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ONE SIZE RESOLVABILITY OF GRAPHS

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Abstract: For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the code of v with respect to W is the k-vector

$$C_W(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k)).$$

The set W is a one size resolving set for G if (1) the size of subgraph $\langle W \rangle$ induced by W is one and (2) distinct vertices of G have distinct code with respect to W. The minimum cardinality of a one size resolving set in graph G is the one size resolving number, denoted by $\text{or}(G)$. A one size resolving set of cardinality $\text{or}(G)$ is called an or-set of G. We study the existence of or-set in graphs and characterize all nontrivial connected graphs G of order n with $\text{or}(G) = n$ and $n - 1$.

Keywords: Resolving set, one size resolving set.
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1. Introduction

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

$$C_W(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

as the code of v with respect to W. The set W is called a *resolving set* for G if distinct vertices of G have distinct codes. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for G. The *dimension*, $\dim(G)$, is the number of vertices in a basis for G. If $\dim(G) = k$, then G is called a *k-dimensional graph*.

The concept of resolving set and minimum resolving set have previously appeared in the literature. In the decisive paper [10], Slater first introduced these notions using locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its

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location number, and the code of a vertex with respect to W as the W-location of that vertex. With respect to a tree T, with leaves $L_1, L_2, \cdots, L_k$, where e_i is the number of branch paths in T that are in L_i, and where E is the set of endpoints, with $|E| \geq 3$, Slater proved that

$$\dim(T) = |E| - k\,,$$

and S is a basis for T if and only if it consists of exactly one vertex for each L_i $(1 \leq i \leq k)$. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations.

Harary and Melter [8] discovered these concepts independently as well but used the term metric dimension rather than location number, the terminology that we have adopted. These authors showed that every tree has a metric basis containing endvertices only, and established an algorithm for finding a metric basis of a tree producing a formula for its metric dimension.

Later on, in another valuable paper [11], Slater return to this topic. For a a graph G, he called a subset of vertices, D, as a locating-dominating set, or simply an LD-set, if for any two vertices v and w not in D, $N_D(v)$ is distinct from $N_D(w)$, where $S_D(v)$ denotes the set of neighbors of v in D. The order of the smallest LD-set in G was called the location-domination number and by denoted by $\mathrm{RD}(G)$. A reference-dominating set, or simply an RD-set, is an LD-set with $\mathrm{RD}(G)$ elements. Then he provided several results pertaining to that parameter, presenting some sharp Nordhaus-Gaddum type results. If the order of G is $p \geq 2$, Slater proved that $\mathrm{RD}(G) + \mathrm{RD}(\overline{G}) \leq 2p - 1$ and $\mathrm{RD}(G)\mathrm{RD}(\overline{G}) \leq p(p-1)$. Some additional results include a general bound for $\mathrm{RD}(G)$ depending on the order of the graph and the maximum degree as well as the location-domination number for several classes of graphs.

Some of those concepts were rediscovered by Johnson [9] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound is represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. This is the subject of the papers [1, 2, 4]. It was noted in [7, cf. p. 204] that determining the dimension of a graph is an NP-complete problem. Directed graphs with dimension 1 are characterized by Chartrand, Raines and Zhang in [5]. They proved that if the outdegree of every vertex of a connected oriented graph D of order n is at least 2 and $\dim(D)$ is defined, then $\dim(D) \leq n-3$ and this bound is sharp.

We refer to the reader-friendly book [3] for graph theory notation and terminology not described here.

If G is a nontrivial connected graph of order n, then $1 \leq \dim(G) \leq n-1$. Connected graphs of order $n \geq 2$ with dimension 1 or $n-1$ are characterized in [8, 10, 11].

Theorem A *Let G be a connected graph of order $n \geq 2$.*

(a) *Then $\dim(G) = 1$ if and only if $G = P_n$, the path of order n.*
(b) *Then $\dim(G) = n-1$ if and only if $G = K_n$, the complete graph of order n.*

The resolving sets whose vertices are "independent" to one another have been studied in [6]. An *independent resolving set* W in a connected G is both resolving and independent. This is, a resolving set W of G is independent if the subgraph $\langle W \rangle$ induced by W is an empty subgraph of G. The cardinality of a minimum independent resolving set, ir-set, in a graph G is the *independent resolving number* $\mathrm{ir}(G)$.

