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ON POISSON QUASI-NIJENHUIS LIE ALGEBROIDS

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ABSTRACT: We propose a definition of Poisson quasi-Nijenhuis Lie algebroids as a natural generalization of Poisson quasi-Nijenhuis manifolds and show that any such Lie algebroid has an associated quasi-Lie bialgebroid. Therefore, also an associated Courant algebroid is obtained. We introduce the notion of a morphism of quasi-Lie bialgebroids and of the induced Courant algebroids morphism and provide some examples of Courant algebroid morphisms. Finally, we use paired operators to deform doubles of Lie and quasi-Lie bialgebroids and find an application to generalized complex geometry.

Introduction

The notion of Poisson quasi-Nijenhuis manifold was recently introduced by Stiénon and Xu [14]. It is a manifold M together with a Poisson bivector field π, a $(1,1)$-tensor N compatible with π and a closed 3-form ϕ such that $i_N\phi$ is also closed and the Nijenhuis torsion of N, which is nonzero, is expressed by means of ϕ and π. When $\phi = 0$ one obtains a Poisson-Nijenhuis manifold, a concept introduced by Magri and Morosi [11] to study integrable systems and which was extended to the Lie algebroid framework by Kosmann-Schwarzbach [6] and Grabowski and Urbanski [5] who introduced the notion of a Poisson-Nijenhuis Lie algebroid. In this paper we propose a definition of Poisson quasi-Nijenhuis Lie algebroid, which is a straightforward generalization of a Poisson quasi-Nijenhuis manifold.

Quasi-Lie bialgebroids were introduced by Roytenberg [12] who showed that they are the natural framework to study twisted Poisson structures [13]. On the other hand, quasi-Lie bialgebroids are intimately related to Courant algebroids [9], because the double of a quasi-Lie bialgebroid carries a structure of Courant algebroid and conversely, a Courant algebroid E that admits a Dirac subbundle A and a transversal isotropic complement B, can be identified with the Whitney sum $A \oplus A^*$, where A^* is identified with B [12]. Generalizing a result of Kosmann-Schwarzbach [6] for Poisson-Nijenhuis manifolds and Lie bialgebroids, it is proved in [14] that a Poisson quasi-Nijenhuis structure on a manifold M is equivalent to a quasi-Lie bialgebroid

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structure on T^*M. Extending the result of [14], we show that a Poisson quasi-Nijenhuis Lie algebroid has an associated quasi-Lie bialgebroid, so that it has also an associated Courant algebroid.

In an unpublished manuscript, Alekseev and Xu [1], gave the definition of a Courant algebroid morphism between E_1 and E_2 and, in the case where E_1 and E_2 are doubles of Lie bialgebroids (A, A^*) and (B, B^*), i.e $E_1 = A \oplus A^*$ and $E_2 = B \oplus B^*$, they established a relationship with a Lie bialgebroid morphism $A \to B$ [10]. Since doubles of quasi-Lie bialgebroids are Courant algebroids, it seems natural to obtain a relationship between Courant algebroid morphisms and quasi-Lie bialgebroid morphisms. This is the case when considering Courant algebroids associated with a Poisson quasi-Nijenhuis Lie algebroid of a certain type and with a twisted Poisson Lie algebroid, respectively. In a first step towards our result, we give the definition of a morphism of quasi-Lie bialgebroids which is, up to our knowledge, a new concept that includes morphism of Lie bialgebroids as a particular case.

Another aspect of Poisson quasi-Nijenhuis manifolds that is exploited in [14] is the relation with generalized complex structures. We extend to Poisson quasi-Nijenhuis Lie algebroids some of the results obtained in [14] and also discuss the relation of Poisson quasi-Nijenhuis Lie algebroids with paired operators [3].

The paper is divided into three sections. In section 1 we introduce quasi-Lie bialgebroid morphisms and discuss their relationship with Courant algebroid morphisms. Section 2 is devoted to Poisson quasi-Nijenhuis Lie algebroids. We prove that each Poisson quasi-Nijenhuis Lie algebroid has an associated quasi-Lie bialgebroid and, in some particular cases, we construct a morphism of Courant algebroids. In the last section we use paired operators to deform doubles of Lie and quasi-Lie bialgebroids.

1. Quasi-Lie bialgebroids morphisms

1.1. Quasi-Lie bialgebroids. The main subject of this work are quasi-Lie bialgebroids. We begin by recalling the definition and give some examples.

Definition 1.1. [12] A *quasi-Lie bialgebroid* is a Lie algebroid $(A, [\, ,\,]_A, \rho)$ equipped with a degree-one derivation d_* of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [\, ,\,]_A)$ and a 3-section of A, $X_A \in \Gamma(\wedge^3 A)$ such that

$$\mathrm{d}_* X_A = 0 \quad \text{and} \quad \mathrm{d}_*^2 = [X_A, -]_A \,.$$

If X_A is the null section, then d_* defines a structure of Lie algebroid on A^* such that d_* is a derivation of $[\,,\,]_A$. In this case we say that (A, A^*) is a Lie bialgebroid.

Examples of quasi-Lie bialgebroids arise from different well known geometric structures. We will illustrate some of them that will be needed in our work.

Example 1.2. Let $(A, [\,,\,]_A\,, \rho)$ be a Lie algebroid and consider any closed 3-form ϕ. Equipping A^* with the null Lie algebroid structure, $(A^*, \mathrm{d}_A, \phi)$ is canonically a quasi-Lie bialgebroid.

Example 1.3. [Lie algebroid with a twisted Poisson structure] Let $\pi \in \Gamma(\wedge^2 A)$ be a bivector on the Lie algebroid $(A, [\,,\,]_A\,, \rho)$ and denote by $\pi^\sharp$ the usual bundle map

$$\begin{array}{rccc} \pi^\sharp: & A^* & \longrightarrow & A \\ & \alpha & \longmapsto & \pi^\sharp(\alpha) = i_\alpha \pi. \end{array}$$

This map can be extended to a bundle map from $\Gamma(\wedge^\bullet A^*)$ to $\Gamma(\wedge^\bullet A)$, also denoted by $\pi^\sharp$, as follows:

$$\pi^\sharp(f) = f \quad \text{and} \quad \left\langle \pi^\sharp(\mu), \alpha_1 \wedge \ldots \wedge \alpha_k \right\rangle = (-1)^k \mu\left(\pi^\sharp(\alpha_1), \ldots, \pi^\sharp(\alpha_k)\right),$$

for all $f \in C^\infty(M)$ and $\mu \in \Gamma(\wedge^k A^*)$ and $\alpha_1, \ldots, \alpha_k \in \Gamma(A^*)$.

