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PROFINITE RELATIONAL STRUCTURES

GEORGE JANELIDZE AND MANUELA SOBRAL

Dedicated to Jiří Adámek on the occasion of his 60th birthday

Abstract: We show that a topological preorder (on a Stone space) is profinite if and only if it is inter-clopen, i.e. it can be presented as an intersection of closed-and-open preorders on the same space. In particular this provides a new characterization of the so-called Priestley spaces. We then extend this from preorders to general relational structures satisfying some conditions. We also give a stronger condition that has a rather clear model-theoretic meaning.

Keywords: ordered (preordered) topological spaces, Priestley space, Stone space, fibration, topological functor, profinite, relational structure, quasi-variety.

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0. Introduction

We prove that profinite is the same as inter-clopen for preorders. Here and below "profinite" means "a limit of finite topologically-discrete" and "inter-clopen" means "that it can be presented as an intersection of closed-and-open preorders".

The same is true for equivalence relations (which can either be deduced from results of A. Carboni, G. Janelidze and A. R. Magid in [2] or seen as a special case of our Theorem 2.4 below), and this is important in the Galois theory of commutative rings. In the case of Stone spaces they are precisely the effective equivalence relations and this effectiveness plays an important role in categorical Galois theory, namely in the characterization of the effective descent morphisms (see e.g. [5] and Section 2 of [4]).

This is not true for orders, but gives a new characterization of profinite orders, since (profinite order) = (profinite preorder) + (order).

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These special cases suggest to investigate relational structures in general and we conclude that profinite = inter-clopen also in this context under some conditions. In fact we show that, under the very reasonable assumption of having the forgetful functor topological, the problem reduces to Condition 2.3, which is not only sufficient but also necessary.

Next, although we easily show that Condition 2.3 holds for preorders, it would be nice to have its equivalent (or "nearly equivalent") syntactical reformulation in general. Not having such a reformulation, we propose instead a stronger condition with a reasonably clear model-theoretic meaning. At the end we make it still stronger, very simple, and purely syntactical indeed – in order to make obvious that it holds in many familiar quasi-varieties of models.

1. Profinite preorders

Extending Stone duality, Priestley duality is an equivalence between the dual category of distributive lattices (with 0 and 1) and the category $\mathcal{P}ro\mathcal{F}in(\mathcal{O}rd)$ of profinite ordered topological spaces, also called Priestley spaces. While Stone spaces, which are to be identified with order-discrete Priestley spaces, can be characterized as compact topological spaces in which every two distinct points can be separated by a clopen (=closed-and-open) subset, it is also well-known that the Priestley spaces can be characterized as follows:

An ordered compact topological space X belongs to $\mathcal{P}ro\mathcal{F}in(\mathcal{O}rd)$ if and only if for every two points $x' \nleq x$ in X there exists a clopen decreasing subset U in X, that is a clopen subset such that $y \leq u \in U \Rightarrow y \in U$, with $x \in U$ and $x' \notin U$.

As shown in [3], this can be repeated for preorders as follows:

Theorem 1.1. *The following conditions on a preordered topological space $(X, \leq)$ are equivalent:*

(a) $(X, \leq)$ is a limit of finite topologically-discrete preordered spaces;

(b) X is a Stone space in which for every two points x and x' with $x' \nleq x$, there exists a clopen decreasing subset U in X with $x \in U$ and $x' \notin U$.

Note also that, whenever an ordered topological space can be presented as a limit of finite topologically-discrete preordered spaces, it can also be presented as a limit of finite topologically-discrete ordered spaces.

We shall write $\mathcal{P}ro\mathcal{F}in(\mathcal{P}reord)$ for the category of such preordered topological spaces; accordingly, the objects of $\mathcal{P}ro\mathcal{F}in(\mathcal{P}reord)$ might be called either profinite preorders or Priestley preorders (as in [3]).

Theorem 1.2. *A preordered topological space $(X, \leq)$ belongs to $\mathcal{P}ro\mathcal{F}in(\mathcal{P}reord)$ if only if X is a Stone space and the preorder $\leq$ is inter-clopen, i.e. it can be presented as the intersection of a family $(R_i)_{i\in I}$ of preorder relations on X that are clopen subsets in $X \times X$.*

Proof: "Only if": If $(X, \leq) = lim_{i\in I}(X_i, \leq_i)$ is profinite (with finite R_i's), just take R_i's to be the inverse images of $\leq_i$ under the induced maps $X\times X \to X_i \times X_i$.

