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# QUADRATIC QUASILINEAR EQUATIONS WITH GENERAL SINGULARITIES

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Abstract: We study both existence and nonexistence of nonnegative solutions for nonlinear elliptic problems with singular lower order terms that have natural growth with respect to the gradient, whose model is

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\gamma} = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$, $\gamma > 0$ and $f$ is a function which is strictly positive on every compactly contained subset of $\Omega$. As a consequence of our main results, we prove that the condition $\gamma < 2$ is necessary and sufficient for the existence of solutions in $H_0^1(\Omega)$ for every sufficiently regular $f$ as above.

## 1. Introduction

In this paper we are going to study existence and nonexistence of nonnegative solutions for the following boundary value problem

$$\begin{cases} -\operatorname{div}(M(x,u)\nabla u) + g(x,u)|\nabla u|^2 = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $\Omega$ is a bounded, open subset of $\mathbb{R}^N$, $N \geq 3$, $M(x,s) \stackrel{\text{def}}{=} (m_{ij}(x,s))$, $i,j = 1,\dots,N$ is a symmetric matrix whose coefficients $m_{ij} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions (i.e., $m_{ij}(\cdot,s)$ is measurable on $\Omega$ for every $s \in \mathbb{R}$, and $m_{ij}(x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$) such that there exist constants $0 < \alpha \leq \beta$ satisfying

$$\alpha|\xi|^2 \leq M(x,s)\xi\cdot\xi \text{ and } |M(x,s)| \leq \beta, \qquad \text{for a.e. } x \in \Omega,\ \forall (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \tag{2}$$

The function $g : \Omega \times (0,+\infty) \to \mathbb{R}$ is a Carathéodory function (i.e., $g(\cdot,s)$ is measurable on $\Omega$ for every $s \in (0,+\infty)$, and $g(x,\cdot)$ is continuous on $(0,+\infty)$

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for a.e. $x \in \Omega$) such that

$$g(x,s) \geq 0, \quad \text{for a.e. } x \in \Omega, \ \forall s > 0. \tag{3}$$

We will be mainly interested to the case of a function $g$ which is singular near $s = 0$, such as, for example, $g(x,s) = 1/s^\gamma$, $\gamma > 0$. On the datum $f$, we first suppose that it belongs to $L^{\frac{2N}{N+2}}(\Omega)$ and that it satisfies

$$m_\omega(f) \stackrel{\text{def}}{=} \text{ess inf}\{f(x) : x \in \omega\} > 0, \quad \forall \omega \subset\subset \Omega. \tag{4}$$

Note that (4) implies that $f \geq 0$ in $\Omega$ and that $f \not\equiv 0$ in $\Omega$.

There are several papers concerned with existence and nonexistence of solutions for (1). If $g$ is nonsingular, that is if $g$ is a Carathéodory function on $\Omega \times [0, \infty)$, problem (1) has been exhaustively studied by Boccardo, Murat and Puel [15], Bensoussan, Boccardo and Murat [7] and Boccardo, Gallouët [11] with data $f$ in suitable Lebesgue spaces.

On the contrary, as stated before, in this paper we shall focus our attention to lower order terms $g(x,s)$ having a singularity at $s = 0$. Therefore, we need positive solutions of (1). Specifically, for distributional solution for problem (1), we mean a function $u \in W_0^{1,1}(\Omega)$ which solves the equation in the sense of distributions with $u > 0$ almost everywhere in $\Omega$ and $g(x,u)|\nabla u|^2$ in $L^1(\Omega)$. If moreover $u \in H_0^1(\Omega)$, we say that $u$ is a *finite energy solution* for problem (1).

The model problem for our study is the following:

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\gamma} = f & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Recently, existence of solutions for (5) has been proved in [1, 2, 3] for $0 < \gamma \leq 1$. We also quote the even more recent papers [8] and [20]. Specifically, the existence of positive solutions of (1) is proved in [8] provided $0 \not\equiv f \in L^q(\Omega)$ $(q > N/2)$ with $f \geq 0$ and provided $g(x,s) = 1/s^\gamma$ with $\gamma \leq 1$. On the other hand, if $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{x \in \Omega : u(x) > 0\}$, $0 \leq f \in L^\infty(\Omega)$, $\mu \in \mathbb{R}$ and $\lambda, \gamma > 0$, the different equation

$$-\text{div}\,(M(x,u)\nabla u) + \lambda u + \mu \frac{|\nabla u|^2}{u^\gamma}\chi_{\{u>0\}} = f$$

is studied in [20], where the results about existence of nonnegative solutions in $H_0^1(\Omega)$ depend on $\gamma$. Indeed, existence is proved for every $\mu \in \mathbb{R}$ if $\gamma < 1$, while the case $\gamma \geq 1$ requires that $\mu < 0$. Thus, if $\gamma \geq 1$ the term with quadratic dependence in $\nabla u$ is negative (i.e., the opposite assumption with respect to (3)).

The purpose of this paper is twofold. First of all, we will extend the above results to a more general class of nonlinearities both in the principal part of the operator and in the lower order term, as well as to general, possibly $L^1(\Omega)$, data. Then, we will give a sharp range of nonlinearities $g(x,s)$ for which these problems admit a solution for every datum $f \in L^q(\Omega)$, with $q > N/2$, satisfying (4).

In order to prove our results, we will have to strengthen assumption (3). Specifically, for the results of existence of solutions, we will suppose that the function $g(x,s)$ satisfies

$$0 \le g(x,s) \le h(s), \quad \text{for a.e. } x \in \Omega,\ \forall s > 0, \tag{6}$$

where $h : (0,+\infty) \to [0,+\infty)$ is a continuous nonnegative function such that

$$\begin{gathered} \lim_{s\to 0^+} \int_s^1 \sqrt{h(t)}\,dt < +\infty, \\ h(s) \text{ is nonincreasing in a neighborhood of zero.} \end{gathered} \tag{7}$$

Our result of existence of finite energy solutions (proved in Section 2) is the following.

**Theorem 1.1.** *Let $f$ in $L^{\frac{2N}{N+2}}(\Omega)$ be such that (4) holds, and suppose that (2), (6) and (7) hold. Then there exists a finite energy solution $u$ for problem (1). Furthermore, $u\,g(x,u)|\nabla u|^2 \in L^1(\Omega)$.*

Note that the fact $u\,g(x,u)|\nabla u|^2 \in L^1(\Omega)$ implies that the solution $u$ itself is allowed as test function (since $f \in H^{-1}(\Omega)$) in the weak formulation of (1) (see (18) in Section 2). With respect to the proof, due to the fact that the lower order term $g(x,u)|\nabla u|^2$ is (possibly) singular as the solution is near 0, we will approximate the function $g(x,s)$ by nonsingular ones $g_n(x,s)$ in such a way that the corresponding approximated problems have finite energy solutions $u_n$ for every $n$ in $\mathbb{N}$. The main difficulty in the proof of Theorem 1.1 relies on a suitable local uniform estimate from below of these solutions. To do it, it suffices by (6) to prove that any supersolution $z > 0$ for the equation

$$-\operatorname{div}(M(x,z)\nabla z) + h(z)|\nabla z|^2 = f \quad \text{in } \Omega$$

is above some positive constant in every $\omega \subset\subset \Omega$, i.e.

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega > 0 : \quad z(x) \ge c_\omega > 0. \tag{8}$$

This is proved in Proposition 2.3 via a suitable change of variable which turns the goal into a local $L^\infty$ estimate for solutions of quasilinear problems. The local $L^\infty$ estimate is then obtained using a result of [26] (see also the pioneering paper [17] and also [12, 19]) on an equation whose model is

$$-\div(\widetilde{M}(x,v)\nabla v)+f(x)b(v)=0 \quad \text{in } \Omega, \tag{9}$$

where $\widetilde{M}$ satisfies (2) and $b(s)$ is a function with $b(s)/s$ increasing for large $s>0$ and satisfying the Keller-Osserman condition

$$\int^{+\infty}\frac{dt}{\sqrt{2\int_0^t b(\tau)d\tau}}<+\infty.$$

For the convenience of the reader, the exact result that we need is proved in the appendix (see Theorem A.1). For such type of $L^\infty$ estimates we refer to the "classical" literature on the so-called large solutions (see, among others, [5, 29, 30, 35]) and on local estimates (see, among others, [12, 17, 19, 26, 34]).

Combining the above ideas with those in [32] (see also [25]), we handle the case of data $f$ in a more general Lebesgue space. Indeed, in Section 3, we prove the existence of distributional solutions $u$ of (1), with $u$ in $W_0^{1,q}(\Omega)$ for every $q<\frac{N}{N-1}$. More precisely, we have the following result.

**Theorem 1.2.** *Let $f$ in $L^1(\Omega)$ be such that (4) holds and suppose that (2), (6) and (7) hold. Then there exists a distributional solution $u$ of (1), with $u$ in $W_0^{1,q}(\Omega)$, for every $q<\frac{N}{N-1}$. If, in addition, there exist $s_0>0$ and $\mu>0$ such that*

$$g(x,s)\geq\mu \quad \text{for a.e. } x\in\Omega, \quad \forall s\geq s_0, \tag{10}$$

*then $u\in H_0^1(\Omega)$ (i.e., it is a finite energy solution).*

We are also concerned with nonexistence of positive solutions for problem (1) for data $f$ in $L^q(\Omega)$ for some $q>\frac{N}{2}$, with $f\geq 0$ and $f\not\equiv 0$. In contrast with the previous existence results, we will assume in this case that the nonlinearity $g(x,s)$ is above a function $h(s)$ whose square root is not integrable in $(0,1)$. Specifically, we assume that

$$0\leq h(s)\leq g(x,s), \quad \text{for a.e. } x\in\Omega,\ \forall s>0, \tag{11}$$

where $h:(0,+\infty)\to[0,+\infty)$ is a nonnegative continuous function such that

$$\lim_{s\to 0^+}h(s)=+\infty, \quad \lim_{s\to 0^+}\int_s^1\sqrt{h(t)}\,dt=+\infty, \tag{12}$$

and

$$\lim_{s\to 0^+} \sqrt{h(s)}\, \mathrm{e}^{\displaystyle\int_1^s \sqrt{h(t)}dt} = h_0 \geq 0. \tag{13}$$

Among others, we are going to prove in Section 4 that if $\lambda_1(f)$ denotes the first positive eigenvalue of the laplacian operator $-\Delta$ with zero Dirichlet boundary conditions and weight $f \in L^q(\Omega)$, $(q > N/2)$, then the following result holds.

**Theorem 1.3.** *Let $f$ in $L^q(\Omega)$, with $q > \frac{N}{2}$, be such that $f \geq 0$ and $f \not\equiv 0$, and assume that (2), (11), (12), (13) hold. If $\lambda_1(f) > \frac{\beta}{\alpha}$, then (1) does not have any finite energy solution.*

As an easy consequence of Theorem 1.3, we will prove (see Corollary 4.5) that the model problem (5) does not have any finite energy solution provided $\gamma \geq 2$. By gathering together this nonexistence result and Theorem 1.1 we conclude immediately that, in the case of the model problem (5), we have a sharp range of values of $\gamma$ for which there exist solutions. In addition, if $\gamma$ is not in this range, we prove also what happens if we try to approximate problem (5) with a sequence of problems for which solutions exist.

**Theorem 1.4.** *Problem* (5) *has a finite energy solution for every $f \in L^q(\Omega)$ ($q > \frac{N}{2}$) satisfying (4) if and only if $\gamma < 2$. Moreover, let $\lambda_1$ be the first eigenvalue of the laplacian in the $N$-dimensional unit ball (i.e. the first positive zero of the Bessel function $J_m$ with $m = N/2-1$), assume $f \in L^\infty(\Omega)$, and either*

$$\gamma > 2 \quad \text{or} \quad \gamma = 2 \text{ and } \|f\|_{L^\infty(\Omega)} < \frac{\lambda_1}{\mathrm{diam}(\Omega)^2}. \tag{14}$$

*Then the sequence $\{u_n\}$ of solutions of*

$$\begin{cases} -\Delta u_n + \dfrac{|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^\gamma} = f & \text{in } \Omega, \\ \qquad u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

*tends to 0 in $H_0^1(\Omega)$, and the sequence $\frac{|\nabla u_n|^2}{\left(u_n+\frac{1}{n}\right)^\gamma}$ converges to $f$ in the weak$*$ topology of measures.*

To conclude this introduction, some remarks are in order. First, we have to mention that uniqueness of solutions for (5) is proved in [4] for the case

$0 < \gamma \le 1$. Secondly, let us explicitly state that we have chosen to present the results and to perform the proofs in the case $N \ge 3$. However, all the results but Theorem 1.1 hold true also in the case $N = 2$ (with easier proofs). In addition, if $N = 2$ (which implies $\frac{2N}{N+2} = 1$), Theorem 1.1 is also true provided we replace the assumption $f \in L^{\frac{2N}{N+2}}(\Omega)$ with $f \in L^m(\Omega)$, and assume $m > 1$.

The plan of the paper is the following: in Section 2 we will prove a local estimate from below for the solutions, together with Theorem 1.1. Section 3 is devoted to the results concerning $L^1$ data (Theorem 1.2) while in Section 4 we prove the nonexistence result (both theorems 1.3 and 1.4). Finally we present in the Appendix some results related to the local estimate (8). For instance, we show in detail how to get the lower bound for solutions of (1), through a suitable change of variable, proving a local bound from above for solutions of a semilinear equation whose model is (9) (Theorem A.1). Such topic is strictly related to the possibility of constructing estimates for solutions of (9) that do not depend on the behavior at the boundary: and indeed in Theorem A.8 we prove the existence of solutions that blow-up at the boundary (i.e., the so-called "large solutions") for such equations.

