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A RELATIVE THEORY OF UNIVERSAL CENTRAL EXTENSIONS

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ABSTRACT: Basing ourselves on Janelidze and Kelly's general notion of central extension, we study universal central extensions in the context of semi-abelian categories. Thus we unify classical, recent and new results in one conceptual framework. The theory we develop is relative with respect to a chosen Birkhoff subcategory of the category considered: for instance, we consider groups vs. abelian groups, Lie algebras vs. vector spaces, precrossed modules vs. crossed modules and Leibniz algebras vs. Lie algebras. We also examine the interplay between the relative case and the "absolute" theory determined by the Birkhoff subcategory of all abelian objects.

KEYWORDS: categorical Galois theory, semi-abelian category, homology, perfect object, commutator, Baer invariant, Birkhoff subcategory.
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Introduction

Universal central extensions have always been an important tool in low-dimensional homology. In fact the concept was there in Schur's foundational work [37], when homology for algebraic objects was only just starting to be developed; it was Schur who discovered the relations between perfect groups, universal central extensions and the Schur multiplier—the latter of which, as Hopf later realised, may be expressed in terms of integral homology of groups [25].

Given a group G and a normal subgroup N of G, the **commutator** $[N, G]$ is the normal subgroup generated by the elements $ngn^{-1}g^{-1}$ for all $n \in N$ and $g \in G$. A group G is **perfect** when G is equal to its commutator subgroup $[G, G]$. A surjective group homomorphism $f\colon B \to A$ is a **central extension** when the commutator $[K[f], B]$ of the kernel $K[f]$ of f with B is trivial.

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A central extension $u\colon U \to A$ is **universal** when, for any other central extension $f\colon B \to A$, there exists a unique group homomorphism $\overline{f}\colon U \to B$ satisfying $f \circ \overline{f} = u$. If u is a universal central extension then both its domain U and its codomain A are perfect. Conversely, for any perfect group A there exists such a universal central extension u. Computing the second integral homology group $H_2(A, \mathbb{Z})$ of a perfect group A is particularly simple: take a universal central extension $u\colon U \to A$; its kernel $K[u]$ is $H_2(A, \mathbb{Z})$.

Of course the application of those ideas is not limited to the case of groups: universal central extensions have been considered in many other situations as well. For instance, the case of Lie algebras is classical (here the Lie bracket plays the role of commutator), and more recently, similar theories have been worked out for crossed modules, precrossed modules, Leibniz algebras, etc. [1, 10, 11, 21, 22]. The aim of the present work is to make explicit the underlying unity of these results, and to join them in one abstract framework, so that a basic theory of universal central extensions is developed for all those special cases simultaneously.

Our approach is based on categorical Galois theory with, in particular, Janelidze and Kelly's general notion of central extension [28]. Their notion is *relative* in sense that whether or not an extension $f\colon B \to A$ in a Barr exact Mal'tsev category $\mathcal{A}$ is central depends on the choice of a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$. This relative approach is based on (and generalises) the work of the Fröhlich school [19, 35, 20] which focused on varieties of Ω-groups. Recall [24] that such is a variety of universal algebras which has amongst its operations and identities those of the variety of groups but has just one constant; and a Birkhoff subcategory of a variety is the same thing as a subvariety. Although some examples (e.g., groups vs. abelian groups and Lie algebras vs. vector spaces) are *absolute*, meaning that they fit into the theory relative with respect to the subcategory of all abelian objects, others are not: precrossed modules vs. crossed modules, and Leibniz algebras vs. Lie algebras, for instance. In the absolute case, some results were already investigated in [22].

The text is structured as follows. In the first section we develop that part of the theory which does not depend on the existence of either projective objects or short exact sequences. Here we work in the context of pointed Barr exact Mal'tsev categories. We give several characterisations of universal central extensions in terms of perfect objects. The most interesting results, however, are obtained in the setting of semi-abelian categories with enough projectives [4, 29]—which still includes all varieties of Ω-groups. In Section 2 we prove that any perfect object admits a universal central extension; we also make the connections with semi-abelian homology. In the last section we consider an adjunction

$$\mathcal{A} \underset{\supset}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{B} \qquad (\mathbf{A})$$

with the two resulting absolute theories—abelianisation in $\mathcal{A}$ and in $\mathcal{B}$—and focus on the interplay between the different types of universal central extension induced by them.

1. Basic theory

In their article [28], Janelidze and Kelly introduced a general theory of relative central extensions in the context of exact categories. This is the theory we shall be considering here, focusing on the induced relative notion of universal central extension. We give an overview of the needed definitions and prove some preliminary results on the relation between universal central extensions and perfect objects.

1.1. Barr exact Mal'tsev categories. Recall that a **regular epimorphism** is a coequaliser of some pair of arrows. A category is **regular** when it is finitely complete with coequalisers of kernel pairs and with pullback-stable regular epimorphisms. In a regular category, any morphism may be factored as a regular epimorphism followed by a monomorphism, and this **image factorisation** is unique up to isomorphism. A category is **Barr exact** when it is regular and such that any internal equivalence relation is a kernel pair. Since the theory of central extensions becomes somewhat simpler then, we shall restrict ourselves to the case when the categories considered are also **Mal'tsev**, i.e., any internal reflexive relation is an equivalence relation. In fact, a Barr exact category is Mal'tsev if and only if the pushout of a regular epimorphism along a regular epimorphism always exists, and the comparison map to the induced pullback is also a regular epimorphism [8]. See, e.g., [4] for further details.

Examples 1.2. The examples we shall be considering throughout the text are all categories which are (equivalent to) varieties of Ω-groups, and as such are finitely complete Barr exact Mal'tsev categories. They are: the categories Gp of groups and Ab of abelian groups; $\mathsf{Leib}_{\mathbb{K}}$, $\mathsf{Lie}_{\mathbb{K}}$ and $\mathsf{Vect}_{\mathbb{K}}$ of Leibniz algebras, Lie algebras and vector spaces over a field $\mathbb{K}$; and the categories PXMod, XMod and AbXMod of precrossed modules, crossed modules and abelian crossed modules.

Recall [32, 33] that a **Leibniz algebra** $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$ equipped with a bilinear operation $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ that satisfies

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

(the **Leibniz identity**) for all $x,y,z\in\mathfrak{g}$. When $[x,x]=0$ for all $x\in\mathfrak{g}$ the bracket is skew-symmetric and the Leibniz identity is the Jacobi identity, so $\mathfrak{g}$ is a Lie algebra.

