CAUCHY COMPLETENESS, LAX EPIMORPHISMS AND EFFECTIVE DESCENT FOR SPLIT FIBRATIONS

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ABSTRACT. For any suitable monoidal category $\mathcal{V}$, we find that $\mathcal{V}$-fully faithful lax epimorphisms in $\mathcal{V}$-Cat are precisely those $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{B}$ whose induced $\mathcal{V}$-functors $\mathfrak{C}F\colon \mathfrak{C}\mathcal{A} \to \mathfrak{C}\mathcal{B}$ between the Cauchy completions are equivalences. For the case $\mathcal{V} = \mathsf{Set}$, this is equivalent to requiring that the induced functor $F^*\colon \mathsf{CAT}(\mathcal{A}, \mathsf{Cat}) \to \mathsf{CAT}(\mathcal{B}, \mathsf{Cat})$ is an equivalence.

By reducing the study of effective descent functors with respect to the indexed category of split (op)fibrations $\mathcal{F}$ to the study of the codescent factorization, we find that the observations above on fully faithful lax epimorphisms provide us with a characterization of (effective) $\mathcal{F}$-descent morphisms in the category of small categories Cat; namely, we find that they are precisely the (effective) descent morphisms with respect to the indexed categories of *discrete* opfibrations — previously studied by Sobral. We include some comments on the Beck-Chevalley condition and future work.

INTRODUCTION

Let C be a category with pullbacks and $p : e \to b$ a morphism in C. The kernel pair of p induces the internal groupoid $\mathsf{Eq}\,(p)$, whose underlying truncated simplicial object in C is given by the diagram (1) below. In this setting, each indexed category $\mathfrak{F} : \mathsf{C}^{\mathrm{op}} \to \mathsf{CAT}$ has the associated category $\mathsf{Desc}_{\mathfrak{F}}\,(p)$ of internal $\mathfrak{F}$-actions of the internal groupoid $\mathsf{Eq}\,(p)$, also called the *category of* $\mathfrak{F}$*-descent data for* p. This category comes with a factorization given by the diagram (2) below, where $\mathsf{d}^p_{\mathfrak{F}}$ is the forgetful functor that discards descent data.

$$(1)\qquad e\times_b e\times_b e \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; e\times_b e \;\substack{\xrightarrow{\pi_e}\\ \longleftarrow\\ \xrightarrow[\pi^e]{}}\; e$$

$$(2)\qquad \begin{array}{ccc} \mathfrak{F}\,(b) & \xrightarrow{\mathfrak{F}(p)} & \mathfrak{F}\,(e) \\ & {\scriptstyle \mathcal{K}^p_{\mathfrak{F}}}\searrow \qquad \nearrow{\scriptstyle \mathsf{d}^p_{\mathfrak{F}}} & \\ & \mathsf{Desc}_{\mathfrak{F}}\,(p) & \end{array}$$

In the context of Janelidze-Galois Theory (*viz.* [5]) and Grothendieck Descent Theory (*viz.* [10, 14]), one is interested in characterizing the morphisms p in C such that the comparison $\mathcal{K}^p_{\mathfrak{F}}$ is an equivalence (or just fully faithful); that is to say, in characterizing

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the *effective* $\mathfrak{F}$*-descent (respectively,* $\mathfrak{F}$*-descent) morphisms of* C. We refer the reader to [9, 8] for comprehensive introductions.

By the Bénabou-Roubaud Theorem (see [3] or [14, Theorem 8.5] for a generalization), whenever $\mathfrak{F}$ comes from a bifibration satisfying the Beck-Chevalley condition (see, for instance, [15, Section 4]), p is of effective $\mathfrak{F}$-descent if and only if $\mathfrak{F}(p)$ is monadic. This provides us with a way of studying (effective) $\mathfrak{F}$-descent morphisms via the Beck's monadicity theorem.