Let $\phi \in \Gamma(\wedge^3 A^*)$ be a closed 3-form on A. We say that (π, ϕ) defines a *twisted Poisson structure on A* [13] if

$$[\pi, \pi]_A = 2\, \pi^\sharp(\phi).$$

In this case, the bracket on the sections of A^* defined by

$$[\alpha, \beta]_\pi^\phi = \boldsymbol{\mathcal{L}}_{\pi^\sharp \alpha}\beta - \boldsymbol{\mathcal{L}}_{\pi^\sharp \beta}\alpha - \mathrm{d}\left(\pi(\alpha, \beta)\right) + \phi(\pi^\sharp\alpha, \pi^\sharp\beta, -), \quad \forall \alpha, \beta \in \Gamma(A^*),$$

is a Lie bracket and $A^*_{\pi,\phi} = (A^*, [\,,\,]_\pi^\phi\,, \rho \circ \pi^\sharp)$ is a Lie algebroid. The differential of this Lie algebroid is given by

$$\mathrm{d}_\pi^\phi X = [\pi, X]_A - \pi^\sharp(i_X \phi), \quad \forall X \in \Gamma(A).$$

The pair (A, A^*) is not a Lie bialgebroid but when we consider the bracket on $\Gamma(A)$ defined by:

$$[X, Y]' = [X, Y]_A - \pi^\sharp(\phi(X, Y, -)), \quad \forall X, Y \in \Gamma(A),$$

the associated differential d', given by

$$\mathrm{d}' f = \mathrm{d} f \quad \text{and} \quad \mathrm{d}'\alpha = \mathrm{d}\alpha - i_{\pi^\sharp \alpha}\phi, \quad \forall f \in C^\infty(M), \alpha \in \Gamma(A^*),$$

defines on $A^*_{\pi,\phi}$ a structure of quasi-Lie bialgebroid $(A^*_{\pi,\phi}, \mathrm{d}', \phi)$.

One should notice that when $\phi = 0$, π is a Poisson bivector. The Lie algebroid $A^*_{\pi,0}$ is simply denoted by A^*_π, and together with the Lie algebroid A it defines a special kind of Lie bialgebroid called a *triangular Lie bialgebroid*.

Any bundle map $\Phi : A \to B$ induces a map $\Phi^* : \Gamma(B^*) \to \Gamma(A^*)$ which assigns to each section $\alpha \in \Gamma(B^*)$ the section $\Phi^*\alpha$ given by

$$\Phi^*\alpha(X)(m) = \langle \alpha(\phi(m)), \Phi_m X(m) \rangle, \quad \forall m \in M,\ X \in \Gamma(A),$$

where $\phi : M \to N$ is the map induced by Φ on the base manifolds. We denote by the same latter Φ^* the extension of this map to the multisections of B^*, where we set $\Phi^* f = f \circ \phi$, for $f \in C^\infty(N)$.

Let $A \to M$ and $B \to N$ be two Lie algebroids. Recall that a *Lie algebroid morphism* is a bundle map $\Phi : A \to B$ such that $\Phi^* : (\Gamma(\wedge^\bullet B^*), \mathrm{d}_B) \to (\Gamma(\wedge^\bullet A^*), \mathrm{d}_A)$ is a chain map.

Generalizing the notion of Lie bialgebroid morphism we propose the following definition of morphism between quasi-Lie bialgebroids:

Definition 1.4. Let $(A, \mathrm{d}_{A^*}, X_A)$ and $(B, \mathrm{d}_{B^*}, X_B)$ be quasi-Lie bialgebroids over M and N, respectively. A bundle map $\Phi : A \to B$ is a *quasi-Lie bialgebroid morphism* if

1) Φ is a Lie algebroid morphism;
2) Φ^* is compatible with the brackets on the sections of A^* and B^*:
$$[\Phi^*\alpha, \Phi^*\beta]_{A^*} = \Phi^* [\alpha, \beta]_{B^*};$$
3) the vector fields $\rho_{B^*}(\alpha)$ and $\rho_{A^*}(\Phi^*\alpha)$ are ϕ-related:
$$T\phi \cdot \rho_{A^*}(\Phi^*\alpha) = \rho_{B^*}(\alpha) \circ \phi;$$
4) $\Phi X_A = X_B \circ \phi$,

where α, $\beta \in \Gamma(B^*)$ and $\phi : M \to N$ is the smooth map induced by Φ on the base.

Example 1.5. A Lie bialgebroid morphism [10] is a Lie algebroid morphism which is also a Poisson map, when we consider the Lie-Poisson structures induced by their dual Lie algebroids. We can easily see that in case we are dealing with Lie bialgebroids, the definition of quasi-Lie bialgebroid morphism coincides with the one of Lie bialgebroid morphism.

Example 1.6. Consider $(A, \mathrm{d}_{A*}, X_A)$ and $(B, \mathrm{d}_{B*}, X_B)$ two quasi-Lie bialgebroids over the same base manifold M. We can see that a base preserving quasi-Lie bialgebroid morphism (such that $\phi = \mathrm{id}$) is a bundle map $\Phi : A \to B$ such that $\Phi^* \circ \mathrm{d}_B = \mathrm{d}_A \circ \Phi^*$, $\Phi \circ \mathrm{d}_{A^*} = \mathrm{d}_{B^*} \circ \Phi$ and $\Phi X_A = X_B$.

Other examples of quasi-Lie bialgebroid morphisms will appear in the next section associated with quasi-Nijenhuis structures.

1.2. Courant algebroids. A *Courant algebroid* $E \to M$ is a vector bundle over a manifold M equipped with a nondegenerate symmetric bilinear form $\langle \, , \rangle$, a vector bundle map $\rho : E \to TM$ and a bilinear bracket $\circ$ on $\Gamma(E)$ satisfying:

C1) $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$
C2) $e \circ e = \rho^* d \langle e, e \rangle$
C3) $\boldsymbol{\mathcal{L}}_{\rho(e)} \langle e_1, e_2 \rangle = \langle e \circ e_1, e_2 \rangle + \langle e_1, e \circ e_2 \rangle$
C4) $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$
C5) $e_1 \circ f e_2 = f(e_1 \circ e_2) + \boldsymbol{\mathcal{L}}_{\rho(e_1) f} e_2$,

for all $e, e_1, e_2, e_3 \in \Gamma(E)$, $f \in C^\infty(M)$.

Associated with the bracket $\circ$, we can define a skew-symmetric bracket on the sections of E by:

$$[\![e_1, e_2]\!] = \frac{1}{2} (e_1 \circ e_2 - e_2 \circ e_1)$$

and the properties C1)-C5) can be expressed in terms of this bracket.