**Notation**. For any $k > 0$ we set $T_k(s) = \min(k, \max(s, -k))$ and $G_k(s) = s - T_k(s)$. Moreover, for any $q > 1$, $q' = \frac{q}{q-1}$ will be the Hölder conjugate exponent of $q$, while for any $1 < p < N$, $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate exponent of $p$. As usual, $\mathcal{S}$ denotes the best Sobolev constant, i.e.,

$$\mathcal{S} = \sup\{\|u\|_{L^{2^*}(\Omega)} : \|u\|_{H_0^1(\Omega)} = 1\}.$$

In Section 3 we will use some ideas related to Marcinkiewicz spaces; for the convenience of the reader we recall here their definition and some properties. For $s > 1$, we denote by $\mathcal{M}^s(\Omega)$ the space of measurable functions $v : \Omega \to \mathbb{R}$ such that there exists $c > 0$, with

$$\text{meas}\{x \in \Omega : |v(x)| \ge k\} \le \frac{c}{k^s}, \quad \forall k > 0. \tag{15}$$

The space $\mathcal{M}^s(\Omega)$ is a Banach space, and on it can be defined the pseudo-norm

$$\|v\|^s_{\mathcal{M}^s(\Omega)} = \inf\{c > 0 : \text{ (15) holds}\}.$$

We also recall that, since $\Omega$ is bounded, for every $\varepsilon \in (0, s-1]$, there exists a positive constant $C$ such that

$$\begin{aligned} \|v\|_{\mathcal{M}^s(\Omega)} &\leq \|v\|_{L^s(\Omega)}, \quad \forall v \in L^s(\Omega) \\ \|w\|_{L^{s-\varepsilon}(\Omega)} &\leq C\, \|w\|_{\mathcal{M}^s(\Omega)}, \quad \forall w \in \mathcal{M}^s(\Omega). \end{aligned} \tag{16}$$

Finally, following [15], we set $\varphi_\lambda(s) = se^{\lambda s^2}$, $\lambda > 0$; in what follows we will use that for every $a$, $b > 0$ we have

$$a\varphi_\lambda'(s) - b|\varphi_\lambda(s)| \geq 1, \tag{17}$$

if $\lambda > \frac{b^2}{4a^2}$. We will also denote by $\varepsilon(n)$ any quantity that tends to 0 as $n$ diverges.

## 2. Finite energy solutions

In this section we will prove the existence of finite energy solutions for problem (1). Let us recall its definition.

**Definition 2.1.** A *distributional supersolution (resp. subsolution)* for problem (1) is a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that

1) $u > 0$ almost everywhere in $\Omega$,
2) $g(x,u)|\nabla u|^2$ belongs to $L^1(\Omega)$,
3) for every $0 \leq \phi \in C^\infty_c(\Omega)$, it holds

$$\int_\Omega M(x,u)\nabla u \cdot \nabla\phi + \int_\Omega g(x,u)|\nabla u|^2\phi \underset{(\leq)}{\geq} \int_\Omega f\,\phi\,.$$

A function $u \in W^{1,1}_0(\Omega)$ is a *distributional solution* for (1) if it is both a supersolution and a subsolution for such a problem.
If moreover $u \in H^1_0(\Omega)$, we say that $u$ is a *finite energy solution* for problem (1). In this case, we have

$$\int_\Omega M(x,u)\nabla u \cdot \nabla\psi + \int_\Omega g(x,u)|\nabla u|^2\psi = \int_\Omega f\,\psi, \quad \forall \psi \in H^1_0(\Omega) \cap L^\infty(\Omega). \tag{18}$$

The proof of Theorem 1.1 relies on approximating the datum $f \in L^{\frac{2N}{N+2}}(\Omega)$ by its truncature $f_n = T_n(f)$ and the nonlinearity $g$ by a suitable sequence of Carathéodory functions $g_n$ (for $n \in \mathbb{N}$). Specifically, we define

$$g_n(x,s) \stackrel{\text{def}}{=} \begin{cases} g(x,s) & s \geq \dfrac{1}{n}, \\ nh\left(\dfrac{1}{n}\right)\dfrac{s}{h(s)}g(x,s) & 0 < s \leq \dfrac{1}{n}, \\ 0 & s \leq 0. \end{cases}$$

Since $h$ is nonincreasing in a neighborhood of zero, we observe that there exists $n_0 \in \mathbb{N}$, such that $g_n$ satisfies, for a.e. $x \in \Omega$, $\forall s > 0$,

$$\begin{cases} \lim\limits_{n\to+\infty} g_n(x,s) = g(x,s)\,, \\ g_n(x,s) \leq g(x,s)\,, \forall n \geq n_0\,, \\ g_n(x,s) \geq 0\,. \end{cases} \tag{19}$$

Since for fixed $n$ both functions $f_n(x)$ ($x \in \Omega$) and $\frac{|\xi|^2}{1+\frac{1}{n}|\xi|^2}$ ($\xi \in \mathbb{R}^N$) are bounded, classical results (see [28] and [33]) allow us to deduce, that problem

$$\begin{cases} -\mathrm{div}\,(M(x,u_n)\nabla u_n) + g_n(x,u_n)\dfrac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

has a solution $u_n$ that belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$.

We are going to prove now some properties of the sequence $u_n$ that we will use in the sequel.

**Lemma 2.2.** *Assume that* $0 \not\equiv f \in L^{\frac{2N}{N+2}}(\Omega)$ *satisfies* $f \geq 0$ *and that* $M(x,s)$ *satisfies (2). If, for every* $n \in \mathbb{N}$*, the function* $u_n \in H_0^1(\Omega)$ *is a solution of problem (20), then:*

1. *The sequence* $\{u_n\}$ *is bounded in* $H_0^1(\Omega)$ *and*

$$u_n g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \text{ is bounded in } L^1(\Omega).$$

2. *The functions* $u_n$ *are continuous in* $\Omega$ *and* $u_n(x) > 0$ *for every* $x \in \Omega$ *and* $n \in \mathbb{N}$.

*Proof*: 1. Taking $u_n$ as test function in (20) and using Hölder and Sobolev inequalities we obtain that

$$\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla u_n + \int_\Omega g_n(x,u_n)u_n\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} = \int_\Omega f_n u_n$$

$$\leq \mathcal{S}\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}.$$

By the ellipticity condition (2) and the nonnegativeness of $g_n(x,s)s$, we conclude that the sequences $u_n$ and $u_n g_n(x,u_n)\dfrac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}$ are bounded, respectively, in $H_0^1(\Omega)$ and in $L^1(\Omega)$.

2. We take $u_n^- \stackrel{\text{def}}{=} \min(u_n,0)$ as test function in (20), so that, by (2),

$$\alpha\int_\Omega |\nabla u_n^-|^2 + \int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}u_n^- \leq \int_\Omega f_n\, u_n^- .$$

Using that $f_n \geq 0$ and $g_n(x,s)$ is zero for every $s\leq 0$, we obtain

$$\alpha\int_\Omega |\nabla u_n^-|^2 \leq \int_\Omega f_n\, u_n^- \leq 0.$$

Thus $u_n^- \equiv 0$ and so $u_n \geq 0$. Moreover,

$$-\mathrm{div}\,(M(x,u_n)\nabla u_n) = f_n - g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \in L^\infty(\Omega).$$

Hence $u_n$ belongs to the space of the Hölder continuous functions in $\Omega$ (see for instance [24], Theorem 1.1 in Chapter 4).

We are now going to prove that $u_n > 0$ in $\Omega$. Let $C_n > 0$ be such that $g_n(x,s) \leq C_n s$, for $s \in [0, \|u_n\|_\infty]$. Thus the nonnegative function $u_n$ satisfies in $\Omega$

$$-\mathrm{div}\,(M(x,u_n)\nabla u_n) + nC_n u_n \geq$$

$$-\mathrm{div}\,(M(x,u_n)\nabla u_n) + g_n(x,u_n)\tfrac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} = f_n\,.$$

Observing that $f_n$ is nonnegative and not identically zero (since $f\not\equiv 0$), by the strong maximum principle (see [22] for instance) we deduce that $u_n > 0$ in $\Omega$. ■

In the next proposition we will prove that the sequence $\{u_n\}$ is uniformly bounded from below, away from zero, in every compact set in $\Omega$. This result will be crucial in order to prove the existence of a solution for (1).

**Proposition 2.3.** *Suppose that $f \in L^\infty_{\rm loc}(\Omega)$ satisfies (4), and that $g(x,s)$ satisfies (6) (with $h$ such that (7) holds). Let $\omega$ be a compactly contained open subset of $\Omega$. Then there exists a constant $c_\omega > 0$ such that every supersolution $0 < z \in H^1_{\rm loc}(\Omega) \cap C(\Omega)$ of the equation*

$$-{\rm div}\,(M(x,z)\nabla z) + h(z)|\nabla z|^2 = f \quad \text{in } \Omega, \tag{21}$$

*satisfies*

$$z \geq c_\omega \quad \text{in } \omega.$$

**Remark 2.4.** The above proposition will be crucial in the proofs of both Theorem 1.1 and 1.2. In fact, we will use the following consequences:

(i) Let $u_n$ be a solution of (20) with $n \geq n_0$ ($n_0$ given by (19)). Since the inequalities $g_n(x,s) \leq g(x,s) \leq h(s)$ for every $s > 0$ and $f_n \geq f_1$ imply that $u_n > 0$ in $\Omega$ (see Lemma 2.2), $u_n$ is a supersolution for

$$-{\rm div}\,(M(x,z)\nabla z) + h(z)|\nabla z|^2 = f_1 \quad \text{in } \Omega.$$

Therefore, by the above proposition (with $f = f_1$ and $z = u_n \in H^1_0(\Omega) \cap C(\overline{\Omega})$ (Lemma 2.2-2.)) for any $\omega \subset\subset \Omega$ we deduce the existence of a positive constant $c_\omega$ such that $u_n \geq c_\omega$ in $\omega$. Taking $k > 0$ and $m_0 > \max\{n_0, \frac{1}{c_\omega}\}$, we deduce, by the definition of $g_n$, that for all $n \geq m_0$

$$g_n(x,u_n(x)) = g(x,u_n(x)) \leq c_k(\omega) \stackrel{\rm def}{=} \max_{s\in[c_\omega,k]} h(s)\,,$$

for every $x \in \omega$ such that $u_n(x) \leq k$.

(ii) If $0 < u_n \in H^1_{\rm loc}(\Omega) \cap C(\Omega)$ is a solution of

$$-{\rm div}\,(M(x,u_n)\nabla u_n) + g(x,u_n)|\nabla u_n|^2 = f_n \quad \text{in } \Omega,$$

then, using again that $g(x,s) \leq h(s)$ and $f_n \geq f_1$, we derive that $u_n$ is also a supersolution of

$$-{\rm div}\,(M(x,z)\nabla z) + h(z)|\nabla z|^2 = f_1 \quad \text{in } \Omega.$$

Consequently, if $\omega \subset\subset \Omega$ and $c_\omega$ has been defined above (with $f = f_1$), then $u_n \geq c_\omega$ in $\omega$. Therefore,

$$g(x,u_n(x)) \leq c_k(\omega) \stackrel{\rm def}{=} \max_{s\in[c_\omega,k]} h(s)\,,$$

for every $x \in \omega$ such that $u_n(x) \leq k$.

*Proof of Proposition 2.3*: Let $z > 0$ be a supersolution of (21). We are going to consider a suitable change of variable. In order to make it, since in general the function $h$ may be integrable in $(0,1)$, we set $\widetilde{h}(s) = h(s) + \dfrac{\alpha}{s}$, and define, for $s > 0$, the nondecreasing function

$$H(s) = \int_1^s \widetilde{h}(t)dt = \int_1^s h(t)dt + \log s^\alpha,$$

and the nonincreasing function

$$\psi(s) = \int_s^1 \mathrm{e}^{-\frac{H(t)}{\alpha}} dt = \int_s^1 t^{-1} \mathrm{e}^{-\frac{\int_1^t h(\tau)d\tau}{\alpha}} dt.$$

Observing that

$$\lim_{s\to 0^+} \psi(s) = +\infty, \qquad \lim_{s\to+\infty} \psi(s) = \psi_\infty \in [-\infty, 0),$$

we can define

$$v \stackrel{\text{def}}{=} \psi(z).$$

Since $z$ is continuous and strictly positive in $\Omega$, we get that $z$ is bounded away from zero (with the bound depending on $z$) in every open set $\omega$ compactly contained in $\Omega$. Consequently, by the chain rule, we have

$$\nabla v = -\mathrm{e}^{-\frac{H(z)}{\alpha}} \nabla z \in L^2(\omega) \quad \forall \omega \subset\subset \Omega, \tag{22}$$

and thus $v \in H^1(\omega)$ for every $\omega \subset\subset \Omega$, i.e., $v \in H^1_{\mathrm{loc}}(\Omega)$.

