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A general insertion theorem for uniform locales ☆

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ABSTRACT

A general insertion theorem due to Preiss and Vilimovský is extended to the category of locales. More precisely, given a preuniform structure on a locale we provide necessary and sufficient conditions for a pair $f \geq g$ of localic real functions to admit a uniformly continuous real function in-between. As corollaries, separation and extension results for uniform locales are proved. The proof of the main theorem relies heavily on (pre-)diameters in locales as a substitute for classical pseudometrics. On the way, several general properties concerning these (pre-)diameters are also shown.

1. Introduction

In classical point-set topology, one finds in the literature a large number of insertion-type results that provide conditions under which two comparable real-valued functions belonging to given classes admit a continuous function in-between (see the different variants in [5,12,15,18,25,26]). However, besides the paper by Preiss and Vilimovský [22], the literature on insertion results for uniform structures is scarce. The main insertion theorem from [22] can be stated as follows:

Topological insertion theorem for uniform spaces. *Let X be a uniform space and $f, g\colon X \to \mathbb{R}$ two maps with $f \geq g$. Then the following are equivalent.*

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(i) *There is a uniformly continuous* $h\colon X \to \mathbb{R}$ *such that* $f \geq h \geq g$;
(ii) *For every* $\delta > 0$ *there is a uniform cover* $\mathfrak{U}$ *of* X *such that for all* $n \in \mathbb{N}$ *the subspaces* $f^{-1}(-\infty, r]$ *and* $g^{-1}[s, +\infty)$ *are* $\mathsf{St}^n(\mathfrak{U})$*-far whenever* $s - r > (n+1)\delta$.

The main goal of the present paper is to extend this result to the category of locales. Even more generally, we prove the result for *preuniform* locales (i.e., locales equipped with uniformities but no compatibility condition between them) so that, in particular, complete regularity of the underlying locale is not required.

Throughout the paper, we make use of the theory of general localic real functions launched in [9], because it provides a useful representation of arbitrary — i.e., not necessarily continuous — real functions on locales. As is well known, this representation allows one to phrase and prove point-free counterparts of the Katětov-Tong theorem for normal spaces [9], of the general insertion result of Blair and Lane [8], and many others (see e.g. [1,6,10]).

In this paper we show that the notion of uniform continuity can be recasted via these general maps of locales in such a way that one obtains a convenient setting to phrase and prove the uniform insertion, extension and separation results for uniform locales.

In certain aspects, the localic approach diverges significantly from the classical one. For example, as a substitute of classical pseudometrics, we make use of the notion of *localic diameter*, which was introduced by Pultr in the eighties for extending metric structures to the category of locales. In particular, we partially improve some results from [24].

This paper is organized as follows. In Section 2 we provide specific preliminaries concerning real functions and uniform locales. Section 3 concerns (pre-)diameters and contains the proof of the main technical lemma for the uniform insertion theorem. In Section 4 we discuss the notion of farness for sublocales, and we introduce uniform continuity in the setting of arbitrary localic real functions. In particular, we prove that this notion coincides with the usual notion of a uniform homomorphism. Section 5 is devoted to proving the main result of the paper — the uniform insertion theorem for locales. We also outline an easier proof of the insertion theorem for the bounded case by using a technique due to Katětov. In Section 6, we prove a separation result for sublocales and an extension result as consequences of the insertion theorem.

2. Preliminaries

Our notation and terminology regarding the categories of frames and locales will be that of [19]. The Heyting operator in a frame L, right adjoint to the meet operator, will be denoted by $\to$; for each $a \in L$, $a^* = a \to 0$ is the pseudocomplement of a. Furthermore, an element b is rather below a (written $b \prec a$) if $b^* \vee a = 1$. A *sublocale* of a locale L is a subset $S \subseteq L$ closed under arbitrary meets such that

$$\forall a \in L, \quad \forall s \in S, \quad a \to s \in S.$$

These are precisely the subsets of L for which the embedding $j_S\colon S \hookrightarrow L$ is a morphism of locales.

The system $\mathsf{S}(L)$ of all sublocales of L, partially ordered by inclusion, is a coframe [19, Theorem III.3.2.1], that is, its dual lattice is a frame. Infima and suprema are given by

$$\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i, \quad \bigvee_{i\in I} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i\in I} S_i\}.$$

The least element is the sublocale $\mathsf{O} = \{1\}$ and the greatest element is the entire locale L.