Example 1.7. [Standard Courant algebroid] Let $(A, [\, ,]_A, \rho_A)$ be a Lie algebroid. The double $A \oplus A^*$ equipped with the skew-symmetric bracket

$$[\![X + \alpha, Y + \beta]\!] = [X, Y]_A + \left(\boldsymbol{\mathcal{L}}_X \beta - \boldsymbol{\mathcal{L}}_Y \alpha + \frac{1}{2} \mathrm{d}(\alpha(Y) - \beta(X)) \right),$$

the pairing $\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X)$ and the anchor $\rho(X + \alpha) = \rho_A(X)$ is a Courant algebroid.

A standard Courant algebroid is a simple example of a Courant algebroid which is a the double of a Lie bialgebroid. The construction of Courant algebroids as doubles of Lie bialgebroids is implicit in the next example, where we explicit the construction of the double of a quasi-Lie bialgebroid.

Example 1.8. [Double of a quasi-Lie bialgebroid] Let (A, d_*, X_A) be a quasi-Lie bialgebroid. Its double $E = A \oplus A^*$ is a Courant algebroid if it is equipped with the pairing $\langle X+\alpha, Y+\beta\rangle = \alpha(Y)+\beta(X)$, the anchor $\rho=\rho_A+\rho_{A^*}$ and the bracket

$$\llbracket X+\alpha, Y+\beta\rrbracket = [X,Y]_A + \boldsymbol{\mathcal{L}}^*_\alpha Y - \boldsymbol{\mathcal{L}}^*_\beta X - \frac{1}{2}\mathrm{d}_*(\alpha(Y)-\beta(X)) + X_A(\alpha,\beta,-) + \left([\alpha,\beta]_* + \boldsymbol{\mathcal{L}}_X\beta - \boldsymbol{\mathcal{L}}_Y\alpha + \frac{1}{2}\mathrm{d}(\alpha(Y)-\beta(X))\right),$$

Taking $X_A=0$ we have the Courant algebroid structure of a double of a Lie bialgebroid.

Another particular case that worths to be mentioned is the double of the quasi-Lie bialgebroid (A^*, d, ϕ) illustrated in Example 1.2. In this case the anchor is simply $\rho_E=\rho_A$ and the skew-symmetric bracket is a twisted version of the standard Courant bracket given by:

$$\llbracket X+\alpha, Y+\beta\rrbracket^\phi = [X,Y]_A + \boldsymbol{\mathcal{L}}_X\beta - \boldsymbol{\mathcal{L}}_Y\alpha + \frac{1}{2}\mathrm{d}(\alpha(Y)-\beta(X)) + \phi(X,Y,-). \tag{1}$$

1.3. Dirac structures supported on a submanifold. Dirac structures play an important role in the theory of Courant algebroids. Let us recall them before proceed.

A *Dirac structure on a Courant algebroid* E is a subbundle $A\subset E$, which is maximal isotropic with respect to the pairing $\langle\,,\,\rangle$ and it is integrable in the sense that the space of the sections of A is closed under the bracket on $\Gamma(E)$. Restricting the skew-symmetric bracket of E and the anchor to A, we endow the Dirac structure with a Lie algebroid structure $(A, \llbracket\,,\,\rrbracket_{|_A}, \rho_{E_{|_A}})$. A Courant algebroid together with a Dirac structure is called a *Manin pair*.

As a way to generalize Dirac structures we have the concept of generalized Dirac structures or Dirac structures supported on a submanifold of the base manifold.

Definition 1.9. [1] On a Courant algebroid $E\to M$, a *Dirac structure supported on a submanifold* P *of* M or a *generalized Dirac structure* is a subbundle F of $E_{|_P}$ such that:

D1) for each $x\in P$, F_x is maximal isotropic;
D2) F is compatible with the anchor, i.e. $\rho_{|_P}(F)\subset TP$;
D3) For each $e_1,e_2\in\Gamma(E)$, such that $e_{1|_P}, e_{2|_P}\in\Gamma(F)$, we have $(e_1\circ e_2)_{|_P}\in\Gamma(F)$.

Obviously, a Dirac structure supported on the whole base manifold M is an usual Dirac structure of the Courant algebroid.

Generalizing the Theorem 6.11 on [1] to quasi-Lie bialgebroids we have:

Theorem 1.10. *Let $E = A \oplus A^*$ be the double of a quasi-Lie bialgebroid (A, d_*, X_A) over the manifold M, $L \to P$ a vector subbundle of A over a submanifold P of M and $F = L \oplus L^\perp$. Then F is a Dirac structure supported on P if and only if the following conditions hold:*

1) *L is a Lie subalgebroid of A;*
2) *$L^\perp$ is closed for the bracket on A^* defined by d_*;*
3) *$L^\perp$ is compatible with the anchor, i.e., $\rho_{A^*|_P}(L^\perp) \subset TP$;*
4) *$X_{A|_{L^\perp}} = 0$.*

Proof: Since $F = L \oplus L^\perp$, this is a Lagrangian subbundle of E. Suppose F is a Dirac structure supported on P. By definition, we immediately deduce that L is a Lie subalgebroid of A and, for α, β sections of A^* such that $\alpha_{|_P}$, $\beta_{|_P} \in \Gamma(L^\perp)$, we have

$$(\alpha \circ \beta)_{|_P} = X_A(\alpha, \beta, -)_{|_P} + [\alpha, \beta]_{A^*|_P} \in \Gamma(L \oplus L^\perp),$$

and this means that $[\alpha, \beta]_{A^*|_P} \in L^\perp$ and $X_A(\alpha, \beta, -)_{|_P} \in L$, or equivalently, $L^\perp$ is closed with respect to the bracket of E and $X_{A|_{L^\perp}} = 0$.

Moreover, since F is compatible with the anchor,

$$\rho_{A^*|_P}(\alpha_{|_P}) = \rho_{A^*}(\alpha)_{|_P} = \rho_E(\alpha)_{|_P} \in TP,$$

so $L^\perp$ is compatible with ρ_{A^*}.