Let $0 \le \phi \in C_c^\infty(\Omega)$, and take $\mathrm{e}^{-\frac{H(z)}{\alpha}}\phi$ as test function in (21) to deduce from the inequality $h(s) \le \widetilde{h}(s)$ that

$$-\int_\Omega M(x,z)\nabla z\cdot\nabla z\,\frac{\widetilde{h}(z)}{\alpha}\mathrm{e}^{-\frac{H(z)}{\alpha}}\phi + \int_\Omega M(x,z)\nabla z\cdot\nabla\phi\,\mathrm{e}^{-\frac{H(z)}{\alpha}} + \int_\Omega \widetilde{h}(z)|\nabla z|^2 \mathrm{e}^{-\frac{H(z)}{\alpha}}\phi \ge \int_\Omega f \mathrm{e}^{-\frac{H(z)}{\alpha}}\phi,$$

and using (2) together with (22) we get,

$$-\int_\Omega M(x,z)\nabla\psi(z)\cdot\nabla\phi \ge \int_\Omega f\mathrm{e}^{-\frac{H(z)}{\alpha}}\phi \ge \int_\Omega \left(\mathrm{e}^{-\frac{H(z)}{\alpha}} - 1\right) f\phi.$$

If we define $\widetilde{M}(x,s) = M(x,\psi^{-1}(s))$ and $b(s) = \mathrm{e}^{-\frac{H(\psi^{-1}(s))}{\alpha}} - 1$ for every $s \in (\psi_\infty, +\infty)$, then $v$ is subsolution of

$$-\mathrm{div}\,(\widetilde{M}(x,v)\nabla v) + f(x)\,b(v) = 0 \quad \text{in } \Omega.$$

Observe that $\frac{b(s)}{s}$ is nondecreasing for large $s > 0$; indeed, this is equivalent to prove that $\Upsilon(t) = \frac{\mathrm{e}^{-\frac{H(t)}{\alpha}}-1}{\psi(t)}$ is nonincreasing in a neighborhood of $t = 0$. To show this, let $w_0 \in (0,1)$ be such that $\widetilde{h}(t)$ is nonincreasing in $(0, w_0]$, and, note that

$$-\mathrm{e}^{\frac{H(t)}{\alpha}}\psi^2(t)\Upsilon'(t) = \frac{\widetilde{h}(t)}{\alpha}\psi(t) - (\mathrm{e}^{-\frac{H(t)}{\alpha}} - 1) = \int_t^1 \frac{[\widetilde{h}(t) - \widetilde{h}(s)]}{\alpha}\mathrm{e}^{-\frac{H(s)}{\alpha}}ds$$

$$\geq \int_{w_0}^1 \frac{[\widetilde{h}(t) - \widetilde{h}(s)]}{\alpha}\mathrm{e}^{-\frac{H(s)}{\alpha}}ds = \widetilde{h}(t)M_1 - M_2$$

where

$$M_1 = \frac{1}{\alpha}\int_{w_0}^1 \mathrm{e}^{-\frac{H(s)}{\alpha}}ds \quad \text{and} \quad M_2 = \frac{1}{\alpha}\int_{w_0}^1 \widetilde{h}(s)\mathrm{e}^{-\frac{H(s)}{\alpha}}ds.$$

Thus, if $t$ belongs to the interval $(0, \widetilde{h}^{-1}(\min\{w_0, M_2/M_1\}))$, then the right hand side of the above inequality is positive, and consequently $\Upsilon(t)$ is nonincreasing in this interval.

We also claim now that since $\int_0^1 \sqrt{h(s)}ds < +\infty$ and $h$ is nonincreasing in a neighborhood of zero, then the function $b(s)$ satisfies the well-known Keller-Osserman condition (see [23] and [31] for instance), i.e., there exists $t_0 > 0$ such that

$$\int_{t_0}^{+\infty} \frac{dt}{\sqrt{2\int_0^t b(s)ds}} < +\infty. \tag{23}$$

We postpone the proof of the claim for the moment, and we show how to conclude the proof by using the claim. Indeed, by applying [26, Theorem 7] (see also Theorem A.1 in the Appendix where, for the convenience of the reader, we have also included a proof of the precise result that we need here) we derive that for every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that

$$v \leq C_\omega \quad \text{in } \omega.$$

Therefore, undoing the change

$$z \geq \psi^{-1}(C_\omega) = c_\omega > 0 \quad \text{in } \omega\,,$$

as desired.

Consequently, to conclude the proof it suffices to show (23) or, equivalently, that

$$\int_{t_0}^{+\infty} \frac{dt}{\sqrt{2\int_0^t \mathrm{e}^{-\frac{H(\psi^{-1}(s))}{\alpha}} ds}} < +\infty.$$

Using the change $\tau = \psi^{-1}(s)$, we obtain

$$\int_{t_0}^{+\infty} \frac{dt}{\sqrt{2\int_0^t \mathrm{e}^{-\frac{H(\psi^{-1}(s))}{\alpha}} ds}} = \int_{t_0}^{+\infty} \frac{dt}{\sqrt{2\int_{\psi^{-1}(t)}^{\psi^{-1}(0)} \mathrm{e}^{-2\frac{H(\tau)}{\alpha}} d\tau}}.$$

Now we apply the change $w = \psi^{-1}(t)$ to deduce that

$$\int_{t_0}^{+\infty} \frac{dt}{\sqrt{2\int_0^t \mathrm{e}^{-\frac{H(\psi^{-1}(s))}{\alpha}} ds}} \le \int_0^{w_0} \frac{dw}{\sqrt{2\int_w^{w_0} \mathrm{e}^{\frac{2}{\alpha}[H(w)-H(\tau)]} d\tau}},$$

with $0 < w_0 = \psi^{-1}(t_0) < 1 = \psi^{-1}(0)$ since $\psi$ is nonincreasing, and we choose $t_0 >> 1$ such that $h$ is nonincreasing in $(0, w_0]$.

Since $h$ satisfies (7), also $\widetilde{h}$ satisfies it, so that we conclude the proof if we show that there exists a positive contant $c_0$ such that

$$\widetilde{h}(w) \int_w^{w_0} \mathrm{e}^{\frac{2}{\alpha}[H(w)-H(\tau)]}\, d\tau \ge c_0 > 0, \quad \forall w \in (0, w_0). \tag{24}$$

Indeed, the only difficulty is near zero. To overcome it, we use that $h$ (hence $\widetilde{h}$) is nonincreasing in $(0, w_0]$, to obtain

$$\begin{aligned} \widetilde{h}(w) \int_w^{w_0} \mathrm{e}^{\frac{2}{\alpha}[H(w)-H(\tau)]}\, d\tau &\ge \int_w^{w_0} \widetilde{h}(\tau) \mathrm{e}^{\frac{2}{\alpha}[H(w)-H(\tau)]}\, d\tau \\ &= -\frac{\alpha \mathrm{e}^{\frac{2}{\alpha}H(w)}}{2} \int_w^{w_0} -\frac{2}{\alpha}\widetilde{h}(\tau)\mathrm{e}^{-\frac{2}{\alpha}H(\tau)}\, d\tau \\ &= -\frac{\alpha \mathrm{e}^{\frac{2}{\alpha}H(w)}}{2} \left[\mathrm{e}^{-\frac{2}{\alpha}H(\tau)}\right]_w^{w_0} = -\frac{\alpha}{2}\frac{\mathrm{e}^{\frac{2}{\alpha}H(w)}}{\mathrm{e}^{\frac{2}{\alpha}H(w_0)}} + \frac{\alpha}{2}. \end{aligned}$$

Using the above inequality and the fact that $\mathrm{e}^{\frac{2}{\alpha}H(w)}$ is close to zero for $w$ small enough, we can choose $\overline{w} \in (0, w_0)$ such that

$$\widetilde{h}(w) \int_w^{w_0} \mathrm{e}^{\frac{2}{\alpha}[H(w)-H(\tau)]}\, d\tau \ge \frac{\alpha}{4},$$

for $0 < w < \overline{w}$. Thus the existence of $c_0$ such that (24) holds is deduced. ■

**Remark 2.5.** If $h$ is such that

$$\lim_{s\to 0^+}\int_s^1 h(t)\,dt = +\infty,$$

there is no need to define the above function $\widetilde{h}$. Indeed, in this case, the proof of the above theorem works by using directly $h$ instead of $\widetilde{h}$.

*Proof of Theorem 1.1*: We are going to prove that, up to a subsequence, the sequence $\{u_n\}$ of finite energy solutions of (20) converges to a finite energy solution of (1).

By Case 1. of Lemma 2.2 we obtain

$$\|u_n\|_{H_0^1(\Omega)} \le C_1, \quad \int_\Omega u_n g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \le C_2. \tag{25}$$

Thus, up to a subsequence, we can assume that $u_n$ converges to some $u \in H_0^1(\Omega)$ weakly in $H_0^1(\Omega)$ and, by Rellich's Theorem, strongly in $L^2(\Omega)$ and a.e. in $\Omega$.

Choosing $\frac{1}{\varepsilon}T_\varepsilon(u_n)$ as test function in (20) and taking into account that $f_n \le f$ in $\Omega$, we deduce that

$$\int_\Omega \frac{T_\varepsilon(u_n)}{\varepsilon} g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \le \int_\Omega f_n \le \int_\Omega f\,.$$

If we take the limit as $\varepsilon$ tends to zero, and we use that, by Lemma 2.2, $u_n > 0$ in $\Omega$, we get

$$\int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} = \int_{\{u_n>0\}} g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \le \int_\Omega f\,. \tag{26}$$

The proof will be concluded by proving the following steps:
**Step 1.** For every $k > 0$, $T_k(u_n) \to T_k(u)$ strongly in $H^1_{\mathrm{loc}}(\Omega)$.
**Step 2.** $u_n$ is strongly convergent in $H^1_{\mathrm{loc}}(\Omega)$.
**Step 3.** We pass to the limit in (20).

**Step 1.** Here we want to prove that

$$\lim_{n\to+\infty}\int_\Omega |\nabla(T_k(u_n)-T_k(u))|^2\phi = 0, \quad \forall \phi \in C_c^\infty(\Omega) \text{ with } \phi \ge 0. \tag{27}$$

Considering the function $\varphi_\lambda(s)$ defined in (17) and taking $\varphi_\lambda(T_k(u_n) - T_k(u))\phi$ as test function in (20), we have

$$\int_\Omega M(x,u_n)\nabla u_n \cdot \nabla(T_k(u_n) - T_k(u))\varphi_\lambda'(T_k(u_n) - T_k(u))\phi$$
$$+ \int_\Omega M(x,u_n)\nabla u_n \cdot \nabla\phi\, \varphi_\lambda(T_k(u_n) - T_k(u))$$
$$+ \int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\varphi_\lambda(T_k(u_n) - T_k(u))\phi = \int_\Omega f_n\, \varphi_\lambda(T_k(u_n) - T_k(u))\phi\,.$$

Since $T_k(u_n) \to T_k(u)$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, we note that

$$\int_\Omega f_n\, \varphi_\lambda(T_k(u_n) - T_k(u))\phi - \int_\Omega M(x,u_n)\nabla u_n \cdot \nabla\phi\, \varphi_\lambda(T_k(u_n) - T_k(u)) = \varepsilon(n).$$

Moreover, choosing $\omega_\phi \subset\subset \Omega$ with $\operatorname{supp}\phi \subset \omega_\phi$, we deduce, by Case (i) of Remark 2.4 and by the nonnegativeness of both $g_n$ and $\varphi_\lambda(k - T_k(u))$, that

$$\int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\varphi_\lambda(T_k(u_n) - T_k(u))\phi$$
$$\geq \int_{\{u_n\leq k\}} g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\varphi_\lambda(T_k(u_n) - T_k(u))\phi$$
$$\geq -c_k(\omega_\phi)\int_\Omega |\nabla T_k(u_n)|^2|\varphi_\lambda(T_k(u_n) - T_k(u))|\phi\,.$$

Thus

$$\begin{aligned}&\int_\Omega M(x,u_n)\nabla u_n \cdot \nabla(T_k(u_n) - T_k(u))\varphi_\lambda'(T_k(u_n) - T_k(u))\phi\\ &\quad -c_k(\omega_\phi)\int_\Omega |\nabla T_k(u_n)|^2|\varphi_\lambda(T_k(u_n) - T_k(u))|\phi \leq \varepsilon(n).\end{aligned} \tag{28}$$

Note that

$$\int_\Omega M(x,u_n)\nabla u_n \cdot \nabla(T_k(u_n) - T_k(u))\varphi_\lambda'(T_k(u_n) - T_k(u))\phi\chi_{\{u_n\geq k\}}$$
$$= -\int_\Omega M(x,u_n)\nabla u_n \cdot \nabla T_k(u)\varphi_\lambda'(k - T_k(u))\phi\chi_{\{u_n\geq k\}} = \varepsilon(n),$$

so that, adding

$$-\int_\Omega M(x,u_n)\nabla T_k(u) \cdot \nabla(T_k(u_n) - T_k(u))\varphi_\lambda'(T_k(u_n) - T_k(u))\phi = \varepsilon(n)$$

in both sides of (28) and since

$$\begin{aligned}
&\int_\Omega |\nabla T_k(u_n)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))|\phi \\
&\le 2\int_\Omega |\nabla(T_k(u_n) - T_k(u))|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))|\phi \\
&\quad +2\int_\Omega |\nabla T_k(u)|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))|\phi \\
&= 2\int_\Omega |\nabla(T_k(u_n) - T_k(u))|^2 |\varphi_\lambda(T_k(u_n) - T_k(u))|\phi + \varepsilon(n),
\end{aligned}$$

we find, using also (2), (we omit the arguments of the functions $\varphi_\lambda$ and $\varphi'_\lambda$ for brevity)

$$\int_\Omega |\nabla(T_k(u_n) - T_k(u))|^2 \Big[\alpha\varphi'_\lambda - 2c_k(\omega_\phi)|\varphi_\lambda|\Big]\phi \le \varepsilon(n).$$

Choosing $\lambda$ such that (17) holds with $a = \alpha$ and $b = 2c_k(\omega_\phi)$, we obtain (27).