Recall that a **precrossed module** (T,G,∂) is a group homomorphism $\partial\colon T\to G$ together with an action of G on T, denoted ${}^{g}t$ for $g\in G$ and $t\in T$, satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ for all $g\in G$ and $t\in T$. If in addition if verifies the **Peiffer identity** ${}^{\partial(t)}t' = tt't^{-1}$ for all t, $t'\in T$, we say that (T,G,∂) is a **crossed module**. A morphism of (pre)crossed modules $(f_1,f_0)\colon (T,G,\partial)\to(T',G',\partial')$ consists of group homomorphisms $f_1\colon T\to T'$ and $f_0\colon G\to G'$ such that $\partial'\circ f_1 = f_0\circ\partial$ and the action is preserved. The categories PXMod and XMod are equivalent to varieties of Ω-groups; see, e.g., [29], [30] or [31]. The category AbXMod consists of **abelian** crossed modules, i.e., (T,G,∂) such that T and G are abelian groups and the action of G on T is trivial.

From now on, $\mathcal{A}$ will denote a chosen Barr exact Mal'tsev category.

1.3. Birkhoff subcategories. The notion of central extension introduced in [28] is *relative*, being defined with respect to a chosen subcategory $\mathcal{B}$ of the category $\mathcal{A}$ considered.

A **Birkhoff subcategory** $\mathcal{B}$ of $\mathcal{A}$ is a full and reflective subcategory which is closed under subobjects and regular quotients. We write the induced adjunction as in (**A**) and denote its unit $\eta\colon 1_{\mathcal{A}}\Rightarrow I$. A Birkhoff subcategory of a variety of universal algebras is the same thing as a subvariety. If $\mathcal{A}$ is finitely complete Barr exact Mal'tsev then so is any Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$.

For a given full and reflective subcategory $\mathcal{B}$ closed under subobjects, the Birkhoff property of $\mathcal{B}$ (i.e., closure under quotients) is equivalent to the following condition: given any regular epimorphism $f\colon B\to A$ in $\mathcal{A}$, the induced square of regular epimorphisms

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ IB & \xrightarrow[If]{} & IA \end{array} \tag{\textbf{B}}$$

is a pushout.

The examples considered in 1.2 form categories with a chosen Birkhoff subcategory in the following ways.

Example 1.4 (Groups)**.** The left adjoint to the inclusion of Ab in Gp is the **abelianisation functor**, denoted $\mathsf{ab}\colon \mathsf{Gp} \to \mathsf{Ab}$; it sends a group G to its abelianisation $G/[G,G]$.

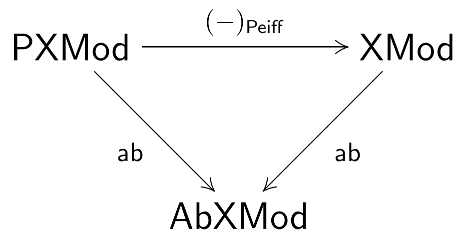
Example 1.5 (Leibniz algebras and Lie algebras)**.** Here there are three inclusions of Birkhoff subcategories, of which the left adjoints form the next commutative triangle.

$$\begin{array}{ccc} \mathsf{Leib}_{\mathbb{K}} & \xrightarrow{(-)_{\mathsf{Lie}}} & \mathsf{Lie}_{\mathbb{K}} \\ & {\scriptstyle \mathsf{ab}}\searrow \quad \swarrow{\scriptstyle \mathsf{ab}} & \\ & \mathsf{Vect}_{\mathbb{K}} & \end{array}$$

The left adjoint $(-)_{\mathsf{Lie}}\colon \mathsf{Leib}_{\mathbb{K}} \to \mathsf{Lie}_{\mathbb{K}}$ (which is usually called the **Liesation functor**) takes a Leibniz algebra $\mathfrak{g}$ and maps it to the quotient $\mathfrak{g}/\mathfrak{g}^{\mathsf{Ann}}$, where $\mathfrak{g}^{\mathsf{Ann}}$ is the two-sided ideal (i.e., normal subalgebra) of $\mathfrak{g}$ generated by all elements $[x,x]$ for $x \in \mathfrak{g}$.

$\mathsf{Vect}_{\mathbb{K}}$ may be considered as a subvariety of $\mathsf{Lie}_{\mathbb{K}}$ by equipping a vector space with the trivial Lie bracket; the left adjoint $\mathsf{ab}\colon \mathsf{Lie}_{\mathbb{K}} \to \mathsf{Vect}_{\mathbb{K}}$ to the inclusion $\mathsf{Vect}_{\mathbb{K}} \subset \mathsf{Lie}_{\mathbb{K}}$ takes a Lie algebra $\mathfrak{g}$ and maps it to the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, where $[\mathfrak{g},\mathfrak{g}]$ is generated by the elements $[x,y] \in \mathfrak{g}$ for all x, $y \in \mathfrak{g}$.

Example 1.6 ((Pre)crossed modules)**.** As in the previous example, we obtain a commutative triangle of left adjoint functors.



Given two normal precrossed submodules (M, H, ∂) and (N, K, ∂) of a precrossed module (T, G, ∂), the **Peiffer commutator** $\langle M, N\rangle$ is the normal subgroup of T generated by the Peiffer elements $\langle m, n\rangle = mnm^{-1}({}^{\partial(m)}n)^{-1}$ and $\langle n, m\rangle = nmn^{-1}({}^{\partial(n)}m)^{-1}$ for $m \in M$, $n \in N$ [13]. We denote by

$$\langle (M, H, \partial), (N, K, \partial)\rangle$$

the precrossed module $(\langle M, N\rangle, 1, 1)$; it may be considered as a normal precrossed submodule of (T, G, ∂). The precrossed module

$$\langle (T, G, \partial), (T, G, \partial)\rangle = (\langle T, T\rangle, 1, 1)$$

is the smallest one that makes the quotient $(T, G, \partial)/\langle (T, G, \partial), (T, G, \partial)\rangle$ a crossed module. This defines a functor $(-)_{\mathsf{Peiff}} \colon \mathsf{PXMod} \to \mathsf{XMod}$, left adjoint to the inclusion of XMod in PXMod.

Given a precrossed module (T, G, ∂), the commutator $[G, T]$ is the normal subgroup of T generated by the elements ${}^{g}tt^{-1}$ for $g \in G$ and $t \in T$. The left adjoint functor $\mathsf{ab} \colon \mathsf{PXMod} \to \mathsf{AbXMod}$ takes a precrossed module (T, G, ∂) and maps it to $(T/[T, T][G, T], G/[G, G], \overline{\partial})$, where $\overline{\partial}$ is the induced group homomorphism. The functor $\mathsf{ab} \colon \mathsf{XMod} \to \mathsf{AbXMod}$ is given by $\mathsf{ab}(T, G, \partial) = (T/[G, T], G/[G, G], \overline{\partial})$.

From now on, $\mathcal{B}$ will be a fixed Birkhoff subcategory of a Barr exact Mal'tsev category $\mathcal{A}$.

1.7. Extensions and central extensions. An **extension** in $\mathcal{A}$ is a regular epimorphism. A morphism of extensions is a commutative square between them, and thus we obtain the category $\mathsf{Ext}\mathcal{A}$ of extensions in $\mathcal{A}$.