In this paper, we study those resolving sets of graph whose vertices are "almost independent" to one another.

Definition 1.1. *A set W of G is a* one size resolving set *if*

(1): *the size of subgraph $\langle W \rangle$ induced by W is one and*
(2): *distinct vertices of G have distinct code with respect to W.*

The minimum cardinality of a one size resolving set in graph G is the *one size resolving number*, denoted by $\mathrm{or}(G)$. A one size resolving set of cardinality $\mathrm{or}(G)$ is called an *or-set* of G. Let G be a connected graph of order n containing an or-set. Since the size of induced subgraph by or-set is one, it follows that

$$2 \leq \mathrm{or}(G) \leq n. \tag{1}$$

To illustrate this concept, consider the graph G of Figure 1. The set $\{u, v\}$ where u is one of $\{v_1, v_2\}$ and v is one of $\{v_4, v_5\}$ is a basis for G and so $\dim(G) = 2$. However, $\{u, v\}$ is not a one size resolving set for G. In fact, the set $W = \{v_1, v_4, v_5\}$ is an or-set with codes of the vertices of G with respect to W as

$$C_W(v_1) = (0, 2, 2), C_W(v_2) = (2, 2, 2), C_W(v_3) = (1, 1, 1),$$
$$C_W(v_4) = (2, 0, 1), C_W(v_5) = (2, 1, 0)$$

which the size of $\langle W \rangle$ induced by W is one. We can show that G contains no 2-element or-set and so $\text{or}(G) = 3$.

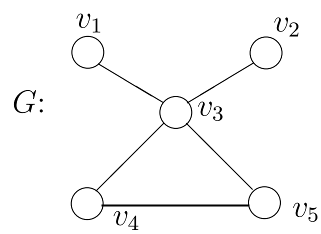

FIGURE 1. A graph G with $\dim(G) = 2$ and $\text{or}(G) = 3$.

Also, as an example, the only induced subgraph of the complete graph K_4 of size one consists of two vertices. Thus $\text{or}(K_4)$ is not defined. Figure 2 shows the 3-regular graphs $K_4, K_{3,3}$ and the Peterson graph P. A resolving set of $K_{3,3}$ contains at least two vertices from each partite sets of $K_{3,3}$. Since the induced subgraph of at least two vertices from each partite sets of $K_{3,3}$ has a size at least four, it follows that $\text{or}(K_{3,3})$ does not exist, however $\text{or}(P)$ exists. In Figure 2, the solid vertices represent an or-set for the Peterson graph P.

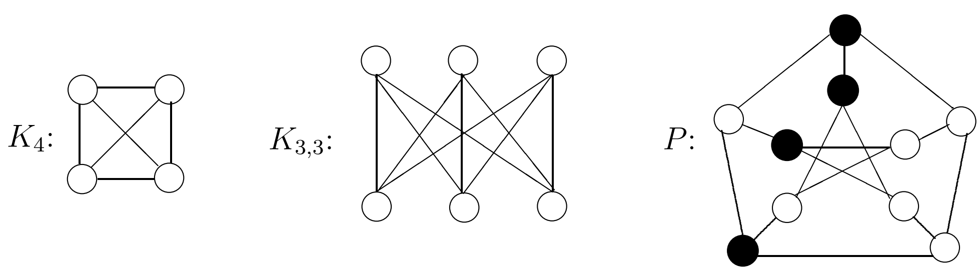

FIGURE 2. Three 3-regular graphs

Two vertices u and v in a connected graph G are *distance similar* if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. For a vertex v in a graph G, let $N[v]$ be the set $N(v) \cup \{v\}$, where $N(v)$ simply denotes $N_G(v)$. Two vertices u and v in a connected graph are distance similar if and only if

(1): $uv \in E(G)$ and $N(u) = N(v)$ or
(2): $uv \in E(G)$ and $N[u] = N[v]$.

Distance similarity in graph G is an equivalence relation on $V(G)$. As we will see, the following observation turns out quite useful.