Conversely, suppose L is a Lie subalgebroid of A, $L^\perp \subset A^*$ is closed for $[\,,\,]_{A^*}$, $\rho_{A^*|_P}(L^\perp) \subset TP$ and $Q_{A|_{L^\perp}} = 0$. Obviously F is compatible with the anchor. We are left to prove that F is closed with respect to the bracket on E. Let $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$ such that $X + \alpha$ and $Y + \beta$ restricted to P are sections of F, then

$$(X + \alpha) \circ (Y + \beta)_E = [X, Y]_A + i_\alpha \mathrm{d}_* Y - i_\beta \mathrm{d}_* X + \mathrm{d}_*(\alpha(Y)) + X_A(\alpha, \beta, -) + [\alpha, \beta]_{A^*} + \mathcal{L}_X \beta - i_Y \mathrm{d}\alpha.$$

By hypothesis, we immediately have that

$$[X, Y]_{A|_P} = \left[X_{|_P}, Y_{|_P}\right]_L \in \Gamma(L),$$

$$[\alpha, \beta]_{A^*|_P} = \left[\alpha_{|_P}, \beta_{|_P}\right]_{L^\perp} \in \Gamma(L^\perp)$$

and

$$X_A(\alpha,\beta,-)_{|_P} \in \Gamma(L).$$

Now, notice that $\alpha(Y)_{|_P} = 0$, so $d\alpha(Y)_{|_P} \in \nu^*(P) = (TP)^0$. Since $\rho_{A^*|_P}(L^\perp) \subset TP$, we have that

$$\mathrm{d}_*\alpha(Y)_{|_P} = \rho^*_{A^*|_P} d\alpha(Y) \in \Gamma(L).$$

Analogously, $\mathrm{d}\alpha(Y)_{|_P} = \rho^*_A d\alpha(Y)_{|_P} \in \Gamma(L^\perp)$.

Also,

$$\mathrm{d}_*Y(\alpha,\beta)_{|_P} = \left(\rho_{A^*}(\alpha)\cdot\beta(Y) - \rho_{A^*}(\beta)\cdot\alpha(Y) - [\alpha,\beta]_{A^*}(Y)\right)_{|_P} = 0,$$

so $i_\alpha \mathrm{d}_* Y \in \Gamma(L)$. Analogously, $i_X \mathrm{d}\beta \in \Gamma(L^\perp)$.

All these conditions allow us to say that $(X+\alpha)\circ(Y+\beta) \in \Gamma(L\oplus L^\perp)$ and, consequently, F is a Dirac structure supported on P. ∎

Corollary 1.11. [1] *Let $E = A\oplus A^*$ be the double of a Lie bialgebroid then $F = L\oplus L^\perp$ is a Dirac structure supported on P if and only if L and $L^\perp$ are Lie subalgebroids of A and A^*.*

Notice that when $P = M$ we obtain Proposition 7.1 of [9].

Corollary 1.12. [1] *Let $E = TM\oplus T^*M$ be the standard Courant algebroid twisted by the 3-form $\phi\in\Omega^3(M)$ (see equation (1) in Example 1.8). For any submanifold P of M, $F = TP\oplus\nu^*P$ is a Dirac structure supported on M iff $i^*\phi = 0$, where $i : P \hookrightarrow M$ is the inclusion map.*

Like Lie bialgebroids morphisms, quasi-Lie bialgebroid morphisms give rise to Courant algebroid morphisms. Let us recall what is a Courant algebroid morphism.

Definition 1.13. [1] A *Courant algebroid morphism* between two Courant algebroids $E\to M$ and $E'\to M'$ is a Dirac structure in $E\times\overline{E}'$ supported on graph ϕ, where $\phi : M\to M'$ is a smooth map and $\overline{E}'$ denotes the Courant algebroid obtained from E' by changing the sign of the bilinear form.

Theorem 1.14. *Let $E_1 = A\oplus A^*$ and $E_2 = B\oplus B^*$ be doubles of quasi-Lie bialgebroids $(A, \mathrm{d}_{A*}, X_A)$ and $(B, \mathrm{d}_{B*}, X_B)$ and $(\Phi,\phi) : A\to B$ a quasi-Lie bialgebroid morphism, then*

$$F = \{(a+\Phi^*b^*, \Phi a + b^*) | a\in A \text{ and } b^*\in B^* \text{over compatible fibers}\} \subset E_1\times\overline{E_2}$$

is a Dirac structure supported on graph ϕ, *i.e. F is a Courant algebroid morphism.*

Proof: The idea of the proof is analogous to the idea of the proof of Theorem 6.10 in [1] for Lie bialgebroid morphisms.

Consider M and N the base manifolds of A and B, respectively. Consider the following subbundles over graph ϕ

$$L = \text{graph}\,\Phi = \{(a, \Phi a) |\, a \in A\} \subset A \times B$$

and

$$L^{\perp} = \{(\Phi^* b^*, -b^*) |\, b^* \in B^*\} \subset A^* \times B^*.$$

Since Φ is a Lie algebroid morphism, L is clearly a Lie subalgebroid of $A \times B$. Analogously, we can also conclude that $L^{\perp}$ is closed for the bracket on $A^* \times \overline{B^*}$ (where $\overline{B^*}$ denotes the bundle B^* with bracket $[\,,\,]_{\overline{B^*}} = -[\,,\,]_{B^*}$) and it is compatible with the anchor $\rho_{A^* \times \overline{B^*}} = (\rho_{A^*}, -\rho_{B^*})$. Also, since $\Phi X_A = X_B \circ \phi$, we have that $(X_A, X_B)_{|L^{\perp}} = 0$. So, Theorem 1.10 guarantees that $L \oplus L^{\perp}$ is a Dirac structure supported on graph ϕ of the double $A \times B \oplus A^* \times \overline{B^*}$ which is the Courant algebroid $A \oplus A^* \times B \oplus \overline{B^*}$. Finally, observe that the bundle morphism $b + b^* \mapsto b - b^*$ induces a canonical isomorphism between F and $L \oplus L^{\perp}$ and the result follows. ■

2. Poisson Quasi-Nijenhuis Lie algebroids

Let $(A, [\,,\,], \rho)$ be a Lie algebroid over a manifold M. The torsion of a bundle map $N : A \to A$ (over the identity) is defined by

$$\mathcal{T}_N(X, Y) := [NX, NY] - N[X, Y]_N, \quad X, Y \in \Gamma(A), \tag{2}$$

where $[\,,\,]_N$ is given by:

$$[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y], \quad X, Y \in \Gamma(A).$$

When $\mathcal{T}_N = 0$, the bundle map N is called a *Nijenhuis operator*, the triple $A_N = (A, [\,,\,]_N, \rho_N = \rho \circ N)$ is a new Lie algebroid and $N : A_N \to A$ is a Lie algebroid morphism.