**Step 2.** We prove now that the sequence $u_n$ is strongly convergent in $H^1_{\mathrm{loc}}(\Omega)$.

Let us choose $G_k(u_n)$ as test function in (20) and drop the positive integral involving the lower order term. By using (2), and Hölder and Sobolev inequalities, we have

$$\int_\Omega |\nabla G_k(u_n)|^2 \le \frac{\mathcal{S}^2}{\alpha^2}\left(\int_{\{u_n \ge k\}} f^{\frac{2N}{N+2}}\right)^{1+\frac{2}{N}},$$

and the right hand side of the previous inequality is arbitrarily small if $k$ is large enough. Therefore, $|\nabla G_k(u_n)|^2$ is equiintegrable. Moreover by Lemma 1 of [9] (see also [14]), we deduce that, up to (not relabeled) subsequences, $\nabla u_n$ converges to $\nabla u$ a.e. in $\Omega$, so that by Vitali theorem

$$G_k(u_n) \to G_k(u) \qquad \text{in } H^1_0(\Omega).$$

Combining this and Step 1 we deduce that

$$u_n \to u \qquad \text{in } H^1_{\mathrm{loc}}(\Omega).$$

**Step 3.** Let us observe that, by applying Fatou lemma in (25) and (26), we deduce that

$$\int_\Omega u g(x,u)|\nabla u|^2 \le C_2 \quad \text{and} \quad \int_\Omega g(x,u)|\nabla u|^2 \le \int_\Omega f\,,$$

respectively. Therefore, to conclude the proof we only have to prove that $u$ is a distributional solution of the problem (1). We begin by passing to the limit on $n$ in the equation satisfied by $u_n$, i.e., in

$$\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla\phi+\int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\phi=\int_\Omega f_n\phi\,,\ \forall\phi\in C_c^\infty(\Omega).$$

First of all, the weak convergence of $u_n$ to $u$ and the weak-$*$ convergence of $M(x,u_n)$ to $M(x,u)$ in $L^\infty(\Omega)$ implies that

$$\lim_{n\to+\infty}\int_\Omega M(x,u_n)\nabla u_n\nabla\phi=\int_\Omega M(x,u)\nabla u\nabla\phi\,,\quad\forall\phi\in C_c^\infty(\Omega). \tag{29}$$

On the other hand, if we fix $\omega\subset\subset\Omega$, then, by Remark 2.4,

$$g_n(x,u_n(x))\le c_k(\omega),\quad\forall n>>1,\ \text{and}\ \forall x\in\omega\ \text{satisfying}\ u_n(x)\le k.$$

Consequently, if $E\subset\subset\omega$ we have

$$\begin{aligned}&\int_E|g_n(x,u_n(x))|\frac{|\nabla u_n(x)|^2}{1+\frac{1}{n}|\nabla u_n(x)|^2}\\ &\le\int_{E\cap\{u_n\le k\}}g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}+\int_{E\cap\{u_n\ge k\}}g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\\ &\le c_k(\omega)\int_{E\cap\{u_n\le k\}}|\nabla T_k(u_n)|^2+\int_{\{u_n\ge k\}}g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\,.\end{aligned} \tag{30}$$

Let $\varepsilon>0$ be fixed. Observe that if, for $k>1$, we use $T_1(G_{k-1}(u_n))$ as test function in (20) and drop positive terms, we deduce that

$$\int_{\{u_n\ge k\}}g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\le\int_{\{u_n\ge k-1\}}f_n\le\int_{\{u_n\ge k-1\}}f\,.$$

Thus, since the right hand side tends to 0 uniformly in $n$ as $k$ diverges, we obtain the existence of $k_0>1$ such that

$$\int_{\{u_n\ge k\}}g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\le\frac{\varepsilon}{2},\quad\forall k\ge k_0,\ \forall n\in\mathbb{N}.$$

Moreover, since $T_k(u_n)$ is strongly compact in $H^1_{\mathrm{loc}}(\Omega)$, there exist $n_\varepsilon$, $\delta_\varepsilon$ such that for every $E\subset\subset\Omega$ with $\mathrm{meas}\,(E)<\delta_\varepsilon$ we have

$$\int_{E\cap\{u_n\le k\}}|\nabla T_k(u_n)|^2<\frac{\varepsilon}{2c_k(\omega)},\quad\forall n\ge n_\varepsilon.$$

In conclusion, by (30), taking $k \geq k_0$ we see that meas$(E) < \delta_\varepsilon$ implies

$$\int_E |g_n(x, u_n(x))| \frac{|\nabla u_n(x)|^2}{1 + \frac{1}{n}|\nabla u_n(x)|^2} \leq \varepsilon, \quad \forall n \geq n_\varepsilon.$$

i.e., the sequence $g_n(x, u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}$ is equiintegrable. This, together with its a.e. convergence to $g(x, u)|\nabla u|^2$, implies by Vitali theorem that

$$\lim_{n \to +\infty} \int_\Omega g_n(x, u_n) \frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2} \phi = \int_\Omega g(x, u)|\nabla u|^2 \phi \quad \forall \phi \in C_c^\infty(\Omega).$$

Therefore, using the above limit, (29) and since $f_n$ tends to $f$ strongly in $L^1(\Omega)$ we conclude that

$$\int_\Omega M(x, u)\nabla u \nabla \phi + \int_\Omega g(x, u)|\nabla u|^2 \phi = \int_\Omega f\phi \quad \forall \phi \in C_c^\infty(\Omega).$$

■

**Remark 2.6.** In addition, if $f \in L^q(\Omega)$ with $q > N/2$, then the solution $u$ given by Theorem 1.1 is continuous in $\Omega$. Indeed, by using $\psi = T_m(G_k(u))$, with $m > k$, as test function in (18), it is easy to adapt the idea of Stampacchia ([33]) in order to obtain that $u \in L^\infty(\Omega)$. Now, consider a function $\zeta \in C^\infty(\Omega)$ with $0 \leq \zeta(x) \leq 1$, for every $x \in \Omega$ and compact support in a ball $B_\rho$ of radius $\rho > 0$, and set $A_{k,\rho} = \{x \in K_\rho \cap \Omega : u(x) > k\}$. Following the idea of the proof of Theorem 1.1 of Chapter 4 in [24], take $\phi = G_k(u)\zeta^2$ as test function in (18) to deduce by (2) and Hölder's inequality that

$$\int_{A_{k,\rho}} |\nabla u|^2 \zeta^2 \leq \frac{\|f\|_{L^q(\Omega)}\|u\|_{L^\infty(\Omega)}}{\alpha} (\text{meas } A_{k,\rho})^{1-\frac{1}{q}} + 2\frac{\beta}{\alpha} \int_{A_{k,\rho}} |\nabla u||\nabla \zeta|\zeta G_k(u).$$

Using again Young's inequality we get

$$\int_{A_{k,\rho}} |\nabla u|^2 \zeta^2 \leq \frac{2\|f\|_{L^q(\Omega)}\|u\|_{L^\infty(\Omega)}}{\alpha} (\text{meas } A_{k,\rho})^{1-\frac{1}{q}} + \frac{4\beta}{\alpha^2} \int_{A_{k,\rho}} |\nabla \zeta|^2 G_k^2(u).$$

In particular, if for $\sigma \in (0, 1)$ we choose $\zeta$ such that it is constantly equal to 1 in the concentric ball $B_{\rho-\sigma\rho}$ (to $B_\rho$) of radius $\rho - \sigma\rho$ and $|\nabla \zeta| < \frac{1}{\sigma\rho}$, we obtain

$$\int_{A_{k,\rho-\sigma\rho}} |\nabla u|^2 \leq \gamma \left(1 + \frac{1}{\sigma^2 \rho^{2(1-\frac{N}{2q})}} \max_{A_{k,\rho}} (u-k)^2\right) (\text{meas } A_{k,\rho})^{1-\frac{1}{q}},$$

where $\gamma = \max\left\{\frac{2\|f\|_{L^q(\Omega)}\|u\|_{L^\infty(\Omega)}}{\alpha}, \frac{4\beta}{\alpha^2}\omega_N^{\frac{1}{q}}\right\}$ with $\omega_N$ denoting the measure of the unit ball of $\mathbb{R}^N$.

This means that for $\delta > 0$ small enough and every $M \geq \|u\|_{L^\infty(\Omega)}$, the function $u$ belongs to the class $\mathcal{B}_2(\Omega, M, \gamma, \delta, \frac{1}{2q})$ with $2q > N$ (see [24], pag. 81). Applying Theorem 6.1 of [24] we deduce that $u$ is Hölder continuous in $\Omega$.

# 3. Existence for data in $L^1(\Omega)$

In this section we prove Theorem 1.2. In this case, taking advantage of Theorem 1.1, we approximate problem (1) by

$$\begin{cases} -\mathrm{div}\,(M(x,u_n)\nabla u_n) + g(x,u_n)|\nabla u_n|^2 = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{31}$$

where $f_n = T_n(f)$.

Note that the existence of a nonnegative finite energy solution $u_n \in H_0^1(\Omega)\cap C(\Omega)$ such that $g(x,u_n)|\nabla u_n|^2 \in L^1(\Omega)$ follows from Theorem 1.1 and Remark 2.6.

**Lemma 3.1.** *If $f \in L^1(\Omega)$ satisfies (4), $g(x,s)$ satisfies (6) (with $h(s)$ such that (7) holds), and $u_n$ is a solution of (31), then*

(i) *the sequence $u_n$ is bounded in $\mathcal{M}^{\frac{N}{N-2}}(\Omega)$ and $|\nabla u_n|$ is bounded in $\mathcal{M}^{\frac{N}{N-1}}(\Omega)$;*
(ii) *up to subsequences, the sequence $u_n$ is weakly convergent to some $u$ in $W_0^{1,q}(\Omega)$ for every $q \in [1, \frac{N}{N-1})$;*
(iii) *for any $k > 0$ and for any $\omega \subset\subset \Omega$,*

$$T_k(u_n) \to T_k(u) \quad \text{in} \quad H^1(\omega).$$

*Proof*: (i) Taking $T_k(u_n)$ as test function in (31) and using (2), we have

$$\alpha\int_\Omega |\nabla T_k(u_n)|^2 + \int_\Omega g(x,u_n)T_k(u_n)|\nabla u_n|^2 \leq k\|f_n\|_{L^1(\Omega)}.$$

Since $0 \leq f_n \leq f$ and $g(x,u_n) \geq 0$, we have

$$\alpha\int_\Omega |\nabla T_k(u_n)|^2 \leq k\|f\|_{L^1(\Omega)}. \tag{32}$$

Standard estimates (see [6, Lemmas 4.1 and 4.2]) imply that $u_n$ is bounded in $\mathcal{M}^{\frac{N}{N-2}}(\Omega)$ and that $|\nabla u_n|$ is bounded in $\mathcal{M}^{\frac{N}{N-1}}(\Omega)$.
(ii) Let $1 \le q < \frac{N}{N-1}$. By the preceding case and by the embedding (16), we deduce that $u_n$ is bounded in $W_0^{1,q}(\Omega)$ and thus, passing to a subsequence if necessary, there exists $u$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$.
(iii) Our aim is to show that

$$\lim_{n\to+\infty} \int_\Omega |\nabla[T_k(u_n) - T_k(u)]|^2 \phi = 0\,, \qquad \forall \phi \in C_c^\infty(\Omega)\,,\ \phi \ge 0.$$

Here we adapt to our case a technique to obtain the strong convergence of truncations first introduced in [25] (see also [32]). Let us choose $\varphi_\lambda(w_n)\phi$ as test function in (31) where $\varphi_\lambda(s)$ has been defined in (17) and

$$w_n = T_{2k}[u_n - T_l(u_n) + T_k(u_n) - T_k(u)], \qquad 0 < k < l.$$

Thus we have

$$\begin{aligned} &\int_\Omega M(x,u_n)\nabla u_n \cdot \nabla w_n \varphi_\lambda'(w_n)\phi + \int_\Omega M(x,u_n)\nabla u_n \cdot \nabla\phi\varphi_\lambda(w_n) \\ &\quad + \int_\Omega g(x,u_n)|\nabla u_n|^2 \varphi_\lambda(w_n)\phi = \int_\Omega f_n\,\phi\varphi_\lambda(w_n)\,. \end{aligned} \tag{33}$$

Observing that $\nabla T_k(u_n) = 0$ if $u_n > k$ and $\nabla w_n \equiv 0$ if $u_n \ge 2k + l \equiv \mathcal{K}$ (we recall that $l > k$), we have

$$\begin{aligned} &\int_\Omega M(x,u_n)\nabla u_n \cdot \nabla w_n \varphi_\lambda'(w_n)\phi \\ &\quad = \int_\Omega M(x,u_n)\nabla T_k(u_n) \cdot \nabla(T_k(u_n) - T_k(u))\varphi_\lambda'(w_n)\phi \\ &\qquad + \int_{\{u_n \ge k\}} M(x,u_n)\nabla T_{\mathcal{K}}(u_n) \cdot \nabla T_{2k}(G_l(u_n) + k - T_k(u))\varphi_\lambda'(w_n)\phi\,. \end{aligned}$$

