, we obtain

$$
\begin{aligned}
&[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta} \left(\frac{\delta^4 Z[J,\bar{\eta},\eta]}{\delta J^{\rho}(z)\,\delta J^{\nu}(w)\,\delta\eta_{\theta}(y)\,\delta\bar{\eta}_{\beta}(x)} \right) \\
&\quad + \; i\,g\,(\gamma^{\mu})_{\alpha\beta} \left(\frac{\delta^5 Z[J,\bar{\eta},\eta]}{\delta J^{\rho}(z)\,\delta J^{\nu}(w)\,\delta J^{\mu}(x)\,\delta\eta_{\theta}(y)\,\delta\bar{\eta}_{\beta}(x)} \right) \\
&\qquad + \; \delta_{\alpha\theta}\,\delta(x-y) \left(\frac{\delta^2 Z[J,\bar{\eta},\eta]}{\delta J^{\rho}(z)\,\delta J^{\nu}(w)} \right) + \; \eta_{\alpha}(x) \left(\frac{\delta^3 Z[J,\bar{\eta},\eta]}{\delta J^{\rho}(z)\,\delta J^{\nu}(w)\,\delta\eta_{\theta}(y)} \right) \;=\; 0\;. \qquad (3.34)
\end{aligned}
$$

Following the usual procedure, we can write this in terms of the connected Green's functions generating functional W, Eq. (2.19). Thus, after relabelling and setting the sources to zero, it becomes

$$
\begin{aligned}
&[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta} \left(\frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \\
&+\; g\,(\gamma^\mu)_{\alpha\beta'} \Bigg[i \left(\frac{\delta^5 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta J^\nu(w)\,\delta J^\mu(x)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) \\
&\qquad -\; \left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) \left(\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta J^\mu(x)} \right) \\
&\qquad -\; \left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) \left(\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^\mu(x)} \right) \\
&\qquad -\; \left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta J^\nu(w)\,\delta J^\mu(x)} \right) \left(\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) \Bigg]\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \\
&+\; \Bigg[i\,(i\,\partial\!\!\!/_x \;-\; m)_{\alpha\beta'} \left(\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) \;-\; g\,(\gamma^\mu)_{\alpha\beta'} \left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\mu(x)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) \\
&\qquad +\; \delta_{\alpha\beta}\,\delta(x-y) \Bigg]\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \left(\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta J^\nu(w)} \right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \;=\; 0\;, \qquad (3.35)
\end{aligned}
$$

where, using Eq. (3.6), the expression inside the brackets in the last line vanish. Furthermore, applying Furry's theorem[1], the term proportional to the three-photon one-particle irreducible vertex also vanishes for QED. Thus, after identifying the photon propagators, see Eq. (2.30), we obtain

$$
\begin{aligned}
&[i\,\partial\!\!\!/_x \;-\; m]_{\alpha\beta} \left(\frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \\
&+\; g\,(\gamma^\mu)_{\alpha\beta'} \Bigg[i \left(\frac{\delta^5 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta J^\nu(w)\,\delta J^\mu(x)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) \\
&\qquad +\; \left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) D_{\mu\rho}(x-z) \\
&\qquad -\; \left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\rho(z)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(x)} \right) D_{\mu\nu}(x-w) \Bigg]\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \;=\; 0\;. \qquad (3.36)
\end{aligned}
$$

Using the results obtained in Appendix A, see Eqs. (A.5), (A.10) and (A.12), we can replace the connected Green's functions by their one-particle irreducible decompositions. Specifically

[1] Furry's theorem states that, in QED, the contribution of a diagram that consists of a closed fermion loop and an odd number of vertices vanishes.

for the 5-point connected Green's function, its complete decomposition, given in Eq. (A.12), includes higher-order one-particle irreducible functions that have not been considered so far and would therefore require the introduction of the Dyson-Schwinger equations associated with these new vertices. As mentioned previously, the Dyson-Schwinger equations form an infinite tower of coupled non-linear integral equations and consequently it is impossible to handle all Green's functions simultaneously. Hence, we need to introduce a truncation to obtain a closed set of equations. In our case, we will ignore the contributions of the five-point 1PI Green's function and the four-photon 1PI Green's function:

$$\left(\frac{\delta^5 \Gamma[A_{cl}, \psi_{cl}, \bar{\psi}_{cl}]}{\delta A_{cl,\,\rho}(x_5)\, \delta A_{cl,\,\nu}(x_4)\, \delta A_{cl,\,\mu}(x_3)\, \delta \psi_{cl,\,\beta'}(x_2)\, \delta \bar{\psi}_{cl,\,\alpha}(x_1)} \right) \approx 0 \,, \tag{3.37}$$

$$\left(\frac{\delta^4 \Gamma[A_{cl}, \psi_{cl}, \bar{\psi}_{cl}]}{\delta A_{cl,\,\rho}(x_4)\, \delta A_{cl,\,\nu}(x_3)\, \delta A_{cl,\,\mu}(x_2)\, \delta A_{cl,\,\sigma}(x_1)} \right) \approx 0 \,. \tag{3.38}$$

Nevertheless, by doing the decomposition of the four-point connected Green's function following the same procedure done in Appendix A, we find that it consists of a sum of the term containing the four-photon irreducible vertex and other terms that are proportional to the three-photon irreducible vertex, which vanishes in QED according to Furry's theorem. Thus, by neglecting the contribution of the four-photon 1PI Green's function, the contribution of the four-photon connected Green's function can also be disregarded. Then, applying that truncation and replacing the three- and four-point connected Green's function by their decompositions, the five-point connected Green's function in momentum space reads

$$\left(\frac{\delta^5 W[J,\bar{\eta},\eta]}{\delta J^\rho(s)\,\delta J^\nu(w)\,\delta J^\mu(z)\,\delta\eta_\beta(y)\,\delta\bar{\eta}_\alpha(x)} \right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} =$$

$$= \; g^3 \int \frac{d^4k_1}{(2\,\pi)^4}\,\frac{d^4k_2}{(2\,\pi)^4}\,\frac{d^4k_3}{(2\,\pi)^4}\,\frac{d^4p_1}{(2\,\pi)^4}\,\frac{d^4p_2}{(2\,\pi)^4}\,e^{-i\,(k_1\,s\;+\;k_2\,w\;+\;k_3\,z\;+\;p_1\,x\;-\;p_2\,y)}$$

$$(2\,\pi)^4\,\delta(k_1\;+\;k_1\;+\;k_2\;+\;k_3\;+\;p_1\;-\;p_2)$$

$$D_{\rho\rho'}(k_1)\,D_{\nu\nu'}(k_2)\,D_{\mu\mu'}(k_3)$$

$$\Big\{ S(p_1) \Big[\Gamma^{\rho'}(p_1,-p_1-k_1;k_1)\,S(p_1+k_1)\,\Gamma^{\nu'}(p_1+k_1,-p_2+k_3;k_2)$$

$$S(p_2-k_3)\,\Gamma^{\mu'}(p_2-k_3,-p_2;k_3)$$

$$+\;\Gamma^{\rho'}(p_1,-p_1-k_1;k_1)\,S(p_1+k_1)\,\Gamma^{\mu'}(p_1+k_1,-p_2+k_2;k_3)$$

$$S(p_2-k_2)\,\Gamma^{\nu'}(p_2-k_2,-p_2;k_2)$$

$$+\;\Gamma^{\nu'}(p_1,-p_1-k_2;k_2)\,S(p_1+k_2)\,\Gamma^{\rho'}(p_1+k_2,-p_2+k_3;k_1)$$

$$S(p_2-k_3)\,\Gamma^{\mu'}(p_2-k_3,-p_2;k_3)$$

$$+\;\Gamma^{\nu'}(p_1,-p_1-k_2;k_2)\,S(p_1+k_2)\,\Gamma^{\mu'}(p_1+k_2,-p_2+k_1;k_3)$$

$$S(p_2-k_1)\,\Gamma^{\rho'}(p_2-k_1,-p_2;k_1)$$

$$+\;\Gamma^{\mu'}(p_1,-p_1-k_3;k_3)\,S(p_1+k_3)\,\Gamma^{\rho'}(p_1+k_3,-p_2+k_2;k_1)$$

$$S(p_2-k_2)\,\Gamma^{\nu'}(p_2-k_2,-p_2;k_2)$$

$$+\;\Gamma^{\mu'}(p_1,-p_1-k_3;k_3)\,S(p_1+k_3)\,\Gamma^{\nu'}(p_1+k_3,-p_2+k_1;k_2)$$

$$S(p_2-k_1)\,\Gamma^{\nu'}(p_2-k_1,-p_2;k_1)$$

$$+\;\Gamma^{\rho'}(p_1,-p_1-k_1;k_1)\,S(p_1+k_1)\,\Gamma^{\nu'\mu'}(p_1+k_1,-p_2;k_2,k_3)$$

$$+\;\Gamma^{\nu'}(p_1,-p_1-k_2;k_2)\,S(p_1+k_2)\,\Gamma^{\rho'\mu'}(p_1+k_2,-p_2;k_1,k_3)$$

$$+\;\Gamma^{\mu'}(p_1,-p_1-k_3;k_3)\,S(p_1+k_3)\,\Gamma^{\rho'\nu'}(p_1+k_3,-p_2;k_1,k_2)$$

$$+\;\Gamma^{\nu'\mu'}(p_1,-p_2+k_1;k_2,k_3)\,S(p_2-k_1)\,\Gamma^{\rho'}(p_2-k_1,-p_2;k_1)$$

$$+\;\Gamma^{\rho'\mu'}(p_1,-p_2+k_2;k_1,k_3)\,S(p_2-k_2)\,\Gamma^{\nu'}(p_2-k_2,-p_2;k_2)$$

$$+\;\Gamma^{\rho'\nu'}(p_1,-p_2+k_3;k_1,k_2)\,S(p_2-k_3)\,\Gamma^{\mu'}(p_2-k_3,-p_2;k_3)\;\Big]\,S(p_2)\Big\}_{\alpha\beta}\,, \qquad (3.39)$$

where the irreducible vertices are defined in Eqs. (2.32) and (2.33). Then, proceeding as usual, we can replace the connected Green's functions in Eq. (3.36) by their decompositions and perform the necessary Fourier transformations. After multiplying the resulting equation by the inverse of the fermion and photon propagators and relabelling, it becomes

$$
\begin{aligned}
&(\not p \; - \; m)\, S(p) \Big[\Gamma^{\rho\nu}(p, -p-k-q; k, q) \\
&\qquad\qquad + \; \Gamma^{\rho}(p, -p-k; k)\, S(p+k)\, \Gamma^{\nu}(p+k, -p-k-q; q) \\
&\qquad\qquad + \; \Gamma^{\nu}(p, -p-q; q)\, S(p+q)\, \Gamma^{\rho}(p+q, -p-k-q; k) \Big] \\
&- \; \gamma^{\rho}\, S(p+k)\, \Gamma^{\nu}(p+k, -p-k-q; q) \\
&- \; \gamma^{\nu}\, S(p+q)\, \Gamma^{\rho}(p+q, -p-k-q; k) \\
&- \; i\, g^2 \int \frac{d^4 w}{(2\,\pi)^4}\, D_{\mu\mu'}(w)\, \gamma^{\mu}\, S(p-w) \\
&\quad \Big\{ \Gamma^{\rho}(p-w, -p-k+w; k)\, S(p+k-w) \\
&\qquad \Big[\Gamma^{\nu}(p+k-w, -p-k-q+w; q)\, S(p+k+q-w) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \Gamma^{\mu'}(p+k+q-w, -p-k-q; w) \\
&\qquad + \; \Gamma^{\mu'}(p+k-w, -p-k; w)\, S(p+k)\, \Gamma^{\nu}(p+k, -p-k-q; q) \Big] \\
&\quad + \; \Gamma^{\nu}(p-w, -p-q+w; q)\, S(p+q-w) \\
&\qquad \Big[\Gamma^{\rho}(p+q-w, -p-k-q+w; q)\, S(p+k+q-w) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \Gamma^{\mu'}(p+k+q-w, -p-k-q; w) \\
&\qquad + \; \Gamma^{\mu'}(p+q-w, -p-q; w)\, S(p+q)\, \Gamma^{\rho}(p+q, -p-k-q; k) \Big] \\
&\quad + \; \Gamma^{\mu'}(p-w, -p; w)\, S(p) \\
&\qquad \Big[\Gamma^{\rho}(p, -p-k; k)\, S(p+k)\, \Gamma^{\nu}(p+k, -p-k-q; q) \\
&\qquad + \; \Gamma^{\nu}(p, -p-q; q)\, S(p+q)\, \Gamma^{\rho}(p+q, -p-k-q; k) \Big] \\
&\quad + \; \Gamma^{\rho}(p-w, -p-k+w; k)\, S(p+k-w)\, \Gamma^{\nu\mu'}(p+k-w, -p-k-q; q, w)
\end{aligned}
$$

$$+ \; \Gamma^{\nu}(p-w,-p-q+w;q)\, S(p+q-w)\, \Gamma^{\rho\mu'}(p+q-w,-p-k-q;k,w)$$

$$+ \; \Gamma^{\mu'}(p-w,-p;w)\, S(p)\, \Gamma^{\rho\nu}(p,-p-k-q;k,q)$$

$$+ \; \Gamma^{\rho\nu}(p-w,-p-k-q+w;k,q)\, S(p+k+q-w)\, \Gamma^{\mu'}(p+k+q-w,-p-k-q;w)$$

$$+ \; \Gamma^{\rho\mu'}(p-w,-p-k;k,w)\, S(p+k)\, \Gamma^{\nu}(p+k,-p-k-q;q)$$

$$+ \; \Gamma^{\nu\mu'}(p-w,-p-q;q,w)\, S(p+q)\, \Gamma^{\nu}(p+q,-p-k-q;k) \Big\} \; = \; 0 \; . \tag{3.40}$$

Next, using the fermion gap equation, given in Eq. (3.9), we can simplify the terms that are multiplied by $(\not{p} \; - \; m)$ together with the appropriated integral terms, i.e.

$$\left[(\not{p} \; - \; m) \; - \; i\, g^2 \int \frac{d^4 w}{(2\,\pi)^4}\, D_{\mu\mu'}(w)\, \gamma^{\mu}\, S(p-w) \right] S(p)$$

$$\Big[\Gamma^{\rho\nu}(p,-p-k-q;k,q)$$

$$+ \; \Gamma^{\rho}(p,-p-k;k)\, S(p+k)\, \Gamma^{\nu}(p+k,-p-k-q;q)$$

$$+ \; \Gamma^{\nu}(p,-p-q;q)\, S(p+q)\, \Gamma^{\rho}(p+q,-p-k-q;k) \Big] \; =$$

$$= \; \Gamma^{\rho\nu}(p,-p-k-q;k,q)$$

$$+ \; \Gamma^{\rho}(p,-p-k;k)\, S(p+k)\, \Gamma^{\nu}(p+k,-p-k-q;q)$$

$$+ \; \Gamma^{\nu}(p,-p-q;q)\, S(p+q)\, \Gamma^{\rho}(p+q,-p-k-q;k) \; . \tag{3.41}$$