"If": First suppose that the original preorder relation $\leq$ is clopen. Then, for each $x \in X$, the set $\downarrow x = \{u \in U | u \leq x\}$ is clopen, since it is the inverse image of $\leq$ under the continuous map $X \to X \times X$ sending u to (u, x). Consequently $(X, \leq)$ satisfies condition (b) of Theorem 1.1.

After this, instead of proving that $(X, \leq)$ is a limit of finite discrete preorders, it suffices to prove that $(X, \leq)$ is a limit of clopen preorders (on Stone spaces). However, this is obvious: just take the diagram formed by all identity maps $(X, R_i) \to (X, X \times X)$, where R_i's are clopen preorder relation on X whose intersection is the original preorder relation. ■

Our next corollary is in fact an easy consequence:

Corollary 1.3. *An ordered topological space $(X, \leq)$ belongs to $\mathcal{P}ro\mathcal{F}in(\mathcal{O}rd)$, i.e. is a Priestley space, if only if X is compact and the relation $\leq$ is inter-clopen (as a preorder).*

Remark 1.4. (a) It seems that Corollary 1.3 might suggest considering "order-inter-clopen" order relations, i.e. those order relations that are intersections of clopen order relations. In fact not, simply because there are no such relations! More precisely, the following four conditions on an ordered topological space $(X, \leq)$ are equivalent:

(1) $\leq$ is an order-inter-clopen subset in $X \times X$;
(2) $\leq$ is an clopen subset in $X \times X$;
(3) $\leq$ is an open subset in $X \times X$;
(4) X is discrete as a topological space.

(b) On the other hand inter-clopen equivalence relations are important: in the case of Stone spaces they are precisely the effective equivalence relations, and they are useful in Galois theory of commutative rings [2], which in fact was our original motivation for considering inter-clopen preorders.

2. Profinite and inter-clopen models

A first order (finitary, one-sorted) language $\mathbb{L}$ is determined by its set $F(\mathbb{L})$ of functional symbols and its set $P(\mathbb{L})$ of predicate symbols both equipped with arity maps into the set $\{0, 1, 2, \cdots\}$ of natural numbers. A model (or a structure) for such a language $\mathbb{L}$ is a pair $A = (A_0, \nu_A)$ in which A_0 is a set and ν_A a map that associates an n-ary operation on A_0 to each $F \in F(\mathbb{L})$ and an n-ary relation on A_0 to each $P \in P(\mathbb{L})$; we then simply write F_A and P_A instead of $\nu_A(F)$ and $\nu_A(P)$ respectively. A homomorphism $f : A \to B$ of models is a map $f : A_0 \to B_0$ with

$$fF_A = F_B f^n \text{ and } f^n(P_A) \subseteq P_B$$

for all natural n, n-ary F in $F(\mathbb{L})$, and n-ary P in $P(\mathbb{L})$; here f^n denotes the map $(A_0)^n \to (B_0)^n$ induced by f. The category of models for $\mathbb{L}$ and their homomorphisms will be denoted by $Mod(\mathbb{L})$; we will freely use various well-known properties of this category. In particular we will use the fact that the forgetful functor $U_{\mathbb{L}} : Mod(\mathbb{L}) \to Alg(\mathbb{L})$ is a fibration; here by $Alg(\mathbb{L})$ we denote the category of models of the language obtained from $\mathbb{L}$ by removing all predicate symbols. Recall, however, that for A in $Alg(\mathbb{L})$, B in $Mod(\mathbb{L})$, and a morphism $f : A \to U_{\mathbb{L}}(B)$ in $Alg(\mathbb{L})$, the cartesian lifting $f^*(A) \to B$ has:

- $f^*(A)_0$ and all $F_{f*(A)}(F \in F(L))$ are as in A;
- $P_{f^*(A)} = (f^n)^{-1}(P_B)$ for all natural n and each n-ary P in $P(\mathbb{L})$.