Moreover, using that

$$\begin{aligned} \nabla T_{\mathcal{K}}(u_n) \cdot \nabla(G_l(u_n) - T_k(u)) &= \nabla T_{\mathcal{K}}(u_n) \cdot \nabla G_l(u_n) - \nabla T_{\mathcal{K}}(u_n)\nabla T_k(u) \\ &\ge -\nabla T_{\mathcal{K}}(u_n) \cdot \nabla T_k(u), \end{aligned}$$

we have

$$\int_{\{u_n>k\}\cap\{G_l(u_n)+k-T_k(u)\le 2k\}} M(x,u_n)\nabla T_{\mathcal{K}}(u_n)\cdot\nabla(G_l(u_n)-T_k(u))\varphi_\lambda'(w_n)\phi$$
$$\ge -\int_{\{G_l(u_n)+k-T_k(u)\le 2k\}} |M(x,u_n)\nabla T_{\mathcal{K}}(u_n)\cdot\nabla T_k(u)|\varphi_\lambda'(w_n)\phi\chi_{\{u_n>k\}}\,,$$

and thus, since the above integral tends to zero as $n$ diverges,

$$\begin{aligned}&\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla w_n\varphi_\lambda'(w_n)\phi\\ &\qquad\ge\int_\Omega M(x,u_n)\nabla T_k(u_n)\cdot\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(w_n)\phi+\varepsilon(n).\end{aligned}\tag{34}$$

On the other hand, since $G_l(u_n)+k-T_k(u)\ge 0$,

$$\int_\Omega g(x,u_n)|\nabla u_n|^2\varphi_\lambda(w_n)\phi\ \ge\int_{\{u_n\le k\}} g(x,u_n)|\nabla u_n|^2\varphi_\lambda(w_n)\phi\ .$$

Thanks to Case (ii) of Remark 2.4 applied to a subset $\omega_\phi\subset\subset\Omega$ with $\operatorname{supp}\phi\subset\omega_\phi$, we have $g(x,u_n(x))\le c_k(\omega_\phi)$ for every $x\in\omega$ with $u_n(x)\le k$. Then, we get

$$\begin{aligned}&\left|\int_{\{u_n\le k\}} g(x,u_n)|\nabla u_n|^2\varphi_\lambda(T_k(u_n)-T_k(u))\phi\right|\\
&\le c_k(\omega_\phi)\int_\Omega|\nabla T_k(u_n)|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\\
&\le 2c_k(\omega_\phi)\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\\
&\qquad+2c_k(\omega_\phi)\int_\Omega|\nabla T_k(u)|^2|\varphi_\lambda(T_k(u_n)-T_k(u))|\phi\,.\end{aligned}$$

Note that the last integral tends to 0 as $n$ diverges since $\varphi_\lambda(T_k(u_n)-T_k(u))$ converges to zero in the weak-$*$ topology of $L^\infty(\Omega)$ and $T_k(u)\in H_0^1(\Omega)$.

Therefore, we deduce from this, (33) and (34) that

$$\int_\Omega M(x,u_n)\nabla T_k(u_n)\cdot\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(w_n)\phi$$

$$-2c_k(\omega_\phi)\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2|\varphi_\lambda(w_n)|\phi$$

$$\leq\int_\Omega f_n\phi\varphi_\lambda(w_n)\ -\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla\phi\varphi_\lambda(w_n)\ +\varepsilon(n),$$

and adding to both sides of the previous inequality

$$-\int_\Omega M(x,u_n)\nabla T_k(u)\cdot\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(w_n)\phi\ =\varepsilon(n),$$

we find from (2),

$$\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2\Big[\alpha\varphi'_\lambda(w_n)-2c_k(\omega_\phi)|\varphi_\lambda(w_n)|\Big]\phi$$
$$\leq\int_\Omega f_n\phi\varphi_\lambda(w_n)\ -\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla\phi\,\varphi_\lambda(w_n)\ +\varepsilon(n).$$

Choosing $\lambda$ such that $\varphi_\lambda$ satisfies (17) with $a=\alpha$ and $b=2c_k(\omega_\phi)$, we get

$$\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2\phi$$

$$\leq\int_\Omega f_n\phi\varphi_\lambda(w_n)\ -\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla\phi\,\varphi_\lambda(w_n)\ +\varepsilon(n).$$

Moreover, $w_n$ a.e. (and weakly-$*$ in $L^\infty(\Omega)$) converges to $w=T_{2k}(G_l(u))$ and thus, recalling that $\nabla u_n\to\nabla u$ weakly in $(L^q(\Omega))^N$, $q<N/(N-1)$,

$$\lim_{n\to+\infty}\int_\Omega f_n\phi\varphi_\lambda(w_n)\ -\int_\Omega M(x,u_n)\nabla u_n\cdot\nabla\phi\varphi_\lambda(w_n)$$

$$=\int_\Omega f\phi\varphi_\lambda(w)\ -\int_\Omega M(x,u)\nabla u\cdot\nabla\phi\varphi_\lambda(w)\,.$$

Consequently, using (2)

$$\int_\Omega|\nabla(T_k(u_n)-T_k(u))|^2\phi\leq\int_\Omega f\phi\varphi_\lambda(w)\ -\int_\Omega M(x,u)\nabla u\cdot\nabla\phi\,\varphi_\lambda(w)\ +\varepsilon(n)$$

$$\leq \varphi_\lambda(2k)\int_{\{u\geq l\}}(f+\beta|\nabla u||\nabla\phi|)\,+\varepsilon(n).$$

Since the last integral tends to zero as $l$ diverges, (iii) is proved. ■

Now, we prove our main result concerning $L^1(\Omega)$ data:

*Proof of Theorem 1.2*: We begin by proving the first part of the theorem, i.e. that there exists a solution $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, of problem (1). We first observe that we deduce from the results of [14] that $\nabla u_n \to \nabla u$ a.e., and from Lemma 3.1 the estimates on $u_n$ and $|\nabla u_n|$ in $\mathcal{M}^{\frac{N}{N-2}}(\Omega)$ and $\mathcal{M}^{\frac{N}{N-1}}(\Omega)$ respectively. Thus $u_n \to u$ strongly in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Arguing as in the proof of Theorem 1.1, we can show that, choosing $\frac{1}{\varepsilon}T_\varepsilon(u_n)$ as test function in (31) and applying Fatou lemma, we have $g(x,u)|\nabla u|^2 \in L^1(\Omega)$.

In order to prove that for all $\omega \subset\subset \Omega$, $\{g(x,u_n)|\nabla u_n|^2\}$ is strongly convergent in $L^1(\omega)$ to $g(x,u)|\nabla u|^2$, it suffices to show the local uniform equiintegrability of such sequence. To prove the claim, we choose $T_1(G_{k-1}(u_n))$ (for $k > 1$) as test function in the equation (31) and we deduce, by dropping the first positive term (in virtue of (2)), and since $f_n \leq f$, that

$$\int_{\{u_n\geq k\}} g(x,u_n)|\nabla u_n|^2 \leq \int_{\{u_n\geq k-1\}} f\,. \tag{35}$$

By a similar argument to the one used in Step 3 of the proof of Theorem 1.1, we prove the claim. Indeed, let $E \subset \omega \subset\subset \Omega$ be a measurable set. By Remark 2.4-(ii) and (35), we have, $\forall k > 1$,

$$\begin{aligned}\int_E g(x,u_n)|\nabla u_n|^2 &= \int_{E\cap\{u_n\leq k\}} g(x,u_n)|\nabla u_n|^2\\
&+\int_{E\cap\{u_n\geq k\}} g(x,u_n)|\nabla u_n|^2 \leq c_k(\omega)\int_{E\cap\{u_n\leq k\}}|\nabla T_k(u_n)|^2\\
&+\int_{\{u_n\geq k\}} g(x,u_n)|\nabla u_n|^2 \leq c_k(\omega)\int_E|\nabla T_k(u_n)|^2+\int_{\{u_n\geq k-1\}} f\,.\end{aligned}$$

Since meas $(\{x\in\Omega\,:\,u_n\geq k-1\})$ tends to zero (uniformly with respect to $n$) as $k$ tends to $+\infty$ (because of the boundedness of $\{u_n\}$ in the space $\mathcal{M}^{N/(N-2)}(\Omega)$ by Lemma 3.1-(ii)), we obtain that the last integral in the above inequalities tends to zero as $k$ goes to $+\infty$. This, and the local equiintegrability of $|\nabla T_k(u_n)|^2$ (by Lemma 3.1-(iii)), then show the local equiintegrability of $\{g(x,u_n)|\nabla u_n|^2\}$.

Using moreover that $\nabla u_n \to \nabla u$ a.e., we conclude by Vitali theorem that

$$g(x,u_n)|\nabla u_n|^2 \to g(x,u)|\nabla u|^2 \quad \text{in } L^1(\omega)\,,\ \forall \omega \subset\subset \Omega\,. \tag{36}$$

Now, using (36) and the strong convergence of $\nabla u_n$ to $\nabla u$ in $(L^q(\Omega))^N$, for every $q < \frac{N}{N-1}$, we can pass to the limit in (31) to show that $u$ is a solution for (1).

In order to prove the second part of the theorem, we simply note that we can fix $k \geq \max\{s_0, 1\}$ so that (10) and (35) imply

$$\mu \int_\Omega |\nabla G_k(u_n)|^2 = \mu \int_{\{u_n \geq k\}} |\nabla u_n|^2 \leq \int_{\{u_n \geq k-1\}} f \leq \|f\|_{L^1(\Omega)}. \tag{37}$$

Hence, taking into account both (32) and (37), we have

$$\int_\Omega |\nabla u_n|^2 = \int_\Omega |\nabla T_k(u_n)|^2 + \int_\Omega |\nabla G_k(u_n)|^2 \leq \left(\frac{k}{\alpha} + \frac{1}{\mu}\right) \|f\|_{L^1(\Omega)},$$

i.e., the boundedness of the sequence $\{u_n\}$ in $H_0^1(\Omega)$. This implies that the solution $u$, which is the limit of (a subsequence of) $\{u_n\}$, belongs to $H_0^1(\Omega)$. ■

**Remark 3.2.** Actually, if (10) holds, it is possible to prove, in this latter case, that the approximate sequence $u_n$ is strongly convergent to $u$ in $H^1(\omega)$, for every $\omega \subset\subset \Omega$. Indeed, due to the a.e. convergence of $\nabla u_n$ to $\nabla u$ in $\Omega$, it suffices to check the equiintegrability of $|\nabla u_n|^2$ in every $\omega \subset\subset \Omega$. To do that, we take a measurable set $E \subset \omega \subset\subset \Omega$, and we observe that, thanks to (37), for any $k \geq \max\{s_0, 1\}$, we can write

$$\begin{aligned} \int_E |\nabla u_n|^2 &= \int_E |\nabla T_k(u_n)|^2 + \int_E |\nabla G_k(u_n)|^2 \\ &\leq \int_E |\nabla T_k(u_n)|^2 + \frac{1}{\mu} \int_{\{u_n \geq k-1\}} f\,. \end{aligned} \tag{38}$$

Therefore, using again both the boundedness of $u_n$ in $\mathcal{M}^{\frac{N}{N-2}}(\Omega)$ and the equiintegrability of $|\nabla T_k(u_n)|^2$ in $\omega$ given by Lemma 3.1, we see that (38) yields the desired result.

## 4. Nonexistence results

This section is devoted to study nonexistence of solutions for (1). We begin by observing that if the function $g(x,s)$ satisfies condition (11) with $h$ such that (12) and (13) hold, then we can change $h$ by a smaller function $\overline{h}$ which, in addition to (12) and (13), also satisfies $\overline{h}(s)=0$ for every $s>1$. Indeed, if $s_0$ is the point where $h$ attains its minimum value in $[\frac{1}{2},1]$, then it suffices to define

$$\overline{h}(s)=\begin{cases}(h(s)-h(s_0))^+ & \text{if } s\in(0,s_0],\\ 0 & \text{if } s>s_0.\end{cases}$$

Consequently, without loss of generality, we will assume in the following that condition (11) holds with $h$ satisfying (12), (13), and

$$h(s)=0,\quad \forall s\geq 1. \tag{39}$$

Let us consider the function $G:(0,+\infty)\to(0,+\infty)$ given by

$$G(s)=\mathrm{e}^{\displaystyle\int_1^s \frac{h(t)}{\beta}dt}\qquad \text{for every } s>0,$$

where $\beta$ is given by (2). Observe that, by (12), the function $G$ can be continuously extended to $[0,+\infty)$ setting $G(0)=0$. Moreover, we also define the function $\sigma:[0,+\infty)\to[0,+\infty)$ by setting $\sigma(0)=0$ and

$$\sigma(s)=\mathrm{e}^{\displaystyle\int_1^s \sqrt{h(t)}dt}\qquad \text{for every } s>0.$$

Observe that, thanks to (12) and (13), we have that $\sigma\in C^1([0,+\infty))$, $\sigma'(0)=h_0$ and $\sigma(s)=0$ if and only if $s=0$. As a consequence of (39), $\sigma(s)=1$ for every $s>1$ and $\sigma(s)\leq 1$ for every $s\geq 0$.