If $\mathfrak{F}$ does not satisfy the Beck-Chevalley condition, the equivalence will not necessarily hold. A particular prominent example of this setting is the indexed category of discrete (op)fibrations $\mathcal{F}_D = \mathsf{CAT}(-, \mathsf{Set})$, thoroughly studied by Sobral in [17] (see also [15, Remark 4.8]).

The main point of this work is to extend Sobral's techniques and viewpoint on discrete (op)fibrations [17] to other settings. In this paper, we study the case of the indexed category $\mathcal{F}\colon \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}, e \mapsto \mathsf{CAT}(\mathsf{e}, \mathsf{Cat})$; by Grothendieck's construction, is essentially the indexed category of split opfibrations, which is another glaring example of an indexed category that does not satisfy the Beck-Chevalley condition.

Recall that a lax epimorphism $p\colon e \to b$ in $\mathcal{V}$-Cat is a $\mathcal{V}$-functor such that

$$\mathcal{V}\text{-}\mathsf{Cat}(p, x)\colon \mathcal{V}\text{-}\mathsf{Cat}(b, x) \to \mathcal{V}\text{-}\mathsf{Cat}(e, x) \tag{3}$$

is fully faithful for any small $\mathcal{V}$-category x. Underlying Sobral's study of effective $\mathcal{F}_D$-descent morphisms p, there are two fundamental steps. The first step is to construct a factorization of p such that its image is (isomorphic to) (2). The second step of [17] relies on the characterization of lax epimorphisms in Cat, that is, the functors p such that $\mathsf{Cat}(p, c)$ is fully faithful for any c.

We revisit these two fundamental steps of [17], giving a systematic view over them, in Sections 1 and 2. Since it is suitable for our future work, we do that in the $\mathcal{V}$-enriched setting. The reader, however, can opt to always consider the case $\mathcal{V} = \mathsf{Set}$.

In Section 1, we show how we can reduce the problem of studying effective $\mathcal{F}$-descent morphisms to the study of the codescent factorization induced by (1): namely, the factorization given by the universal property of the weighted colimit called *codescent category* (see, for instance, [12] or [13, Lemma 3.3]). This observation leads to a straightforward formal result that characterizes effective $\mathfrak{F}$-descent morphisms whenever the domain of $\mathfrak{F}$ is a 2-category with codescent objects and $\mathfrak{F}$ preserves suitable two-dimensional limits: namely, descent objects.

In Section 2, we thoroughly study the characterization of fully faithful lax epimorphisms in $\mathcal{V}$-Cat. Inspired by its relation with the (co)presheaf categories, we show its relation with the Cauchy completion pseudofunctor, which we denote by $\mathfrak{C}$. We prove the equivalence of the following statements (Theorem 2.5) for monoidal categories $\mathcal{V}$:

- p is a fully faithful lax epimorphism.
- $\mathfrak{C}p$ is an equivalence.
- $\mathcal{V}$-$\mathsf{CAT}(p, \mathcal{V})$ is an equivalence.

In Section 3, the characterization of fully faithful lax epimorphisms together with the formal result of Section 1 provide us with a proof that a morphism is of *(effective)* $\mathcal{F}$*-descent if and only if it is of (effective)* $\mathcal{F}_D$*-descent.* This means that Sobral's characterization can be plainly extended to the case of the indexed category of split fibrations.

We finish Section 3 by discussing a straightforward example related to the well-known fact that our indexed category indeed does not satisfy the Beck-Chevalley condition.

We end the paper with Section 4. It gives a brief account of some problems following this line of work: for example, we state open problems in the enriched setting and the $(\mathcal{T}, \mathcal{V})$-categorical setting (*viz.* [7, 6]).

1. Grothendieck Descent and Codescent

In this section, we consider an arbitrary 2-category $\mathcal{A}$ with lax codescent objects, but the reader may safely assume $\mathcal{A} = \mathsf{Cat}$, which is the scope of our main results in Sect. 3 (see, for instance, [12] or [13, p. 42] for definitions of the *two-dimensional colimit* known as *lax codescent category*).