Remark 2.1. Let A be a Lie algebroid and ϕ a closed 3-form. We have that (A^*, d, ϕ) is quasi-Lie bialgebroid (see Example 1.2). If $N : A \to A$ is a Nijenhuis operator, then $A_N = (A, [\,,\,]_N, \rho \circ N)$ is a Lie algebroid and $N : A_N \to A$ is a Lie algebroid morphism. So,

$$\mathrm{d}_N N^* \phi = N^* \mathrm{d}\phi = 0,$$

$(A^*, \mathrm{d}_N, N^*\phi)$ is a quasi-Lie bialgebroid and $N^* : (A^*, \mathrm{d}, \phi) \to (A^*, \mathrm{d}_N, N^*\phi)$ is a quasi-Lie bialgebroid morphism.

Definition 2.2. On a Lie algebroid A with a Poisson structure $\pi \in \Gamma(\wedge^2 A)$, we say that a bundle map $N : A \to A$ is *compatible* with π if $N\pi^\sharp = \pi^\sharp N^*$ and the *Magri-Morosi concomitant* vanishes:

$$\mathcal{C}(\pi, N)(\alpha, \beta) = [\alpha, \beta]_{N\pi} - [\alpha, \beta]_\pi^{N^*} = 0,$$

where $[\,,\,]_{N\pi}$ is the bracket defined by the bivector field $N\pi \in \Gamma(\wedge^2 A)$, and $[\,,\,]_\pi^{N^*}$ is the Lie bracket obtained from the Lie bracket $[\,,\,]_\pi$ by deformation along the tensor N^*.

As a straightforward generalization of the definition of quasi-Poisson Nijenhuis manifolds presented in [14], we have:

Definition 2.3. A *Poisson quasi-Nijenhuis Lie algebroid* (A, π, N, ϕ) is a Lie algebroid A equipped with a Poisson structure π, a bundle map $N : A \to A$ compatible with π and a closed 3-form $\phi \in \Gamma(\wedge^3 A^*)$ such that

$$\mathcal{T}_N(X, Y) = -\pi^\sharp\left(i_{X\wedge Y}\phi\right) \text{ and } \mathrm{d}i_N\phi = 0.$$

Theorem 2.4. *If (A, π, N, ϕ) is a Poisson quasi-Nijenhuis Lie algebroid then $(A^*_\pi, \mathrm{d}_N, \phi)$ is a quasi-Lie bialgebroid.*

Proof: First notice that $\mathrm{d}\phi = 0$ and $\mathrm{d}i_N\phi = 0$ imply that

$$\mathrm{d}_N\phi = [i_N, \mathrm{d}]\,\phi = i_N\mathrm{d}\phi - \mathrm{d}i_N\phi = 0.$$

Secondly, we notice that since the bundle morphism N and the Poisson structure π are compatible, then d_N is a derivation of the Lie bracket $[\,,\,]_\pi$. In fact, first one directly sees that d is a derivation of $[\,,\,]_{N\pi}$ and, since $\mathcal{C}(\pi, N)$ vanishes and

$$\begin{aligned}\mathrm{d}(\mathcal{C}(\pi, N)(\alpha, \beta)) &= \mathrm{d}_N\,[\alpha, \beta]_\pi - [\mathrm{d}_N\alpha, \beta]_\pi - [\alpha, \mathrm{d}_N\beta]_\pi \\ &\quad - \mathrm{d}\,[\alpha, \beta]_{N\pi} + [\mathrm{d}\alpha, \beta]_{N\pi} + [\alpha, \mathrm{d}\beta]_{N\pi}\,,\end{aligned}$$

we immediately conclude that d_N is a derivation of $[\,,\,]_\pi$ (the particular case where $A = TM$ can be found in [6]).

It remains to prove that $\mathrm{d}_N^2 = [\phi, -]_\pi$. Using the definition of d_N, we have:

$$\mathrm{d}_N^2\alpha(X, Y, Z) = \mathcal{T}_N(X, Y)\,\langle\alpha, Z\rangle - \langle\alpha, [\mathcal{T}_N(X, Y), Z] + \mathcal{T}_N([X, Y]\,, Z)\rangle + \text{c.p.}$$

The fact that $\mathcal{T}_N(X,Y) = -\pi^\sharp i_{X\wedge Y}\phi$ yields:

$$\begin{aligned}
\mathrm{d}_N^2\alpha(X,Y,Z) &= -\phi(X,Y,\pi^\sharp \mathrm{d}\langle\alpha,Z\rangle) - \left\langle \alpha, \boldsymbol{\pounds}_Z\left(\pi^\sharp i_{X\wedge Y}\phi\right) - \pi^\sharp i_{[X,Y]\wedge Z}\phi\right\rangle + \text{c.p.}\\
&= -\phi(X,Y,\pi^\sharp \mathrm{d}\langle\alpha,Z\rangle) - \left\langle \alpha, \left(\boldsymbol{\pounds}_Z\pi\right)^\sharp i_{X\wedge Y}\phi + \pi^\sharp\left(\boldsymbol{\pounds}_Z i_{X\wedge Y}\phi\right)\right\rangle\\
&\quad +\phi([X,Y],Z,\pi^\sharp\alpha) + \text{c.p.}\\
&= -\phi(X,Y,\pi^\sharp \mathrm{d}\langle\alpha,Z\rangle) - \left\langle \alpha, \left(\boldsymbol{\pounds}_Z\pi\right)^\sharp i_{X\wedge Y}\phi + \pi^\sharp\left(i_{X\wedge Y}\boldsymbol{\pounds}_Z\phi\right) - \pi^\sharp i_{[Z,X\wedge Y]}\phi\right\rangle\\
&\quad +\phi([X,Y],Z,\pi^\sharp\alpha) + \text{c.p.}\\
&= -\phi(X,Y,\pi^\sharp \mathrm{d}\langle\alpha,Z\rangle) - \phi(X,Y,\left(\boldsymbol{\pounds}_Z\pi\right)^\sharp\alpha) - \boldsymbol{\pounds}_Z\phi(X,Y,\pi^\sharp\alpha)\\
&\quad -\phi(X,[Z,Y],\pi^\sharp\alpha) + \text{c.p.}.
\end{aligned}$$

Since

$$\begin{aligned}
[\phi,\alpha]_\pi(X,Y,Z) &= -\boldsymbol{\pounds}_{\pi^\sharp(\alpha)}\phi(X,Y,Z)\\
&\quad - \left\{\phi(X,Y,\pi^\sharp \mathrm{d}\langle\alpha,Z\rangle) - \phi(X,Y,\boldsymbol{\pounds}_Z\pi^\sharp(\alpha)) + \text{c.p.}\right\},
\end{aligned}$$

and by hypothesis, ϕ is closed, we finally have that

$$(\mathrm{d}_N^2\alpha - [\phi,\alpha]_\pi)(X,Y,Z) = -\mathrm{d}\phi(X,Y,Z,\pi^\sharp\alpha) = 0.$$

■

Suppose (A,π,N,ϕ) is a Poisson quasi-Nijenhuis Lie algebroid. The double of the quasi-Lie bialgebroid $(A^*_\pi, \mathrm{d}_N, \phi)$ is a Courant algebroid (see Example 1.8) that we denote by E^ϕ_π.