Claim 1.2. *If U is a distance similar equivalence class in a connected graph G with $|U| = p \geq 2$, then every resolving of G contains at least $p-1$ vertices from U. Then if G has k distance similar equivalence classes and* $\mathrm{or}(G)$ *is defined, therefore*

$$n - k \leq \dim(G) \leq \mathrm{or}(G).$$

There exist graphs G such that every or-set of G must contain all vertices of some distance similar equivalence class. For example, let G be the graph as shown in Figure 3 obtained form $K_{2,p}$, whose partite sets are $\{x, y\}$ and $U = \{u_1, u_2, \cdots, u_p\}$ with $p \geq 4$, by adding vertices $v_1, v_2, \cdots, v_{p'}$ with $p' \geq 4$ and for $1 \leq i \leq p'$, the pendant edges xv_i and edge v_1v_2. Then G contains three distance similar equivalence classes of cardinality at least 2 namely $U, V = \{v_1, v_2\}$ and $V' = \{v_3, v_4, \cdots, v_{p'}\}$. Since every or-set of G has the form $(U \cup V \cup V') - \{w\}$, for some $w \in U \cup V'$, it follows that every or-set of G contains V and either U or V'.

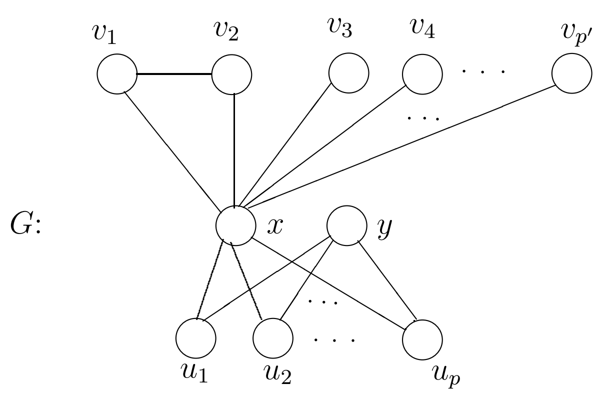

FIGURE 3. The graph G with or-set $(U \cup V \cup V') - \{w\}$ for some $w \in U \cup V'$

If U is a distance similar equivalence class of a connected graph G, then either U is an independent set in G or the subgraph $\langle U \rangle$ induced by U is complete in G. Thus we have the following observation.

Claim 1.3. *Let G be a connected graph and let U be a distance similar equivalence class in a connected graph G with $|U| \geq 4$. If U is not independent in G, then* $\mathrm{or}(G)$ *is not defined.*

2. One size resolving sets in some well-known graphs

In this section, we determine the existence of one size resolving sets in some well-known classes of graphs.

Proposition 2.1. *For a path P_n of order $n \geq 2$,* $\mathrm{or}(P_n) = 2$.

Proof. Let $P_n : v_1, v_2, \cdots, v_n$ be a path of order $n \geq 2$ and let $W = \{v_1, v_2\}$. We have $C_W(v_1) = (0, 1), C_W(v_2) = (1, 0), C_W(v_i) = (i-1, i-2)$ for $3 \leq i \leq n$. Therefore, W is a resolving set. Moreover, it follows by (1) that $\mathrm{or}(P_n) = 2$. ■

Proposition 2.2. *For a cycle C_n of order $n \geq 3$,* $\mathrm{or}(C_n) = 2$.

Proof. Let $C_n : v_1, v_2, \cdots, v_n, v_1$ be a cycle of order $n \geq 3$ and let $d = \lfloor \frac{n+1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the smallest integer greater than or equal to x. We consider two cases according the parity of n.

Case 1. *n is odd.* Let $W = \{v_1, v_n\}$. Then subgraph $\langle W \rangle$ induced by W is P_2. Since

$$C_W(v_i) = \begin{cases} (i-1, i) & \text{for } 1 \leq i \leq d-1, \\ (i-1, i-1) & \text{for } i = d, \text{ and} \\ (n-i+1, n-i) & \text{for } d+1 \leq i \leq n, \end{cases}$$

it follows that W is a one size resolving set.

Case 2. *n is even.* Let $W = \{v_1, v_n\}$. Then subgraph $\langle W \rangle$ induced by W is P_2. Since

$$C_W(v_i) = \begin{cases} (i-1, i) & \text{for } 1 \leq i \leq d-1, \\ (i-1, n-i) & \text{for } i = d, \text{ and} \\ (n-i+1, n-i) & \text{for } d+1 \leq i \leq n, \end{cases}$$

it follows that W is a one size resolving set. Thus it implies that $\mathrm{or}(C_n) = 2$. ■

Theorem 2.3. *If G is a complete graph of order $n \geq 3$, then* $\mathrm{or}(G)$ *exists if and only if $G = K_3$. Furthermore,* $\mathrm{or}(K_3) = 2$.