Together with the classes $|\mathsf{Ext}\mathcal{A}|$ and $|\mathsf{Ext}\mathcal{B}|$ of extensions in $\mathcal{A}$ and $\mathcal{B}$, the adjunction (**A**) forms a Galois structure

$$\Gamma = (\mathcal{A} \underset{\supset}{\overset{I}{\rightleftarrows}} \mathcal{B}, |\mathsf{Ext}\mathcal{A}|, |\mathsf{Ext}\mathcal{B}|)$$

in the sense of [26]. With respect to this Galois structure, there are notions of *trivial* and *central* extension, which in the present context amount to the following [7, 28]. An extension $f\colon B \to A$ in $\mathcal{A}$ is **trivial (with respect to $\mathcal{B}$)** or **$\mathcal{B}$-trivial** when the induced square (**B**) is a pullback. And f is **central (with respect to $\mathcal{B}$)** or **$\mathcal{B}$-central** when either one of the projections f_0, f_1 in the kernel pair $(R[f], f_0, f_1)$ of f is $\mathcal{B}$-trivial. That is to say, f is central with respect to $\mathcal{B}$ if and only if in the diagram

$$\begin{array}{ccccc} R[f] & \overset{f_0}{\underset{f_1}{\rightrightarrows}} & B & \xrightarrow{f} & A \\ {\scriptstyle \eta_{R[f]}}\downarrow & & \downarrow{\scriptstyle \eta_B} & & \\ IR[f] & \overset{If_0}{\underset{If_1}{\rightrightarrows}} & IB & & \end{array}$$

either one of the left hand side squares is a pullback. It may be shown that central extensions are pullback-stable, and that a split epimorphism is trivial if and only if it is central [28, Proposition 4.3 and Theorem 4.8].

Examples 1.8 (Classical examples)**.** Some of our examples give rise to classical notions of central extension. In the case of Example 1.4, Gp vs. Ab, an extension $f\colon B \to A$ is central if and only if its kernel $K[f]$ is contained in the centre

$$ZB = \{z \in B \mid [z, b] = 1 \text{ for all } b \in B\}$$

of B. Similarly, the notion obtained in the case of $\mathsf{Lie}_{\mathbb{K}}$ vs. $\mathsf{Vect}_{\mathbb{K}}$ (Example 1.5) is the ordinary notion of central extension of Lie algebras, where the kernel $K[f]$ of $f\colon \mathfrak{b} \to \mathfrak{a}$ should be included in the centre of $\mathfrak{b}$, i.e., in

$$Z\mathfrak{b} = \{z \in \mathfrak{b} \mid [z, b] = 0 \text{ for all } b \in \mathfrak{b}\}.$$

Example 1.9 ($\mathsf{Leib}_{\mathbb{K}}$ vs. $\mathsf{Lie}_{\mathbb{K}}$, Example 1.5)**.** Given a Leibniz algebra $\mathfrak{g}$, its **$\mathsf{Lie}_{\mathbb{K}}$-centre** $Z_{\mathsf{Lie}}(\mathfrak{g})$ is the two-sided ideal generated by

$$\{z \in \mathfrak{g} \mid [g, z] = -[z, g] \text{ for all } g \in \mathfrak{g}\}.$$

For an extension $f\colon \mathfrak{b} \to \mathfrak{a}$ of Leibniz algebras, the following three conditions are now equivalent:

(1) $f\colon \mathfrak{b} \to \mathfrak{a}$ is $\mathsf{Lie}_{\mathbb{K}}$-central;
(2) $R[f]^{\mathsf{Ann}} \cong \mathfrak{b}^{\mathsf{Ann}}$;
(3) $K[f] \subset Z_{\mathsf{Lie}}(\mathfrak{b})$.

Indeed, (1) is equivalent to (2) by definition. Now suppose that (2) holds and consider $k \in K[f]$ and $b \in \mathfrak{b}$. Then both $[(k,0),(k,0)] = ([k,k],[0,0])$ and $[(b-k,b),(b-k,b)] = ([b-k,b-k],[b,b])$ are in $R[f]^{\mathsf{Ann}}$, which implies that $[k,k] = [0,0] = 0$ and $[b-k,b-k] = [b,b]$. Thus we see that $[b,k]+[k,b] = 0$, which implies that (3) holds.

Conversely, consider $[(b,k),(b,k)]$ in $R[f]^{\mathsf{Ann}} \cap K[f_1]$; then $[k,k] = 0$ and $b-k$ is an element of the kernel of f. Now (3) implies that $[b-k,b-k] = 0$, $[b,k] = [b,k]-[k,k] = [b-k,k] = 0$ and $[k,b] = [k,b]-[k,k] = [k,b-k] = 0$, so that also $[b,b] = 0$. It follows that $[(b,k),(b,k)] = 0$. Hence $R[f]^{\mathsf{Ann}} \cong \mathfrak{b}^{\mathsf{Ann}}$ and (2) holds.

Example 1.10 ((Pre)crossed modules, Example 1.6)**.** The results of [16, Section 9.5] imply that an extension of precrossed modules $f\colon B \to A$ is XMod-central if and only if $\langle K[f], B\rangle = 1$; the next characterisation may also be shown directly, parallel to Example 1.9. Given a precrossed module (T,G,∂), its **XMod-centre** $Z_{\mathsf{XMod}}(T,G,\partial)$ is the normal precrossed submodule $(Z_{\mathsf{XMod}}T,G,\partial)$ of (T,G,∂) where

$$Z_{\mathsf{XMod}}T = \{t \in T \mid \langle t,t'\rangle = 1 = \langle t',t\rangle \text{ for all } t' \in T\}.$$

For an extension $(f_1,f_0)\colon (T,G,\partial) \to (T',G',\partial')$ of precrossed modules, the following conditions are equivalent:

(1) (f_1,f_0) is XMod-central;
(2) $\langle (R[f_1],R[f_0],\partial\times\partial),(R[f_1],R[f_0],\partial\times\partial)\rangle \cong \langle (T,G,\partial),(T,G,\partial)\rangle$;
(3) $\langle R[f_1],R[f_1]\rangle \cong \langle T,T\rangle$;
(4) $K[f_1] \subset Z_{\mathsf{XMod}}T$;
(5) $K[(f_1,f_0)] \subset Z_{\mathsf{XMod}}(T,G,\partial)$.

Alternatively, these central extensions may be characterised in terms of group commutators: see [15].

On the other hand, as shown in [7], an extension of crossed modules is central with respect to AbXMod exactly when it is central in the sense of [23]. And an extension of *pre*crossed modules is AbXMod-central if and only if it is central in the sense of [1, 2].

1.11. Perfect objects. An object A of $\mathcal{A}$ is called **perfect (with respect to $\mathcal{B}$)** or **$\mathcal{B}$-perfect** when IA is the terminal object 1 of $\mathcal{B}$. If $f\colon B \to A$ is an extension and B is perfect then so is A, because the reflector I preserves regular epimorphisms, and a regular quotient of a terminal object is terminal.