An interesting case is when the 3-form ϕ is the image by N^* of another closed 3-form ψ:

$$\phi = N^*\psi \quad \text{and} \quad \mathrm{d}\psi = 0.$$

In this case $(A, N\pi, \psi)$ is a twisted Poisson Lie algebroid because

$$[N\pi, N\pi] = 2\pi^\sharp(\phi) = 2\pi^\sharp(N^*\psi) = 2N\pi^\sharp(\psi)$$

and A^* has a structure of Lie algebroid: $A^{*\psi}_{N\pi} = (A^*, [\,,\,]^\psi_{N\pi}, N\pi^\sharp)$ (see Example 1.3). Equipping A with the differential d' given by

$$\mathrm{d}'f = \mathrm{d}f, \quad \text{and} \quad \mathrm{d}'\alpha = \mathrm{d}\alpha - i_{N\pi^\sharp\alpha}\psi,$$

for $f \in C^\infty(M)$ and $\alpha \in \Gamma(A^*)$, we obtain a quasi-Lie bialgebroid: $(A^{*\psi}_{N\pi}, \mathrm{d}', \psi)$. Its double is a Courant algebroid and we denote it by $E^\psi_{N\pi}$.

Theorem 2.5. *Let (A,π,N,ϕ) be a Poisson quasi-Nijenhuis Lie algebroid and suppose that $\phi = N^*\psi$, for some closed 3-form ψ, then*

$$F = \{(a + N^*\alpha, Na+\alpha)\,|a \in A \text{ and } \alpha\in A^*\} \subset E^{\psi}_{N\pi}\times \overline{E^{\phi}_{\pi}}$$

defines a Courant algebroid morphism between $E^{\psi}_{N\pi}$ and E^{ϕ}_{π}.

In order to prove the theorem, we need to remark the following property.

Lemma 2.6. *Let (A,π,N,ϕ) be a Poisson quasi-Nijenhuis Lie algebroid, then*

$$\langle \mathcal{T}_{N^*}(\alpha,\beta), X\rangle = \phi(\pi^\sharp\alpha,\pi^\sharp\beta,X),$$

for all $X\in\Gamma(A)$ and $\alpha,\beta\in\Gamma(A^*)$.

Proof: The compatibility between N and π implies that (see [7])

$$\langle \mathcal{T}_{N^*}(\alpha,\beta), X\rangle = \left\langle \alpha, \mathcal{T}_N(X,\pi^\sharp\beta)\right\rangle,$$

so

$$\langle \mathcal{T}_{N^*}(\alpha,\beta), X\rangle = \left\langle \alpha, -\pi^\sharp\left(i_{X\wedge\pi^\sharp\beta}\phi\right)\right\rangle = -\phi(X,\pi^\sharp\beta,\pi^\sharp\alpha) = \phi(\pi^\sharp\alpha,\pi^\sharp\beta,X).$$

■

Proof of the Theorem: First notice that $N^*: A^{*\psi}_{N\pi} \to A^*_\pi$ is a Lie algebroid morphism because it is obviously compatible with the anchors and

$$\begin{aligned} N^*[\alpha,\beta]^{\psi}_{N\pi} &= N^*[\alpha,\beta]_{N\pi} + N^*\psi(\pi^\sharp\alpha,\pi^\sharp\beta,-)\\ &= [N^*\alpha,N^*\beta]_\pi - \mathcal{T}_{N^*}(\alpha,\beta) + \phi(\pi^\sharp\alpha,\pi^\sharp\beta,-) = [N^*\alpha,N^*\beta]_\pi\,. \end{aligned}$$

Let $[\,,\,]'$ be the bracket on the sections of A induced by the differential d'. Notice that

$$[X,f]' = \langle \mathrm{d}'f, X\rangle = \langle \mathrm{d}f, X\rangle\,,$$

so $N^*\mathrm{d}'f = \mathrm{d}_N f$, for all $f\in C^\infty(M)$ and $X\in\Gamma(A)$.

And since

$$[X,Y]' = [X,Y] - (N\pi)^\sharp(\psi(X,Y,-)),$$

we have:

$$\begin{aligned} N[X,Y]_N &= [NX,NY] - \mathcal{T}_N(X,Y) = [NX,NY] + \pi^\sharp(i_{X\wedge Y}N^*\psi)\\ &= [NX,NY] + \psi(NX,NY,N\pi^\sharp-) = [NX,NY]'\,, \end{aligned}$$

for all $X,Y\in\Gamma(A)$.

This way we conclude that $N^*: A^{*\psi}_{N\pi} \to A^*_\pi$ is a quasi-Lie bialgebroid morphism (see definition 1.4) and the result follows from Theorem 1.14. ■

3. Paired operators

Let $(A, \mathrm{d}_{A^*}, X_A)$ be a quasi-Lie bialgebroid over M and consider a bundle map over the identity, $\mathcal{N} : A \oplus A^* \to A \oplus A^*$. This bundle map can be written in the matrix form $\mathcal{N} = \begin{pmatrix} N & \pi \\ \sigma & N_{A^*} \end{pmatrix}$ with $N : A \to A$, $N_{A^*} : A^* \to A^*$, $\pi : A^* \to A$ and $\sigma : A \to A^*$.

Definition 3.1. The operator $\mathcal{N}$ is called *paired* if

$$\langle X + \alpha, \mathcal{N}(Y + \beta)\rangle + \langle \mathcal{N}(X + \alpha), Y + \beta\rangle = 0,$$

for all $X + \alpha, Y + \beta \in A \oplus A^*$, where $\langle \cdot, \cdot\rangle$ is the usual pairing on the double $A \oplus A^*$.

As it is observed in [3], $\mathcal{N}$ is paired if and only if $\pi \in \Gamma(\wedge^2 A)$, $\sigma \in \Gamma(\wedge^2 A^*)$ and $N_{A^*} = -N^*$.

3.1. Paired operators on the double of Lie bialgebroids. Let us now take the Lie bialgebroid (A, A^*), where A^* has the null Lie algebroid structure. In this case, the double $A \oplus A^*$ is the standard Courant algebroid of Example 1.7.