Proof. It is not difficult to see that the theorem is true for a complete graph of order 3. Let us assume that G is a complete graph of order $n \geq 4$.

Then it follows by Theorem A that the size of subgraph induced by any resolving set of G is greater than one. Thus $\mathrm{or}(G)$ does not exist. ■

Theorem 2.4. *Let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$. Then*

$$\mathrm{or}(K_{r,s}) \text{ exists if and only if } 1 \leq r \leq s \leq 2.$$

Furthermore, $\mathrm{or}(K_{r,s}) = 2$ *for* $1 \leq r \leq s \leq 2$.

Proof. It is an immediate consequence of Propositions 2.1 and 2.2 that $\mathrm{or}(K_{r,s})$ exists for $1 \leq r \leq s \leq 2$. Reciprocally, let $K_{r,s}$ be a complete bipartite graph with $1 \leq r \leq s$ and $r, s \notin \{1, 2\}$. let $U = \{u_1, u_2, \cdots, u_r\}$ and $V = \{v_1, v_2, \cdots, v_s\}$ be partite sets of $K_{r,s}$. Without loss generality, we assume that $s \geq 3$. Let W be an or-set of $K_{r,s}$. Then it follows from Claim 1.2 that W contains at least $s - 1$ vertices from V. Since subgraph $\langle V \rangle$ induced by V is an empty graph, it follows that W contains at least one vertex from U. However, the size of subgraph $\langle W \rangle$ induced by W is greater than one, which is a contradiction. ■

3. Realizable results

As we have noticed, $2 \leq \mathrm{or}(G) \leq n$, for all connected graphs G of order $n \geq 2$ such that $\mathrm{or}(G)$ exist. We are able to characterize all nontrivial connected graphs with one size resolving number n and $n - 1$ as following.

Theorem 3.1. *Let G be a nontrivial connected graph of order n, then*

$$\mathrm{or}(G) = n \text{ if and only if } G = P_2.$$

Proof. Let $G = P_2$. From Proposition 2.1, it follows that $\mathrm{or}(G) = 2 = n$. Conversely, let G be a connected graph of order $n \geq 2$ and $\mathrm{or}(G) = n$ and let W be an or-set of G with $|W| = |V(G)| = n$. Since the size of subgraph $\langle W \rangle$ induced by W is one and G is a connected graph, it follows that $|W| = |V(G)| = 2$. Thus G is a path of order 2. ■

It is an immediate consequence of Theorem 3.1 that if G is a connected graph of order $n \geq 3$, then $\mathrm{or}(G) \leq n - 1$.

Theorem 3.2. *Let G be a nontrivial connected graph of order $n \geq 3$, then*

$$\mathrm{or}(G) = n - 1 \text{ if and only if } G = P_3 \text{ or } K_3.$$

Proof. Let $G = P_3$ or K_3. By Proposition 2.1 and Theorem 2.3, it follows that $\mathrm{or}(G) = 2 = n-1$. To verify the converse, suppose that G is a connected graph of order $n \geq 3$ and $\mathrm{or}(G) = n - 1$. For $n = 3$, it is straightforward to show that $G = P_3$ or K_3. Thus we may assume that $n \geq 4$, Let W be an or-set of G and let $V(G) - W = \{x\}$. Let u, v be adjacent vertices in subgraph $\langle W \rangle$ induced by W. Then x is adjacent to every independent vertex of $W - \{u, v\}$ and at least one of $\{u, v\}$, say u. Let $W' = W - \{x, y\}$ where y is one of $W - \{u, v\}$. Since $d(u, x) = 1$ and $d(u, y) = 2$, it follows that $C_{W'}(x) \neq C_{W'}(y)$ and so W' is a one size resolving set of G with cardinality $n - 2$ which is impossible. ∎

By Theorems 3.1 and 3.2, we have a following consequence.

Corollary 3.3. *Let G be a nontrivial connected graph of order $n \geq 4$, then*

$$2 \leq \mathrm{or}(G) \leq n - 2.$$

Finally, we provide sufficient and conditions for a pair k, n positive integers, with $k \leq n$, to be realizable as the one size resolving number and order of some connected graph, respectively.