Given a 2-functor $\mathfrak{F} : \mathcal{A}^{\mathrm{op}} \to \mathsf{CAT}$ and a morphism $p \colon e \to b$ in $\mathcal{A}$, we consider the image of the diagram (1) by $\mathfrak{F}$: namely, the diagram (5) below. The universal property of the *lax descent object* (5) in CAT induces the *descent factorization* (2) of $\mathfrak{F}(p)$ (see, for instance, [15, Lemma 3.6] for this description via the *two-dimensional limit lax descent object*). *We say that such a morphism p is of effective $\mathfrak{F}$-descent ($\mathfrak{F}$-descent) whenever $\mathcal{K}^{\mathsf{Eq}(p)}$ is an equivalence (resp. fully faithful).*

We get the factorization of p in $\mathcal{A}$, depicted in diagram (4) below, by the *lax codescent object* $\mathsf{CoDesc}(\mathsf{Eq}(p))$ of the diagram (1) in $\mathcal{A}$.

$$\text{(4)} \qquad \begin{array}{ccc} e & \xrightarrow{\quad p \quad} & b \\ & {\scriptstyle \mathsf{d}^{\mathsf{Eq}(p)}} \searrow \quad \nearrow {\scriptstyle \mathcal{K}^{\mathsf{Eq}(p)}} & \\ & \mathsf{CoDesc}(\mathsf{Eq}(p)) & \end{array} \qquad \text{(5)} \qquad \mathfrak{F}(e) \underset{\mathfrak{F}(\pi^e)}{\overset{\mathfrak{F}(\pi_e)}{\rightleftarrows}} \mathfrak{F}(e \times_b e) \rightrightarrows \mathfrak{F}(e \times_b e \times_b e)$$

If $\mathfrak{F} : \mathcal{A}^{\mathrm{op}} \to \mathsf{CAT}$ preserves lax descent objects, then the image of (4) by $\mathfrak{F}$ is isomorphic to (2). In particular:

Lemma 1.1. *Let $\mathfrak{F} : \mathcal{A}^{\mathrm{op}} \to \mathsf{CAT}$ be a 2-functor that preserves two-dimensional limits. A morphism $p : e \to b$ is of effective $\mathfrak{F}$-descent ($\mathfrak{F}$-descent) if, and only if, $\mathfrak{F}\left(\mathsf{K}^{\mathsf{Eq}(p)}\right)$ is an equivalence (fully faithful).*

2. Fully faithful lax epimorphisms

Throughout this section, let $\mathcal{V}$ be a symmetric monoidal closed, complete and cocomplete category. We consider the 2-category $\mathcal{V}$-Cat of small $\mathcal{V}$-categories.

A morphism $p \colon e \to b$ in a 2-category $\mathcal{A}$ is a *lax epimorphism* if $\mathcal{A}(p, c)$ is fully faithful in CAT for any $c \in \mathcal{A}$. This is the dual of the notion of fully faithfulness. In particular, a $\mathcal{V}$-functor $p \colon e \to b$ is a *lax epimorphism* whenever (3) is fully faithful for every small $\mathcal{V}$-category x.

A morphism $p \colon e \to b$ of $\mathcal{V}$-Cat is $\mathcal{V}$-*fully faithful* if, for any $x, y \in e$, the morphism $p : e(x, y) \to b(px, py)$ in $\mathcal{V}$ is invertible. It is easy to see that, if q has an adjoint in $\mathcal{V}$-Cat, q is a $\mathcal{V}$-fully faithful morphism if, and only if, q is fully faithful in the 2-category $\mathcal{V}$-Cat (see [16, Lemma 5.2]).

The main point of this section is to study characterizations of the morphisms that are simultaneously $\mathcal{V}$-fully faithful and lax epimorphic in $\mathcal{V}$-Cat. In particular, we give a characterization in terms of the Cauchy completions of the $\mathcal{V}$-categories. We recall basic aspects about those below.