Additionally, we can also use the Dyson-Schwinger equation for the photon-fermion vertex, Eq. (3.33), to further simplify the equation. Indeed, we can write the two last terms in Eq. (3.41) as

$$
\begin{aligned}
&\Gamma^{\rho}(p,-p-k;k)\,S(p+k)\,\Gamma^{\nu}(p+k,-p-k-q;q) \\
&+\ \Gamma^{\nu}(p,-p-q;q)\,S(p+q)\,\Gamma^{\rho}(p+q,-p-k-q;k)\ = \\
&\quad = \ \Big\{\gamma^{\rho}\ +\ i\,g^2\int\frac{d^4w}{(2\,\pi)^4}D_{\mu\mu'}(w)\,\gamma^{\mu}\,S(p-w)\,\Big[\Gamma^{\rho\mu'}(p-w,-p-k;k,w) \\
&\qquad\qquad +\ \Gamma^{\rho}(p-w,-p-k+w;k)\,S(p+k-w)\,\Gamma^{\mu'}(p+k-w,-p-k;w)\Big]\Big\} \\
&\qquad\qquad\qquad\qquad S(p+k)\,\Gamma^{\nu}(p+k,-p-k-q;q) \\
&\quad +\ \Big\{\gamma^{\nu}\ +\ i\,g^2\int\frac{d^4w}{(2\,\pi)^4}D_{\mu\mu'}(w)\,\gamma^{\mu}\,S(p-w)\,\Big[\Gamma^{\nu\mu'}(p-w,-p-q;q,w) \\
&\qquad\qquad +\ \Gamma^{\nu}(p-w,-p-q+w;q)\,S(p+q-w)\,\Gamma^{\mu'}(p+q-w,-p-q;w)\Big]\Big\} \\
&\qquad\qquad\qquad\qquad S(p+q)\,\Gamma^{\rho}(p+q,-p-k-q;k)\ ,
\end{aligned}
\tag{3.42}
$$

which yields cancellations with other terms of Eq. (3.40). Thus, the truncated Dyson-Schwinger equation for the two-photon-two-fermion vertex in momentum space becomes

$$
\begin{aligned}
&\Gamma^{\mu\nu}(p,-p-k-q;k,q)\ = \\
&\quad =\ i\,g^2\int\frac{d^4w}{(2\,\pi)^4}\,D_{\sigma\sigma'}(w)\,\gamma^{\sigma}\,S(p-w) \\
&\qquad \Big\{\Big[\Gamma^{\mu}(p-w,-p-k+w;k)\,S(p+k-w)\,\Gamma^{\nu}(p+k-w,-p-k-q+w;q) \\
&\qquad +\ \Gamma^{\nu}(p-w,-p-q+w;q)\,S(p+q-w)\,\Gamma^{\mu}(p+q-w,-p-k-q+w;q) \\
&\qquad +\ \Gamma^{\mu\nu}(p-w,-p-k-q+w;k,q)\Big] \\
&\qquad\qquad\qquad S(p+k+q-w)\,\Gamma^{\sigma'}(p+k+q-w,-p-k-q;w) \\
&\qquad +\ \Gamma^{\mu}(p-w,-p-k+w;k)\,S(p+k-w)\,\Gamma^{\nu\sigma'}(p+k-w,-p-k-q;q,w) \\
&\qquad +\ \Gamma^{\nu}(p-w,-p-q+w;q)\,S(p+q-w)\,\Gamma^{\mu\sigma'}(p+q-w,-p-k-q;k,w)\Big\}
\end{aligned}
\tag{3.43}
$$

and it is represented diagrammatically in Fig. 3.4.

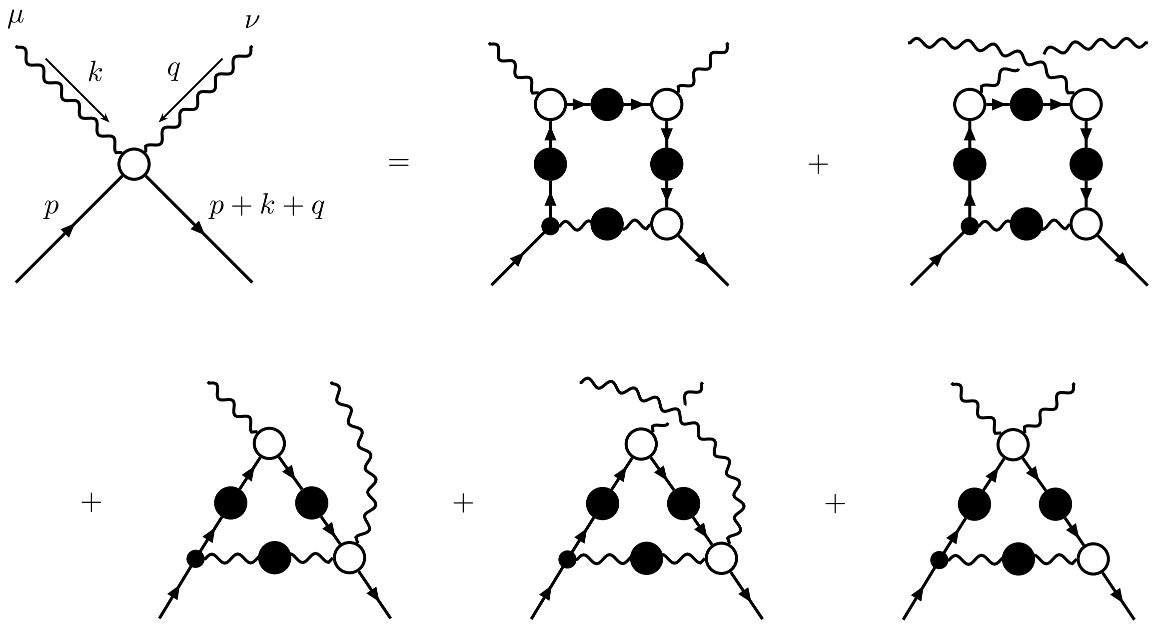


Figure 3.4: Diagrammatic representation of the truncated Dyson-Schwinger equation for the two-photon-two-fermion vertex in momentum space.

3.5 Ward-Takahashi Identities

In Sec. 2.1, the Lagrangian density for QED was built to respect the gauge invariance principle. However, we are forced to introduce the gauge fixing term in order to remove the infinite contribution associated with the gauge freedom of the field A_μ and doing so breaks the gauge invariance of the QED Lagrangian density, Eq. (2.11). Nevertheless, the generating functional Z must be gauge invariant, which leads to the Ward-Takahashi identities (WTIs) [4, 27, 28].

In order to derive the Ward-Takahashi identities, let us consider the generating functional of QED, given by Eq. (2.17),

$$Z[J,\bar{\eta},\eta] \;=\; \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; \exp\Big\{ i \int d^4x \Big[\bar{\psi}\,(i\,\not{D} \;-\; m)\,\psi \;-\; \frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \\ -\; \frac{1}{2\,\xi}(\partial_\mu\,A^\mu)^2 \;+\; J^\mu\,A_\mu \;+\; \bar{\eta}\,\psi \;+\; \bar{\psi}\,\eta \Big] \Big\} \;, \quad (3.44)$$

Where the normalising factor $1/Z[0,0,0]$ is implicit. Performing the infinitesimal local gauge transformations, see Eqs. (2.8) - (2.10),

$$\psi(x) \;\to\; \psi(x) \;+\; i\,\lambda(x)\,\psi(x)\;, \tag{3.45}$$

$$\bar{\psi}(x) \;\to\; \bar{\psi}(x) \;-\; i\,\lambda(x)\,\bar{\psi}(x)\;, \tag{3.46}$$

$$A_\mu(x) \;\to\; A_\mu(x) - \frac{1}{g}\,\partial_\mu\,\lambda(x)\;, \tag{3.47}$$

we can write the generating functional as

$$Z \;+\; \delta Z \;=\; \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\;\exp\Big\{ i \int d^4x \,\Big[\mathcal{L}_{QED}(A_\mu, \bar{\psi}, \psi) \;+\; J^\mu\,A_\mu \;+\; \bar{\eta}\,\psi \;+\; \bar{\psi}\,\eta \\ +\; \Big(\frac{1}{g\,\xi}\,\Box\,(\partial^\mu\,A_\mu) \;+\; \frac{1}{g}\,(\partial_\mu\,J^\mu) \;+\; i\,(\bar{\eta}\,\psi \;-\; \bar{\psi}\,\eta)\Big)\,\lambda\Big]\Big\}\;. \tag{3.48}$$

Then, expanding the exponential of the extra terms to $\mathcal{O}(\lambda)$ the variation δZ becomes

$$\delta Z \;=\; \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\Big[\frac{1}{g\,\xi}\,\Box\,(\partial^\mu\,A_\mu) \;+\; \frac{1}{g}\,(\partial_\mu\,J^\mu) \;+\; i\,(\bar{\eta}\,\psi \;-\; \bar{\psi}\,\eta)\Big]\,\lambda \\ \exp\Big\{ i \int d^4x \,\Big[\mathcal{L}_{QED}(A_\mu, \bar{\psi}, \psi) \;+\; J^\mu\,A_\mu \;+\; \bar{\eta}\,\psi \;+\; \bar{\psi}\,\eta\Big]\Big\} \tag{3.49}$$

and, making the replacements given in Eq. (2.18), we can write the condition $\delta Z \;=\; 0$ as

$$\frac{1}{g\,\xi}\,\Box_x\,\partial_x^\mu\left(\frac{\delta Z[J,\bar{\eta},\eta]}{i\,J^\mu(x)}\right) \;+\; \frac{1}{g}\,(\partial_x^\mu\,J_\mu(x))\,Z[J,\bar{\eta},\eta] \\ +\; \bar{\eta}_\sigma(x)\left(\frac{\delta Z[J,\bar{\eta},\eta]}{\delta\bar{\eta}_\sigma(x)}\right) \;+\; \left(\frac{\delta Z[J,\bar{\eta},\eta]}{\delta\eta_\sigma(x)}\right)\eta_\sigma(x) \;=\; 0\;. \tag{3.50}$$

Since we are interested in the Ward-Takahashi identity for the photon-fermion vertex, we must take additional functional derivatives with respect to $\bar{\eta}_\alpha(y)$ and $\eta_\beta(z)$, which reads

$$-\,\frac{i}{g\,\xi}\,\Box_x\,\partial_x^\mu\left(\frac{\delta^3 Z[J,\bar{\eta},\eta]}{\delta J^\mu(x)\,\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)}\right) \;+\; \frac{1}{g}\,(\partial_x^\mu\,J_\mu(x))\left(\frac{\delta^2 Z[J,\bar{\eta},\eta]}{\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)}\right) \\ +\; \delta(x-y)\left(\frac{\delta^2 Z[J,\bar{\eta},\eta]}{\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(x)}\right) \;+\; \bar{\eta}_\sigma(x)\left(\frac{\delta^3 Z[J,\bar{\eta},\eta]}{\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)\,\delta\bar{\eta}_\sigma(x)}\right) \\ +\; \left(\frac{\delta^3 Z[J,\bar{\eta},\eta]}{\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)\,\delta\eta_\sigma(x)}\right)\eta_\sigma(x) \;+\; \delta(x-z)\left(\frac{\delta^2 Z[J,\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(y)\,\delta\eta_\beta(x)}\right) \;=\; 0\;. \tag{3.51}$$

Using Eq. (2.19), we can rewrite this equation in terms of the connected generating functional W. Then, setting the sources to zero and identifying the fermion propagator, given in Eq. (2.31), we find:

$$\frac{1}{g\,\xi}\,\Box_x\,\partial_x^\mu\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^\mu(x)\,\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)}\right) \;-\; i\,\delta(x-y)\,S_{\alpha\beta}(x-z)$$
$$+\; i\,\delta(x-z)\,S_{\alpha\beta}(y-x) \;=\; 0 \quad (3.52)$$

Finally, replacing the three-point connected Green's function by its decomposition in 1PI functions, given in Eq. (A.5), performing the Fourier transform and multiplying the resulting equation by the inverse of the fermion propagators, we obtain the WTI for the photon-fermion vertex in momentum space:

$$k_\mu\,\Gamma^\mu(p,-p-k;k) \;=\; S^{-1}(p+k) \;-\; S^{-1}(p)\;. \quad (3.53)$$

In order to get the WTI for the two-photon-two-fermion vertex, we need to derivate Eq. (3.51) once more with respect to $J^\nu(w)$, obtaining

$$-\,\frac{i}{g\,\xi}\,\Box_x\,\partial_x^\mu\left(\frac{\delta^4 Z[J,\bar{\eta},\eta]}{\delta J^\nu(w)\,\delta J^\mu(x)\,\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)}\right) \;+\; \frac{1}{g}\,(\partial_\nu^x\,\delta(x-w))\left(\frac{\delta^2 Z[J,\bar{\eta},\eta]}{\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)}\right)$$
$$+\;\delta(x-y)\left(\frac{\delta^3 Z[J,\bar{\eta},\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(x)}\right) \;+\;\delta(x-z)\left(\frac{\delta^3 Z[J,\bar{\eta},\eta]}{\delta J^\nu(w)\,\delta\bar{\eta}_\alpha(y)\,\delta\eta_\beta(x)}\right)$$
$$+\;\cdots\;=\;0\,, \quad (3.54)$$

where $\cdots$ indicates the terms that vanish when the sources are set to zero. Writing this equation in terms of the connected generating functional W, setting the sources to zero and identifying the fermion and photon propagators, we find

$$\frac{1}{g\,\xi}\,\Box_x\,\partial_x^\mu\left[\left(\frac{\delta^4 W[J,\bar{\eta},\eta]}{\delta J^\nu(w)\,\delta J^\mu(x)\,\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(y)}\right)\Bigg|_{J,\bar{\eta},\eta=0} \;+\; i\,S_{\alpha\beta}(y-z)\,D_{\mu\nu}(x-w)\right]$$
$$-\;\frac{i}{g}\,(\partial_\nu^x\,\delta(x-w))\,S_{\alpha\beta}(y-z) \;+\; i\,\delta(x-y)\,\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(z)\,\delta\bar{\eta}_\alpha(x)}\right)\Bigg|_{J,\bar{\eta},\eta=0}$$
$$-\;i\,\delta(x-z)\,\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(x)\,\delta\bar{\eta}_\alpha(y)}\right)\Bigg|_{J,\bar{\eta},\eta=0} \;=\;0\,. \quad (3.55)$$

Furthermore, using the expression for the photon propagator given in Eq. (3.24) together with the transversality condition $k^\mu \, P^T_{\mu\nu}(k) \; = \; 0$, we have that

$$\frac{1}{\xi}\,\Box_x\,\partial^\mu_x\,D_{\mu\nu}(x-w) \;=\; \partial^x_\nu\,\delta(x-w)\;. \tag{3.56}$$

Therefore, the second and third terms in Eq. (3.55) cancel exactly and the equation becomes simply

$$\begin{aligned}
\frac{1}{g\,\xi}\,\Box_x\,\partial^\mu_x\,&\left(\frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^\mu(x)\,\delta\eta_\beta(z)\,\delta\bar\eta_\alpha(y)}\right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \\
&+\; i\,\delta(x-y)\,\left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(z)\,\delta\bar\eta_\alpha(x)}\right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \\
&-\; i\,\delta(x-z)\,\left(\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(x)\,\delta\bar\eta_\alpha(y)}\right)\Bigg|_{J,\,\bar\eta,\,\eta\,=\,0} \;=\; 0\;.
\end{aligned} \tag{3.57}$$

Replacing the three- and four-point connected Green's function by their decompositions given in Eqs. (A.5) and (A.10), respectively, and performing a Fourier transform, we obtain

$$\begin{aligned}
k_\mu\,D_{\nu\nu'}(q)\,S(p)\,\Big[&\Gamma^\mu(p,-p-k;k)\,S(p+k)\,\Gamma^{\nu'}(p+k,-p-k-q;q) \\
&+\;\Gamma^{\nu'}(p,-p-q;q)\,S(p+q)\,\Gamma^\mu(p+q,-p-k-q;k) \\
&+\;\Gamma^{\mu\nu'}(p,-p-k-q;k,q)\Big]\,S(p+k+q) \\
+\;D_{\nu\nu'}(q)\,&S(p+k)\,\Gamma^{\nu'}(p+k,-p-k-q;q)\,S(p+k+q) \\
+\;D_{\nu\nu'}(q)\,&S(p)\,\Gamma^{\nu'}(p,-p-q;q)\,S(p+q) \;=\; 0\;,
\end{aligned} \tag{3.58}$$

where we have ignored the term proportional to the three-photon irreducible vertex, according to Furry's theorem. Then, using the WTI for the photon-fermion vertex to simplify the contractions $k_\mu\,\Gamma^\mu$ and multiplying the equation by the inverse of the photon and fermion propagators, we find the Ward-Takahashi identity for the two-photon-two-fermion vertex in momentum space:

$$k_\mu\,\Gamma^{\mu\nu}(p,-p-k-q;k,q) \;=\; \Gamma^\nu(p,-p-q;q) \;-\; \Gamma^\nu(p+k,-p-k-q;q)\;. \tag{3.59}$$