Examples 1.12. Example 1.5 gives rise to a new notion of perfect object; a Leibniz algebra $\mathfrak{g}$ is $\mathsf{Lie}_{\mathbb{K}}$-perfect if and only if $\mathfrak{g} = \mathfrak{g}^{\mathsf{Ann}}$.

Similarly, a precrossed module (T, G, ∂) is XMod-perfect if and only if $(T, G, \partial) = \langle (T, G, \partial), (T, G, \partial) \rangle$. In particular, then $G = 1$; hence $\langle T, T \rangle = [T, T]$, so that (T, G, ∂) is XMod-perfect exactly when T is Ab-perfect and G is trivial.

In the absolute case, 1.5 and 1.6 give classical notions of perfect object—see also Section 3.

Our first result, Proposition 1.14, shows how an object being perfect may help when composing central extensions.

Lemma 1.13. *If $f\colon B \to A$ is a $\mathcal{B}$-central extension and B is $\mathcal{B}$-perfect, then the kernel pair $(R[f], f_0, f_1)$ of f may be written as $(IR[f] \times B, \mathrm{pr}_B, \mathrm{pr}_B \circ b)$ for some automorphism $b\colon B \to B$ of B.*

Proof: The assumptions on f and B imply that the squares

$$\begin{array}{ccc} R[f] & \xrightarrow{f_0} & B \\ {\scriptstyle \eta_{IR[f]}}\downarrow & & \downarrow \\ IR[f] & \longrightarrow & 1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R[f] & \xrightarrow{f_1} & B \\ {\scriptstyle \eta_{IR[f]}}\downarrow & & \downarrow \\ IR[f] & \longrightarrow & 1 \end{array}$$

are pullbacks. The result follows. ∎

Proposition 1.14. *Let $\mathcal{A}$ be a Barr exact Mal'tsev category and $\mathcal{B}$ a Birkhoff subcategory of $\mathcal{A}$. If $f\colon B \to A$ and $g\colon C \to B$ are $\mathcal{B}$-central extensions and B is a $\mathcal{B}$-perfect object then the extension $f \circ g$ is $\mathcal{B}$-central.*

Proof: Take the kernel pair of f and further pullbacks along g until the next commutative diagram is obtained, in which all the squares are pullbacks.

$$\begin{array}{ccccc} R[f \circ g] & \xrightarrow{\overline{\overline{g}}} & \underline{R[f]} & \xrightarrow{\underline{f_1}} & C \\ {\scriptstyle \underline{\underline{g}}}\downarrow & & {\scriptstyle \underline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{R[f]} & \xrightarrow{\overline{g}} & R[f] & \xrightarrow{f_1} & B \\ {\scriptstyle \overline{f_0}}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & B & \xrightarrow[f]{} & A \end{array}$$

Using Lemma 1.13, the top left square may be written as

$$\begin{array}{ccc} IR[f]\times R[g] & \xrightarrow{1_{IR[f]}\times g_0} & IR[f]\times C \\ {\scriptstyle 1_{IR[f]}\times g_1}\downarrow & & \downarrow{\scriptstyle 1_{IR[f]}\times g} \\ IR[f]\times C & \xrightarrow[1_{IR[f]}\times g]{} & IR[f]\times B. \end{array}$$

Now f_1 is trivial by assumption, hence so is its pullback $\underline{f_1}$: both split epimorphisms and central extensions are pullback-stable. Also g_0 is assumed to be trivial, so that $\overline{\overline{g}}$ is a trivial extension. It follows that the composite $\underline{f_1}\circ\overline{\overline{g}} = (f\circ g)_0$ is trivial, which finishes the proof. ■

1.15. Pointed categories. As the proof of the next lemma indicates, the theory only starts behaving really well when the terminal object 1 is also initial, i.e., when the category $\mathcal{A}$ is **pointed**. In this case, the object $1 = 0$ is called the **zero object** of $\mathcal{A}$. A map f is **zero** when it factors over the zero object.

Since the reflector I always preserves pullbacks of split epimorphisms along split epimorphisms, in the pointed case, it also preserves products.

From now on, $\mathcal{A}$ will be a fixed pointed exact Mal'tsev category.

Lemma 1.16. *Let B' be a $\mathcal{B}$-perfect object and let $f\colon B \to A$ be a $\mathcal{B}$-central extension. If b_0, $b_1\colon B' \to B$ are morphisms such that $f\circ b_0 = f\circ b_1$ then $b_0 = b_1$.*

Proof: The extension f being central means that the square in the diagram

$$\begin{array}{ccccc} R[f] & \xrightarrow{f_0} & B & \xrightarrow{f} & A \\ {\scriptstyle \eta_{R[f]}}\downarrow & & \downarrow{\scriptstyle \eta_B} & & \\ IR[f] & \xrightarrow[If_0]{} & IB & & \end{array}$$

is a pullback. Since $f\circ b_0 = f\circ b_1$, we have an induced map $(b_0, b_1)\colon B' \to R[f]$. Now $f_0\circ(b_0, b_0) = b_0 = f_0\circ(b_0, b_1)$, but also

$$\eta_{R[f]}\circ(b_0, b_0) = I(b_0, b_0)\circ\eta_{B'} = 0 = I(b_0, b_1)\circ\eta_{B'} = \eta_{R[f]}\circ(b_0, b_1).$$

The unicity in the universal property of pullbacks now implies that $(b_0, b_0) = (b_0, b_1)$, so that $b_0 = b_1$. ■

1.17. Universal central extensions. For an object A of $\mathcal{A}$, let $\mathsf{Centr}_{\mathcal{B}}A$ denote the category of all $\mathcal{B}$-central extensions of A, i.e., the full subcategory of the slice category $\mathcal{A} \downarrow A$ determined by the central extensions. A (weakly) initial object of this category $\mathsf{Centr}_{\mathcal{B}}A$ is called a **(weakly) universal central extension** of A. A central extension $u\colon U \to A$ is weakly universal when for every central extension $f\colon B \to A$ there exists a map $\overline{f}$ from u to f, i.e., such that $f\circ\overline{f} = u$. And u is universal when this induced map $\overline{f}$ is, moreover, unique. Note also that, up to isomorphism, an object admits at most one universal central extension.

Examples 1.18. Examples of universal $\mathsf{Vect}_{\mathbb{K}}$-central extensions of Leibniz algebras over a field $\mathbb{K}$ may be found in [11]. The article [1] gives several non-trivial examples of universal AbXMod-central extensions of (pre)crossed modules.