**Lemma 4.1.** *Assume (12) and (13). Then the function*

$$\varphi(s)=\begin{cases}\dfrac{\displaystyle\int_0^s G(t)[\sigma'(t)]^2dt}{G(s)} & \textit{if } s>0,\\[2ex] 0 & \textit{if } s=0,\end{cases} \tag{40}$$

*is a continuously differentiable function on* $[0,+\infty)$ *that satisfies the ordinary differential equation*

$$\begin{cases} \varphi'(s) + \dfrac{h(s)}{\beta}\varphi(s) = [\sigma'(s)]^2, \quad \text{on } [0,+\infty), \\ \varphi(0) = 0. \end{cases} \tag{41}$$

*Moreover, the following inequality holds:*

$$\varphi(s) \le \beta[\sigma(s)]^2, \quad \forall s > 0. \tag{42}$$

*Proof.* The first part of the proof is straightforward except for checking that $\varphi$ is differentiable at zero and $\varphi'$ is continuous at zero. In order to do it, we note firstly that $\varphi$ is continuous at zero. Indeed, since $G$ is nondecreasing and $[\sigma']^2$ is continuous in $[0,+\infty)$ we have

$$0 \le \lim_{s\to 0^+}\varphi(s) = \lim_{s\to 0^+}\frac{\displaystyle\int_0^s G(t)[\sigma'(t)]^2dt}{G(s)} \le \lim_{s\to 0^+}\int_0^s [\sigma'(t)]^2dt = 0.$$

Now we observe that, using the L'Hôpital Rule, (12) and (13),

$$\begin{aligned} \varphi'(0) &= h_0^2 - \lim_{s\to 0^+}\frac{h(s)\displaystyle\int_0^s G(t)\,[\sigma'(t)]^2\,dt}{\beta G(s)} \\ &= h_0^2 - h_0^2\lim_{s\to 0^+}\frac{G(s)[\sigma'(s)]^2}{2\beta\sigma(s)\sigma'(s)G(s) + h(s)[\sigma(s)]^2G(s)} \\ &= h_0^2 - h_0^2\lim_{s\to 0^+}\frac{1}{2\beta\frac{1}{\sqrt{h(s)}} + 1} = h_0^2 - h_0^2 = 0. \end{aligned}$$

Hence $\varphi$ is differentiable at zero and $\varphi'$ is continuous at zero.

In order to prove inequality (42), we first observe that since $[\sigma'(s)]^2 = [\sigma(s)]^2\,h(s)$, then

$$\varphi(s) = \frac{\beta}{G(s)}\int_0^s G(t)\,\frac{h(t)}{\beta}\,[\sigma(t)]^2\,dt.$$

Since

$$G(t)\,\frac{h(t)}{\beta} = \frac{d}{dt}\,G(t),$$

we can integrate by parts to find (recall that $G(0) = \sigma(0) = 0$)

$$\begin{aligned}\varphi(s) &= \frac{\beta}{G(s)}\ \left[G(t)\,[\sigma(t)]^2\right]_{t=0}^{t=s} - \frac{2\beta}{G(s)}\int_0^s G(t)\,\sigma(t)\,\sigma'(t)\,dt\\ &= \beta\,[\sigma(s)]^2 - \frac{2\beta}{G(s)}\int_0^s G(t)\,[\sigma(t)]^2\sqrt{h(t)}\,dt\\ &\le \beta\,[\sigma(s)]^2,\end{aligned}$$

since all the functions in the last integral are nonnegative. ■

*Proof of Theorem 1.3*: Let $u \in H_0^1(\Omega)$ be a positive solution for (1) and let $\varphi \in C^1([0,+\infty))$ be given by (40). Observing that $\varphi(0) = 0$, that $\varphi'$ is bounded and that, by (42) and since $\sigma(s) \le 1$, we have $\varphi(s) \le \beta$, we derive that $\varphi(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, we can take $v = \varphi(u)$ as test function in (18) to obtain, by using (11), that

$$\int_\Omega M(x,u)\nabla u\cdot\nabla u\,\varphi'(u)\ + \int_\Omega h(u)|\nabla u|^2\varphi(u)\ \le \int_\Omega f\,\varphi(u)\ .$$

Thus, adding and subtracting $\displaystyle\frac{1}{\beta}\int_\Omega M(x,u)\nabla u\cdot\nabla u\,h(u)\varphi(u)$ , we derive from (2) and (41) that

$$\begin{aligned}\int_\Omega M(x,u)\nabla u\cdot\nabla u[\sigma'(u)]^2 \ &\le\ \int_\Omega M(x,u)\nabla u\cdot\nabla u\left[\varphi'(u) + \frac{h(u)}{\beta}\varphi(u)\right]\\ &\quad + \int_\Omega\left[I - \frac{M\,(x,u)}{\beta}\right]\nabla u\cdot\nabla u\,h(u)\varphi(u)\\ &\le\ \int_\Omega f\,\varphi(u)\ .\end{aligned}$$

Using now (2), (42) and the fact that $f \ge 0$, we have

$$\alpha\int_\Omega|\nabla\sigma(u)|^2\ = \alpha\int_\Omega|\nabla u|^2[\sigma'(u)]^2\ \le \int_\Omega f\,\varphi(u)\ \le \beta\int_\Omega f\,[\sigma(u)]^2\ . \tag{43}$$

Hence, recalling (see [18]) that, since $f$ belongs to $L^q(\Omega)$ with $q > \frac{N}{2}$, and $f^+ \not\equiv 0$, the first positive eigenvalue $\lambda_1(f)$ of the eigenvalue boundary value problem

$$\begin{cases} -\Delta u = \lambda\,f\,u & \text{in } \Omega,\\ \quad u = 0 & \text{on } \partial\Omega,\end{cases}$$

is such that

$$\lambda_1(f) \int_\Omega f\, v^2 \le \int_\Omega |\nabla v|^2, \quad \forall v \in H_0^1(\Omega).$$

we deduce from (43) that

$$\alpha \int_\Omega |\nabla \sigma(u)|^2 \le \frac{\beta}{\lambda_1(f)} \int_\Omega |\nabla \sigma(u)|^2.$$

Recalling the assumption $\frac{\beta}{\alpha} < \lambda_1(f)$, this implies that

$$\int_\Omega |\nabla \sigma(u)|^2 = 0,$$

which yields

$$\sigma(u) = 0 \quad \text{for a.e. } x \in \Omega.$$

Therefore, recalling that $\sigma(s) = 0$ if and only if $s = 0$, we have $u \equiv 0$, contradicting $u > 0$ in $\Omega$: therefore, there are no positive solutions of (1). ■

**Remark 4.2.** Theorem 1.3 can be extended to more general operators. Specifically, if $a(x, s, \xi)$ is a Carathéodory function such that

$$\exists \alpha > 0 \,:\, a(x, s, \xi) \cdot \xi \ge \alpha |\xi|^2 \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

$$\exists \beta > 0 \,:\, |a(x, s, \xi)| \le \beta |\xi| \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,$$

and $0 \le f \in L^q(\Omega)$ with $q > \frac{N}{2}$ and $f \not\equiv 0$, then problem

$$\begin{cases} -\mathrm{div}\,(a(x, u, \nabla u)) + g(x, u)|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has no finite energy solutions provided $\lambda_1(f) > \dfrac{\beta}{\alpha}$ and conditions (11), (12) and (13) hold.

**Remark 4.3.** Let $0 \le f \in L^q(\Omega)$ with $q > \frac{N}{2}$ and $f \not\equiv 0$. Assume (2) and that $g(s)$ satisfies (11). Observe that if $u \in H_0^1(\Omega)$ is a solution of (1), and $R > 0$, then $v = Ru$ is a solution of

$$\begin{cases} -\mathrm{div}\left(M\left(x, \dfrac{v}{R}\right) \nabla v\right) + \dfrac{1}{R} g\left(x, \dfrac{v}{R}\right) |\nabla v|^2 = R\, f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$\frac{g\left(x, \frac{s}{R}\right)}{R} \geq h_R(s) \stackrel{\text{def}}{=} \frac{1}{R} h\left(\frac{s}{R}\right).$$

Therefore, by Theorem 1.3, and since $\lambda_1(Rf) = \lambda_1(f)/R$, if $h_R(s)$ satisfies conditions (12) and (13), then a necessary condition for the existence of finite energy solutions of (1) is that $\lambda_1(f) \leq R\beta/\alpha$.

In the following result, as a consequence of Theorem 1.3 (and Remark 4.3), we give conditions to assure the nonexistence of solutions of (1) for every datum $f$.

**Corollary 4.4.** *Let $0 \leq f \in L^q(\Omega)$ with $q > \frac{N}{2}$ and $f \not\equiv 0$. Assume (2) and that $g(s)$ satisfies (11). If there exists $R_0 > 0$ such that the function $h_R(s) = \dfrac{1}{R} h\left(\dfrac{s}{R}\right)$ satisfies (12) and (13) for every $R \in (0, R_0)$, then (1) does not have any finite energy solution.* ■

As a consequence of the above results we also have the following.

**Corollary 4.5.** *Let $0 \leq f \in L^q(\Omega)$ with $q > \frac{N}{2}$ and $f \not\equiv 0$. Suppose that (2) holds and that for some constants $s_0, \Lambda > 0$ and $\gamma \geq 2$ we have*

$$\frac{\Lambda}{s^\gamma} \leq g(x, s), \qquad \text{for a.e. } x \in \Omega,\ \forall s \in (0, s_0].$$

*If either*

(i) $\gamma > 2$,

*or*

(ii) $\gamma = 2$ *and* $\lambda_1(f) > \frac{\beta}{\Lambda\alpha}$,

*then (1) does not have any finite energy solution.*

*Proof.* Consider a continuous function $h(s)$ such that

$$h(s) = \begin{cases} \dfrac{\Lambda}{s^\gamma} & \text{if } 0 < s \leq \dfrac{s_0}{2}, \\ \leq \dfrac{\Lambda}{s^\gamma} & \text{if } \dfrac{s_0}{2} < s < s_0, \\ 0 & \text{if } s_0 \leq s. \end{cases}$$

Observing that $h_R(s) = \dfrac{\Lambda R^{\gamma-1}}{s^\gamma}$ for every $s \in (0, \frac{s_0}{2})$, and using that $\gamma \geq 2$, we have that $h_R(s)$ is not integrable in $(0, \frac{s_0}{2})$, i.e., it satisfies (12).

In addition, if $\gamma > 2$, then $h_R(s)$ satisfies (13) for every $R > 0$, so that Corollary 4.4 concludes the proof in this case.

On the other hand, if we assume that $\gamma = 2$, then

$$\begin{aligned}\lim_{s\to 0^+} \sqrt{h_R(s)}\, \mathrm{e}^{\displaystyle\int_1^s \sqrt{h_R(t)}dt} &= \lim_{s\to 0^+} \frac{\sqrt{\Lambda R}}{s}\, \mathrm{e}^{\displaystyle\int_{s_0/2}^s \frac{\sqrt{\Lambda R}}{s}dt + \int_1^{s_0/2} \sqrt{h_R(t)}dt} \\ &= \lim_{s\to 0^+} C s^{\sqrt{\Lambda R}-1} = h_0 \geq 0 \iff R \geq \frac{1}{\Lambda}.\end{aligned}$$

In other words, $h_R(s)$ satisfies (13) if and only if $R \geq \frac{1}{\Lambda}$. Therefore, Remark 4.3 implies the nonexistence of solutions provided that $\lambda_1(f) > \frac{\beta}{\Lambda\alpha}$. ■

As a consequence of this result, we have that the first part of Theorem 1.4 is proved. We are now going to prove the second part of it.

*Proof of Theorem 1.4*: We first note that if $\gamma < 2$, then Theorem 1.2 guarantees the existence of a solution. Conversely, if $\gamma > 2$ or if $\gamma = 2$ and $\|f\|_{L^\infty(\Omega)}$ is large enough, Theorem 1.3 applies and no solutions exist for (5).

On the other hand, if $f \in L^\infty(\Omega)$ and (14) holds, we recall that existence and uniqueness of a solution $u_n$ in $H_0^1(\Omega) \cap C(\overline{\Omega})$ for

$$\begin{cases} -\Delta u_n + \dfrac{|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^\gamma} = f & \text{in } \Omega, \\ \qquad u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{44}$$

(with $\gamma \geq 2$) follows by the results of [4, 16]. Taking $u_n$, $G_k(u_n)$, and $T_\varepsilon(u_n)/\varepsilon$ as test functions and working as in Lemma 2.2 (1.), it is easy to see that $u_n$ is bounded in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$, and that

$$\int_\Omega \frac{|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^\gamma} \leq C.$$

Therefore, up to subsequences, there exists a nonnegative bounded Radon measure $\nu$ such that

$$\frac{|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^\gamma} \text{ converges to } \nu \text{ in the weak-}* \text{ topology of measures.}$$

Since $u_n$ is bounded in $H_0^1(\Omega)$ then it converges, up to subsequences, to some function $u$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in $\Omega$. Moreover, since $f - \frac{|\nabla u_n|^2}{\left(u_n+\frac{1}{n}\right)^\gamma}$ is bounded in $L^1(\Omega)$, the result of [14] yields that (up again to subsequences) $\nabla u_n$ converges to $\nabla u$ almost everywhere in $\Omega$. Then we have, by Fatou lemma, that $\frac{|\nabla u|^2}{u^\gamma}\chi_{\{u>0\}}$ belongs to $L^1(\Omega)$, and that

$$\nu = \frac{|\nabla u|^2}{u^\gamma}\chi_{\{u>0\}} + \nu_0,$$

where $\nu_0$ is a nonnegative bounded Radon measure on $\Omega$. Therefore, $u \in H_0^1(\Omega)$ is a finite energy solution of

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\gamma}\chi_{\{u>0\}} = f - \nu_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note also that since $u_{n+1}$ is a subsolution for (44), we can apply the comparison principle of [4] so that, for every $x \in \Omega$, we have

$$u_n(x) \geq u_{n+1}(x) \geq \ldots \geq u(x),$$

and thus we can assume that $u_n(x)$ is converging to $u(x)$ for every $x \in \Omega$. We claim that $u \equiv 0$, so that $u_n$ converges to zero in $L^2(\Omega)$. Indeed, we divide the proof of this assertion in two steps:

**Step 1.** The case in which $\Omega$ is a ball of radius $R > 0$, $\Omega = B_R$, and $f = T > 0$ is a constant.

**Step 2.** The general case.

**Step 1.** Assume that $\Omega = B_R$ and $f = T > 0$ is a constant. In this case, (14) means that, if $\gamma = 2$, then the first eigenvalue $\lambda_1^{B_R}(T)$ of the Laplacian operator with weight $T$ in $B_R$ is greater than one, i.e., $\lambda_1^{B_R}(T) > 1$. We first observe that $u$ is radially symmetric (and thus continuous for $|x| \neq 0$). Indeed, if we define

$$\psi_n(s) = \int_0^s e^{-H_n(t)}dt, \quad \text{where} \quad H_n(t) = \frac{n^{\gamma-1}}{\gamma-1}\left[1 - (1+nt)^{1-\gamma}\right]$$

and we set $v_n = \psi_n(u_n)$, it is easy to check that $v_n$ is the unique solution of

$$\begin{cases} -\Delta v_n = T\mathrm{e}^{-H_n(\psi_n^{-1}(v_n))} & \text{in } B_R \\ v_n = 0 & \text{on } \partial B_R\,. \end{cases}$$

Since the nonlinearity $0 \leq \mathrm{e}^{-H_n(\psi_n^{-1}(s))}$ is $C^1$, we can apply the result of Gidas, Ni and Nirenberg (see [21]) in order to deduce that $v_n$ is radially symmetric (hence $v_n = v_n(r)$), monotone decreasing with respect to $r$ and such that $v_n'(0) = 0$. Since $\psi_n$ and $H_n$ are smooth and increasing, the functions $u_n$

have the same properties as $v_n$. Passing to the limit with respect to $n$ we deduce that $u$ is radially symmetric and monotone nonincreasing.