Furthermore, we can also write a scalar version of the WTI by contracting Eq. (3.59) with the momentum q_ν:

$$k_\mu \, q_\nu \, \Gamma^{\mu\nu}(p, -p-k-q; k, q) \;=\; S^{-1}(p+k) \;+\; S^{-1}(p+q) \;-\; S^{-1}(p+k+q) \;-\; S^{-1}(p) \; . \quad (3.60)$$

3.6 Renormalisation in QED

Very often when doing calculations in quantum field theories, one encounters divergent quantities. In order to enable the theory to make meaningful physical predictions, one needs to remove these ultraviolet divergences. This can be done by implementing a regularisation procedure and subsequently performing renormalisation. There are several ways of regularising a theory making it finite, such as cutoff regularisation, dimensional regularisation and Pauli-Villars regularisation [3]. The renormalisation procedure for QED [3, 35] consists in introducing the multiplicative factors Z_i as follows

$$\begin{aligned} \psi \;&=\; Z_2^{\frac{1}{2}} \, \psi^{(phys)} \; , \qquad A_\mu \;=\; Z_3^{\frac{1}{2}} \, A_\mu^{(phys)} \; , \\ m \;=\; \frac{Z_0}{Z_2} \, m^{(phys)} \; , \qquad \xi \;&=\; Z_3 \, \xi^{(phys)} \qquad \text{and} \qquad g \;=\; \frac{Z_1}{Z_2 \, Z_3^{\frac{1}{2}}} \, g^{(phys)} \; , \end{aligned} \quad (3.61)$$

where the subscript "*phys*" indicates the physical quantities. Thus, from the definitions given in Eqs. (2.30) - (2.33), the prescription described in Eq. (3.61) leads to the following renormalisation for the fermion and photon propagators:

$$S(p) \;=\; Z_2 \, S^{(phys)}(p) \quad (3.62)$$

$$D_{\mu\nu}(k) \;=\; Z_3 \, D_{\mu\nu}^{(phys)}(k) \; , \quad (3.63)$$

and for the vertices:

$$\Gamma_\mu(p', p; k) \;=\; \frac{1}{Z_1} \, \Gamma_\mu^{(phys)}(p', p; k) \quad (3.64)$$

$$\Gamma_{\mu\nu}(p', p; k, q) \;=\; \frac{Z_2}{Z_1^2} \, \Gamma_{\mu\nu}^{(phys)}(p', p; k, q) \; . \quad (3.65)$$

On the other hand, from the Ward-Takahashi identities, Eqs. (3.53) and (3.59), and the fermion propagator renormalisation, Eq. (3.62), it follows that, respectively,

$$\Gamma_\mu(p',p;k) \;=\; \frac{1}{Z_2}\,\Gamma^{(phys)}_\mu(p',p;k) \tag{3.66}$$

$$\Gamma_{\mu\nu}(p',p;k,q) \;=\; \frac{1}{Z_2}\,\Gamma^{(phys)}_{\mu\nu}(p',p;k,q)\ , \tag{3.67}$$

which implies that

$$Z_1 \;=\; Z_2\ . \tag{3.68}$$

This leaves the renormalisation of the electric charge simply $g \,=\, Z_3^{-\frac{1}{2}}\, g^{(phys)}$ and, consequently, the coupling constant $\alpha \,=\, g^2/(4\pi)$ renormalisation becomes $\alpha \,=\, Z_3^{-1}\,\alpha^{(phys)}$. Thus, we can write the renormalised Dyson-Schwinger equations, in which the subscript "*phys*" will be omitted for the purpose of simplifying the notation. The fermion gap equation, Eq. (3.9), becomes

$$S^{-1}(p) \;=\; Z_2\left[\not p - \frac{Z_0}{Z_2}\,m - \Sigma(p)\right]\ , \tag{3.69}$$

where the self-energy $\Sigma(p)$ is defined in Eq. (3.13) and written in terms of renormalised (physical) fields. Using the decompositions for the inverse fermion propagator and fermion self-energy, Eqs. (3.11) and (3.14) respectively, we find the following renormalised equations for the $\mathcal{A}(p^2)$ and $\mathcal{B}(p^2)$ functions:

$$\mathcal{A}(p^2) = Z_2\left[1 \;-\; \Sigma_v(p^2)\right] \tag{3.70}$$

$$\mathcal{B}(p^2) = Z_0\, m \;+\; Z_2\,\Sigma_s(p^2)\ . \tag{3.71}$$

The renormalised photon DSE in terms of the function $\mathcal{G}(k^2)$, see Eq. (3.28), is given by

$$\frac{1}{\mathcal{G}(k^2)} \;=\; Z_3 \;+\; Z_2\,\Pi(k^2)\ , \tag{3.72}$$

where the scalar photon polarisation $\Pi(k^2)$ is defined in Eq. (3.29) and written in terms of renormalised fields. Finally, the renormalised equations for the photon-fermion and the two-photon-two-fermion vertices, see Eqs. (3.33) and (3.43), are, respectively:

$$\Gamma^{\mu}(p,-p-k;k) \; =$$

$$= \; Z_2 \left\{ \gamma^{\mu} \; + \; i\, g^2 \int \frac{d^4 q}{(2\,\pi)^4} D_{\nu\nu'}(q) \left[\gamma^{\nu}\, S(p-q)\, \Gamma^{\mu\nu'}(p-q,-p-k;k,q) \right.\right.$$

$$\left.\left. + \; \gamma^{\nu}\, S(p-q)\, \Gamma^{\mu}(p-q,-p-k+q;k)\, S(p+k-q)\, \Gamma^{\nu'}(p+k-q,-p-k;q) \right] \right\} , \quad (3.73)$$

$$\Gamma^{\mu\nu}(p,-p-k-q;k,q) \; =$$

$$= \; i\, Z_2\, g^2 \int \frac{d^4 w}{(2\,\pi)^4}\, D_{\sigma\sigma'}(w)\, \gamma^{\sigma}\, S(p-w)$$

$$\left\{ \left[\Gamma^{\mu}(p-w,-p-k+w;k)\, S(p+k-w)\, \Gamma^{\nu}(p+k-w,-p-k-q+w;q) \right.\right.$$

$$+ \; \Gamma^{\nu}(p-w,-p-q+w;q)\, S(p+q-w)\, \Gamma^{\mu}(p+q-w,-p-k-q+w;q)$$

$$\left. + \; \Gamma^{\rho\nu}(p-w,-p-k-q+w;k,q) \right]$$

$$S(p+k+q-w)\, \Gamma^{\sigma'}(p+k+q-w,-p-k-q;w)$$

$$+ \; \Gamma^{\mu}(p-w,-p-k+w;k)\, S(p+k-w)\, \Gamma^{\nu\sigma'}(p+k-w,-p-k-q;q,w)$$

$$\left. + \; \Gamma^{\nu}(p-w,-p-q+w;q)\, S(p+q-w)\, \Gamma^{\rho\mu'}(p+q-w,-p-k-q;k,w) \right\} . \quad (3.74)$$

The renormalisation conditions needed to determine the constants Z_0, Z_2 and Z_3 can be obtained using Eqs. (3.70) - (3.72). Thus, based on the tree-level propagators, we can set

$$\mathcal{A}(\mu_f^2) \; = \; 1 \; , \quad \mathcal{B}(\mu_f^2) \; = \; m \quad \text{and} \quad \mathcal{G}(\mu_{ph}^2) \; = \; 1 \; , \quad (3.75)$$

obtaining

$$Z_2 \; = \; \frac{1}{1 \; - \; \Sigma_v(\mu_f^2)} \; , \quad (3.76)$$

$$Z_0 \; = \; 1 \; - \; Z_2\, \frac{\Sigma_s(\mu_f^2)}{m} \; , \quad (3.77)$$

$$Z_3 \; = \; 1 \; - \; Z_2\, \Pi(\mu_{ph}^2) \; , \quad (3.78)$$

where μ_f and μ_{ph} and the renormalisation scales for the fermion and the photon, respectively. Therefore, we can write the renormalised equations as

$$\mathcal{A}(p^2) \;=\; 1 \;-\; Z_2 \left[\Sigma_v(p^2) \;-\; \Sigma_v(\mu_f^2)\right] , \tag{3.79}$$

$$\mathcal{B}(p^2) \;=\; m + Z_2 \left[\Sigma_s(p^2) \;-\; \Sigma_s(\mu_f^2)\right] , \tag{3.80}$$

$$\frac{1}{\mathcal{G}(k^2)} \;=\; 1 \;+\; Z_2 \left[\Pi(k^2) \;-\; \Pi(\mu_{ph}^2)\right] , \tag{3.81}$$

where Z_2 is determined in Eq. (3.76).

4 The Fermion-Photon Coupled System

In this chapter we will study the fermion-photon coupled system, described by Eqs. (3.9) and (3.23). However, because of the way that the Dyson-Schwinger equations are naturally structured, these equations contain the one-particle irreducible photon-fermion vertex, which is itself dependent on higher order Green's functions. Thus, in order to decouple the fermion and photon equations from the other DSEs, we will approximate the result of the DSE for the photon-fermion vertex, Eq. (3.23), by an *Ansatz*, which we can substitute into the equations for the propagators. The choice of the vertex *Ansatz* can vary greatly, depending of the focus of the study. However, in general the *Ansatz* should meet several criteria [10, 11, 19, 20, 29], such as satisfying the Ward-Takahashi identity and being free of kinetic singularities. In the free-field limit, the vertex $\Gamma^\mu(p', p; k)$ should reduce to γ^μ and they must have the same transformation law under charge conjugation $\mathcal{C}$. Finally, it should also ensure that the DSEs respect local gauge covariance and multiplicative renormalizability. In this study, we will consider mainly the Ball-Chiu vertex *Ansatz*, which is built upon the satisfaction of the Ward-Takahashi identity.

Furthermore, we will also look at the Dyson-Schwinger equations in Euclidean spacetime and analyse some preliminary numerical solutions.

4.1 Ball-Chiu Vertex

As discussed in Sec. 3.5, the Ward-Takahashi identities are a consequence of the gauge invariance of the theory and, therefore, any acceptable *Ansatz* for the photon-fermion vertex should comply with it. Thus, the WTI for the photon-fermion vertex, given in Eq. (3.53),

$$k_\mu \, \Gamma^\mu(p_1, p_2; k) \;=\; S^{-1}(-p_2) \;-\; S^{-1}(p_1) \;, \tag{4.1}$$

where $k = -(p_2 + p_1)$, is going to play a central role in modelling the vertex. We start by writing the vertex in terms of longitudinal and transverse components, relative to the photon momentum:

$$\Gamma^\mu(p_1,p_2;k) \;=\; \Gamma^\mu_L(p_1,p_2;k) \;+\; \Gamma^\mu_T(p_1,p_2;k)\;. \tag{4.2}$$

However, by definition,

$$k_\mu\,\Gamma^\mu_T(p_1,p_2;k) \;=\; 0\;. \tag{4.3}$$

As a result, the transverse part of the vertex is blind to the WTI and Eq. (4.1) becomes

$$k_\mu\,\Gamma^\mu_L(p_1,p_2;k) \;=\; S^{-1}(-p_2) \;-\; S^{-1}(p_1)\;. \tag{4.4}$$

In general, we can write the longitudinal vertex as

$$\Gamma^\mu_L(p_1,p_2;k) \;=\; \sum_{i=1}^{4} \lambda_i(p_1^2,p_2^2,k^2)\,L_i^\mu(p_1,p_2;k)\;, \tag{4.5}$$

where λ_i are Lorentz scalar form factors and L_i^μ are the longitudinal basis vectors determined by Ball and Chiu [24]. The latter are defined as

$$\begin{aligned}
L_1(p_1,p_2;k) \;&=\; \gamma^\mu\;,\\
L_2(p_1,p_2;k) \;&=\; (\not p_2 \;-\; \not p_1)\,(p_2 \;-\; p_1)^\mu\;,\\
L_3(p_1,p_2;k) \;&=\; (p_2 \;-\; p_1)^\mu\;,\\
L_4(p_1,p_2;k) \;&=\; \sigma^{\mu\nu}\,(p_2 \;-\; p_1)_\nu\;,
\end{aligned} \tag{4.6}$$

with $\sigma_{\mu\nu} \;=\; [\gamma_\mu,\gamma_\nu]/2$. Then, using the parametrisation of the fermion propagator given in Eq. (3.11),

$$S^{-1}(p) \;=\; \mathcal{A}(p^2)\,\not p \;-\; \mathcal{B}(p^2)\;, \tag{4.7}$$

and the longitudinal vertex decomposition in Eq. (4.5), we can solve the Ward-Takahashi identity, Eq. (4.1), and determine the form factors λ_i in terms of $\mathcal{A}$ and $\mathcal{B}$:

$$\lambda_1(p_1^2, p_2^2, k^2) \; = \; \frac{1}{2}\left[\mathcal{A}(p_2^2) \; + \; \mathcal{A}(p_1^2)\right] \, , \tag{4.8}$$

$$\lambda_2(p_1^2, p_2^2, k^2) \; = \; \frac{1}{2\,(p_2^2 \; - \; p_1^2)}\left[\mathcal{A}(p_2^2) \; - \; \mathcal{A}(p_1^2)\right] \, , \tag{4.9}$$

$$\lambda_3(p_1^2, p_2^2, k^2) \; = \; \frac{1}{(p_2^2 \; - \; p_1^2)}\left[\mathcal{B}(p_2^2) \; - \; \mathcal{B}(p_1^2)\right] \, , \tag{4.10}$$

$$\lambda_4(p_1^2, p_2^2, k^2) \; = \; 0 \; . \tag{4.11}$$

This fixes the longitudinal component of the vertex and it is known in literature as the Ball-Chiu vertex:

$$\begin{aligned} \Gamma^\mu_L(p_1, p_2; k) \; &= \; \Gamma^\mu_{BC}(p_1, p_2; k) \\ &= \; \frac{1}{2}\left[\mathcal{A}(p_2^2) \; + \; \mathcal{A}(p_1^2)\right]\gamma^\mu \\ &\quad + \; \frac{1}{2\,(p_2^2 \; - \; p_1^2)}\left[\mathcal{A}(p_2^2) \; - \; \mathcal{A}(p_1^2)\right](\not{p}_2 \; - \; \not{p}_1)\,(p_2 \; - \; p_1)^\mu \\ &\quad + \; \frac{1}{(p_2^2 \; - \; p_1^2)}\left[\mathcal{B}(p_2^2) \; - \; \mathcal{B}(p_1^2)\right](p_2 \; - \; p_1)^\mu \; . \end{aligned} \tag{4.12}$$

It is worth noticing that the non-vanishing form factors, Eqs. (4.8) - (4.10), are symmetric under the exchange of the two fermion momenta and therefore this vertex respects the charge conjugation transformation law. Furthermore, assuming that $\mathcal{A}$ and $\mathcal{B}$ are smooth functions of the momentum, in the limit where $p_1 \; \to \; p_2$ it follows that λ_2 and λ_3 are proportional to the derivatives of $\mathcal{A}$ and $\mathcal{B}$, respectively. Thus, these form factors are also free of kinematic singularities.