Since $\mathsf{S}(L)$ is a coframe, every sublocale S of L has a *supplement* denoted by $S^{\#_L}$ (or simply $S^{\#}$ if there is no risk of confusion) which can be characterized as the smallest sublocale of L whose join with S is the entire L. We note that if S is a complemented sublocale of L and T is a sublocale of S, then

$$T^{\#_S} = S \cap T^{\#} \tag{2.1}$$

(see [7, Proposition 4.1 (7)]).

For any $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = \uparrow a = \{x \in L \mid x \geq a\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{a \to b \mid b \in L\}$$

are the *closed* and *open* sublocales of L, respectively (that we shall denote simply by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ when there is no danger of confusion). For each $a \in L$, $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathsf{S}(L)$ and satisfy the identities

$$\bigcap_{i \in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i \in I} a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \tag{2.2}$$
$$\bigvee_{i \in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i \in I} a_i) \quad \text{and} \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b).$$

For any sublocale S of L, the closed (resp. open) sublocales $\mathfrak{c}_S(a)$ (resp. $\mathfrak{o}_S(a)$) of S are precisely the intersections $\mathfrak{c}(a) \cap S$ (resp. $\mathfrak{o}(a) \cap S$) and we have, for any $a \in L$, $\mathfrak{c}(a) \cap S = \mathfrak{c}_S(j_S^*(a))$ and $\mathfrak{o}(a) \cap S = \mathfrak{o}_S(j_S^*(a))$, where $j_S^* \colon L \twoheadrightarrow S$ denotes the left adjoint of the sublocale embedding $j_S \colon S \hookrightarrow L$.

2.1. The frame of reals

Recall the frame of reals $\mathfrak{L}(\mathbb{R})$ from [3]. Here we define it, equivalently, as the frame presented by generators $(r, —)$ and $(—, r)$ for all $r \in \mathbb{Q}$, and relations

(r1) $(p, —) \wedge (—, q) = 0$ if $q \leq p$;
(r2) $(p, —) \vee (—, q) = 1$ if $p < q$;
(r3) $(p, —) = \bigvee_{r > p} (r, —)$;
(r4) $(—, q) = \bigvee_{s < q} (—, s)$;
(r5) $\bigvee_{p \in \mathbb{Q}} (p, —) = 1$;
(r6) $\bigvee_{q \in \mathbb{Q}} (—, q) = 1$.

Further, for rationals $r, s \in \mathbb{Q}$ we denote $(r, s) = (r, —) \wedge (—, s)$.

A *continuous real-valued function* [3] on a frame L is a frame homomorphism $h \colon \mathfrak{L}(\mathbb{R}) \to L$. We denote by $\mathcal{R}(L)$ the collection of all continuous real-valued functions on L — i.e.,

$$\mathcal{R}(L) := \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L).$$

The collection $\mathcal{R}(L)$ is an ℓ-ring partially ordered by

$$f \leq g \iff g(—, r) \leq f(—, r) \text{ for all } r \in \mathbb{Q} \iff f(r, —) \leq g(r, —) \text{ for all } r \in \mathbb{Q}.$$

There is a useful way of specifying continuous real-valued functions with the help of scales ([9, Section 4]). A *descending scale* (resp. *ascending scale*) in L is a family $\{b_r\}_{r \in \mathbb{Q}} \subseteq L$ such that $b_s^* \vee b_r = 1$ ($b_r^* \vee b_s = 1$) whenever $r < s$ and such that $\bigvee_{r \in \mathbb{Q}} b_r = 1 = \bigvee_{r \in \mathbb{Q}} b_r^*$. For each descending (resp. ascending) scale $\{b_r\}_{r \in \mathbb{Q}}$ in L, the formulas

$$h(p, —) = \bigvee_{p < r} b_r \quad \text{and} \quad h(—, q) = \bigvee_{q > s} b_s^*$$
$$(\text{resp. } h(p, —) = \bigvee_{p < r} b_r^* \quad \text{and} \quad h(—, q) = \bigvee_{q > s} b_s)$$

determine an $h \in \mathcal{R}(L)$.