We argue by contradiction assuming that $u$ is not identically zero. In this case, using that $u(r)$ is nonincreasing in $(0, R)$,

$$r_1 = \inf\{0 < r \leq R : u(r) = 0\} > 0,$$

and then

$$u \geq c_\varepsilon := u(r_1 - \varepsilon) \quad \text{in } B_{r_1 - \varepsilon}\,.$$

Therefore, repeating the proof of Theorem 1.1, we prove that

$$\lim_{n\to+\infty} \frac{|\nabla u_n|^2}{\left(u_n + \frac{1}{n}\right)^\gamma} = \frac{|\nabla u|^2}{u^\gamma} \quad \text{strongly in } L^1_{\text{loc}}(B_{r_1}),$$

so that $\nu_0$ is zero on $B_{r_1}$ and, by the continuity of $u$ for $r \neq 0$, $u$ is a solution of

$$\begin{cases} -\Delta u + \dfrac{|\nabla u|^2}{u^\gamma} = T & \text{in } B_{r_1}, \\ \qquad u = 0 & \text{on } \partial B_{r_1}, \end{cases}$$

and this contradicts the result of Theorem 1.3 (note that, if $\gamma = 2$, we have $\lambda_1^{B_{r_1}}(T) > \lambda_1^{B_R}(T) > 1$). Therefore $u \equiv 0$.

**Step 2.** $\Omega$ is an open set and $f$ is nonnegative and belongs to $L^\infty(\Omega)$.

By (14), we can fix $R > \operatorname{diam}\Omega$ with $\lambda_1 > \|f\|_\infty R^2$ provided that $\gamma = 2$. Let $v_n$ be also the solution of

$$\begin{cases} -\Delta v_n + \dfrac{|\nabla v_n|^2}{\left(v_n + \frac{1}{n}\right)^\gamma} = \|f\|_{L^\infty(\Omega)} & \text{in } B_R \\ \qquad v_n = 0 & \text{on } \partial B_R. \end{cases}$$

By definition of $\operatorname{diam}\Omega$, we have $\overline{\Omega} \subset B_R$. Our aim is to prove that $v_n$ is a supersolution for (44). Indeed, let $0 \leq \psi \in C_0^\infty(\Omega)$ and we use it as test function in the formulation of $v_n$. Thus

$$\int_{B_R} \nabla v_n \cdot \nabla \psi + \int_{B_R} \frac{|\nabla v_n|^2}{(\frac{1}{n} + u_n)^\gamma}\psi = \int_{B_R} \|f\|_{L^\infty(B_R)}\psi\,,$$

and since the support of $\psi$ is contained in $\Omega$ we deduce

$$\int_\Omega \nabla v_n \cdot \nabla \psi + \int_\Omega \frac{|\nabla v_n|^2}{(\frac{1}{n} + u_n)^\gamma}\psi = \int_\Omega \|f\|_{L^\infty(B_R)}\psi \geq \int_\Omega f\psi$$

for every nonnegative $\psi$ in $H_0^1(\Omega) \cap L^\infty(\Omega)$ (by an easy density argument). Using again the comparison principle of [4], $u_n \le v_n$ in $\Omega$. Now, observing that by the choice of $R$, if $\gamma = 2$, we have

$$\lambda_1^{B_R}(\|f\|_\infty) = \frac{\lambda_1}{R^2\|f\|_\infty} > 1, \tag{45}$$

we are able to apply the previous Step 1, so that $v_n$ tends to 0 strongly in $L^2(B_R)$, which implies that $u_n$ tends to zero in $L^2(\Omega)$ and the claim has been proved.

Finally, we conclude the proof by taking $u_n$ as test function in (44) and dropping the nonnegative quadratic term to deduce that the convergence to zero is strong in $H_0^1(\Omega)$; using this fact in the weak formulation of (44) then yields that $\nu = f$, as desired.

■

**Remark 4.6.** Let us emphasize that the condition $\|f\|_\infty \le \frac{\lambda_1}{(\operatorname{diam}\Omega)^2}$ imposed in assumption (14) for the case $\gamma = 2$ is not optimal. We use it for the sake of simplicity. However, as shown in the proof of Theorem 1.4 (see (45)), a sharper condition can be used in this case. More precisely, if $\gamma = 2$ and there exists a ball $B_R$ of radius $R > 0$ such that $\overline{\Omega} \subset B_R$ and

$$\|f\|_{L^\infty(\Omega)} < \frac{\lambda_1}{R^2},$$

then the result of Theorem 1.4 holds.

# Appendix A. Local a priori estimates and large solutions

We devote this appendix to recall some results concerning the following equation

$$-\div (a(x,u,\nabla u)) + B(x,u) = F(x), \quad x \in \Omega, \tag{46}$$

where $F \in L^1_{\mathrm{loc}}(\Omega)$ and $a(x,s,\xi), B(x,s)$ are Carathéodory functions. Suppose that there exist constants $\beta \ge \alpha > 0$ such that

$$a(x,s,\xi)\cdot\xi \ge \alpha|\xi|^2, \tag{47}$$

$$|a(x,s,\xi)| \le \beta|\xi|, \tag{48}$$

$$(a(x,s,\xi) - a(x,s,\eta))\cdot(\xi-\eta) > 0, \tag{49}$$

for a.e. $x \in \Omega$, for every $s \in \mathbb{R}$ and for every $\xi$, $\eta \in \mathbb{R}^N$, $\xi \ne \eta$.

Our aim is to prove that if the datum $F$ belongs to $L^q_{\rm loc}(\Omega)$ with $q > \frac{N}{2}$ and there exists a continuous nonnegative function $b : [0, +\infty) \to [0, +\infty)$ such that

$$\begin{aligned}
&b(s) \text{ is increasing and satisfies (23), } b(s)/s \text{ is nondecreasing for large } s,\\
&\text{and for every } \omega \subset\subset \Omega \text{ there exists } m_\omega > 0 \text{ such that:}\\
&\forall s \in \mathbb{R}^+, \quad B(x, s) \geq m_\omega b(s) \geq 0 \quad \text{for a.e. } x \in \omega,
\end{aligned} \tag{50}$$

then the subsolutions of equation (46) are uniformly bounded from above in $\omega \subset\subset \Omega$. This result is essentially contained in [26].

**Theorem A.1.** *Suppose that $a(x, s, \xi)$ satisfies (47)–(49), $B(x, s)$ satisfies (50) and assume that $F \in L^q_{\rm loc}(\Omega)$, $q > \frac{N}{2}$. Let $u \in H^1_{\rm loc}(\Omega)$ be any distributional subsolution for (46) such that $B(x, u^+), u^+B(x, u^+) \in L^1_{\rm loc}(\Omega)$. Then for every $\omega \subset\subset \Omega$ there exists $C_\omega > 0$ such that*

$$u(x) \leq C_\omega, \quad \forall x \in \omega\,.$$

In order to prove this theorem, we need the following two lemmas.

**Lemma A.2** (Lemma 1.1 of [26])**.** *Let $b : [0, +\infty) \to [0, +\infty)$ be a continuous function, satisfying the Keller-Osserman condition (23), such that $\frac{b(s)}{s}$ is nondecreasing for large $s$. Then, for any $C > 0$ and $\gamma \geq 0$, there exists a smooth function $\varphi : [0, 1] \longrightarrow [0, 1]$, with $\varphi(0) = \varphi'(0) = 0$, $\varphi(1) = 1$, depending only on $b$, $C$ and $\gamma$, satisfying*

$$t^{\gamma+1}\,\frac{\varphi'(\tau)^2}{\varphi(\tau)} \leq \frac{1}{C}\,t^\gamma\, b(t)\,\varphi(\tau) + 1, \qquad \forall \tau \in (0, 1],\, \forall t \geq 0.$$

**Remarks A.3.** (1) In Lemma 1.1 of [26] it is imposed that $b(s)$ is increasing, $b(0) = 0$, and the function $\frac{b(s)}{s}$ is nondecreasing in $\mathbb{R}^+$. However, it is easy to see that the proof works by using the weaker assumptions of Lemma A.2.

(2) In addition, also in [26], the Keller-Osserman condition is replaced by the following one:

$$\int^{+\infty} \frac{ds}{\sqrt{sb(s)}} < +\infty.$$

Note that, as a consequence of the monotonicity of $b(s)$ for large $s$, the above assumption is equivalent to (23).

Let us recall a local version of a classical result by Stampacchia we will use in the following.

**Lemma A.4** ([33]). *Let $\sigma(j,\rho)\ :[0,+\infty)\times[0,R_0)\longrightarrow\mathbb{R}$ be a function such that $\sigma(\cdot,\rho)$ is nonincreasing and $\sigma(j,\cdot)$ is nondecreasing. Moreover, suppose that there exist $K_0>0$, $\mu 1$, and $C,\nu,\gamma>0$ satisfying*

$$\sigma(j,\rho)\le C\frac{\sigma(k,R)^{\mu}}{(j-k)^{\nu}(R-\rho)^{\gamma}},\quad \forall j>k>K_0,\ \forall\ 0<\rho<R<R_0.$$

*Then for every $\delta\in(0,1)$, there exists $d>0$ such that:*

$$\sigma(K_0+d,(1-\delta)R_0)=0\,,$$

*where* $d^{\nu}=2^{(\nu+\gamma)\frac{\mu}{\mu-1}}C\frac{(\sigma(K_0,R_0))^{\mu-1}}{\delta^{\gamma}R_0^{\gamma}}$.

*Idea of the Proof of Theorem A.1*: The proof of this result is essentially contained in [26], but for the convenience of the reader, we include here the proof of the exact result that we have used in the proof of Proposition 2.3.

Actually we deal with equation

$$-\mathrm{div}(\widetilde{M}(x,u)\nabla u)+f(x)b(u)=0,\quad \text{in }\Omega\,,$$

where $\widetilde{M}(x,s)$ satisfies (2), $b(s)$ satisfies the Keller-Osserman condition (23), $\frac{b(s)}{s}$ is nondecreasing for $s$ large and $f$ satisfies (4). Consequently all the assumptions of the theorem are satisfied. We remind that the functions $\widetilde{M}(x,s)$, $b(s)$ and $f(x)$, appearing in Proposition 2.3, satisfy the above assumptions.

Suppose now that $b(u^+),u^+b(u^+)\in L^1_{\mathrm{loc}}(\Omega)$. We set $\omega\subset\subset\omega'\subset\subset\Omega$ and a cut-off function $\eta(x)$ such that $0\le\eta\le1$ and

$$\eta(x)=\begin{cases}1, & x\in\omega,\\ 0, & x\in\Omega\backslash\omega'.\end{cases}\tag{51}$$

We fix $C=\|\nabla\eta\|^2_{L^{\infty}(\Omega)}\frac{4\beta^2+\alpha^2}{8\alpha m_{\omega'}}$ and we also consider the function $\varphi$ given by Lemma A.2 with $\gamma=1$ and this constant $C$. Note that if $\xi=\sqrt{\varphi(\eta)}$, then $u\xi^2=u\varphi(\eta)\in H^1_0(\Omega)$ and

$$\nabla(u\xi^2)=\begin{cases}\xi^2\nabla u+2\xi u\nabla\xi, & \text{if }\xi(x)>0,\\ 0, & \text{if }\xi(x)=0,\end{cases}\tag{52}$$

a.e. in $\Omega$. Moreover $f(x)u^+b(u^+) \in L^1_{\rm loc}(\Omega)$ and thus $v = G_k(u^+)\xi^2$ is an admissible test function. Using (2), Young's inequality, and (4), we deduce

$$\frac{\alpha}{2}\int_{\omega'}|\nabla G_k(u^+)|^2\xi^2 + m_{\omega'}\int_{\omega'} b(u^+)G_k(u^+)\xi^2 \le \frac{2\beta^2}{\alpha}\int_{\{\xi(x)>0\}}|\nabla\xi|^2 G_k(u^+)^2 \,.$$

Hence,

$$\frac{\alpha}{4}\int_{\omega'}|\nabla[G_k(u^+)\xi]|^2 + m_{\omega'}\int_{\omega'} b(u^+)G_k(u^+)\xi^2 \le \frac{4\beta^2+\alpha^2}{2\alpha}\int_{\omega'}|\nabla\xi|^2 G_k(u^+)^2.$$