4.2 Equations in Minkowski Spacetime

We can now substitute the Ball-Chiu (BC) vertex, Eq. (4.12), into the DSEs for the fermion and photon propagators, obtaining a closed set of coupled non-linear integral equations that allow us to study the fermion-photon system. Thus, using the renormalised equations, Eqs. (3.79) - (3.81), we have a system of three coupled equations that determine the form factors associated with the fermion ($\mathcal{A}$ and $\mathcal{B}$) and photon ($\mathcal{G}$) propagators:

$$\mathcal{A}(p^2) \;=\; 1 \;-\; Z_2 \left[\Sigma_v^{BC}(p^2) \;-\; \Sigma_v^{BC}(\mu_f^2)\right] \,, \tag{4.13}$$

$$\mathcal{B}(p^2) \;=\; m + Z_2 \left[\Sigma_s^{BC}(p^2) \;-\; \Sigma_s^{BC}(\mu_f^2)\right] \,, \tag{4.14}$$

$$\frac{1}{\mathcal{G}(k^2)} \;=\; 1 \;+\; Z_2 \left[\Pi^{BC}(k^2) \;-\; \Pi^{BC}(\mu_{ph}^2)\right] \,, \tag{4.15}$$

where the fermion self-energy and scalar photon polarisation with the BC vertex are

$$\Sigma_v^{BC}(p^2) \;=\; i\, g^2 \,\frac{1}{4\,p^2}\,\mathrm{Tr}\Big[\not p \int \frac{d^4k}{(2\,\pi)^4}\, D_{\mu\nu}(k)\,\gamma^\mu\, S(p-k)\,\Gamma_{BC}^\nu(p-k,-p;k)\Big] \,, \tag{4.16}$$

$$\Sigma_s^{BC}(p^2) \;=\; i\, g^2 \,\frac{1}{4}\,\mathrm{Tr}\Big[\int \frac{d^4k}{(2\,\pi)^4}\, D_{\mu\nu}(k)\,\gamma^\mu\, S(p-k)\,\Gamma_{BC}^\nu(p-k,-p;k)\Big] \,, \tag{4.17}$$

$$\Pi^{BC}(k^2) \;=\; -i\, N_f\, \frac{g^2}{3}\,\frac{1}{k^2}\int \frac{d^4p}{(2\,\pi)^4}\;\mathrm{Tr}\Big[\gamma_\mu\, S(p)\,\Gamma_{BC}^\mu(p,-p+k;-k)\,S(p-k)\Big] \,. \tag{4.18}$$

Let us first focus our attention on the equations related to the fermion propagators, Eqs. (4.13) and (4.14). Using the BC vertex, Eq. (4.12), we can write

$$\Gamma_{BC}^\nu(p-k,-p;k) \;=\; \lambda_1\,\gamma^\nu \;+\; \lambda_2\,(\not k \;-\; 2\not p)(k \;-\; 2\,p)^\nu \;+\; \lambda_3\,(k \;-\; 2\,p)^\nu \,, \tag{4.19}$$

where $\lambda_i \;\equiv\; \lambda_i((p-k)^2, p^2, k^2)$ are defined in Eqs. (4.8) - (4.11). The fermion and photon propagators, see Eqs. (3.11) and (3.24), are given by

$$\begin{aligned} S(p) \;&=\; \frac{1}{\mathcal{A}^2(p^2)\,p^2 \;-\; \mathcal{B}^2(p^2) \;+\; i\,\epsilon}\left[\mathcal{A}(p^2)\,\not p \;+\; \mathcal{B}(p^2)\right] \\ &=\; Y(p^2)\Big[\mathcal{A}(p^2)\,\not p \;+\; \mathcal{B}(p^2)\Big] \,. \end{aligned} \tag{4.20}$$

$$D_{\mu\nu}(k) \;=\; \frac{\mathcal{G}(k^2)}{k^2}\left(\frac{k_\mu\,k_\nu}{k^2} \;-\; g_{\mu\nu}\right) \;-\; \xi\,\frac{k_\mu\,k_\nu}{k^4} \,, \tag{4.21}$$

Then, the computation of the traces in Eqs. (4.16) and (4.17), see Appendix B for some useful results, leaves us with

$$
\begin{aligned}
\Sigma_v^{BC}(p^2) \;=\; & i\,g^2\,\frac{1}{p^2}\int\frac{d^4k}{(2\,\pi)^4}\,Y((p-k)^2) \\
& \left\{\frac{\mathcal{G}(k^2)}{k^2}\left[\mathcal{A}((p-k)^2)\,\lambda_1\left(\frac{2\,(kp)^2}{k^2}\;-\;3(kp)\;+\;p^2\right)\right.\right. \\
& \quad +\;2\,\mathcal{A}((p-k)^2)\,\lambda_2\left(2\,\frac{p^2\,(kp)^2\;-\;(kp)^3}{k^2}\;+\;(kp)^2\right. \\
& \qquad \left. +\;2\,p^2\,(kp)\;-\;k^2\,p^2\;-\;2\,p^4\right) \\
& \quad \left. +\;2\,\mathcal{B}((p-k)^2)\,\lambda_3\left(p^2\;-\;\frac{(kp)^2}{k^2}\right)\right] \\
& +\;\frac{\xi}{k^2}\left[\mathcal{A}((p-k)^2)\,\lambda_1\left((kp)\;+\;p^2\;-\;\frac{2\,(kp)^2}{k^2}\right)\right. \\
& \quad +\;\mathcal{A}((p-k)^2)\,\lambda_2\left(4\,\frac{(kp)^3\;-\;p^2\,(kp)^2}{k^2}\;-\;4\,(kp)^2\right. \\
& \qquad \left. +\;4\,p^2\,(kp)\;+\;k^2\,(kp)\;-\;k^2\,p^2\right) \\
& \quad \left.\left. +\;\mathcal{B}((p-k)^2)\,\lambda_3\left(\frac{2\,(kp)^2}{k^2}\;-\;(kp)\right)\right]\right\}
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
\Sigma_s^{BC}(p^2) \;=\; & i\,g^2\int\frac{d^4k}{(2\,\pi)^4}\,Y((p-k)^2) \\
& \left\{\frac{\mathcal{G}(k^2)}{k^2}\left[-\,3\,\mathcal{B}((p-k)^2)\,\lambda_1\right.\right. \\
& \quad +\;4\,\mathcal{B}((p-k)^2)\,\lambda_2\left(\frac{(kp)^2}{k^2}\;-\;p^2\right) \\
& \quad \left. +\;2\,\mathcal{A}((p-k)^2)\,\lambda_3\left(p^2\;-\;\frac{(kp)^2}{k^2}\right)\right] \\
& +\;\frac{\xi}{k^2}\left[-\,\mathcal{B}((p-k)^2)\,\lambda_1\right. \\
& \quad +\;\mathcal{B}((p-k)^2)\,\lambda_2\left(-\,\frac{4\,(kp)^2}{k^2}\;+\;4\,(kp)\;-\;k^2\right) \\
& \quad \left.\left. +\;\mathcal{A}((p-k)^2)\,\lambda_3\left(\frac{2\,(kp)^2}{k^2}\;-\;3\,(kp)\;+\;k^2\right)\right]\right\}\,,
\end{aligned}
\tag{4.23}
$$

where (kp) is the scalar product between the two momenta. However it is possible to write these equations in another way. By writing the photon propagator as

$$D_{\mu\nu}(k) \;=\; -g_{\mu\nu}\,\frac{\mathcal{G}(k^2)}{k^2} \;+\; \Big(\mathcal{G}(k^2) \;-\; \xi\Big)\,\frac{k_\mu\,k_\nu}{k^4}\;, \tag{4.24}$$

we can use the WTI, Eq. (4.1), to simplify the contraction between the second term of Eq. (4.24) and the photon-fermion vertex. This leaves the integral in Eqs. (4.16) and (4.17) as

$$\begin{aligned}\int \frac{d^4k}{(2\,\pi)^4}\, D_{\mu\nu}(k)\,\gamma^\mu\, S(p-k)\,\Gamma^\nu_{BC}(p-k,-p;k) \;&=\\ &= \;-\;\frac{\mathcal{G}(k^2)}{k^2}\,\gamma_\nu\, S(p-k)\,\Gamma^\nu_{BC}(p-k,-p;k)\\ &+\;\frac{1}{k^4}\,\Big(\mathcal{G}(k^2)-\xi\Big)\,\not{k}\Big[S(p-k)\;-\;S^{-1}(p)\;-\;1\Big]\,.\end{aligned} \tag{4.25}$$

Calculating the respective traces, see Appendix B, the simplified equations for $\Sigma^{BC}_v(p^2)$ and $\Sigma^{BC}_s(p^2)$ become

$$\begin{aligned}\Sigma^{BC}_v(p^2) \;=\; i\,g^2\,Z_2\,\frac{1}{p^2}\int\frac{d^4k}{(2\,\pi)^4}\,\Bigg\{&Y((p-k)^2)\,\frac{\mathcal{G}(k^2)}{k^2}\\ &\Bigg[2\,\mathcal{A}((p-k)^2)\,\lambda_1\Big(p^2\;-\;(kp)\Big)\\ &+\;\mathcal{A}((p-k)^2)\,\lambda_2\Bigg(\Big(-\,2\,(kp)\;+\;8\,p^2\;+\;k^2\Big)\,(kp)\\ &\qquad-\;\Big(3\,k^2\;+\;4\,p^2\Big)\,p^2\Bigg)\\ &+\;\mathcal{B}((p-k)^2)\,\lambda_3\Big(2\,p^2\;-\;(kp)\Big)\Bigg]\\ &+\;\frac{1}{k^2}\Big(\mathcal{G}(k^2)\;-\;\xi\Big)\\ &\Bigg[Y((p-k)^2)\,\mathcal{A}(p^2)\,\mathcal{A}((p-k)^2)\,p^2\left(\frac{(kp)}{k^2}\;-\;1\right)\\ &-\;Y((p-k)^2)\,\mathcal{B}(p^2)\,\mathcal{B}((p-k)^2)\,\frac{(kp)}{k^2}\\ &-\;\frac{(kp)}{k^2}\Bigg]\Bigg\}\end{aligned} \tag{4.26}$$

and

$$
\begin{aligned}
\Sigma_s^{BC}(p^2) \;=\; i\,g^2 \int \frac{d^4k}{(2\,\pi)^4}\, Y((p-k)^2) \\
\Bigg\{ \frac{\mathcal{G}(k^2)}{k^2} \Bigg[-\,4\,\mathcal{B}((p-k)^2)\,\lambda_1 \\
+\; \mathcal{B}((p-k)^2)\,\lambda_2 \Big(4\,(kp) \;-\; 4\,p^2 \;-\; k^2 \Big) \\
+\; \mathcal{A}((p-k)^2)\,\lambda_3 \Big(-\,3\,(kp) \;+\; 2\,p^2 \;+\; k^2 \Big) \Bigg] \\
+\; \frac{1}{k^2} \Big(\mathcal{G}(k^2) \;-\; \xi \Big) \Bigg[\mathcal{A}((p-k)^2)\,\mathcal{B}(p^2) \left(1 \;-\; \frac{(kp)}{k^2} \right) \\
+\; \mathcal{A}(p^2)\,\mathcal{B}((p-k)^2)\,\frac{(kp)}{k^2} \Bigg] \Bigg\} \,. \qquad (4.27)
\end{aligned}
$$

Finally, we can now look at the equation for the photon, Eq. (4.15). Using once again the BC *Ansatz*, Eq. (4.12), we can write the vertex contained in the definition of $\Pi^{BC}(k^2)$, see Eq. (4.18), as

$$
\Gamma^{\nu}_{BC}(p,-p+k;-k) \;=\; \lambda_1'\,\gamma^{\nu} \;+\; \lambda_2'\,(\not{k} \;-\; 2\not{p})(k \;-\; 2\,p)^{\nu} \;+\; \lambda_3'\,(k \;-\; 2\,p)^{\nu}\;, \qquad (4.28)
$$

where $\lambda_i' \;\equiv\; \lambda_i(p^2,(p-k)^2,k^2)$. Writing the fermion propagator one more time as Eq. (4.20) and computing the trace in Eq. (4.18), see Appendix B, it becomes

$$
\begin{aligned}
\Pi^{BC}(k^2) \;=\; & -\,i\,N_f\,\frac{4\,g^2}{3}\,\frac{1}{k^2}\int\frac{d^4p}{(2\,\pi)^4}\,Y(p^2)\,Y((p-k)^2) \\
& \Bigg\{\lambda_1'\Bigg[2\,\mathcal{A}(p^2)\,\mathcal{A}((p-k)^2)\Bigg((kp)\;-\;p^2\Bigg) \\
& \quad +\;4\,\mathcal{B}(p^2)\,\mathcal{B}((p-k)^2)\Bigg] \\
& +\;\lambda_2'\Bigg[\mathcal{A}(p^2)\,\mathcal{A}((p-k)^2)\Bigg(\Big(2(kp)\;-\;k^2\;-\;8\,p^2\Big)\,(kp) \\
& \quad +\;\Big(3\,k^2\;+\;4\,p^2\Big)\,p^2\Bigg) \\
& \quad +\;\mathcal{B}(p^2)\,\mathcal{B}((p-k)^2)\Bigg(-\,4\,(kp)\;+\;k^2\;+\;4\,p^2\Bigg)\Bigg] \\
& +\;\lambda_3'\Bigg[\mathcal{A}(p^2)\,\mathcal{B}((p-k)^2)\Bigg((kp)\;-\;2\,p^2\Bigg) \\
& \quad +\;\mathcal{A}((p-k)^2)\,\mathcal{B}(p^2)\Bigg(3\,(pk)\;-\;k^2\;-\;2\,p^2\Bigg)\Bigg]\Bigg\}\,. \qquad (4.29)
\end{aligned}
$$

4.3 Equations in Euclidean Spacetime

So far all equations presented here were in Minkowski spacetime, however, it is easier to perform the integrations if we write them in Euclidean spacetime. To do this, we need to effectuate a Wick rotation [25, 3, 18], which consists in transforming the four-momentum as $p_0 \to i\,p_0$ and $\vec{p} \to \vec{p}$. This transforms the original Minkowski metric $p^2 = p_0^2 - \vec{p}^{\,2}$ into an Euclidean metric $-p^2 = -(p_0^2 + \vec{p}^{\,2})$. In practice, it is equivalent to making the following replacements:

$$
\begin{aligned}
p^2 &\to -p^2 & \qquad \mathcal{A}(p^2) &\to \mathcal{A}(p^2) \\
(pk) &\to -(pk) & \qquad \mathcal{B}(p^2) &\to \mathcal{B}(p^2) \\
\int d^4k &\to i\int d^4k & \qquad \mathcal{G}(p^2) &\to \mathcal{G}(p^2)
\end{aligned} \qquad (4.30)
$$

In order to keep the definitions given in Eqs. (4.8) - (4.10) also valid in Euclidean spacetime, we also consider

$$
\begin{aligned}
\lambda_1(p'^2, p^2, k^2) &\rightarrow \lambda_1(p'^2, p^2, k^2) \;, \\
\lambda_2(p'^2, p^2, k^2) &\rightarrow -\lambda_2(p'^2, p^2, k^2) \;, \\
\lambda_3(p'^2, p^2, k^2) &\rightarrow -\lambda_3(p'^2, p^2, k^2) \;.
\end{aligned}
\tag{4.31}
$$

Furthermore, in Euclidean spacetime $Y(p^2)$ is defined as

$$
Y(p^2) \;=\; \frac{1}{\mathcal{A}^2(p^2)\, p^2 \;+\; \mathcal{B}^2(p^2)} \;. \tag{4.32}
$$