When B is a terminal object in $Mod(\mathbb{L})$, we will write A^* instead of $f^*(A)$; we will also write B^* instead of $(U_{\mathbb{L}}(B))^*$ for an arbitrary B in $Mod(\mathbb{L})$. Note that $P_{A^*} = (A_0)^n$ for all natural n and each n-ary P in $P(\mathbb{L})$.

Throughout this paper: $\mathbf{Q}$ *will denote a fixed full subcategory in* $Alg(\mathbb{L})$ *closed under products and subobjects, in particular it could be any quasi-variety;* $\mathbf{C}$ *will denote a full subcategory of* $Mod(\mathbb{L})$*, satisfying the following conditions:*

- *The forgetful functor* $U_{\mathbb{L}} : Mod(\mathbb{L}) \to Alg(\mathbb{L})$ *induces a functor* $U_{\mathbf{C},\mathbf{Q}} : \mathbf{C} \to \mathbf{Q}$*;*
- $\mathbf{C}$ *is closed in* $Mod(\mathbb{L})$ *under limits and cartesian liftings with respect to the functor* $U_{\mathbf{C},\mathbf{Q}}$.

Note that this makes the forgetful functor $U_{\mathbf{C},\mathbf{Q}} : \mathbf{C} \to \mathbf{Q}$ what is called a *topological functor* in categorical topology (see e.g. G. C. L. Brümmer [1]).

We will also consider:

- the category $\mathcal{T}op(\mathbf{C})$ whose objects are $A = (A_0, \nu_A)$ in $\mathbf{C}$ equipped with a topology on A_0, making F_A continuous for each F in $F(\mathbb{L})$;
- the full subcategory $\mathcal{F}in(\mathbf{C})$ in $\mathcal{T}op(\mathbf{C})$ with objects all $A = (A_0, \nu_A)$ with finite discrete A_0 (of course $\mathcal{F}in(\mathbf{C})$ can also be considered as the full subcategory in $\mathbf{C}$ with objects all finite models from $\mathbf{C}$);
- the full subcategory $\mathcal{P}ro\mathcal{F}in(\mathbf{C})$ in $\mathcal{T}op(\mathbf{C})$ defined as the limit completion of $\mathcal{F}in(\mathbf{C})$ in $\mathcal{T}op(\mathbf{C})$;
- the full subcategory $\mathcal{P}ro\mathcal{F}in(\mathbf{Q})$ in $\mathcal{T}op(\mathbf{Q})$ defined (in a similar way) as the limit completion of $\mathcal{F}in(\mathbf{Q})$ in $\mathcal{T}op(\mathbf{Q})$.

Definition 2.1. *For an object A in $\mathcal{T}op(\mathbf{C})$, we say that*
(a) A is closed if P_A is a closed subset in P_{A^*} for each P in $P(\mathbb{L})$;
(b) A is open if P_A is an open subset in P_{A^*} for each P in $P(\mathbb{L})$;
(c) A is clopen if it is closed and open at the same time;
(d) A is inter-clopen if A^* has a set $\mathcal{S}$ of clopen subobjects, such that

$$S_0 = A_0 \text{ for all } S \in \mathcal{S}, \text{ and } P_A = \cap_{S\in\mathcal{S}} P_S$$

for each P in $P(\mathbb{L})$.

Lemma 2.2. *Every object in $\mathcal{P}ro\mathcal{F}in(\mathbf{C})$ is inter-clopen.*

Proof: Let A be an object in $\mathcal{P}ro\mathcal{F}in(\mathbf{C})$ and let $D : X \to \mathcal{F}in(\mathbf{C})$ be a diagram, whose limit is A with the limit projections $p_x : A \to D(x)$, $x \in X$. For each object x in X, let $A[x]$ be the object in $\mathbf{C}$ defined via the cartesian lifting $A[x] \to D(x)$ of $U_{\mathbf{C},\mathbf{Q}}(p_x) : U_{\mathbf{C},\mathbf{Q}}(A) \to U_{\mathbf{C},\mathbf{Q}}D(x)$. Assuming $A[x]$ to be equipped with the topology of A, we just take $\mathcal{S}$ of 2.1(d) to be the set of all such objects $A[x] (x \in X)$. ■