Lemma 1.19. *If $u\colon U \to A$ is a universal $\mathcal{B}$-central extension then the objects U and A are $\mathcal{B}$-perfect.*

Proof: We know that the first projection $\mathrm{pr}_A\colon A \times IU \to A$ is a central extension, because the square

$$\begin{array}{ccc} A \times IU \times IU & \xrightarrow{1_A \times \mathrm{pr}_0} & A \times IU \\ {\scriptstyle \eta_{A\times IU\times IU}}\downarrow & & \downarrow{\scriptstyle \eta_{A\times IU}} \\ IA \times IU \times IU & \xrightarrow[I(1_A \times \mathrm{pr}_0)]{} & IA \times IU \end{array}$$

is a pullback, which means that the first projection $(\mathrm{pr}_A)_0 = 1_A \times \mathrm{pr}_0$ in the kernel pair of pr_A is a trivial extension. Hence there exists just one morphism $(u, v)\colon U \to A \times IU$ such that $\mathrm{pr}_A\circ(u, v) = u$. But then $0\colon U \to IU$ is equal to $\eta_U\colon U \to IU$, and $IU = 0$. This implies that both U and A are perfect. ■

Proposition 1.20. *Let $\mathcal{A}$ be a pointed Barr exact Mal'tsev category and $\mathcal{B}$ a Birkhoff subcategory of $\mathcal{A}$. Let $u\colon U \to A$ be a $\mathcal{B}$-central extension. Then the following are equivalent:*

(1) *u is universal;*
(2) *u is weakly universal and the object U is $\mathcal{B}$-perfect;*
(3) *the object U is $\mathcal{B}$-perfect and every central extension of U splits.*

Proof: We start by proving that (1) implies (3). If u is universal then it follows from Lemma 1.19 that U is perfect. Given a central extension $g\colon C \to U$, also the composite $u \circ g \colon C \to A$ is central by Proposition 1.14. The weak universality of u now yields a map $h \colon U \to C$ such that $u \circ g \circ h = u$. But also $u \circ 1_U = u$, so that $g \circ h = 1_U$ by the universality of u, and the central extension g splits.

Now suppose that (3) holds. To prove (2), let $f \colon B \to A$ be a central extension of A. Then its pullback $u^* f \colon \overline{B} \to U$ along u is still central; hence $u^* f$ admits a splitting $s \colon U \to \overline{B}$, and $(f^* u) \circ s$ is the needed map $u \to f$.

Finally, (2) implies (1) by Lemma 1.16. ∎

Proposition 1.21. *Let $\mathcal{A}$ be a pointed Barr exact Mal'tsev category and $\mathcal{B}$ a Birkhoff subcategory of $\mathcal{A}$. Let $f \colon B \to A$ and $g \colon C \to B$ be $\mathcal{B}$-central extensions. Then $f \circ g$ is a universal $\mathcal{B}$-central extension if and only if so is g.*

Proof: If g is universal then B is a perfect object by Lemma 1.19; moreover, Proposition 1.14 implies that $f \circ g$ is a central extension of A. In order to prove its universality, let $h \colon D \to A$ be another central extension of A. Then the pullback $f^* h \colon \overline{D} \to B$ of h along f is a central extension of B, and as such induces a unique map $\overline{h}$ from g to $f^* h$. The composite $(h^* f) \circ \overline{h} \colon C \to D$ is the needed unique map $f \circ g \to h$.

Conversely, when $f \circ g$ is a universal central extension, Lemma 1.19 implies that the object C, and hence also the object B, is perfect. Consider a central extension $h \colon D \to B$ of B. Then by Proposition 1.14 also the extension $f \circ h$ is central, so that there exists a unique map $\overline{f \circ h}$ from $f \circ g$ to $f \circ h$. It remains to prove that $h \circ \overline{f \circ h} = g$; but this follows from the universality of $f \circ g$ and the fact that both $h \circ \overline{f \circ h}$ and g are maps from $f \circ g$ to f. ∎

2. The universal central extension construction

Our aim is now to prove a converse to Lemma 1.19: Theorem 2.9, which essentially states that every perfect object admits a universal central extension. To do so, a richer categorical context is needed; for instance, a good notion of short exact sequence will be crucial in what follows. We switch to the context semi-abelian categories. Also the existence of projective objects will become important now.

2.1. Semi-abelian categories. A pointed and regular category is **Bourn protomodular** when the **(Regular) Short Five Lemma** holds: this means that for any commutative diagram

$$\begin{array}{ccccc} K[f'] & \xrightarrow{\mathsf{Ker}\, f'} & B' & \xrightarrow{f'} & A' \\ {\scriptstyle k}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ K[f] & \xrightarrow[\mathsf{Ker}\, f]{} & B & \xrightarrow[f]{} & A \end{array} \qquad (\mathbf{C})$$

such that f and f' are regular epimorphisms, k and a being isomorphisms implies that b is an isomorphism. A **semi-abelian** category is pointed, Barr exact and Bourn protomodular with binary coproducts [29]. A variety of Ω-groups is always a semi-abelian category. A semi-abelian category is always Mal'tsev.

Since, in a semi-abelian category, a regular epimorphism is always the cokernel of its kernel, an appropriate notion of **short exact sequence** exists. Such will be any sequence

$$K \xrightarrow{k} B \xrightarrow{f} A$$

that satisfies $k = \mathsf{Ker}\, f$ and $f = \mathsf{Coker}\, k$. We denote this situation

$$0 \longrightarrow K \xrightarrow{k} B \xrightarrow{f} A \longrightarrow 0. \qquad (\mathbf{D})$$

Lemma 2.2. [5, 6] *Consider a morphism of short exact sequences as* (**C**) *above.*

1. *The right hand side square $f \circ b = a \circ f'$ is a pullback iff k is iso.*
2. *The left hand side square $\mathsf{Ker}\, f \circ k = b \circ \mathsf{Ker}\, f'$ is a pullback iff a is mono.* ∎

The first statement implies that any pullback square between regular epimorphisms (i.e., any square as $f \circ b = a \circ f'$ in (**C**)) is a pushout. It is also well-known that the regular image of a kernel is a kernel [29]. In any semi-abelian category, the classical homological lemma's like the Snake Lemma and the 3×3 Lemma are valid; for further details and many other results we refer the reader to the article [29] and the monograph [4].

From now on, $\mathcal{A}$ will be a chosen semi-abelian category and $\mathcal{B}$ a Birkhoff subcategory of $\mathcal{A}$.

2.3. Commutators and centralisation. The kernel μ of the unit η of the adjunction (**A**) gives rise to a "zero-dimensional" commutator as follows: for any object A of $\mathcal{A}$,

$$0 \longrightarrow [A,A]_{\mathcal{B}} \xrightarrow{\mu_A} A \xrightarrow{\eta_A} IA \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$; hence A is an object of $\mathcal{B}$ if and only if $[A,A]_{\mathcal{B}} = 0$. On the other hand, an object A of $\mathcal{A}$ is $\mathcal{B}$-perfect precisely when $[A,A]_{\mathcal{B}} = A$. This construction defines a functor $[-,-]_{\mathcal{B}}\colon \mathcal{A} \to \mathcal{A}$ and a natural transformation $\mu\colon [-,-]_{\mathcal{B}} \Rightarrow 1_{\mathcal{A}}$. The functor $[-,-]_{\mathcal{B}}$ preserves regular epimorphisms: see [17, Section 5].