Then, in Euclidean spacetime, Eqs. (4.22) and (4.23) become

$$
\begin{aligned}
\Sigma_v^{BC}(p^2) \;=\; & -g^2\,\frac{1}{p^2} \int \frac{d^4k}{(2\,\pi)^4}\, Y((p-k)^2) \\
& \left\{ \frac{\mathcal{G}(k^2)}{k^2} \left[\mathcal{A}((p-k)^2)\,\lambda_1 \left(\frac{2\,(kp)^2}{k^2} \;-\; 3(kp) \;+\; p^2 \right) \right.\right. \\
& \quad +\; 2\,\mathcal{A}((p-k)^2)\,\lambda_2 \left(2\,\frac{p^2\,(kp)^2 \;-\; (kp)^3}{k^2} \;+\; (kp)^2 \right. \\
& \qquad \left. +\; 2\,p^2\,(kp) \;-\; k^2\,p^2 \;-\; 2\,p^4 \right) \\
& \quad \left. +\; 2\,\mathcal{B}((p-k)^2)\,\lambda_3 \left(\frac{(kp)^2}{k^2} \;-\; p^2 \right) \right] \\
& +\; \frac{\xi}{k^2} \left[\mathcal{A}((p-k)^2)\,\lambda_1 \left((kp) \;+\; p^2 \;-\; \frac{2\,(kp)^2}{k^2} \right) \right. \\
& \quad +\; \mathcal{A}((p-k)^2)\,\lambda_2 \left(4\,\frac{(kp)^3 \;-\; p^2\,(kp)^2}{k^2} \;-\; 4\,(kp)^2 \right. \\
& \qquad \left. +\; 4\,p^2\,(kp) \;+\; k^2\,(kp) \;-\; k^2\,p^2 \right) \\
& \quad \left.\left. +\; \mathcal{B}((p-k)^2)\,\lambda_3 \left((kp) \;-\; \frac{2\,(kp)^2}{k^2} \right) \right] \right\}
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
\Sigma_s^{BC}(p^2) \;=\; & g^2 \int \frac{d^4k}{(2\,\pi)^4}\, Y((p-k)^2) \\
& \left\{ \frac{\mathcal{G}(k^2)}{k^2} \left[3\,\mathcal{B}((p-k)^2)\,\lambda_1 \right.\right. \\
& \qquad + \; 4\,\mathcal{B}((p-k)^2)\,\lambda_2 \left(p^2 \;-\; \frac{(kp)^2}{k^2} \right) \\
& \qquad \left. + \; 2\,\mathcal{A}((p-k)^2)\,\lambda_3 \left(\frac{(kp)^2}{k^2} \;-\; p^2 \right) \right] \\
& + \; \frac{\xi}{k^2} \left[\mathcal{B}((p-k)^2)\,\lambda_1 \right. \\
& \qquad + \; \mathcal{B}((p-k)^2)\,\lambda_2 \left(\frac{4\,(kp)^2}{k^2} \;-\; 4\,(kp) \;+\; k^2 \right) \\
& \qquad \left.\left. + \; \mathcal{A}((p-k)^2)\,\lambda_3 \left(-\,\frac{2\,(kp)^2}{k^2} \;+\; 3\,(kp) \;-\; k^2 \right) \right] \right\} \,, \qquad (4.34)
\end{aligned}
$$

while the simplified version using the WTI, Eqs. (4.26) and (4.27), become

$$
\begin{aligned}
\Sigma_v^{BC}(p^2) \;=\; -g^2\,\frac{1}{p^2}\int\frac{d^4k}{(2\,\pi)^4}\Bigg\{ & Y((p-k)^2)\,\frac{\mathcal{G}(k^2)}{k^2} \\
& \Bigg[2\,\mathcal{A}((p-k)^2)\,\lambda_1\Big(p^2\;-\;(kp)\Big) \\
& +\;\mathcal{A}((p-k)^2)\,\lambda_2\Big(\Big(-\,2\,(kp)\;+\;8\,p^2\;+\;k^2\Big)\,(kp) \\
& \qquad -\;\Big(3\,k^2\;+\;4\,p^2\Big)\,p^2\Big) \\
& +\;\mathcal{B}((p-k)^2)\,\lambda_3\Big((kp)\;-\;2\,p^2\Big)\Bigg] \\
& +\frac{1}{k^2}\Big(\mathcal{G}(k^2)\;-\;\xi\Big) \\
& \Bigg[Y((p-k)^2)\,\mathcal{A}(p^2)\,\mathcal{A}((p-k)^2)\,p^2\left(\frac{(kp)}{k^2}\;-\;1\right) \\
& +\;Y((p-k)^2)\,\mathcal{B}(p^2)\,\mathcal{B}((p-k)^2)\,\frac{(kp)}{k^2} \\
& -\;\frac{(kp)}{k^2}\Bigg]\Bigg\}
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
\Sigma_s^{BC}(p^2) \;=\; g^2\int\frac{d^4k}{(2\,\pi)^4}\,Y((p-k)^2) & \\
\Bigg\{\frac{\mathcal{G}(k^2)}{k^2}\Bigg[& 4\,\mathcal{B}((p-k)^2)\,\lambda_1 \\
& +\;\mathcal{B}((p-k)^2)\,\lambda_2\Big(-\,4\,(kp)\;+\;4\,p^2\;+\;k^2\Big) \\
& +\;\mathcal{A}((p-k)^2)\,\lambda_3\Big(3\,(kp)\;-\;2\,p^2\;-\;k^2\Big)\Bigg] \\
+\frac{1}{k^2}\Big(\mathcal{G}(k^2)\;-\;\xi\Big)\Bigg[& \mathcal{A}((p-k)^2)\,\mathcal{B}(p^2)\left(\frac{(kp)}{k^2}\;-\;1\right) \\
& -\;\mathcal{A}(p^2)\,\mathcal{B}((p-k)^2)\,\frac{(kp)}{k^2}\Bigg]\Bigg\}\,.
\end{aligned}
\tag{4.36}
$$

Lastly, the scalar photon polarisation, Eq. (4.29), in Euclidean spacetime is

$$
\begin{aligned}
\Pi^{BC}(k^2) \;=\; & \frac{4}{3} Z_2\, N_f\, g^2\, \frac{1}{k^2} \int \frac{d^4p}{(2\,\pi)^4}\, Y(p^2)\, Y((p-k)^2) \\
& \Bigg\{ \lambda_1' \Bigg[2\,\mathcal{A}(p^2)\,\mathcal{A}((p-k)^2) \Big((kp) \;-\; p^2 \Big) \\
& \qquad -\; 4\,\mathcal{B}(p^2)\,\mathcal{B}((p-k)^2) \Bigg] \\
& +\; \lambda_2' \Bigg[\mathcal{A}(p^2)\,\mathcal{A}((p-k)^2) \Bigg(\Big(2(kp) \;-\; k^2 \;-\; 8\,p^2 \Big)\,(kp) \\
& \qquad\qquad +\; \Big(3\,k^2 \;+\; 4\,p^2 \Big)\,p^2 \Bigg) \\
& \qquad +\; \mathcal{B}(p^2)\,\mathcal{B}((p-k)^2) \Big(4\,(kp) \;-\; k^2 \;-\; 4\,p^2 \Big) \Bigg] \\
& +\; \lambda_3' \Bigg[\mathcal{A}(p^2)\,\mathcal{B}((p-k)^2) \Big(2\,p^2 \;-\; (kp) \Big) \\
& \qquad +\; \mathcal{A}((p-k)^2)\,\mathcal{B}(p^2) \Big(k^2 \;+\; 2\,p^2 \;-\; 3\,(pk) \Big) \Bigg] \Bigg\} \;. \qquad (4.37)
\end{aligned}
$$

With the equations now written in Euclidean spacetime, we can define the four-dimensional spherical coordinate system. We choose the external momentum, associated with the incoming particle, as $p^\mu \;=\; (p, 0, 0, 0)$ and the internal momentum, associated with the loop, as

$$
k^\mu \;=\; (k\,\cos\theta,\; k\,\sin\theta\,\cos\phi,\; k\,\sin\theta\,\sin\phi\,\cos\psi,\; k\,\sin\theta\,\sin\phi\,\sin\psi) \;, \qquad (4.38)
$$

where $k \;=\; (k_0^2 + k_1^2 + k_2^2 + k_3^2)^{1/2}$ and the integration ranges are

$$
k \;\in\; [0, +\infty[\,, \qquad \theta, \phi \;\in\; [0, \pi] \qquad \text{and} \qquad \psi \;\in\; [0, 2\pi] \;. \qquad (4.39)
$$

The volume element d^4k then becomes

$$
d^4k \;=\; k^3\, sin^2\theta\, \sin\phi\, dk\, d\theta\, d\phi\, d\psi \;. \qquad (4.40)
$$

Since θ is the angle between the two momenta, it follows that $(kp) \;=\; kp\,\cos\theta$ and there is no dependence on ϕ and ψ. Therefore, we can write the momentum and angular integration as

$$\int_0^\infty k^3\,dk \int_0^\pi \sin^2\theta\,d\theta \int_0^\pi \sin\phi\,d\phi \int_0^{2\pi} d\psi \;=\; 4\,\pi \int_0^\infty k^3\,dk \int_{-1}^1 \sin\theta\,d(\cos\theta)$$
$$=\; 4\,\pi \int_0^\infty k^3\,dk \int_{-1}^1 \sqrt{1\;-\;x^2}\,dx\;, \qquad (4.41)$$

where $x \;=\; \cos\theta$.

4.4 Numerical solutions

Finally, we can use the results obtained so far in this chapter to solve numerically the equations for the fermion and photon propagators in Euclidean spacetime. In order to do this, we followed the usual iterative method using the Gauss-Legendre quadrature [30] to approximate the angular and momentum integrals, see Eq. (4.41). The upper limit of the integral is also replaced by an ultraviolet cutoff. The convergence criteria utilised was based in the value of the relative error between solutions obtained in consecutive iterations, defined as

$$\Delta F \;=\; \frac{\|F_{i-1}\;-\;F_i\|}{\|F_i\|}\;, \qquad (4.42)$$

where F is the function in question and the index i represents the iteration number. In the case of this preliminary study, we considered that the solutions had converged when $\Delta F \leq 1\times10^{-6}$. In these circumstances, we find that $\mathcal{G}(k^2)$ has a singularity in the infrared limit which can be removed by renormalising the photon propagator at zero momentum. In this study, we used the same renormalisation scale for the fermion and the photon, so that $\mu_{ph}^2 \;=\; \mu_f^2 \;=\; 0$. Furthermore, we considered the coupling constant $\alpha \;\approx\; 1/137$ and, in the massive case, the electron mass $m \;\approx\; 0.511\,\text{MeV}$ [31]. If not stated otherwise, we consider the Ball-Chiu *Ansatz* for the photon-fermion vertex.

Thus, Figs. 4.1 - 4.3 show the numerical solutions obtained for the $\mathcal{A}(p^2)$, $\mathcal{B}(p^2)$ and $\mathcal{G}(p^2)$ functions. It compares the results obtained for the two different set of equations, with and without the simplification due to the WTI, see Eqs. (4.35), (4.36), (4.37) and Eqs. (4.33), (4.34) (4.37), respectively. They were computed in the Landau gauge and with $N_f \;=\; 1$. The relative error between the solutions obtained with the two different sets of equations, calculated analogously to Eq. (4.42), is smaller than the precision considered in this study. This was also verified for the solutions in the Feynman gauge and there was no relevant difference in efficiency regarding the computational time. Therefore, using either set of equations yields equivalent results and from now on we will only consider the solutions obtained without the WTI.

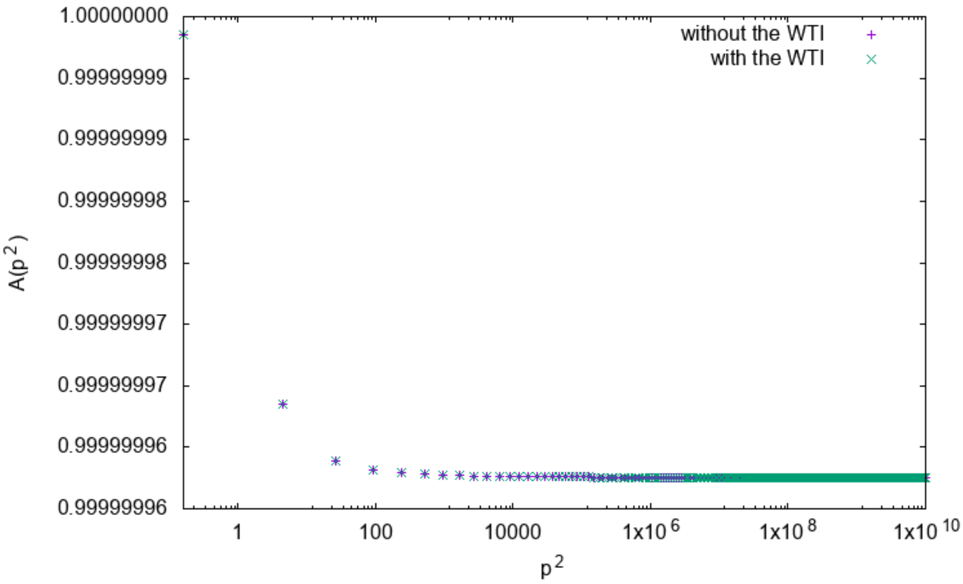


Figure 4.1: Function $\mathcal{A}(p^2)$ obtained using the Ball-Chiu vertex in the Landau gauge ($\xi = 0$), with and without the WTI.

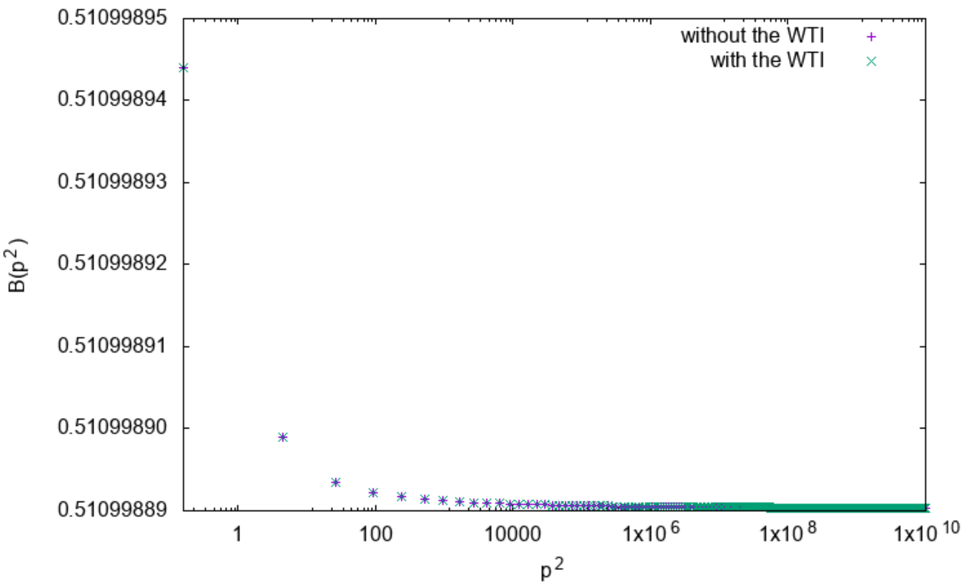


Figure 4.2: Function $\mathcal{B}(p^2)$ obtained using the Ball-Chiu vertex in the Landau gauge ($\xi = 0$), with and without the WTI.

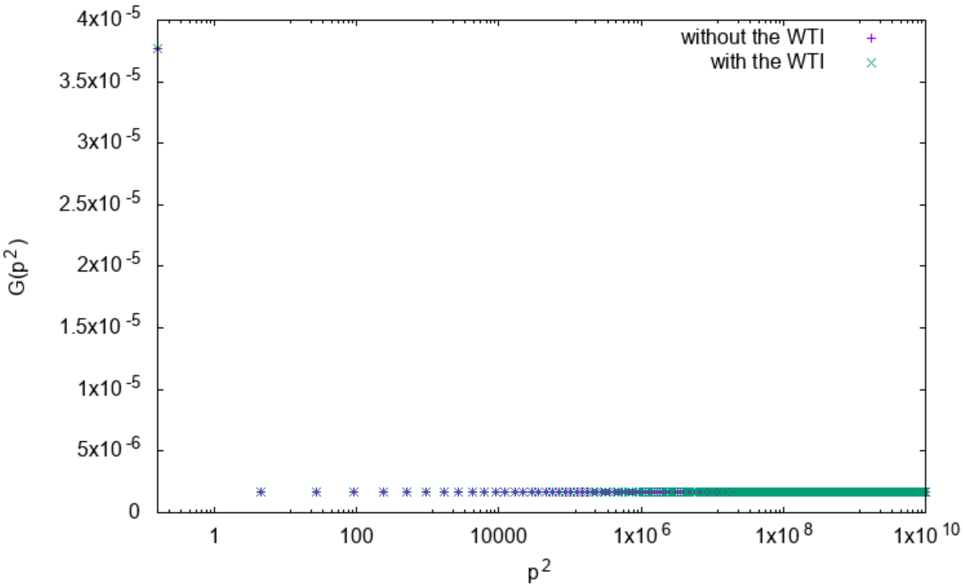


Figure 4.3: Function $\mathcal{G}(p^2)$ obtained using the Ball-Chiu vertex in the Landau gauge ($\xi = 0$), with and without the WTI.