Let us now impose

Condition 2.3. *If A is a clopen object in $\mathcal{T}op(\mathbf{C})$ with A^* in $\mathcal{P}ro\mathcal{F}in(\mathbf{C})$ (or, equivalently, with $U_{\mathbf{C},\mathbf{Q}}(A)$ in $\mathcal{P}ro\mathcal{F}in(\mathbf{Q})$), then A belongs to $\mathcal{P}ro\mathcal{F}in(\mathbf{C})$.*

Theorem 2.4. *Under Condition 2.3, the following conditions on an object A in $\mathcal{T}op(\mathbf{C})$ are equivalent:*

(a) *A belongs to $\mathcal{P}ro\mathcal{F}in(\mathbf{C})$;*

(b) *A^* belongs to $\mathcal{ProFin}(\mathbf{C})$ and A is inter-clopen;*
(c) *$U_{\mathbf{C},\mathbf{Q}}(A)$ belongs to $\mathcal{ProFin}(\mathbf{Q})$ and A is inter-clopen.*

Proof: (a)⇒(c) follows from Lemma 2.2, (b)⇔ (c) is obvious, and (b)⇒(a) follows from the fact that an intersection is a special case of a "wide pullback": in the situation of 2.1(d), A is the limit of the diagram formed by all $S \to A^*(S \in \mathcal{S})$ with A^* and S $(S \in \mathcal{S})$ profinite, by the assumption and by Condition 2.3 respectively. ■

That is, under the assumption that $U_{\mathbf{C},\mathbf{Q}}$ is topological, the problem reduces to Condition 2.3. Note that Condition 2.3, which is not only sufficient but also necessary, holds not only in the special cases considered - preorders and equivalence relations - but also in a wide class of familiar quasi-varieties as we show next.

It is interesting that the concept of topological functor has become relevant, and it is exactly the relevant difference between preorders and orders: the forgetful functor $\mathcal{Preord} \to \mathcal{Set}$ is topological while the forgetful functor $\mathcal{Ord} \to \mathcal{Set}$ is not.

3. A non-topological condition that implies Cond. 2.3

Let $\mathbb{L}, \mathbf{Q}$ and $\mathbf{C}$ be as above; in this section instead of requiring Condition 2.3 we will deduce it from

Condition 3.1. *(a) Let $A \to B$ be a morphism in $Mod(\mathbb{L})$ that is a regular epimorphism and is cartesian with respect to the forgetful functor $U_{\mathbb{L}} : Mod(\mathbb{L}) \to Alg(\mathbb{L})$; then $A \in \mathbf{C}$ and $U_{\mathbb{L}}(B) \in \mathbf{Q}$ imply that $B \in \mathbf{C}$.*
(b) The set $P(\mathbb{L})$ is finite.

Let A be a clopen object in $\mathcal{Top}(\mathbf{C})$ (whose underlying space will also be denoted by A) with A^* in $\mathcal{ProFin}(\mathbf{C})$ (assuming that A^* has the same topology as A), and let $D : X \to \mathcal{Fin}(\mathbf{Q})$ be a functor, for which $U_{\mathbb{L}}(A) = limD$.

Without loss of generality we can assume that X is an ordered set with

$$\forall x \in X \forall\, y \in X \exists\, z \in X (z \leq x \text{ and } z \leq y), \tag{1}$$

and that the limit projections $\pi_x : A \to D(x)(x \in X)$ are surjective maps; in particular (1) implies:

Lemma 3.2. *For every finite set $\mathcal{U}$ of clopen subsets in A, there exists $x \in X$ with*

$$\forall U \in \mathcal{U} (a \in U \ \textit{and} \ \ \pi_x(a) = \pi_x(a') \Rightarrow a' \in U),$$

where $\pi_x : A \to D(x)$ is the limit projection corresponding to x.