2.1. **Cauchy completion.** An object a of a (possibly large) $\mathcal{V}$-category C is *tiny* if the $\mathcal{V}$-functor $\mathsf{C}(a,-)$ preserves colimits (see tiny in [4], or *small-projective* in [11]). For a small $\mathcal{V}$-category e, we denote by $\mathfrak{C}\,e$ the full $\mathcal{V}$-subcategory of $\mathcal{V}\text{-}\mathsf{CAT}[e,\mathcal{V}]$ consisting of the tiny objects of $\mathcal{V}\text{-}\mathsf{CAT}[e,\mathcal{V}]$.

Henceforth, we assume that $\mathcal{V}$ is such that $\mathfrak{C}\,e$, called the Cauchy completion of e, is essentially small for any $e \in \mathcal{V}\text{-}\mathsf{Cat}$. This is true for many base categories $\mathcal{V}$. We are mainly interested in the cases $\mathcal{V} = \mathsf{Set}$ and $\mathcal{V} = \mathsf{Cat}$; other examples include the extended real line, or even more generally any small quantale.

Recall that *tiny objects are preserved by equivalences*. More generally, we have:

Lemma 2.1. *Let $(F \dashv G) : \mathsf{C} \to \mathsf{D}$ be a $\mathcal{V}$-adjunction between (possibly large) $\mathcal{V}$-categories. If G is colimit-preserving, then F preserves tiny objects.*

Proof. Indeed, if a is a tiny object, then $\mathsf{D}(Fa,-) \cong \mathsf{C}(a, G(-))$ is colimit-preserving since it is a composite of colimit-preserving functors. □

For a functor $p\colon e \to b$ in $\mathcal{V}\text{-}\mathsf{Cat}$, we denote by $\mathfrak{C}\,p\colon \mathfrak{C}\,e \to \mathfrak{C}\,b$ the $\mathcal{V}$-functor induced by the restriction of the left Kan extension $\mathsf{LKan}_p\colon \mathcal{V}\text{-}\mathsf{CAT}[e,\mathcal{V}] \to \mathcal{V}\text{-}\mathsf{CAT}[b,\mathcal{V}]$ to the tiny objects. It is clear that $\mathfrak{C}$ naturally extends to a pseudofunctor $\mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$.

2.2. **The characterization.** We start by establishing characterizations of $\mathcal{V}$-fully faithful morphisms and lax epimorphisms of $\mathcal{V}\text{-}\mathsf{Cat}$ in terms of the induced morphism between the Cauchy completions, and in terms of the induced functor between the categories of $\mathcal{V}$-presheaves.

For a small c, we denote by η_c the full inclusion (induced by the Yoneda embedding) of the $\mathcal{V}$-category c into its Cauchy completion. This defines a natural transformation $\mathrm{Id} \to \mathfrak{C}$. Moreover, recall that, by the universal property of the Cauchy completion, the vertical arrows of diagram (6) below are equivalences. Therefore:

Lemma 2.2. *If p is a morphism of $\mathcal{V}\text{-}\mathsf{Cat}$, then the induced functor between the categories of $\mathcal{V}$-presheaves $\mathcal{V}\text{-}\mathsf{CAT}(p,\mathcal{V})$ is fully faithful (resp. lax epimorphic) if and only if $\mathcal{V}\text{-}\mathsf{CAT}(\mathfrak{C}\,p,\mathcal{V})$ is fully faithful (resp. lax epimorphic) as well.*

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathsf{CAT}(\mathfrak{C}\,b,\mathcal{V}) & \xrightarrow{\mathcal{V}\text{-}\mathsf{CAT}(\mathfrak{C}\,p,\mathcal{V})} & \mathcal{V}\text{-}\mathsf{CAT}(\mathfrak{C}\,e,\mathcal{V}) \\ {\scriptstyle \mathcal{V}\text{-}\mathsf{CAT}(\eta_b,\mathcal{V})}\downarrow & & \downarrow{\scriptstyle \mathcal{V}\text{-}\mathsf{CAT}(\eta_e,\mathcal{V})} \\ \mathcal{V}\text{-}\mathsf{CAT}(b,\mathcal{V}) & \xrightarrow[\mathcal{V}\text{-}\mathsf{CAT}(p,\mathcal{V})]{} & \mathcal{V}\text{-}\mathsf{CAT}(e,\mathcal{V}) \end{array} \tag{6}$$