4.4.1 Dependence on the Number of Fermion Flavours

In the DSE for the photon propagator, given in Eq. (3.23), the number of fermion flavours multiplies the term containing the fermion loop. Thus, if $N_f = 0$, then $\mathcal{G}(p^2) = 1$ and the photon propagator reduces to its tree-level form. This is known as the quenched case. In order to explore the effects of varying the number of fermion flavours, we can analyse the solutions obtained in the massless[1] case for $N_f = 0, 1, 2, 3$. As expected, the main difference appears in the photon propagator, as shown in Fig. 4.4. This only includes the solutions for the unquenched case, since $\mathcal{G}(p^2)$ is trivial otherwise, and it is possible to see a non-negligible shift for different values of N_f. On the other hand, Figs. 4.5 and 4.6 show the indirect effect that the quenched approximation has on the fermion solutions. The solutions for $N_f = 2, 3$ are not represented, since the relative errors with respect to the default solution with $N_f = 1$ in these cases are smaller than the convergence precision.

[1]To avoid numerical problems, the fermion was only approximately massless, with $m \approx 10^{-17}$.

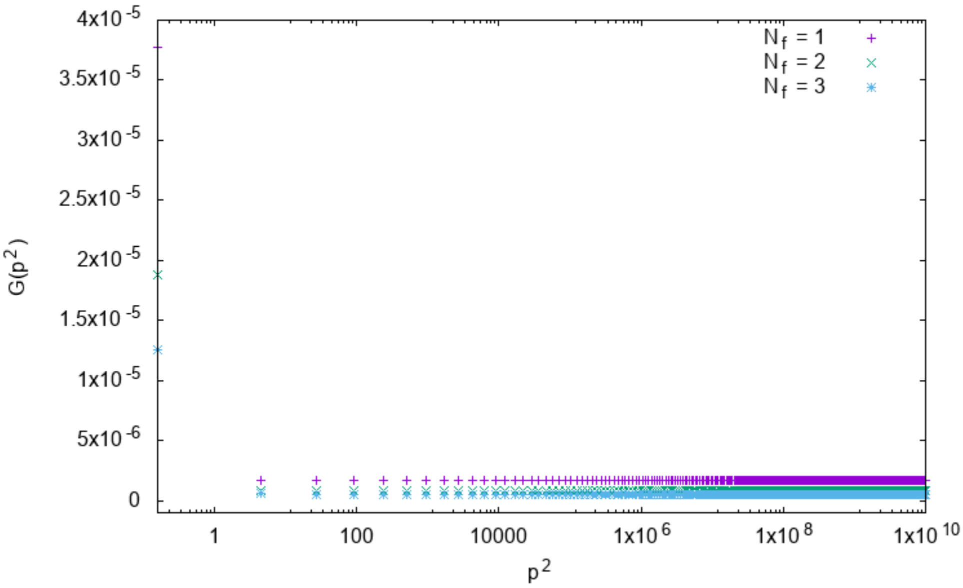


Figure 4.4: Function $\mathcal{G}(p^2)$ in the Landau gauge ($\xi = 0$), in the unquenched case with the number of fermion flavours varying from 1 to 3.

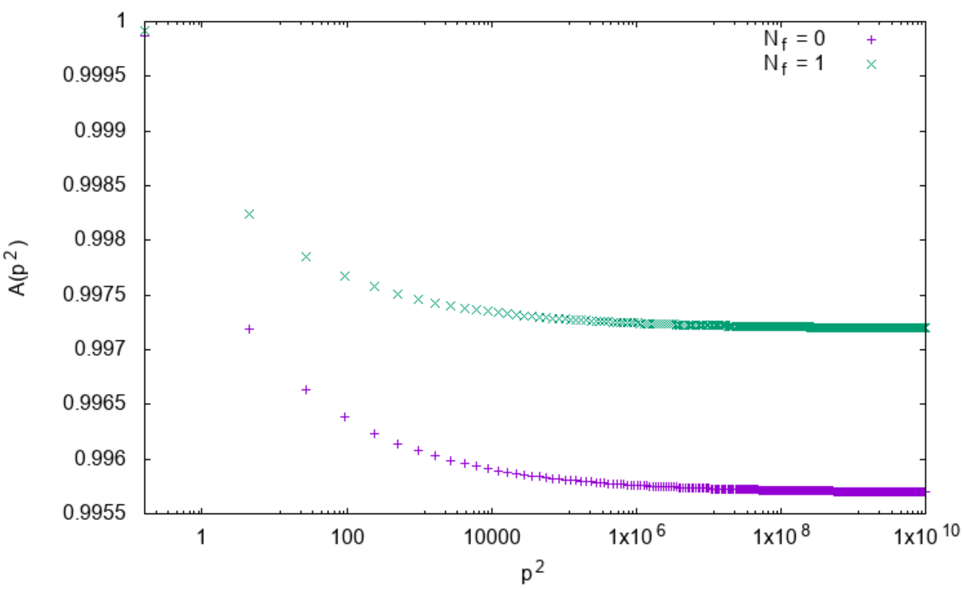


Figure 4.5: Function $\mathcal{A}(p^2)$ in the Landau gauge ($\xi = 0$), in the quenched ($N_f = 0$) and unquenched ($N_f = 1$) cases.

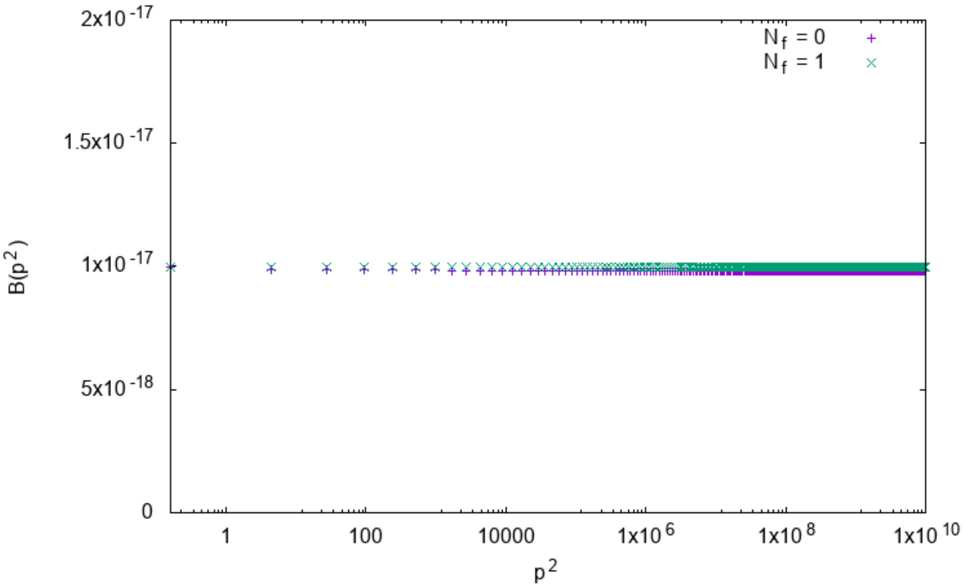


Figure 4.6: Function $\mathcal{B}(p^2)$ in the Landau gauge ($\xi = 0$), in the quenched ($N_f = 0$) and unquenched ($N_f = 1$) cases.

4.4.2 Tree-level Vertex

In the perturbative limit, the photon-fermion vertex is given simply by γ^μ, which can be obtained from the Ball-Chiu vertex by setting $\lambda_1 \approx 1$ and $\lambda_2 = \lambda_3 \approx 0$. However, from the behaviour observed so far for the $\mathcal{A}(p^2)$ and $\mathcal{B}(p^2)$ functions, the overall variation tends to be small, and therefore, from the definitions of the BC form factors given in Eqs. (4.8) - (4.10), we expect that the solutions obtained using the tree-level and the Ball-Chiu vertex will be very similar. Indeed, this is observed in Figs. 4.7 - 4.9.

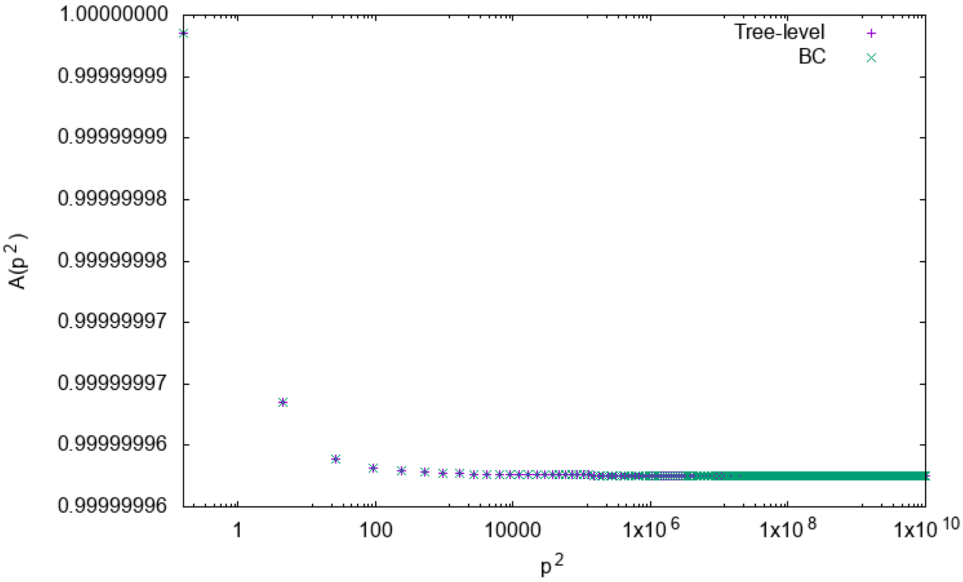


Figure 4.7: Function $\mathcal{A}(p^2)$ in the Landau gauge ($\xi = 0$) obtained using the tree-level and the Ball-Chiu vertex.

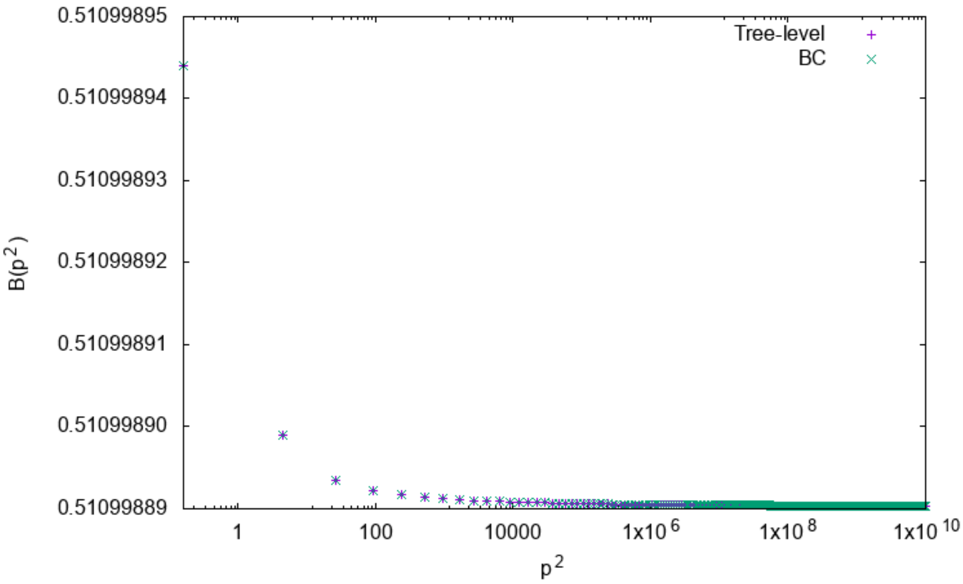


Figure 4.8: Function $\mathcal{B}(p^2)$ in the Landau gauge ($\xi = 0$) obtained using the tree-level and the Ball-Chiu vertex.

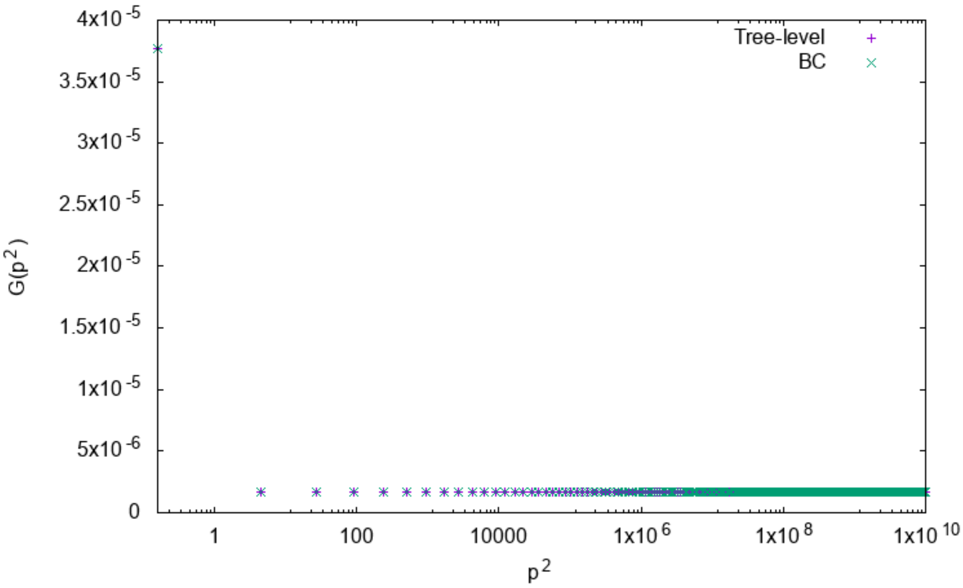


Figure 4.9: Function $\mathcal{G}(p^2)$ in the Landau gauge ($\xi = 0$) obtained using the tree-level and the Ball-Chiu vertex.

4.4.3 Gauge Dependence

The main characteristic of the Ball-Chiu vertex, as discussed in Sec. 4.1, is that it satisfies the Ward-Takahashi identity for the photon-fermion vertex and therefore it affects the gauge dependency of the solution. To analyse the behaviour of the fermion and photon propagator in different gauges, Figs. 4.10 - 4.12 show solutions obtained for each function in the Landau, Feynman and Yennie gauges, with $\xi = 0, 1, 3$, respectively.

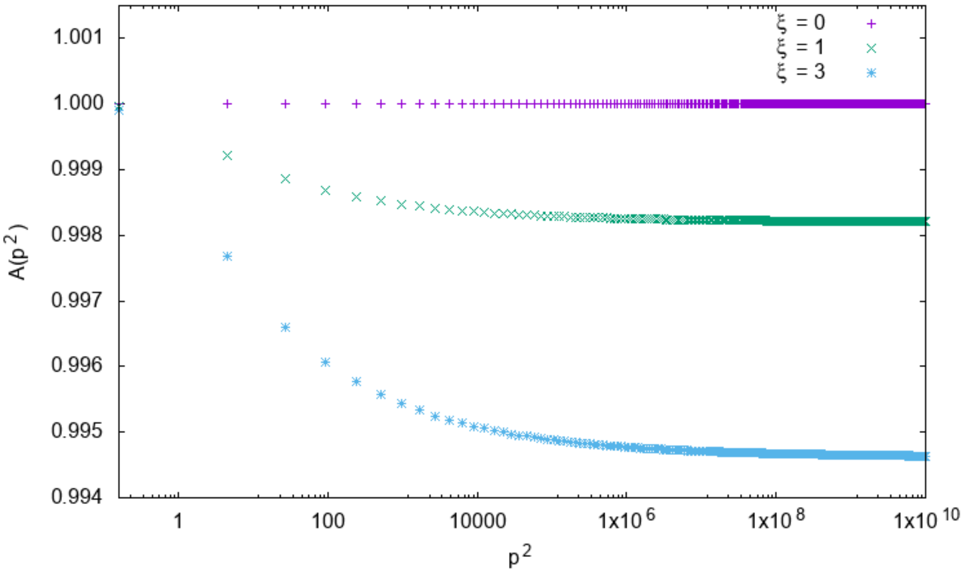


Figure 4.10: Function $\mathcal{A}(p^2)$ obtained in different gauges, with $\xi = 1, 2, 3$.