Theorem 3.4. *For each pair k, n of positive integers with $k \leq n$, there exists a connected graph G of order n with $\mathrm{or}(G) = k$ if and only if $(k, n) = (n-1, 3)$ or $(n, 2)$ or $2 \leq k \leq n - 2$.*

Proof. By Theorems 3.1 and 3.2 and Corollary 3.3, it only remains to show that, for $n \geq 4$ and $2 \leq k \leq n-2$, that there exists a connected graph G of order n with $\mathrm{or}(G) = k$. For $k = 2$, let $G = C_n$ and so $\mathrm{or}(G) = 2$. We now assume that $n \geq 5$ and $3 \leq k \leq n-2$. Let G be a graph obtained from paths $P : u_1, u_2$ and $Q : v_1, v_2, \cdots, v_{n-k}$ and $k - 2$ new vertices $w_1, w_2, \cdots, w_{k-2}$ by joining each u_i and w_j to v_1 for $i = 1, 2$ and $1 \leq j \leq k - 2$. Thus G is a connected graph of order n as shown in Figure 4.

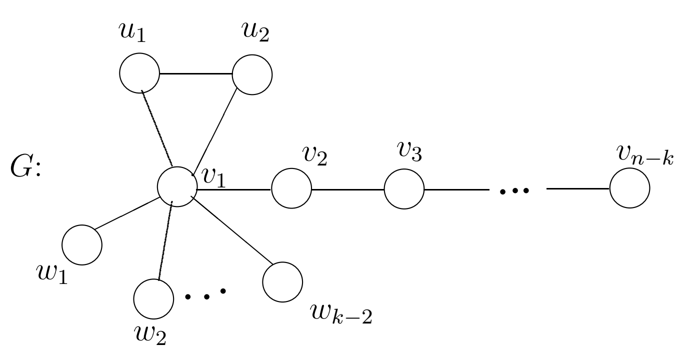

FIGURE 4. Graph G

First, we show that $\text{or}(G) \leq k$. Let $W = \{u_1, u_2, w_1, w_2, \cdots, w_{k-2}\}$. Since the size of subgraph $\langle W \rangle$ induced by W is one and $C_W(v_i) = (i, i, \cdots, i)$ for $1 \leq i \leq n-k$, it follows that W is a one size resolving set of G. We now continue showing that $\text{or}(G) \geq k$. Assume to the contrary, that $\text{or}(G) \leq k-1$. Let W' be an or-set of G. Then $|W'| \leq k-1$. Since $V(P)$ is a distance similar equivalence class in a connected graph G, it follows by Claim 1.2 that W contains at least one vertex from $V(P)$, say u_1. We consider two cases according the parity of k.

Case 1. $k = 3$. Then $|W'| \leq k - 1 = 2$. Since the size of subgraph $\langle W' \rangle$ induced by W' must be one and $|W'| \leq k-1$, it follows that W' is either $V(P)$ or $\{u_1, v_1\}$. However, $d(v_2, w) = d(w_1, w)$ for each w in W', which is a contradiction.

Case 2. $k \geq 4$. Since $\{w_1, w_2, \cdots, w_{k-2}\}$ is a distance similar equivalence class in a connected graph G, it follows by Claim 1.2 that W' contains at least $k-3$ from $\{w_1, w_2, \cdots, w_{k-2}\}$, with out loss of generality, say w_i for $1 \leq i \leq k-3$. Since the size of subgraph $\langle W' \rangle$ induced by W' must be one and $|W'| \leq k-1$, it follows that $W' = V(P) \cup \{w_1, w_2, \cdots, w_{k-3}\}$. However, $C_{W'}(v_2) = C_{W'}(w_{k-2}) = (2, 2, \cdots, 2)$, which is a contradiction.

From the previews two cases, it follows by cases 1 and 2 that $\text{or}(G) \geq k$ and therefore $\text{or}(G) = k$. ■

4. Open questions

After the introduction of the concept of $\text{or}(G)$ some questions naturally arise, specially the ones connected with well-known graph parameters. Among others, we leave to the reader the following open problems.

4.1.: For a pair of integers a and b with $1 \leq a \leq b$, is there a connected graph G with $\dim(G) = a$ and $\text{or}(G) = b$?

4.2.: What is relationship between $\text{ir}(G)$ and $\text{or}(G)$?

4.3.: What is boundary of $\text{or}(G)$ in terms of other parameters such as clique numbers, diameter of graph?

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