Next, for each natural n and each n-ary P in $P(\mathbb{L})$, we present P_A as a finite union

$$P_A = (U_{11} \times \cdots \times U_{1n}) \cup \cdots \cup (U_{k1} \times \cdots \times U_{kn})$$

with all U_{ij} clopen, and take

$$\mathcal{U}_P = \{U_{ij} | i = 1, \cdots, k \text{ and } j = 1, \cdots, n\} \text{ and } \mathcal{U} = \cup_{P \in P(\mathbb{L})} \mathcal{U}_P. \qquad (2)$$

After that, using Condition 3.1(b), we choose x as in Lemma 3.2, put

$$Y = \{y \in X | y \leq x\},$$

and define $E : Y \to \mathcal{F}in(\mathbf{Q})$ as the restriction of D on Y. As follows from (1), we still have

$$U_{\mathbb{L}}(A) = limE.$$

We then use cocartesian liftings to construct the functor $F : Y \to \mathcal{F}in(\mathbf{C})$, which, composed with the forgetful functor $\mathcal{F}in(\mathbf{C}) \to \mathcal{F}\mathbf{in}(\mathbf{Q})$, gives E. Explicitly, for y in Y, we define $F(y)$ as the object in $\mathbf{C}$ with $U_{\mathbb{L}}(F(y)) = E(y)$ and, for each P in $P(\mathbb{L})$,

$$P_{F(y)} = \cap\{P_B | U_{\mathbb{L}}(B) = E(y) \text{ and } \pi_y \text{ determines a morph. } A \to B \text{ in } \mathbf{C}\}.$$

Lemma 3.3. *In the notation above, $A = limF$.*

Proof: In the notation above, let $G(y)$ be the object in $Mod(\mathbb{L})$ with

$$U_{\mathbb{L}}(G(y)) = E(y) \text{ and } P_{G(y)} = (\pi_y)^n(P_A),$$

for each natural n and each n-ary P in $P(\mathbb{L})$. Then π_y becomes a regular epimorphism from A to $G(y)$, and at the same time Lemma 3.2, applied to $\mathcal{U}$ of (2), easily implies that it is a cartesian morphism with respect to the forgetful functor $U_{\mathbb{L}} : Mod(\mathbb{L}) \to Alg(\mathbb{L})$. Therefore $G(y)$ is in $\mathbf{C}$ by Condition 3.1(a), and the universal property of $F(y)$ implies that $G(y)$ coincides with $F(y)$. Hence G is a functor from Y to $\mathcal{F}in(\mathbf{C})$, and since $P_A = ((\pi_y)^n)^{-1}(P_{F(y)})$ (since π_y is cartesian) for each natural n and each n-ary P in $P(\mathbb{L})$, we obtain $A = limG = limF$. ■

That is, we obtain:

Proposition 3.4. *Condition 2.3 follows from Condition 3.1.*

Remark 3.5. It is clear that the quasi-variety (in model-theoretic sense) of preordered sets satisfy Condition 3.1. More generally, let **C** be a quasi-variety determined by a set Φ of quasi-atomic formulas, i.e. formulas of the form

$$(\varphi_1 \wedge \cdots \wedge \varphi_n) \Rightarrow \varphi \tag{3}$$

where $\varphi_1, \cdots, \varphi_n, \varphi$ are atomic formulas, i.e. formulas of the form

$$P(t_1, \cdots, t_n) \tag{4}$$

and

$$t = u \tag{5}$$

where P in $P(\mathbb{L})$ and P is n-ary, and $t_1, \cdots, t_n, t$ and u are terms in the same language $\mathbb{L}$. Then **C** satisfies Condition 3.1 whenever (its set of predicate symbols is finite and) for every formula (3) from Φ, either each of $\varphi_1, \cdots, \varphi_n$ is of the form (4), or each of $\varphi_1, \cdots, \varphi_n, \varphi$ is of the form (5).

This observation provides a wide class of examples of familiar quasi-varieties satisfying Condition 3.1 of course. The reason for having no restrictions on the algebraic atomic formulas here is that we do not touch the algebraic part of the story assuming that the model A in Condition 2.3 and Theorem 2.4 to be "algebraically profinite", i.e. having $U_{\mathbb{L}}(A)$ in $\mathcal{P}ro\mathcal{F}in(\mathbf{Q})$.

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George Janelidze
Dept. of Math. and Appl. Math., University of Cape Town, Rondebosh 7701, Cape Town, South Africa
E-mail address: janelidg@maths.uct.ac.za

Manuela Sobral
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: sobral@mat.uc.pt