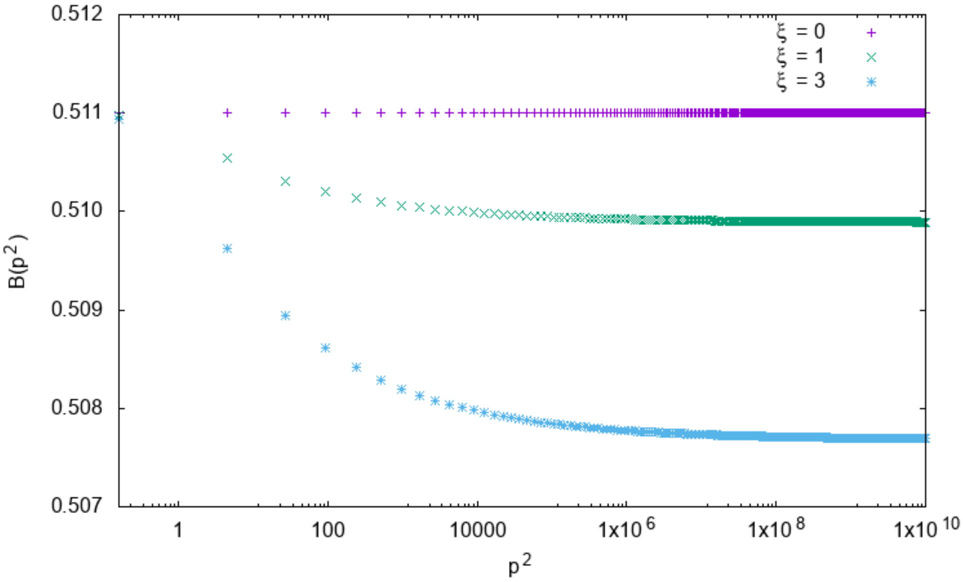


Figure 4.11: Function $\mathcal{B}(p^2)$ obtained in different gauges, with $\xi = 1, 2, 3$.

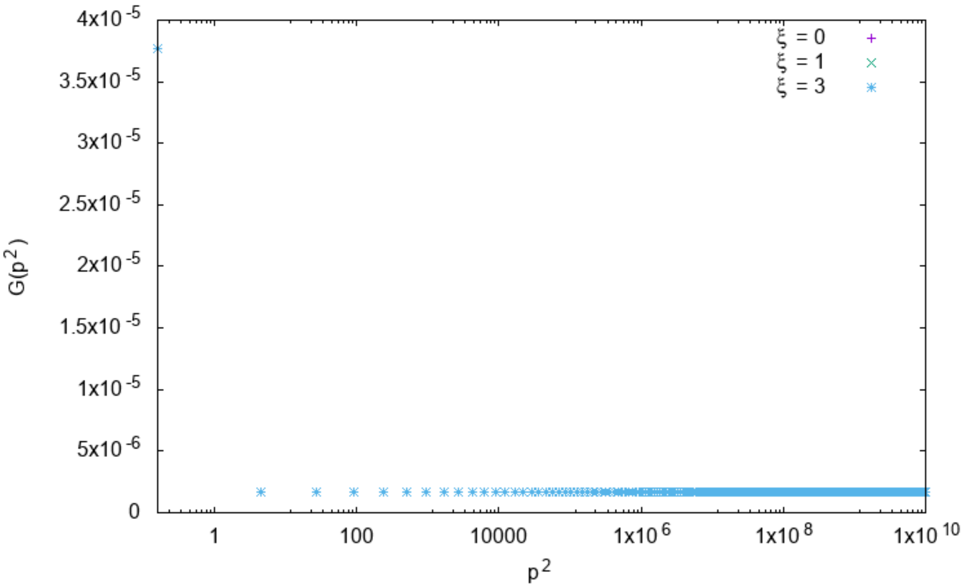


Figure 4.12: Function $\mathcal{G}(p^2)$ obtained in different gauges, with $\xi = 1, 2, 3$.

According to the Landau-Khalatnikov-Fradkin transformations [5, 11, 23], in QED the photon wave-function renormalisation $\mathcal{G}(p^2)$ should be gauge invariant. However, although small, it is possible to observe some gauge dependence in Fig. 4.13. This can be attributed to the fact that the WTI only acts upon the longitudinal part of the vertex, and therefore by itself it is not sufficient to guarantee that the solution respects gauge invariance. Nonetheless, when compared to the solution obtained with the tree-level vertex, the solution with the Ball-Chiu vertex is less gauge dependent, as can be seen in Figs. 4.13 and 4.14.

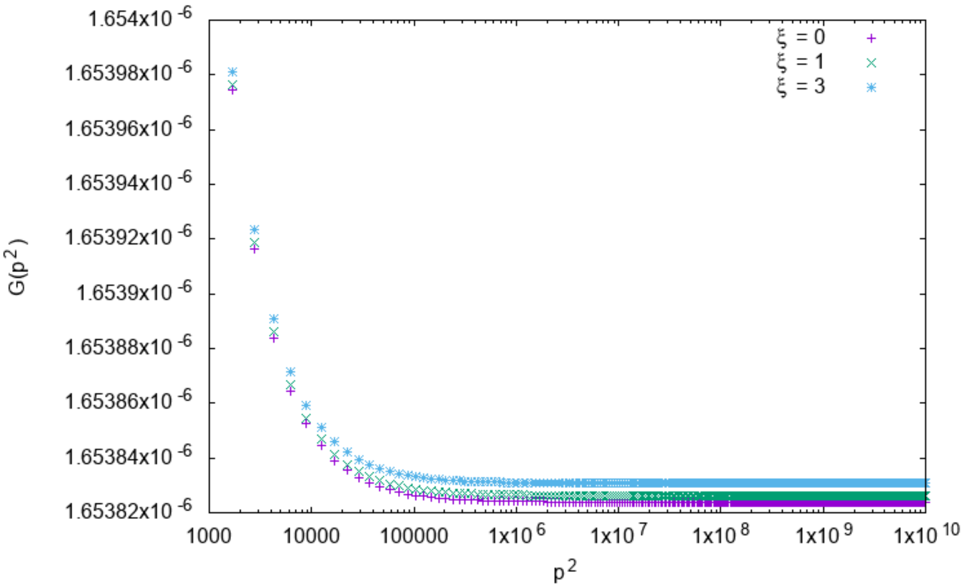


Figure 4.13: Function $\mathcal{G}(p^2)$ in the region with $p^2 > 1000\,\mathrm{MeV}^2$ obtained in different gauges, with $\xi = 1, 2, 3$.

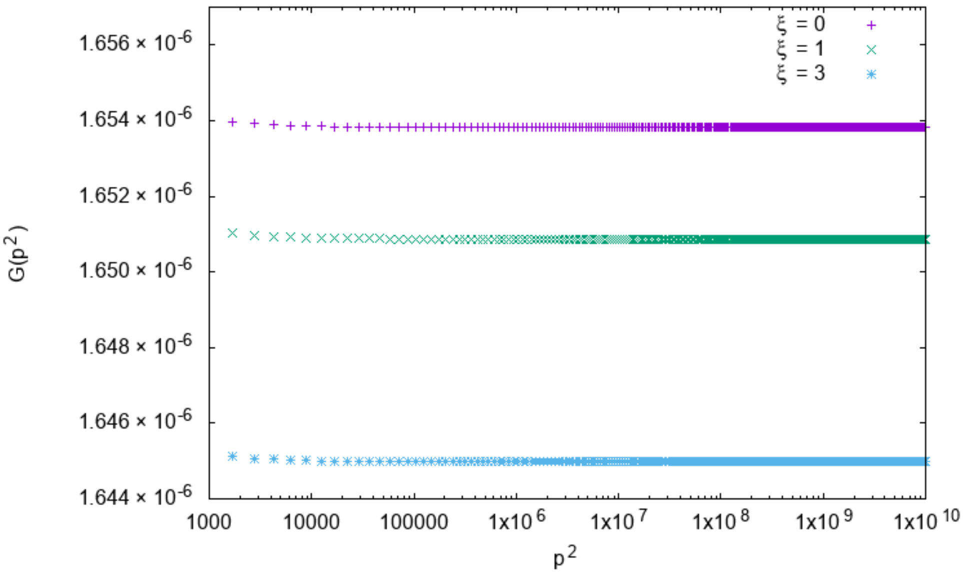


Figure 4.14: Function $\mathcal{G}(p^2)$ in the region with $p^2 > 1000\,\mathrm{MeV}^2$ obtained in different gauges using the tree-level vertex, with $\xi = 1, 2, 3$.

5 Conclusions

Throughout this work, we explored the Dyson-Schwinger equations formalism applied to QED in a general linear covariant gauge. In order to build a minimal set of equations for the photon and fermion propagators and the photon-fermion vertex, the DSEs for the two- and three-point functions in Minkowski spacetime were derived exactly. However, since the equation for the latter includes the 1PI two-photon-two-fermion vertex, we also considered the approximated equation for the four-point Green's function to build a closed set of equations. The DSEs constitute an infinite tower of coupled equations, hence it becomes necessary to introduce a truncation in order to investigate only a subset of equations.

Along with the Dyson-Schwinger equations, the Ward-Takahashi identities for the photon-fermion and the two-photon-two-fermion vertices were also derived. These identities are a direct consequence of the gauge invariance of the theory and can be used to determine the longitudinal part of these vertices.

No attempts were made to solve the complete set of equation that was derived due to its complexity. Nevertheless, a preliminary numerical study was made for the coupled photon-fermion system using mainly the Ball-Chiu *Ansatz* to replace the photon-fermion vertex. In this study, it was possible to observe an improvement of the Ball-Chiu vertex in comparison to the lowest order perturbative solution regarding the gauge dependence of the theory. Furthermore, there was no difference in the solutions nor in the computational efficiency regarding the two different set of equations obtained with and without the additional simplification using the WTI. The dependence on the number of fermion flavours was also investigated, where the photon propagator is the most affected. On the other hand, there is a difference in the fermion propagator regarding the quenched and unquenched cases, but it remains unaltered when varying the number of fermion flavours from 1 to 3.

Further work may include a more extensive numerical study of the photon-fermion coupled system, in which we could investigate the origin of the infrared pole observed in the photon propagator as well as the dynamical mass generation in QED. It would also be interesting to solve the equation for the photon-fermion vertex, along with the previous propagators. To do this, we could use the longitudinal vertex $\Gamma_L^{\mu\nu}$ obtained in [18] from the solution of the two-photon-two-fermion vertex WTI to model this vertex. In this case, in order to guarantee that the photon-fermion vertex would still satisfy the WTI, we could fix the longitudinal vertex component as the Ball-Chiu vertex and solve the transverse projection of the photon-fermion vertex DSE, i.e., contracting Eq. (3.33) with the transverse projection operator given in Eq. (3.25). Finally, the research developed in this work can be extended to non-Abelian gauge theories, such as QCD, see e.g. [32, 33, 34] for some applications of the Dyson-Schwinger equations.

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Appendix A

Decomposition of the Connected Green's Function

Throughout the derivation of the various DSEs, it is necessary on many occasions to write the connected Green's functions in terms of one-particle irreducible vertices. In this Appendix, we discuss the decomposition of different order Green's functions.

A.1 Three-point Green's function decomposition

To obtain the decomposition of the three-point connected Green's function, we start from Eq. (2.29) and take two additional functional derivatives, the first one with respect to $\eta_\beta(y)$ and the second with respect to $A_{cl,\,\mu}(z)$. Step by step, the first derivative reads

$$
\begin{aligned}
\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta\eta_\beta(y)\,\delta\bar{\psi}_{cl,\,\alpha}(x)} &= \frac{\delta\eta_\alpha(x)}{\delta\eta_\beta(y)} \\
\Leftrightarrow \int d^4u_1\,\frac{\delta\psi_{cl,\,\beta'}(u_1)}{\delta\eta_\beta(y)}\,\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)} &= -\,\delta_{\alpha\beta}\,\delta(x-y) \\
\Leftrightarrow \int d^4u_1\,\frac{\delta^2W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}\,\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)} &= -\,\delta_{\alpha\beta}\,\delta(x-y)\ , \qquad \text{(A.1)}
\end{aligned}
$$

where Eq. (2.22) was used to go from the second to the third line. Then, taking the second derivative, we obtain

$$\frac{\delta}{\delta A_{cl,\,\mu}(z)}\left[\int d^4u_1\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)}\,\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar\psi_{cl,\,\alpha}(x)}\right] = 0$$

$$\Leftrightarrow \int d^4u_1\,d^4u_2\,\frac{\delta J_{\mu'}(u_2)}{\delta A_{cl,\,\mu}(z)}\,\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J_{\mu'}(u_2)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)}\,\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta\eta_{\beta'}(u_1)\,\delta\bar\psi_{cl,\,\alpha}(x)}$$
$$+\int d^4u_1\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)}\,\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar\psi_{cl,\,\alpha}(x)} = 0$$

$$\Leftrightarrow \int d^4u_1\,d^4u_2\,\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta A_{cl}^{\mu'}(u_2)}\,\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta\eta_{\beta'}(u_1)\,\delta\bar\psi_{cl,\,\alpha}(x)}\,\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J_{\mu'}(u_2)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)} =$$
$$= \int d^4u_1\,\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar\psi_{cl,\,\alpha}(x)}\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)}\,, \tag{A.2}$$

where Eq. (2.27) was used in the last step. This equation can be solved using the orthogonality relations given in Eqs. (2.35) and (2.36). Thus, multiplying Eq. (A.2) by the appropriate terms and integrating it, the left-hand side becomes

$$\int d^4u_1\,d^4u_2\,d^4z\,d^4x\,\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta A_{cl}^{\mu'}(u_2)}\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^{\bar\mu}(v_2)\,\delta J^{\mu}(z)}$$
$$\frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta\eta_{\beta'}(u_1)\,\delta\bar\psi_{cl,\,\alpha}(x)}\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\alpha(x)\,\delta\bar\eta_{\bar\alpha}(v_1)}$$
$$\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J_{\mu'}(u_2)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)}$$
$$= \int d^4u_1\,d^4u_2\,\Big(-\,g_{\bar\mu\mu'}\,\delta(v_2-u_2)\Big)\Big(-\,\delta_{\bar\alpha\beta'}\,\delta(v_1-u_1)\Big)\,\frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J_{\mu'}(u_2)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)}$$
$$= \frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^{\bar\mu}(v_2)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\bar\alpha}(v_1)}\,. \tag{A.3}$$

While the right-hand side is given by

$$\int d^4u_1\,d^4z\,d^4x\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^{\bar\mu}(v_2)\,\delta J^{\mu}(z)}\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\alpha(x)\,\delta\bar\eta_{\bar\alpha}(v_1)}\,\frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_1)}$$
$$\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar\psi_{cl,\,\alpha}(x)}\,. \tag{A.4}$$