By the above and [16, Theorem 5.6], we get, then, a full characterization of lax epimorphisms in terms of $\mathfrak{C}$. More precisely:

Proposition 2.3. *The following conditions are equivalent for a $\mathcal{V}$-functor $p\colon e \to b$ between small $\mathcal{V}$-categories:*

i. *p is a lax epimorphism.*
ii. *$\mathfrak{C}\,p$ is a lax epimorphism.*
iii. *$\mathcal{V}\text{-}\mathsf{CAT}(p,\mathcal{V})$ is fully faithful.*

Proof. The equivalence of *i.* and *iii.* was already established by (b) $\leftrightarrow$ (c) of [16, Theorem 5.6]. The equivalence of *i.* and *ii.* follows from Lemma 2.2 and the equivalence *i.* $\leftrightarrow$ *iii.*. □

Although it does not follow from (plain) duality, the counterpart of Proposition 2.3 holds for fully faithful morphisms.

We start by recalling that *the unit of an adjunction* $f \dashv g$ *in a* 2*-category* $\mathcal{A}$ *is invertible iff* f *is fully faithful iff* g *is a lax epimorphism.* Moreover, by coduality, the counit is an isomorphism iff f is a lax epimorphism iff g is fully faithful. In particular, we have that, *assuming that a morphism p has an adjoint, it is an equivalence if, and only if, it is a fully faithful lax epimorphism.* All these statements hold for $\mathcal{A} = \mathcal{V}\text{-}\mathsf{Cat}$ when replacing fully faithfulness by $\mathcal{V}$-fully faithfulness since, for adjoints, both are equivalent (see [16, Examples 2.4, (3)] and [16, Lemma 5.2]).

Proposition 2.4. *The following conditions are equivalent for a* $\mathcal{V}$*-functor* $p\colon e \to b$ *between small* $\mathcal{V}$*-categories:*

i. p *is* $\mathcal{V}$*-fully faithful.*
ii. $\mathfrak{C}p$ *is* $\mathcal{V}$*-fully faithful.*
iii. $\mathcal{V}\text{-}\mathsf{CAT}(p, \mathcal{V})$ *is a lax epimorphism.*

Proof. As in the case of Proposition 2.3, by Lemma 2.2 it is enough to prove that *i.* and *iii.* are equivalent.

Recall that we have an adjunction $\mathsf{LKan}_p \dashv \mathcal{V}\text{-}\mathsf{CAT}(p, \mathcal{V})$. Hence, $\mathcal{V}\text{-}\mathsf{CAT}(p, \mathcal{V})$ is a lax epimorphism if and only if LKan_p is fully faithful. We complete the proof by observing that, as a consequence of the Yoneda Lemma, p is $\mathcal{V}$-fully faithful if and only if LKan_p is fully faithful (see, for instance, [11, Proposition 4.23]). □

Finally, combining the characterizations of $\mathcal{V}$-fully faithful morphisms and lax epimorphisms, we get:

Theorem 2.5. *The following conditions are equivalent for a* $\mathcal{V}$*-functor* $p\colon e \to b$ *between small* $\mathcal{V}$*-categories:*

i. p *is a* $\mathcal{V}$*-fully faithful lax epimorphism.*
ii. $\mathfrak{C}p$ *is a* $\mathcal{V}$*-fully faithful lax epimorphism.*
iii. $\mathcal{V}\text{-}\mathsf{CAT}(p, \mathcal{V})$ *is a fully faithful lax epimorphism.*
iv. $\mathfrak{C}p$ *is an equivalence.*
v. $\mathcal{V}\text{-}\mathsf{CAT}(p, \mathcal{V})$ *is an equivalence.*

Proof. Combining Propositions *2.4* and *2.3*, we obtain the equivalence of i., ii. and iii..