Now we consider, on the sections of $A \oplus A^*$, the bracket deformed by $\mathcal{N}$,

$$\begin{aligned} [\![X + \alpha, Y + \beta]\!]_{\mathcal{N}} = {} & [\![\mathcal{N}(X + \alpha), Y + \beta]\!] + [\![X + \alpha, \mathcal{N}(Y + \beta)]\!] \\ & - \mathcal{N}\, [\![X + \alpha, Y + \beta]\!] \end{aligned}$$

and the Courant-Nijenhuis torsion of $\mathcal{N}$,

$$\mathcal{T}_{\mathcal{N}}(X + \alpha, Y + \beta) := [\![\mathcal{N}(X + \alpha), \mathcal{N}(Y + \beta)]\!] - \mathcal{N}\, [\![X + \alpha, Y + \beta]\!]_{\mathcal{N}} .$$

A simple computation shows that for all $\alpha, \beta \in \Gamma(A^*)$,

$$[\![\alpha, \beta]\!]_{\mathcal{N}} = [\alpha, \beta]_{\pi}.$$

Proposition 3.2. *Let $\mathcal{N}$ be a paired operator on $A \oplus A^*$. If $\mathcal{T}_{\mathcal{N}\,|A^*} = 0$, then the vector bundle A^* is equipped with the Lie algebroid structure A^*_{π}.*

Proof: A straightforward computation shows that

$$\mathcal{T}_{\mathcal{N}}(\alpha, \beta) = 0 \;\Rightarrow\; [\pi^{\#}\alpha, \pi^{\#}\beta] = \pi^{\#}[\alpha, \beta]_{\pi},$$

for all sections α and β of A^*. This means that π is a Poisson bivector on A and the result follows. ■

Now we give sufficient conditions for a paired operator to define a Poisson quasi-Nijenhuis structure on a Lie algebroid.

Theorem 3.3. *Let* $\mathcal{N} = \begin{pmatrix} N & \pi \\ \sigma & -N^* \end{pmatrix}$ *be a paired operator on* $A \oplus A^*$ *such that*

$$N\pi^\# = \pi^\# N^* \quad \textit{and} \quad i_{NX}\sigma = N^*(i_X\sigma), \quad \forall X \in \Gamma(A).$$

If $\mathcal{T}_{\mathcal{N}\,|A^*} = 0$ *and* $\mathcal{T}_{\mathcal{N}\,|A} = 0$*, then* $(A, \pi, N, \mathrm{d}\sigma)$ *is a Poisson quasi-Nijenhuis Lie algebroid.*

Proof: First, notice that the condition $i_{NX}\sigma = N^*(i_X\sigma)$ means that

$$\sigma(NX, Y) = \sigma(X, NY), \quad \forall X, Y \in \Gamma(A),$$

and implies that $N\sigma$ defined by $N\sigma(X, Y) = \sigma(NX, Y)$ is a 2-form on M. The condition $N\pi^\# = \pi^\# N^*$ ensures that $N\pi$ is a bivector field on A.

For all $\alpha, \beta \in \Gamma(A^*)$,

$$\mathcal{T}_{\mathcal{N}}(\alpha, \beta) = 0 \quad \textit{iff} \quad \pi \quad \text{is a Poisson bivector} \quad \text{and} \quad [\alpha, \beta]_\pi^{N^*} = [\alpha, \beta]_{N\pi}.$$

So we have that π and N are compatible. On the other hand, if X and Y are sections of A, then

$$\mathcal{T}_{\mathcal{N}}(X, Y) = 0 \quad \textit{iff} \quad \mathcal{T}_N(X, Y) = \pi^\#(\mathrm{d}\sigma(X, Y, -)) \quad \text{and} \quad \mathrm{d}(N\sigma) = i_N \mathrm{d}\sigma.$$

According to Definition 2.3, $(A, \pi, N, \mathrm{d}\sigma)$ is a Poisson quasi-Nijenhuis Lie algebroid. ■

From Theorem 2.4, we obtain:

Corollary 3.4. $(A^*_\pi, \mathrm{d}_N, \mathrm{d}\sigma)$ *is a quasi-Lie bialgebroid.*

Remark 3.5. We note that a paired operator $\mathcal{N}$ that satisfies $\mathcal{N}^2 = -\mathrm{Id}_{A\oplus A^*}$, also satisfies

$$\langle \mathcal{N}(X + \alpha), \mathcal{N}(Y + \beta)\rangle = \langle X + \alpha, Y + \beta\rangle, \quad \forall X + \alpha, Y + \beta \in \Gamma(A \oplus A^*).$$

In this case $\mathcal{N}$ defines a *generalized complex structure on the Lie algebroid* A. From $\mathcal{N}^2 = -\mathrm{Id}_{A\oplus A^*}$ we deduce that $N\pi^\# = \pi^\# N^*$, $N^2X + \pi^\#(i_X\sigma) = -X$ and $i_{NX}\sigma = N^*(i_X\sigma)$, with $X \in \Gamma(A)$.

Let us denote by $(A \oplus A^*)_{\mathcal{N}}$ the vector bundle map equipped with the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{N}}$ given by

$$\langle X + \alpha, Y + \beta\rangle_{\mathcal{N}} = \langle \mathcal{N}(X + \alpha), \mathcal{N}(Y + \beta)\rangle,$$

the bundle map $\rho_{\mathcal{N}}$ given by $\rho_{\mathcal{N}}(X+\alpha)=a(NX)+\pi^{\#}(\alpha)$ and the bracket $[\![\;,\;]\!]_{\mathcal{N}}$ on its space of sections.

We can now establish a result that generalizes the one of [14], for the case where the Lie algebroid A is TM.

Theorem 3.6. *Let $\mathcal{N}=\begin{pmatrix} N & \pi \\ \sigma & -N^* \end{pmatrix}$ be a paired operator on $A\oplus A^*$ such that $\mathcal{N}^2=-Id_{A\oplus A^*}$. If $\mathcal{T}_{\mathcal{N}\,|A^*}=0$ and $\mathcal{T}_{\mathcal{N}\,|A}=0$, then $(A\oplus A^*)_{\mathcal{N}}$ is a Courant algebroid and it is identified with the double of the quasi-Lie bialgebroid $(A^*_\pi,\mathrm{d}_N,\mathrm{d}\sigma)$.*

Proof: From Corollary 3.4 and Remark 3.5 we have a Courant algebroid $E^{\mathrm{d}\sigma}_{\pi}$ which is the double of the quasi-Lie bialgebroid $(A^*_\pi,\mathrm{d}_N,\mathrm{d}\sigma)$. An easy computation shows that the bracket on $\Gamma(E^{\mathrm{d}\sigma}_{\pi})$ coincides with the bracket $[\![\;,\;]\!]_{\mathcal{N}}$ on $\Gamma((A\oplus A^*)_{\mathcal{N}})$, the anchor of $E^{\mathrm{d}\sigma}_{\pi}$ is $\rho_{\mathcal{N}}$ and the nondegenerate bilinear form on $E^{\mathrm{d}\sigma}_{\pi}$ is exactly $\langle\cdot,\cdot\rangle_{\mathcal{N}}$.