Lemma 2.2 implies that an extension f as in (**D**) is $\mathcal{B}$-central if and only if either one of the maps $[f_0,f_0]_{\mathcal{B}}$, $[f_1,f_1]_{\mathcal{B}}$ is an isomorphism, which happens exactly when they coincide, $[f_0,f_0]_{\mathcal{B}} = [f_1,f_1]_{\mathcal{B}}\colon [[R[f],R[f]]_{\mathcal{B}} \to [B,B]_{\mathcal{B}}$. Hence the kernel $[K,B]_{\mathcal{B}}$ of $[f_0,f_0]_{\mathcal{B}}$ measures how far f is from being central: indeed, f is $\mathcal{B}$-central if and only if $[K,B]_{\mathcal{B}}$ is zero. (Which explains, for instance, why a subobject of a central extension is central.) This "one-dimensional" commutator $[K,B]_{\mathcal{B}}$ may be considered as a normal subobject of B via the composite $\mu_B \circ [f_1,f_1]_{\mathcal{B}} \circ \mathsf{Ker}\,[f_0,f_0]_{\mathcal{B}}\colon [K,B]_{\mathcal{B}} \to B$.

Thus the Galois structure Γ mentioned in 1.7 induces a new adjunction

$$\mathsf{Ext}\mathcal{A} \underset{\supset}{\overset{I_1}{\rightleftarrows}} \mathsf{CExt}_{\mathcal{B}}\mathcal{A},$$

where $\mathsf{CExt}_{\mathcal{B}}\mathcal{A}$ is the full reflective subcategory of $\mathsf{Ext}\mathcal{A}$ determined by the $\mathcal{B}$-central extensions. Given an extension $f\colon B \to A$ with kernel K, its **centralisation** $I_1 f\colon B/[K,B]_{\mathcal{B}} \to A$ is obtained through the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [K,B]_{\mathcal{B}} & \longrightarrow & B & \longrightarrow & \frac{B}{[K,B]_{\mathcal{B}}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I_1 f} & & \\ & & 0 & \longrightarrow & A & = & A & \longrightarrow & 0. \end{array}$$

Considering this diagram as a short exact sequence

$$0 \longrightarrow K[\eta^1_f] \xrightarrow{\mu^1_f} f \xrightarrow{\eta^1_f} I_1 f \longrightarrow 0$$

in the semi-abelian category of arrows $\mathsf{Arr}\mathcal{A}$ (morphisms here are commutative squares) we obtain a description of the unit η^1 of the adjunction and its kernel μ^1.

2.4. Baer invariants. Recall [17, 19] that a **Baer invariant** is a functor $F\colon \mathsf{Ext}\mathcal{A} \to \mathcal{A}$ that makes **homotopic** morphisms of extensions equal: such are (b_0, a_0) and $(b_1, a_1)\colon f' \to f$

$$\begin{array}{ccc} B' & \overset{b_0}{\underset{b_1}{\rightrightarrows}} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \overset{a_0}{\underset{a_1}{\rightrightarrows}} & A \end{array}$$

satisfying $a_0 = a_1$. Such a functor F sends homotopically equivalent extensions to isomorphic objects. The functor $\mathsf{Ext}\mathcal{A} \to \mathcal{A}$ that maps an extension

$$0 \longrightarrow K \overset{k}{\rightarrowtail} B \overset{f}{\twoheadrightarrow} A \longrightarrow 0 \tag{E}$$

to the quotient $[B, B]_{\mathcal{B}}/[K, B]_{\mathcal{B}}$ is an example of a Baer invariant, as is the functor which maps this extension to the quotient $(K \cap [B, B]_{\mathcal{B}})/[K, B]_{\mathcal{B}}$. See [17] for further details.

2.5. Existence of a weakly universal central extension. From now on we suppose that $\mathcal{A}$ has enough (regular) projectives, i.e., given any object A of $\mathcal{A}$, there exists a regular epimorphism $f\colon B \to A$ with B projective, a **(projective) presentation** of A.

Let A be an object of $\mathcal{A}$ and $f\colon B \to A$ a projective presentation with kernel K. The induced objects

$$\frac{[B, B]_{\mathcal{B}}}{[K, B]_{\mathcal{B}}} \qquad \text{and} \qquad \frac{K \cap [B, B]_{\mathcal{B}}}{[K, B]_{\mathcal{B}}}$$

are independent of the chosen projective presentation of A as explained above. It makes sense to call the latter object the **second homology object** or the **Schur multiplier** of A **(relative to $\mathcal{B}$)** and denote it $H_2(A, \mathcal{B})$. We shall denote the former object $U(A, \mathcal{B})$, and $H_1(A, \mathcal{B})$ will be the reflection IA of A into $\mathcal{B}$.

When $\mathcal{A}$ has enough projectives and A is an object of $\mathcal{A}$, the category $\mathsf{Centr}_{\mathcal{B}}A$ always has a weakly initial object: given a projective presentation $f\colon B \to A$ with kernel K, such is its centralisation $I_1 f\colon B/[K, B]_{\mathcal{B}} \to A$. Indeed, any other central extension $g\colon C \to A$ induces a morphism $I_1 f \to g$ in $\mathsf{Centr}_{\mathcal{B}}A$, the object B being projective. So in presence of enough projectives, every object admits a weakly universal central extension.

Remark 2.6. The objects $H_2(A,\mathcal{B})$ and $H_1(A,\mathcal{B})$ are genuine homology objects: when $\mathcal{A}$ is a semi-abelian monadic category, they may be computed using comonadic homology as in [18]; and in any case, they fit into the homology theory worked out in [14]. Theorem 5.9 in [17] states that any short exact sequence (**D**) induces a five-term exact sequence

$$H_2(B,\mathcal{B}) \longrightarrow H_2(A,\mathcal{B}) \longrightarrow \frac{K}{[K,B]_{\mathcal{B}}} \longrightarrow H_1(B,\mathcal{B}) \longrightarrow H_1(A,\mathcal{B}) \longrightarrow 0.$$

This is a relative generalisation of the Stallings-Stammbach sequence for groups, a categorical version of the similar results considered in [19, 20, 35].

Examples 2.7. In the case of groups vs. abelian groups, $H_2(A,\mathsf{Ab})$ is the second integral homology group of a group A.

Given a Leibniz algebra $\mathfrak{g}$, the homology vector space $H_2(\mathfrak{g},\mathsf{Vect}_{\mathbb{K}})$ is the Leibniz homology developed in [34]; see also [12, 36]. As far as we know, $H_2(\mathfrak{g},\mathsf{Lie}_{\mathbb{K}})$ has not been studied before, but certainly the theories referred to in Remark 2.6 apply to it. If $\mathfrak{g}$ is a Lie algebra, the vector space $H_2(\mathfrak{g},\mathsf{Vect}_{\mathbb{K}})$ is the classical Chevalley-Eilenberg homology.

As to Example 1.6, the absolute homology crossed module

$$H_2((T,G,\partial),\mathsf{AbXMod})$$

was studied in [3] in case (T,G,∂) is a precrossed module, and in [9] in case (T,G,∂) is a crossed module. For a precrossed module (T,G,∂), the relative $H_2((T,G,\partial),\mathsf{XMod})$ was characterised in [16].