Of course, v. $\to$ iv., since tiny objects are preserved by equivalences. Moreover, we have iv. $\to$ ii. and v. $\to$ iii. *a fortiori*, taking into account that, for adjoints in $\mathcal{V}\text{-}\mathsf{Cat}$, fully faithfulness coincides with $\mathcal{V}$-fully faithfulness.

We have iii. $\to$ v., as $\mathsf{LKan}_p \dashv \mathcal{V}\text{-}\mathsf{CAT}(p, \mathcal{V})$, and $\mathcal{V}\text{-}\mathsf{CAT}(p, \mathcal{V})$ is a fully faithful lax epimorphism. □

3. Discrete and Split Fibrations

Sobral provided a characterization of effective $\mathsf{CAT}(-, \mathsf{Set})$-descent and $\mathsf{CAT}(-, \mathsf{Set})$-descent functors [17]. We show, herein, how we can extend her characterization to the case of split fibrations. We start by extending our characterization of fully faithful and lax epimorphic morphisms in $\mathsf{Set}\text{-}\mathsf{Cat}$.

Proposition 3.1. *A functor $p\colon e \to b$ between small categories is fully faithful (resp. lax epimorphic) if, and only if,* $\mathsf{CAT}(p, \mathsf{Cat})$ *is lax epimorphic (resp. fully faithful).*

Proof. The 2-functor $J\colon \mathsf{Set}\text{-}\mathsf{CAT} \to \mathsf{Cat}\text{-}\mathsf{CAT}$, taking every category to the corresponding locally discrete 2-category, is a full 2-functor, and it has left and right 2-adjoints. Hence, it preserves and reflects fully faithful morphisms and lax epimorphisms by [16, Lemma 2.8] and [16, Remark 2.5].

By Propositions 2.4 and 2.3, $J(p)$ is fully faithful (resp. lax epimorphic) if and only if $\mathsf{Cat}\text{-}\mathsf{CAT}\,(J(p), \mathsf{Cat}) \cong \mathsf{CAT}\,(p, \mathsf{Cat})$ is lax epimorphic (resp. fully faithful). □

Theorem 3.2. *Let $p\colon e \to b$ be a functor between small categories. Denoting by $\mathsf{K}^{\mathsf{Eq}(p)}$ the comparison functor of the codescent category of the factorization* (4)*, the following statements are equivalent.*

i. $\mathsf{K}^{\mathsf{Eq}(p)}$ *is lax epimorphic (resp. fully faithful lax epimorphic);*
ii. *p is of* $\mathsf{CAT}(-, \mathsf{Set})$*-descent (resp. effective* $\mathsf{CAT}(-, \mathsf{Set})$*-descent);*
iii. *p is of* $\mathsf{CAT}(-, \mathsf{Cat})$*-descent (resp. effective* $\mathsf{CAT}(-, \mathsf{Cat})$*-descent);*
iv. $\mathfrak{C}\,\mathsf{K}^{\mathsf{Eq}(p)}$ *is lax epimorphic (resp. an equivalence)*

Proof. By Lemma 1.1, putting $\mathcal{F} = \mathsf{CAT}(-, \mathsf{Cat}) : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$, we know that p is of $\mathsf{CAT}(-, \mathsf{Cat})$-descent (resp. effective $\mathsf{CAT}(-, \mathsf{Cat})$-descent) if and only if $\mathsf{CAT}(\mathsf{K}^{\mathsf{Eq}(p)}, \mathsf{Cat})$ is fully faithful (resp. an equivalence). By Proposition 2.3 and Theorem 2.5, this is equivalent to i., and also to iv. Using a similar argumentation for the functor $\mathcal{F} = \mathsf{CAT}(-, \mathsf{Set})$ and Proposition 3.1, we conclude that ii. and iii. are equivalent. □

Remark 3.3 (A word on Beck-Chevalley)**.** The indexed category of discrete fibrations (equivalently, the indexed category of split fibrations) is particularly interesting because it provides us with a source of counterexamples in descent theory – it is a fruitful example of an indexed category (coming from a bifibration) that does not satisfy the Beck-Chevalley condition.