Thus, after setting the sources to zero and relabelling, we can use Eqs. (2.30), (2.31) and (2.32) to write the three-point connected Green's function in terms of propagators and the 1PI vertex:

$$\left(\frac{\delta^3 W[J, \bar{\eta}, \eta]}{\delta J^\mu(z)\, \delta\eta_\beta(y)\, \delta\bar{\eta}_\alpha(x)} \right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} = g \int d^4u_1\, d^4u_2\, d^4u_3\, D_{\mu\mu'}(z - u_1) \left[S(x - u_2)\, \Gamma^{\mu'}(u_2, u_3; u_1)\, S(u_3 - y) \right]_{\alpha\beta} . \quad \text{(A.5)}$$

Using Eq. (2.34) we can write this in momentum space, which reads

$$\left(\frac{\delta^3 W[J, \bar{\eta}, \eta]}{\delta J^\mu(z)\, \delta\eta_\beta(y)\, \delta\bar{\eta}_\alpha(x)} \right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} = g \int \frac{d^4p_1}{(2\,\pi)^4} \frac{d^4p_2}{(2\,\pi)^4} \frac{d^4k}{(2\,\pi)^4}\, e^{-i\,(k\,z\,+\,p_1\,x\,-\,p_2\,y)} (2\,\pi)^4\, \delta(p_1\;\; p_2 + k)\, D_{\mu\mu'}(k) \left[S(p_1)\, \Gamma^{\mu'}(p_1, -p_2; k)\, S(p_2) \right]_{\alpha\beta} . \quad \text{(A.6)}$$

A.2 Four-point Green's function decomposition

The four-point connected Green's function can be decomposed in terms of 1PI functions following the same procedure as for the three-point function. Starting from Eqs. (A.3) and (A.4), we can perform an additional functional derivative with respect to $A_{cl,\,\nu}(w)$, arriving at

$$\int d^4u_3 \, \frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\nu}(w)\,\delta A_{cl}^{\nu'}(u_3)} \, \frac{\delta^4 W[J,\bar{\eta},\eta]}{\delta J_{\nu'}(u_3)\,\delta J^{\bar{\mu}}(v_2)\,\delta\eta_\beta(y)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)} \; =$$

$$= \int d^4u_1\, d^4u_3\, d^4z\, d^4x \, \frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\nu}(w)\,\delta A_{cl}^{\nu'}(u_3)} \, \frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}$$

$$\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J_{\nu'}(u_3)\,\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)} \, \frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)} \, \frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$

$$+ \int d^4u_1\, d^4u_3\, d^4z\, d^4x \, \frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\nu}(w)\,\delta A_{cl}^{\nu'}(u_3)} \, \frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}$$

$$\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)} \, \frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J_{\nu'}(u_3)\,\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)} \, \frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$

$$+ \int d^4u_1\, d^4u_3\, d^4z\, d^4x \, \frac{\delta^2\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\nu}(w)\,\delta A_{cl}^{\nu'}(u_3)} \, \frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}$$

$$\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)} \, \frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)} \, \frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J_{\nu'}(u_3)\,\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$

$$- \int d^4u_1\, d^4z\, d^4x \, \frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)} \, \frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)} \, \frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$

$$\frac{\delta^4\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\nu}(w)\,\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)} \; . \tag{A.7}$$

Following the same procedure as before, we can use Eq. (2.35) to solve this equation. This leaves us with

$$\frac{\delta^4 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\nu}}(v_3)\,\delta J^{\bar{\mu}}(v_2)\,\delta\eta_\beta(y)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)} =$$

$$= \int d^4u_1\,d^4z\,d^4x\,\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\nu}}(v_3)\,\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)}\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)}\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$
$$\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}$$

$$+ \int d^4u_1\,d^4z\,d^4x\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)}\,\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\nu}}(v_3)\,\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)}\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$
$$\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}$$

$$+ \int d^4u_1\,d^4z\,d^4x\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)}\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)}\,\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\nu}}(v_3)\,\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$
$$\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}$$

$$+ \int d^4u_1\,d^4w\,d^4z\,d^4x\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\nu}}(v_3)\,\delta J^{\nu}(w)}\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta J^{\bar{\mu}}(v_2)\,\delta J^{\mu}(z)}\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\alpha(x)\,\delta\bar{\eta}_{\bar{\alpha}}(v_1)}\,\frac{\delta^2 W[J,\bar{\eta},\eta]}{\delta\eta_\beta(y)\,\delta\bar{\eta}_{\beta'}(u_1)}$$
$$\frac{\delta^4\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta A_{cl,\,\nu}(w)\,\delta A_{cl,\,\mu}(z)\,\delta\psi_{cl,\,\beta'}(u_1)\,\delta\bar{\psi}_{cl,\,\alpha}(x)}\,. \qquad (A.8)$$

The first term is proportional to the three-photon 1PI vertex, since, after setting the sources to zero,

$$\left(\frac{\delta^3 W[J,\bar{\eta},\eta]}{\delta J^{\mu}(x)\,\delta J^{\nu}(y)\,\delta J^{\rho}(z)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} = g\int d^4u_1\,d^4u_2\,d^4u_3$$
$$D_{\mu\mu'}(x-u_1)\,D_{\nu\nu'}(y-u_2)\,D_{\rho\rho'}(z-u_3)$$
$$\left(\frac{\delta^3\Gamma[A_{cl},\psi_{cl},\bar{\psi}_{cl}]}{\delta J_{\mu'}(u_1)\,\delta J_{\nu'}(u_2)\,\delta J_{\rho'}(u_3)}\right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0}\,, \qquad (A.9)$$

and therefore this vanishes in QED, according to Furry's theorem. Using the decomposition given in Eq. (A.5) for the three-point photon-fermion Green's function, as well as the definitions in Eqs. (2.30), (2.31) and (2.33), the equation becomes, after some relabelling,

$$\left(\frac{\delta^4 W[J,\bar{\eta},\eta]}{\delta J^\mu(z)\,\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar{\eta}_\alpha(x)} \right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} =$$

$$\begin{aligned}
&= \; -g^2 \int d^4u_1\, d^4u_2\, d^4u_3\, d^4u_4\, D_{\mu\mu'}(z-u_1)\, D_{\nu\nu'}(w-u_2) \\
&\qquad \Big[S(x-u_3)\, \Gamma^{\mu'\nu'}(u_3,u_4;u_1,u_2)\, S(u_4-y) \Big]_{\alpha\beta} \\
&\quad - g^2 \int d^4u_1\, d^4u_2\, d^4u_3 d^4u_4\, d^4u_5\, d^4u_6\, D_{\mu\mu'}(z-u_1)\, D_{\nu\nu'}(w-u_2) \\
&\qquad \Big[S(x-u_3)\, \Gamma^{\nu'}(u_3,u_4;u_2)\, S(u_4-u_5)\, \Gamma^{\mu'}(u_5,u_6;u_1)\, S(u_6-y) \Big]_{\alpha\beta} \\
&\quad - g^2 \int d^4u_1\, d^4u_2\, d^4u_3 d^4u_4\, d^4u_5\, d^4u_6\, D_{\mu\mu'}(z-u_1)\, D_{\nu\nu'}(w-u_2) \\
&\qquad \Big[S(x-u_3)\, \Gamma^{\mu'}(u_3,u_4;u_1)\, S(u_4-u_5)\, \Gamma^{\nu'}(u_5,u_6;u_2)\, S(u_6-y) \Big]_{\alpha\beta} \, . \qquad \text{(A.10)}
\end{aligned}$$

We can write this is momentum space using Eq. (2.34), obtaining

$$\left(\frac{\delta^4 W[J,\bar{\eta},\eta]}{\delta J^\mu(z)\,\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar{\eta}_\alpha(x)} \right)\Bigg|_{J,\,\bar{\eta},\,\eta\,=\,0} =$$

$$\begin{aligned}
&= \; -g^2 \int \frac{d^4p_1}{(2\,\pi)^4}\, \frac{d^4p_2}{(2\,\pi)^4}\, \frac{d^4k_1}{(2\,\pi)^4}\, \frac{d^4k_2}{(2\,\pi)^4}\, e^{-\,i\,(k_1\,z\,+\,k_2\,w\,+\,p_1\,x\,-\,p_2\,y)}\, (2\,\pi)^4\, \delta(p_1-p_2+k_1+k_2) \\
&\qquad D_{\mu\mu'}(k_1)\, D_{\nu\nu'}(k_2) \Big\{ S(p_1) \Big[\Gamma^{\mu'\nu'}(p_1,-p_2;k_1,k_2) \\
&\qquad\quad +\; \Gamma^{\nu'}(p_1,-p_1-k_2;k_2)\, S(p_1+k_2)\, \Gamma^{\mu'}(p_1+k_2,-p_2;k_1) \\
&\qquad\quad +\; \Gamma^{\mu'}(p_1,-p_1-k_1;k_1)\, S(p_1+k_1)\, \Gamma^{\nu'}(p_1+k_1,-p_2;k_2) \Big] S(p_2) \Big\}_{\alpha\beta} \, . \qquad \text{(A.11)}
\end{aligned}$$

A.3 Five-point Green's function decomposition

Finally, to decompose the five-point connected Green's function, we follow the same procedure as before, taking now an additional derivative with respect to $A_{cl,\,\rho}(s)$. Then, after some straightforward algebra, we arrive at the decomposition of the five-point Green's function in terms of 1PI functions in coordinate space:

$$
\left(\frac{\delta^5 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^\nu(w)\,\delta J^\mu(z)\,\delta\eta_\beta(y)\,\delta\bar\eta_\alpha(x)} \right) =
$$

$$
\begin{aligned}
&= \int d^4u_1\,d^4u_2\,d^4u_3\,d^4u_4\,d^4u_5\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^{\rho'}(u_1)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^{\nu'}(u_2)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\mu(z)\,\delta J^{\mu'}(u_3)} \\
&\qquad \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_4)\,\delta\bar\eta_\alpha(x)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_5)} \\
&\qquad \frac{\delta^5 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\rho'}(u_1)\,\delta A_{cl,\,\nu'}(u_2)\,\delta A_{cl,\,\mu'}(u_3)\,\delta\psi_{cl,\,\beta'}(u_5)\,\delta\bar\psi_{cl,\,\alpha'}(u_4)} \\
&+ \int d^4u_1\,d^4u_2\,d^4u_3\,d^4u_4\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^{\rho'}(u_1)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^{\nu'}(u_2)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_3)\,\delta\bar\eta_\alpha(x)} \\
&\qquad \frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\mu(z)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_4)} \frac{\delta^4 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\rho'}(u_1)\,\delta A_{cl,\,\nu'}(u_2)\,\delta\psi_{cl,\,\beta'}(u_4)\,\delta\bar\psi_{cl,\,\alpha'}(u_3)} \\
&+ \int d^4u_1\,d^4u_2\,d^4u_3\,d^4u_4\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^{\rho'}(u_1)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\mu(z)\,\delta J^{\mu'}(u_2)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_3)\,\delta\bar\eta_\alpha(x)} \\
&\qquad \frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_4)} \frac{\delta^4 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\rho'}(u_1)\,\delta A_{cl,\,\mu'}(u_2)\,\delta\psi_{cl,\,\beta'}(u_4)\,\delta\bar\psi_{cl,\,\alpha'}(u_3)} \\
&+ \int d^4u_1\,d^4u_2\,d^4u_3\,d^4u_4\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^{\nu'}(u_1)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\mu(z)\,\delta J^{\mu'}(u_2)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_3)\,\delta\bar\eta_\alpha(x)} \\
&\qquad \frac{\delta^3 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_4)} \frac{\delta^4 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\nu'}(u_1)\,\delta A_{cl,\,\mu'}(u_2)\,\delta\psi_{cl,\,\beta'}(u_4)\,\delta\bar\psi_{cl,\,\alpha'}(u_3)} \\
&+ \int d^4u_1\,d^4u_2\,d^4u_3\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^{\rho'}(u_1)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_2)\,\delta\bar\eta_\alpha(x)} \frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^\mu(z)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_3)} \\
&\qquad \frac{\delta^3 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\rho'}(u_1)\,\delta\psi_{cl,\,\beta'}(u_3)\,\delta\bar\psi_{cl,\,\alpha'}(u_2)} \\
&+ \int d^4u_1\,d^4u_2\,d^4u_3\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\nu(w)\,\delta J^{\nu'}(u_1)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_2)\,\delta\bar\eta_\alpha(x)} \frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^\mu(z)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_3)} \\
&\qquad \frac{\delta^3 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\nu'}(u_1)\,\delta\psi_{cl,\,\beta'}(u_3)\,\delta\bar\psi_{cl,\,\alpha'}(u_2)} \\
&+ \int d^4u_1\,d^4u_2\,d^4u_3\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta J^\mu(z)\,\delta J^{\mu'}(u_1)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_2)\,\delta\bar\eta_\alpha(x)} \frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^\nu(w)\,\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_3)} \\
&\qquad \frac{\delta^3 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\mu'}(u_1)\,\delta\psi_{cl,\,\beta'}(u_3)\,\delta\bar\psi_{cl,\,\alpha'}(u_2)} \\
&+ \int d^4u_1\,d^4u_2\,d^4u_3\, \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_{\alpha'}(u_1)\,\delta\bar\eta_\alpha(x)} \frac{\delta^2 W[J,\bar\eta,\eta]}{\delta\eta_\beta(y)\,\delta\bar\eta_{\beta'}(u_2)} \frac{\delta^4 W[J,\bar\eta,\eta]}{\delta J^\rho(s)\,\delta J^\nu(w)\,\delta J^\mu(z)\,\delta J^\sigma(u_3)} \\
&\qquad \frac{\delta^3 \Gamma[A_{cl},\psi_{cl},\bar\psi_{cl}]}{\delta A_{cl,\,\sigma'}(u_3)\,\delta\psi_{cl,\,\beta'}(u_2)\,\delta\bar\psi_{cl,\,\alpha'}(u_1)}
\end{aligned}
\tag{A.12}
$$

where the terms proportional to the three-photon irreducible vertex vanished, due to Furry's theorem.

Appendix B

Trace calculations

In order to obtain the expressions for the propagator's form factors, one needs to calculate traces involving gamma matrices. In this appendix, we list some useful results that can be obtained using the properties of these matrices and of the trace operator:

$$\text{The trace of any product of an odd number of } \gamma^\mu \text{ is zero} \quad , \tag{B.1}$$

$$\text{Tr}\,[\gamma^\mu\,\gamma_\mu] \;=\; 16\;, \tag{B.2}$$

$$\text{Tr}\,[\gamma^\mu\,\gamma^\nu] \;=\; 4\,g^{\mu\nu}\;, \tag{B.3}$$

$$\text{Tr}\,[\gamma^\mu\,\gamma^\nu\,\gamma_\mu\,\gamma^\rho] \;=\; -\,8\,g^{\nu\rho}\;, \tag{B.4}$$

$$\text{Tr}\,[\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\rho] \;=\; 4\,\left(g^{\mu\nu}\,g^{\sigma\rho} \;-\; g^{\mu\sigma}\,g^{\nu\rho} \;+\; g^{\mu\rho}\,g^{\nu\sigma}\right)\;. \tag{B.5}$$

These properties are easily generalised, for example:

$$\begin{aligned}
\text{Tr}\,[\gamma^\mu\,\not{a}\,\not{b}\,\not{c}] \;&=\; \text{Tr}\,[\gamma^\mu\,\gamma^\nu\,\gamma^\sigma\,\gamma^\rho]\,a_\nu\,b_\sigma\,c_\rho \\
&=\; 4\,\left(g^{\mu\nu}\,g^{\sigma\rho} \;-\; g^{\mu\sigma}\,g^{\nu\rho} \;+\; g^{\mu\rho}\,g^{\nu\sigma}\right)\,a_\nu\,b_\sigma\,c_\rho \\
&=\; 4\,[a^\mu\,(b\cdot c) \;-\; b^\mu\,(a\cdot c) \;+\; c^\mu\,(a\cdot b)]\;.
\end{aligned} \tag{B.6}$$