■

3.2. Paired operators on the double of quasi-Lie bialgebroids. Now we consider the quasi-Lie bialgebroid (A^*,d_A,ϕ) of Example 1.2 and the Courant algebroid structure on its double: the standard Courant bracket twisted by ϕ, $[\![\;,\;]\!]^{\phi}$, and the anchor ρ_A. Let $\mathcal{N}=\begin{pmatrix} N & \pi \\ \sigma & -N^* \end{pmatrix}$ be a paired operator and consider the bracket on $\Gamma(A\oplus A^*)$ deformed by $\mathcal{N}$:

$$\begin{aligned}[\![X+\alpha,Y+\beta]\!]^{\phi}_{\mathcal{N}}=&[\![\mathcal{N}(X+\alpha),Y+\beta]\!]^{\phi}+[\![X+\alpha,\mathcal{N}(Y+\beta)]\!]^{\phi}\\ &-\mathcal{N}\,[\![X+\alpha,Y+\beta]\!]^{\phi}.\end{aligned}$$

The Theorem 3.6 admits a direct extension for the case of quasi-Lie bialgebroids.

Theorem 3.7. *Let $\mathcal{N}$ be a paired operator on the double $A\oplus A^*$ of the quasi-Lie bialgebroid (A^*,d_A,ϕ), such that $\mathcal{N}^2=-Id_{A\oplus A^*}$. If $\mathcal{T}_{\mathcal{N}\,|A^*}=0$ and $\mathcal{T}_{\mathcal{N}\,|A}=0$, then $(A\oplus A^*)^{\phi}_{\mathcal{N}}=(A\oplus A^*,[\![\;,\;]\!]^{\phi}_{\mathcal{N}},\rho_{\mathcal{N}},\langle\cdot,\cdot\rangle_{\mathcal{N}})$ is a Courant algebroid and it is identified with the double of the quasi-Lie bialgebroid $(A^*_\pi,\mathrm{d}',\mathrm{d}\sigma+i_N\phi)$, where d' the differential given by $\mathrm{d}'f=\mathrm{d}_N f$ and $\mathrm{d}'\alpha=\mathrm{d}_N\alpha-i_{\pi^{\#}(\alpha)}\phi$, for $f\in\mathcal{C}^\infty(M)$ and $\alpha\in\Gamma(A^*)$.*

Proof: Let $\alpha, \beta \in \Gamma(A^*)$ and $X, Y \in \Gamma(A)$. Then,

$$\mathcal{T}_{\mathcal{N}}(\alpha,\beta) = 0 \quad \textit{iff} \quad \begin{cases} \pi \quad \text{is a Poisson bivector} \\ \\ [\alpha,\beta]_{N\pi} - [\alpha,\beta]_{\pi}^{N^*} = \phi(\pi^{\#}(\alpha), \pi^{\#}(\beta), -) \end{cases}$$

and $\mathcal{T}_{\mathcal{N}}(X,Y) = 0$ *iff*

$$\begin{cases} \mathcal{T}_N(X,Y) = \pi^{\#}((\mathrm{d}\sigma + i_N\phi)(X,Y,-)) - N\pi^{\#}(\phi(X,Y,-)) \\ \\ \mathrm{d}(N\sigma)(X,Y,-) + \phi(X,Y,-) \\ = \phi(NX,NY,-) + \phi(NX,Y,N-) + \phi(X,NY,N-) + (i_N\mathrm{d}\sigma)(X,Y,-). \end{cases}$$

A straightforward generalization for Lie algebroids of the results presented in [15] and in [8] in the case of a manifold, establishes that the four equations corresponding to $\mathcal{T}_{\mathcal{N}|A^*} = 0$ and $\mathcal{T}_{\mathcal{N}|A} = 0$ are equivalent to the vanishing of the Courant-Nijenhuis torsion of $\mathcal{N}$ with respect to the bracket $[\![\ ,\ ]\!]^{\phi}$. Therefore, we have a new Courant algebroid structure on the vector bundle $A \oplus A^*$, $(A \oplus A^*)^{\phi}_{\mathcal{N}} = (A \oplus A^*, [\![\ ,\ ]\!]^{\phi}_{\mathcal{N}}, \rho_{\mathcal{N}}, \langle \cdot, \cdot \rangle_{\mathcal{N}})$.

The restriction of the bracket $[\![\ ,\ ]\!]^{\phi}_{\mathcal{N}}$ to the sections of A^* is the bracket $[\ ,\ ]_{\pi}$ and since $\mathcal{T}_{\mathcal{N}|A^*} = 0$, we have that A^*_{π} is a Dirac structure of the Courant algebroid $(A \oplus A^*)^{\phi}_{\mathcal{N}}$. On the other hand, the restriction of the bracket $[\![\ ,\ ]\!]^{\phi}_{\mathcal{N}}$ to the sections of A gives

$$[\![X,Y]\!]^{\phi}_{\mathcal{N}} = [X,Y]_N - \pi^{\#}(\phi(X,Y,-)) + \mathrm{d}\sigma(X,Y,-) + i_N\phi(X,Y,-)$$

and the anchor $\rho_{\mathcal{N}}$ restricted to $\Gamma(A)$ is $\rho_A \circ N$. If we consider the bracket

$$[X,Y]' = [X,Y]_N - \pi^{\#}(\phi(X,Y,-))$$

on the sections of A and the bundle map $\rho_A \circ N$, the differential corresponding to this structure on A is d' given by,

$$\mathrm{d}'f = \mathrm{d}_N f \quad \text{and} \quad \mathrm{d}'\alpha = \mathrm{d}_N\alpha - i_{\pi^{\#}(\alpha)}\phi,$$

with $f \in \mathcal{C}^{\infty}(M)$ and $\alpha \in \Gamma(A^*)$.

The vector bundle A is obviously a transversal isotropic complement of A^*, so that $(A^*_{\pi}, \mathrm{d}', \mathrm{d}\sigma + i_N\phi)$ is a quasi-Lie bialgebroid [12]. Finally, a simple computation shows that the double of this quasi-Lie bialgebroid is naturally identified with the Courant algebroid $(A \oplus A^*)^{\phi}_{\mathcal{N}}$. ∎

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