2.8. Existence of a universal central extension. The Baer invariants from 2.4 may now be considered with respect to all weakly universal $\mathcal{B}$-central extensions of an object A: indeed, any two such extensions of A are always homotopically equivalent. Since for any weakly universal $\mathcal{B}$-central extension (**E**) the commutator $[K,B]_{\mathcal{B}}$ is zero, the objects

$$[B,B]_{\mathcal{B}} \qquad \text{and} \qquad K\cap[B,B]_{\mathcal{B}}$$

are independent of the chosen weakly universal central extension of A. (Here, as in [27], the Hopf formula becomes $H_2(A,\mathcal{B}) = K\cap[B,B]_{\mathcal{B}}$. Also note that $U(A,\mathcal{B}) = [B,B]_{\mathcal{B}}$.)

We are now ready to prove that, if A is $\mathcal{B}$-perfect, then a universal $\mathcal{B}$-central extension of A does exist. This is a relative version of Proposition 4.1 in [22].

Theorem 2.9. *Let $\mathcal{A}$ be a semi-abelian category with enough projectives and $\mathcal{B}$ a Birkhoff subcategory of $\mathcal{A}$. An object A of $\mathcal{A}$ is $\mathcal{B}$-perfect if and only if it admits a universal $\mathcal{B}$-central extension. Moreover, this universal $\mathcal{B}$-central extension may be chosen in such a way that it occurs in a short exact sequence*

$$0 \longrightarrow H_2(A,\mathcal{B}) \rightarrowtail U(A,\mathcal{B}) \xrightarrow{u_A^{\mathcal{B}}} A \longrightarrow 0.$$

Proof: If an object admits a universal $\mathcal{B}$-central extension then it is $\mathcal{B}$-perfect by Lemma 1.19. Conversely, let (**E**) be a weakly universal central extension of a $\mathcal{B}$-perfect object A. Then μ_A is a regular epimorphism, hence so is the map $f\circ\mu_B = \mu_A\circ[f,f]_{\mathcal{B}}$ in the induced diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K\cap[B,B]_{\mathcal{B}} & \rightarrowtail & [B,B]_{\mathcal{B}} & \xrightarrow{f\circ\mu_B} & A & \longrightarrow & 0\\
 & & \downarrow & \lrcorner & \downarrow{\scriptstyle \mu_B} & & \| & & \\
0 & \longrightarrow & K & \rightarrowtail & B & \xrightarrow[f]{} & A & \longrightarrow & 0.
\end{array}$$

The extension $f\circ\mu_B$ is central as a subobject of the central extension f; its weak universality is clear. By Proposition 1.20, it is also universal: indeed, the object $[B,B]_{\mathcal{B}}$ is $\mathcal{B}$-perfect, because the extensions $f\circ\mu_B$ and f are homotopically equivalent, so that $[B,B]_{\mathcal{B}} \cong [[B,B]_{\mathcal{B}},[B,B]_{\mathcal{B}}]_{\mathcal{B}}$. ■

Corollary 2.10. *A $\mathcal{B}$-central extension $u\colon U\to A$ is universal if and only if $H_1(U,\mathcal{B})$ and $H_2(U,\mathcal{B})$ are zero.*

Proof: The object U is $\mathcal{B}$-perfect because $IU = H_1(U,\mathcal{B}) = 0$; since also $H_2(U,\mathcal{B})$ is zero, the universal $\mathcal{B}$-central extension $u_U^{\mathcal{B}}\colon U(U,\mathcal{B})\to U$ of U induced by Theorem 2.9 is an isomorphism. Proposition 1.20 now implies that every $\mathcal{B}$-central extension of $U\cong U(U,\mathcal{B})$ splits. Another application of Proposition 1.20 shows that also u is a universal $\mathcal{B}$-central extension.

Conversely, if $u\colon U\to A$ is a universal $\mathcal{B}$-central extension then again by Proposition 1.20, $H_1(U,\mathcal{B}) = IU = 0$ and every $\mathcal{B}$-central extension of U splits. This implies that $1_U\colon U\to U$ is a universal $\mathcal{B}$-central extension of U: it is weakly universal, because for any $\mathcal{B}$-central extension $g\colon C\to U$ of U, the induced splitting is the needed map $1_U\to g$; U being $\mathcal{B}$-perfect, it is universal by Proposition 1.20. Theorem 2.9 now tells us that $H_2(U,\mathcal{B})=0$. ■

3. Remarks on the absolute case

An *absolute* version of the theory is obtained when the Birkhoff subcategory $\mathsf{Ab}\mathcal{A}$ of *abelian objects* of $\mathcal{A}$ is considered. This situation is interesting in its own right, because it covers many of the classical examples; but there are also connections with the relative theory which occur in practice.

3.1. Abelian objects. The generalised Eckmann-Hilton argument shows that an object of a semi-abelian category admits at most one internal abelian group structure. An object that does admit such a structure is called an **abelian object**. It turns out that the full subcategory $\mathsf{Ab}\mathcal{A}$ of $\mathcal{A}$ determined by all abelian objects is an abelian Birkhoff subcategory. The reflector $\mathsf{ab}\colon \mathcal{A} \to \mathsf{Ab}\mathcal{A}$ may be described as follows (see, e.g., [4]): given an object A of $\mathcal{A}$, its abelianisation $\mathsf{ab}A$ is the coequaliser

$$A \underset{(1_A,0)}{\overset{(0,1_A)}{\rightrightarrows}} A \times A \xrightarrow{q_A} \mathsf{ab}A$$

of $(0,1_A)$ and $(1_A,0)$, and the A-component $\eta^{\mathsf{Ab}\mathcal{A}}_A\colon A \to \mathsf{ab}A$ of the unit $\eta^{\mathsf{Ab}\mathcal{A}}$ of the adjunction is the composite $q_A\circ(0,1_A)$.

Examples 3.2. Of course $\mathsf{AbGp} = \mathsf{Ab}$. It is also well-known that the category $\mathsf{Vect}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$, considered as a subcategory of $\mathsf{Lie}_{\mathbb{K}}$, is $\mathsf{AbLie}_{\mathbb{K}}$. Moreover, an abelian Leibniz algebra is always a Lie algebra, so that $\mathsf{AbLeib}_{\mathbb{K}}$ coincides with $\mathsf{AbLie}_{\mathbb{K}} \cong \mathsf{Vect}_{\mathbb{K}}$.

Similarly, an abelian crossed module is the same thing as an abelian object in XMod (or PXMod), so that the notation AbXMod makes sense.

On the other hand, the reflections $(-)_{\mathsf{Lie}}$ and $(-)_{\mathsf{Peiff}}$ from Example 1.5 and Example 1.6 are not determined by abelianisation.