Let $\mathsf{S}(L)^{op} = (\mathsf{S}(L), \leq)$, with $\leq \equiv \supseteq$, be the dual lattice of $\mathsf{S}(L)$. Now, a *real-valued function* on L is a frame homomorphism $h\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ (see [9]). We denote by $\mathsf{F}(L)$ the collection of all real-valued functions on L — i.e.,

$$\mathsf{F}(L) := \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathsf{S}(L)^{op}).$$

By the identities (2.2), the set $\mathfrak{c}L$ of all closed sublocales of L is a subframe of $\mathsf{S}(L)^{op}$ isomorphic to the given L. Using this isomorphism $L \simeq \mathfrak{c}L$, the collection $\mathcal{R}(L)$ of continuous real-valued functions on L can be identified with the set of all $f \in \mathsf{F}(L)$ such that $f(r,—)$ and $f(—,r)$ are closed for every $r \in \mathbb{Q}$; since we want to distinguish notationally both collections, the latter will be denoted by $\mathsf{C}(L)$ — i.e.,

$$\mathsf{C}(L) = \{\, f \in \mathsf{F}(L) \mid f(r,—) \text{ and } f(—,r) \text{ are closed for all } r \in \mathbb{Q}\,\}.$$

In other words, if $f \in \mathsf{F}(L)$ then one has that $f \in \mathsf{C}(L)$ if and only if f factors through $\mathfrak{c}\colon L \to \mathsf{S}(L)^{op}$ (the frame homomorphism that sends a to $\mathfrak{c}(a)$).

The ℓ-ring $\mathsf{F}(L)$ is an extension of $\mathcal{R}(L)$, and so it is partially ordered by

$$f \leq g \iff f(—,r) \subseteq g(—,r) \text{ for all } r \in \mathbb{Q} \iff g(r,—) \subseteq f(r,—) \text{ for all } r \in \mathbb{Q}. \tag{2.3}$$

Remarks 2.1. The following properties are easy to check:

(1) If $f \in \mathcal{R}(L)$, then

$$\begin{aligned} f(s,—) \leq f(—,s)^* \leq f(s',—) &\quad \text{for any } s' < s, \text{ and} \\ f(—,r) \leq f(r,—)^* \leq f(—,r') &\quad \text{for any } r' > r \end{aligned}$$

(2) If $f \in \mathsf{F}(L)$, then

$$\begin{aligned} f(s',—) \subseteq f(—,s)^\# \subseteq f(s,—) &\quad \text{for any } s' < s, \text{and} \\ f(—,r') \subseteq f(r,—)^\# \subseteq f(—,r) &\quad \text{for any } r' > r. \end{aligned}$$

Examples 2.2.

(1) For every $p \in \mathbb{Q}$ we have the constant function $\mathbf{p} \in \mathsf{F}(L)$ given by

$$\mathbf{p}(r,—) = \begin{cases} \mathsf{O} & \text{if } r < p, \\ L & \text{if } r \geq p, \end{cases} \quad \text{and} \quad \mathbf{p}(—,r) = \begin{cases} L & \text{if } r \leq p, \\ \mathsf{O} & \text{if } r > p. \end{cases}$$

Notice that $\mathbf{p} \in \mathsf{C}(L)$ for every $p \in \mathbb{Q}$.

(2) For each complemented sublocale S of L we define the *characteristic function* $\chi_S \in \mathsf{F}(L)$ of S given by

$$\chi_S(r,—) = \begin{cases} \mathsf{O} & \text{if } r < 0, \\ S^\# & \text{if } 0 \leq r < 1, \\ L & \text{if } r \geq 1, \end{cases} \quad \text{and} \quad \chi_S(—,r) = \begin{cases} L & \text{if } r \leq 0, \\ S & \text{if } 0 < r \leq 1, \\ \mathsf{O} & \text{if } r > 1. \end{cases}$$

Notice that $\mathbf{0} \leq \chi_S \leq \mathbf{1}$.

We say that an $f \in \mathsf{F}(L)$ is *bounded* if there are $\alpha, \beta \in \mathbb{Q}$ such that $\boldsymbol{\alpha} \leq f \leq \boldsymbol{\beta}$. By (2.3) it is easy to check that $\boldsymbol{\alpha} \leq f \leq \boldsymbol{\beta}$ holds if and only if for every $r, s \in \mathbb{Q}$ with $r < \alpha$ and $s > \beta$ we have

$$f(r, \text{—}) = \mathsf{O} \quad \text{and} \quad f(\text{—}, s) = \mathsf{O}, \tag{2.4}$$

or equivalently if for every $r, s \in \mathbb{Q}$ with $r \leq \alpha$ and $s \geq \beta$ we have

$$f(s, \text{—}) = L \quad \text{and} \quad f(\text{—}, r) = L. \tag{2.5}$$

2.2. Uniform locales via covers

Some general references for uniformities in locale theory are [4,23,24] and Chapters VIII–XII in [19]. In this paper, we adopt the "Tukey-style" approach via open covers (cf. also [13,14]), and the preliminaries contained in [2] will be enough for our purposes. In what follows, we recall briefly some of the basic notions needed.