Although it is simple to directly verify that $\mathsf{CAT}\,(-, \mathsf{Set})$ and $\mathsf{CAT}\,(-, \mathsf{Cat})$ do not satisfy the Beck-Chevalley condition, we can do that indirectly: by showing that the Bénabou-Roubaud theorem does not hold for these cases (and, hence, BC does not hold as well).

More precisely, we know that every effective $\mathsf{CAT}\,(-, \mathsf{Cat})$-descent morphism induces a monadic functor (by [15, Theorem 4.7]). However, we can show that there are functors p such that $\mathsf{CAT}\,(p, \mathsf{Cat})$ is monadic but p is not of effective $\mathsf{CAT}\,(-, \mathsf{Cat})$-descent.

For example, let $p\colon \mathsf{1} \to b$ be a functor, where $\mathsf{1}$ is the terminal category. Note that $p^* = \mathsf{Cat}(p, \mathsf{Cat})$ is monadic whenever b has only one object, but p^* is an effective $\mathsf{CAT}\,(-, \mathsf{Cat})$-descent morphism if and only if p is an equivalence. To see this, note that p is of effective $\mathsf{CAT}(-, \mathsf{Cat})$-descent iff $\mathsf{CAT}(p, \mathsf{Cat})$ is an equivalence.[1] Therefore, by Proposition 3.1, we conclude that $p : \mathsf{1} \to b$ is of effective $\mathsf{CAT}(-, \mathsf{Cat})$-descent if and only if p is fully faithful lax epimorphic: and, of course, since the domain of p is terminal, this is equivalent to p being an equivalence.

[1] This actually holds more generally. See, for instance, [15, Proposition 4.3].

We refer to [17, Remark 3] and [15, Remark 4.8] for more examples of morphisms inducing monadic functors that are not of effective $\mathsf{CAT}\,(-,\mathsf{Cat})$-descent (effective $\mathsf{CAT}\,(-,\mathsf{Set})$-descent).

4. Future work

The main contribution of the present work was showing that, from the formal observations on codescent of Section 1 and the characterization of fully faithful lax epimorphisms of Sections 2 and 3, we were able to extend Sobral's characterization of discrete fibrations for the case of split fibrations.

The authors also believe that the present approach can be insightful towards the study of effective descent morphisms w.r.t. some other interesting indexed categories defined in 2-categories. We give two examples below.

The most natural line of work following this would be the study of effective descent morphisms in $\mathcal{V}$-Cat. By the observations of the present paper, this would solely rely on the thorough study of the codescent object of (4) in $\mathcal{V}$-Cat and its Cauchy completion.

More interestingly, discrete fibrations in the context of $(\mathcal{T},\mathcal{V})$-categories (and, more precisely, in the context of [7, 6]) provides us with the indexed category $\mathcal{E}$ of étale morphisms (see, for instance, [1, 2]). Even for $\mathcal{V}$ thin, the study and characterization of effective $\mathcal{E}$-descent morphisms in this setting is still generally an open problem.

By the present work, we can extend Sobral's techniques to this general setting. Although we leave it to future work, we roughly sketch the general ideas below (we refer to [6] for the basic definitions related to these brief comments).

The first step would be to characterize the lax epimorphisms in that context. We conjecture that these are precisely the $(\mathcal{T},\mathcal{V})$-functors $(X,a)\to(Y,b)$ such that $f_*\cdot f^*=b$ (following the notation of [6, p. 188]) for $(\mathcal{T},\mathcal{V})$-bimodules. A next step would be to fully study the codescent factorization in the 2-category of $(\mathcal{T},\mathcal{V})$-categories. More precisely, this study consists of constructing the suitable codescent objects (if/when it exists). Moreover, provided with the work developed in [6], we can also study the relation of this characterization with the Cauchy completion in this setting.

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