3.3. A triangle of adjunctions. Next to the adjunction (**A**) we shall now also consider the adjunctions induced by abelianisation of the objects of $\mathcal{A}$ and $\mathcal{B}$. We shall be especially interested in the case where every abelian object of $\mathcal{A}$ is also an object of $\mathcal{B}$, so that $\mathsf{Ab}\mathcal{A} = \mathsf{Ab}\mathcal{B}$; we obtain the next commutative triangle of left adjoint functors. (All right adjoints are inclusions.)

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{I} & \mathcal{B} \\ & {\scriptstyle \mathsf{ab}^{\mathcal{A}}} \searrow \quad \swarrow {\scriptstyle \mathsf{ab}^{\mathcal{B}}} & \\ & \mathsf{Ab}\mathcal{A} = \mathsf{Ab}\mathcal{B} & \end{array}$$

Examples 3.4. In the article [21] we find the situation described in Example 1.5. The case of (pre)crossed modules, considered in Example 1.6, occurs in [1].

Lemma 3.5. *Under the given circumstances:*

(1) *an object of $\mathcal{B}$ is $\mathsf{Ab}\mathcal{B}$-perfect if and only if it is $\mathsf{Ab}\mathcal{A}$-perfect;*
(2) *an extension of $\mathcal{B}$ is $\mathsf{Ab}\mathcal{B}$-central if and only if it is $\mathsf{Ab}\mathcal{A}$-central.*

Proof: If B is an object of $\mathcal{B}$ then $\mathsf{ab}^{\mathcal{A}}B = \mathsf{ab}^{\mathcal{B}}IB = \mathsf{ab}^{\mathcal{B}}B$, which proves the first statement. As to the second statement, an extension $f\colon B \to A$ in $\mathcal{B}$ is central with respect to $\mathsf{Ab}\mathcal{B}$ if and only if in the diagram

$$\begin{array}{ccccc} R[f] & \xrightarrow{f_0} & B & \xrightarrow{f} & A \\ {\scriptstyle \eta^{\mathsf{Ab}\mathcal{B}}_{R[f]}} \downarrow & & \downarrow {\scriptstyle \eta^{\mathsf{Ab}\mathcal{B}}_{B}} & & \\ \mathsf{ab}^{\mathcal{B}}R[f] & \xrightarrow[\mathsf{ab}^{\mathcal{B}}f_0]{} & \mathsf{ab}^{\mathcal{B}}B & & \end{array}$$

the square is a pullback. Now the inclusion of $\mathcal{B}$ in $\mathcal{A}$ preserves and reflects all limits and moreover $\mathsf{ab}^{\mathcal{B}}f_0 = \mathsf{ab}^{\mathcal{A}}f_0$, so that f being $\mathsf{Ab}\mathcal{B}$-central is equivalent to f being central with respect to $\mathsf{Ab}\mathcal{A}$. ■

We shall be interested in the **absolute homology** of an object of $\mathcal{B}$, i.e., the homology with respect to abelianisation, when this object is considered either as an object of $\mathcal{B}$ or as an object of $\mathcal{A}$. Note that it makes sense to talk about homology in $\mathcal{B}$, because the left adjoint I preserves projective objects and regular epimorphisms, so that the category $\mathcal{B}$ has enough projectives if $\mathcal{A}$ has.

Lemma 3.6. *For any object B of $\mathcal{B}$, the adjunction* (**A**) *restricts to an adjunction*

$$\mathsf{Centr}_{\mathsf{Ab}\mathcal{A}}B \underset{\supset}{\overset{I}{\rightleftarrows}}{\scriptstyle \perp}\ \mathsf{Centr}_{\mathsf{Ab}\mathcal{B}}B.$$

Hence the functor I preserves universal central extensions:

$$I(u_B^{\mathsf{Ab}\mathcal{A}}\colon U(B,\mathsf{Ab}\mathcal{A}) \to B) \cong u_B^{\mathsf{Ab}\mathcal{B}}\colon U(B,\mathsf{Ab}\mathcal{B}) \to B,$$

for any $\mathsf{Ab}\mathcal{B}$-perfect object B.

Proof: By Lemma 3.5, $\mathsf{Centr}_{\mathsf{Ab}\mathcal{B}}B$ is a subcategory of $\mathsf{Centr}_{\mathsf{Ab}\mathcal{A}}B$.

Suppose that $g\colon C \to B$ is an $\mathsf{Ab}\mathcal{A}$-central extension of B. Applying the functor I, we obtain the extension $Ig = g\circ\eta_C\colon IC \to B$, which is $\mathsf{Ab}\mathcal{A}$-central as a quotient of g. Being an extension of $\mathcal{B}$, Ig is $\mathsf{Ab}\mathcal{B}$-central by Lemma 3.5.

Finally, as any left adjoint functor, I preserves initial objects. ■

Proposition 3.7. *When B is an $\mathsf{Ab}\mathcal{B}$-perfect object of $\mathcal{B}$, there is the exact sequence*

$$0 \longrightarrow H_2(U(B,\mathsf{Ab}\mathcal{B}),\mathsf{Ab}\mathcal{A}) \rightarrowtail H_2(B,\mathsf{Ab}\mathcal{A}) \twoheadrightarrow H_2(B,\mathsf{Ab}\mathcal{B}) \longrightarrow 0$$

relating the two types of absolute homology of B. Moreover,

$$[U(B,\mathsf{Ab}\mathcal{A}),U(B,\mathsf{Ab}\mathcal{A})]_{\mathcal{B}} = H_2(U(B,\mathsf{Ab}\mathcal{B}),\mathsf{Ab}\mathcal{A}),$$

and $u_B^{\mathsf{Ab}\mathcal{A}} = u_B^{\mathsf{Ab}\mathcal{B}}$ if and only if $H_2(B,\mathsf{Ab}\mathcal{A}) \cong H_2(B,\mathsf{Ab}\mathcal{B})$.

Proof: By Lemma 3.6 and Theorem 2.9, when B is an $\mathsf{Ab}\mathcal{B}$-perfect object of $\mathcal{B}$, the comparison map between the induced universal central extensions gives rise to the next 3×3 diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_2(U(B,\mathsf{Ab}\mathcal{B}),\mathsf{Ab}\mathcal{A}) & = & H_2(U(B,\mathsf{Ab}\mathcal{B}),\mathsf{Ab}\mathcal{A}) & \twoheadrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_2(B,\mathsf{Ab}\mathcal{A}) & \rightarrowtail & U(B,\mathsf{Ab}\mathcal{A}) & \xrightarrow{u_B^{\mathsf{Ab}\mathcal{A}}} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \eta_{U(B,\mathsf{Ab}\mathcal{A})}} & & \| & & \\
0 & \longrightarrow & H_2(B,\mathsf{Ab}\mathcal{B}) & \rightarrowtail & U(B,\mathsf{Ab}\mathcal{B}) & \xrightarrow[u_B^{\mathsf{Ab}\mathcal{B}}]{} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The result follows. ■

If we consider Proposition 3.7 for the case given in Example 1.5, we recover the result in [21]; in case of Example 1.6 we regain the result in [1].

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