Lemma A.2 applied with $\gamma = 1$ together to the monotonicity of $b(s)$ yields

$$\frac{\alpha}{4}\int_{\omega'}|\nabla(G_k(u^+)\xi)|^2 \le C_0 \text{meas}\,\{x\in\omega' : u(x)\ge k\}\,,$$

where $C_0 = \|\nabla\eta\|^2_{L^\infty(\Omega)}\frac{4\beta^2+\alpha^2}{2\alpha}$. We deduce by Sobolev inequality that

$$\left(\int_{\omega}|G_k(u^+)\xi|^{2*}\right)^{\frac{2}{2*}} \le \frac{4\mathcal{S}^2}{\alpha}C_0 \text{ meas}\,\{x\in\omega' : u(x)\ge k\}.$$

Hence, using that if $u(x)\ge j>k$ we have $G_k(u)\ge j-k$, we conclude that

$$(j-k)^2 \text{ meas}\,\{x\in\omega : u(x)\ge j\}^{\frac{2}{2^*}} \le \frac{4\mathcal{S}^2}{\alpha}C_0 \text{ meas}\,\{x\in\omega' : u(x)\ge k\}\,. \quad (53)$$

Now, if $\omega\subset\subset\Omega$ is fixed, we consider $R = \text{dist}\,(\omega,\partial\Omega)/2$ and the sets

$$\omega_r = \{x\in\Omega : \text{dist}\,(x,\omega) < r\}\subset\subset\Omega$$

for every $r\in(0,R]$. Taking $\omega=\omega_r$ and $\omega'=\omega_R$ in (53) and defining

$$\sigma(k,r) = \text{meas}\,\{x\in\omega_r : u(x)\ge k\}$$

we deduce, since we can choose $\eta$ such that $\|\nabla\eta\|_{L^\infty(\Omega)} \le \frac{c}{R-r}$, that there exists $c_1>0$ such that

$$(j-k)^2\sigma(j,r)^{2/2^*} \le c_1\frac{\sigma(k,R)}{(R-r)^2}$$

and the proof is concluded by applying Lemma A.4. ∎

**Remark A.5.** By adding a condition on the function $b(s)$ for negative $s$ and using similar ideas to these ones in the above proof, it is possible to give also a priori estimates of the whole $L^\infty$ norm of the solution in every compact subset $\omega$ of $\Omega$. More precisely, if, in addition to the hypotheses of Theorem A.1, we strengthen (50) by imposing that

$$\begin{array}{l} b(s) \text{ is increasing and satisfies (23), } b(s)/s \text{ is nondecreasing,} \\ \text{ for large } s \text{ and for every } \omega \subset\subset \Omega \text{ there exists } m_\omega > 0 \text{ such that:} \\ \forall s \in \mathbb{R}, \quad B(x,s)\,\mathrm{sign}\, s \geq m_\omega b(|s|) \geq 0 \quad \text{for a.e. } x \in \omega, \end{array} \tag{54}$$

then for every $\omega \subset\subset \Omega$ there exists $C_\omega > 0$ such that

$$|u(x)| \leq C_\omega, \quad \forall x \in \omega\,.$$

Theorem A.1 is an extension to quasilinear equations of the well-known local a priori estimate of Keller [23] and Osserman [31] (see also [5], [29], [30], [34], [35] and the references cited therein) for semilinear operators. This semilinear a priori estimate was the crucial tool in order to prove the existence of a large solution, i.e., a solution $u$ of the semilinear equation satisfying $u = +\infty$ at $\partial\Omega$ in the sense that

$$\lim_{\mathrm{dist}(x,\partial\Omega)\to 0} u(x) = +\infty.$$

Thus, it is natural to ask whether it is also possible to prove the existence of a large solution for (46). Clearly, in this nonlinear framework we have to specify the meaning we give to "infinity"at $\partial\Omega$, since it has no sense pointwise. Actually we will assume such a condition in a weak sense, through a condition on the trace on the boundary of the truncation of the solution. Specifically, our definition of a distributional large solution for equation (46) is the following.

**Definition A.6.** An a.e. finite function $u(x)$ such that $T_k(u) \in H^1(\Omega)$ $\forall k > 0$ is a *distributional large solution* for (46) with $F \in L^1_{\mathrm{loc}}(\Omega)$, if:
i) $|a(x,u,\nabla u)| \in L^1_{\mathrm{loc}}(\Omega)$, $B(x,u) \in L^1_{\mathrm{loc}}(\Omega)$;
ii)

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi + \int_\Omega B(x,u)\varphi = \int_\Omega F\,\varphi\,, \qquad \forall \varphi \in C_c^\infty(\Omega)\,;$$

iii) $\forall k > 0$, $k - T_k(u) \in H^1_0(\Omega)$.

**Remark A.7.** In the above definition, iii) has the meaning of "infinity at $\partial\Omega$". We mention that this definition of explosive boundary condition has already been introduced in [27], for a different class of nonlinear elliptic equations involving nonlinear "coercive" gradient terms.

We conclude by observing that even if not explicitly written in [26], all the estimates that we need in order to prove the existence of large solutions for (46) have been proved and thus we have the following result.

**Theorem A.8.** *Suppose that $a(x,s,\xi)$ and $B(x,s)$ satisfy (47), (48), (49), (54) and*

$$\sup_{|s|\le k} |B(x,s)| \in L^1(\Omega), \quad \forall k>0. \tag{55}$$

*Assume also that $F \in L^1_{\mathrm{loc}}(\Omega)$ with $F^- \in L^1(\Omega)$. Then there exists a distributional large solution for* (46).

*Proof.* We consider the following sequence of problems

$$\begin{cases} -\mathrm{div}\ a(x,u_n,\nabla u_n) + B(x,u_n) = F_n \ \text{ in } \ \Omega, \\ u_n - n \in H^1_0(\Omega), \end{cases}$$

where $F_n = T_n(F)$. Since $B(x,s+n)s \ge 0$ for large $|s|$, the existence of a weak solution $u_n \in H^1(\Omega)\cap L^\infty(\Omega)$ is a consequence of [6] (Theorem 6.1), i.e. $u_n - n \in H^1_0(\Omega)$ and it satisfies

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v + \int_\Omega B(x,u_n)v = \int_\Omega F_n v\,,\ \forall v \in H^1_0(\Omega)\cap L^\infty(\Omega). \tag{56}$$

Observing that for any $n \ge k$, $k - T_k(u_n) \in H^1_0(\Omega)\cap L^\infty(\Omega)$, we can choose $v = k - T_k(u_n)$ as test function in (56) and we obtain,

$$-\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n) + \int_\Omega B(x,u_n)[k-T_k(u_n)] = \int_\Omega F_n[k-T_k(u_n)]\,.$$

Using (47), and (50) and (55) we have:

$$\alpha\int_\Omega |\nabla T_k(u_n)|^2 \le 2k\int_\Omega \sup_{|s|\le k}|B(x,s)| + 2k\|F_n^-\|_{L^1(\Omega)}.$$

Thus, for every $k\in\mathbb{N}$, we can now extract a subsequence (not relabeled) of $\{T_k(u_n)\}_{n\in\mathbb{N}}$ that weakly converges in $H^1(\Omega)$ and, by Rellich theorem, strongly in $L^2(\Omega)$.

Now, consider any sets $\omega \subset\subset \omega' \subset\subset \Omega$, a cut-off function $\eta(x)$ chosen as in (51) and $\xi = \sqrt{\varphi(\eta)}$. Arguing as in (52), we deduce that $v = T_k(u_n\xi^2)$ is an admissible test function for (56). Let us set, now, $A_k = \big\{x\in\Omega : |u_n|\xi^2 \le k \text{ and } \xi(x)>0\big\}$, we get

$$\int_{A_k} a(x,u_n,\nabla u_n)\cdot\nabla[u_n\xi^2] + \int_\Omega B(x,u_n)T_k(u_n\xi^2) \le k\|F\|_{L^1(\omega')}\,,$$

and so, using (47) and (54),

$$\alpha\int_{A_k}|\nabla u_n|^2\xi^2 + m_{\omega'}\int_{A_k}|b(u_n)|T_k(u_n\xi^2)$$
$$\leq k\|F\|_{L^1(\omega')} + 2\beta\int_{A_k}|\nabla u_n||\nabla\xi|\, u_n\xi\,.$$

By applying Young inequality, (48) and Lemma A.2 (with $\gamma = 1$ and $C > \frac{\alpha^2+4\beta^2}{8\alpha m_{\omega'}}\|\nabla\eta\|_{L^\infty(\omega')}$ and taking into account Remark A.5) we deduce that there exists $c > 0$ such that

$$\int_\Omega |\nabla T_k(u_n\xi^2)|^2 \leq c(k+1).$$

Then, using that $\xi = 1$ in $\omega$, by Lemmas 4.1 and 4.2 of [6] it follows that $u_n$ and $|\nabla u_n|$ are bounded respectively in $\mathcal{M}^{\frac{N}{N-2}}(\omega)$ and $\mathcal{M}^{\frac{N}{N-1}}(\omega)$, for any $\omega \subset\subset \Omega$. Combining this information with the strong convergence of $T_k(u_n)$ in $L^2(\Omega)$ we deduce that $u_n$ is a Cauchy sequence in measure and so, up to subsequences (not relabeled), it converges for a.e. $x \in \Omega$ to a function $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$. This, in particular, implies that

$$\lim_{n\to+\infty} k - T_k(u_n) = k - T_k(u) \quad \text{weakly in } H^1_0(\Omega)\,,$$

i.e. $u$ satisfies the boundary condition.

On the other hand, we prove that the lower order term is bounded in $L^1_{\mathrm{loc}}(\Omega)$; indeed, if, for $\varepsilon > 0$, we take $v = \frac{1}{\varepsilon}T_\varepsilon(u_n)\xi$ as test function in (56) (as before, such a function it is admissible). Thus, by (47), (48), and dropping positive terms, we get

$$\int_\Omega B(x,u_n)\frac{T_\varepsilon(u_n)}{\varepsilon}\xi \leq \|F\|_{L^1(\omega')} + \beta\|\nabla\xi\|_{L^\infty(\omega')}\int_{\omega'}|\nabla u_n|\,.$$

Since the right hand side is bounded being $\{|\nabla u_n|\}$ bounded in $M^{\frac{N}{N-1}}_{\mathrm{loc}}(\Omega)$ and $F \in L^1_{\mathrm{loc}}(\Omega)$, letting $\varepsilon \to 0$, we deduce by Fatou lemma that there exists $c_\omega > 0$ such that

$$\int_\omega |B(x,u_n)| \leq c_\omega.$$

On the other hand, choosing $v = T_1(G_h(u_n\xi^2))$ as test function, where $\xi^2 = \varphi(\eta)$ we have, by using (47), (48), (54) and (55),

$$\frac{\alpha}{2}\int_{h\leq|u_n\xi^2|\leq h+1}|\nabla u_n|^2\xi^2 + \frac{1}{2}\int_\Omega B(x,u_n)T_1(G_h(u_n\xi^2))$$

$$\leq \int_{\omega'\cap\{u_n\xi^2\geq h\}} |F_n| + \frac{2\beta^2}{\alpha}\|\nabla\eta\|^2_{L^\infty(\Omega)} \text{ meas}\{x\in\omega' \, : \, \xi^2|u_n|\geq h\}.$$

By the strong compactness of $\{F_n\}$ in $L^1(\omega')$ and the local uniform estimate of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}^{\frac{N}{N-2}}_{\mathrm{loc}}(\Omega)$, we derive then that

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{x\in\omega:\,|u_n|\geq h\}} |B(x,u_n)| \, = 0.$$

As a consequence of Vitali theorem we deduce that $\{|B(x,u_n)|\}_{n\in\mathbb{N}}$ is strongly compact in $L^1(\omega')$, where $\omega'\subset\subset\Omega$ is arbitrary. Moreover, since the lower order term is bounded in $L^1_{\mathrm{loc}}(\Omega)$, we can apply Lemma 1 in [10] in order to prove that $\nabla u_n$ converges to $\nabla u$ a.e. in $\Omega$. This, and the weak convergence of $u_n$ in $W^{1,q}(\omega')$, $\forall\omega'\subset\subset\Omega$, imply

$$u_n \longrightarrow u \quad \text{in} \quad W^{1,q}(\omega), \quad \forall 1\leq q<\frac{N}{N-1}, \quad \forall\omega\subset\subset\Omega,$$

and, thanks to (48), we also have that

$$a(x,u_n,\nabla u_n)\longrightarrow a(x,u,\nabla u) \quad \text{in} \quad L^1(\omega)^N, \quad \forall\omega\subset\subset\Omega. \tag{57}$$

Now we can pass to the limit in the distributional formulation: indeed choosing any $\phi\in C^\infty_c(\Omega)$ in (56) we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\phi + \int_\Omega B(x,u_n)\phi \, = \int_\Omega F_n\phi\,.$$

Using (57) we deduce that

$$\lim_{n\to+\infty}\int_{\text{supp }\phi} a(x,u_n,\nabla u_n)\cdot\nabla\phi \, = \int_{\text{supp }\phi} a(x,u,\nabla u)\cdot\nabla\phi\,.$$

Moreover, by the strong convergence of $\{B(x,u_n)\}$ and $\{F_n\}$ in $L^1_{\mathrm{loc}}(\Omega)$, we deduce that

$$\lim_{n\to+\infty}\int_{\text{supp }\phi} F_n\,\phi \, = \int_{\text{supp }\phi} F\,\phi$$

and

$$\lim_{n\to+\infty}\int_{\text{supp }\phi} B(x,u_n)\phi \, = \int_{\text{supp }\phi} B(x,u)\phi$$

and this concludes the proof. ∎

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