2.2.1. Basic properties of covers

A *cover* of a frame L is a subset $U \subseteq L$ such that $\bigvee U = 1$. A cover U *refines* (or is a *refinement* of) a cover V, written, $U \leq V$, if for any $u \in U$ there is some $v \in V$ such that $u \leq v$. For covers U, V we have the largest common refinement $U \wedge V = \{ u \wedge v \mid u \in U, v \in V \}$.

For any $U \subseteq L$ and any $a \in L$ the *star* of a in U is the element

$$U \cdot a = \bigvee \{ u \in U \mid u \wedge a \neq 0 \}.$$

For any $U, V \subseteq L$, set

$$U \cdot V = \{ U \cdot v \mid v \in V \}.$$

One usually denotes Ua and UV instead of $U \cdot a$ and $U \cdot V$. Since this operation is neither associative nor commutative, we will also use parentheses when needed.

Proposition 2.3. *For any covers $U, V \subseteq L$ and any frame homomorphism $h\colon L \to M$, we have*:

(1) *UV is a cover of L;*
(2) *$a \leq Ua$;*
(3) *$Ua \leq b$ implies $a \prec b$;*
(4) *$U \leq UU$;*
(5) *$U \leq V$ and $a \leq b$ imply $Ua \leq Vb$;*
(6) *$U(Va) \leq (UV)a = U(V(Ua))$;*
(7) *$U\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} Ua_i$;*
(8) *$h[U]\, h(a) \leq h(Ua)$.*

For a cover U, define a cover U^n for $n \geq 1$ inductively by setting

$$U^1 = U \quad \text{and} \quad U^{n+1} = U \cdot U^n.$$

Following [2], given a cover U of L we define a map $\mathsf{S}_U\colon L \to L$ given by $\mathsf{S}_U(a) = Ua$ for each $a \in L$. We denote by S_U^n the result of composing S_U with itself n times. Notice that, in general, $\mathsf{S}_{U^n} \neq \mathsf{S}_U^n$. We shall need the following technical properties:

Lemma 2.4. *Let L be a locale, U a cover of L and $n, m \in \mathbb{N}$. Then*:

(1) *If $n > 1$, then $a \in U^n$ if and only if there is a $u \in U$ such that $a = \mathsf{S}_U^{n-1}(u)$;*
(2) $\mathsf{S}_{U^n} = \mathsf{S}_U^{2n-1}$;
(3) $U^n U = U^{2n}$;
(4) $U^{nm} \leq (U^n)^m$.

Proof. (1) and (2) are proved in [2, Fact 5.1] and [2, Eq. 5.1.2] respectively.

(3) By definition $a \in U^n U$ if and only if there is a $u \in U$ with $a = \mathsf{S}_{U^n}(u) = \mathsf{S}_U^{2n-1}(u)$ but by (1) the latter is equivalent to $a \in U^{2n}$.

(4) We may assume $n, m > 1$. By an application of (1) one has $a \in U^{nm}$ if and only if $a = \mathsf{S}_U^{nm-1}(u)$ for some $u \in U$. Further, by (1), $b \in (U^n)^m$ if and only if $b = \mathsf{S}_{U^n}^{m-1}(v)$ for some $v \in U^n$. By another application of (1), the latter is equivalent to the existence of a $w \in U$ such that $b = \mathsf{S}_{U^n}^{m-1}(\mathsf{S}_U^{n-1}(w)) = \mathsf{S}_U^{(m-1)(2n-1)}(\mathsf{S}_U^{n-1}(w)) = \mathsf{S}_U^{(m-1)(2n-1)+n-1}(w)$. The result thus follows from the obvious fact that $nm \leq (m-1)(2n-1) + n$. $\square$

2.2.2. Stronger notions of refinements

We shall be interested in certain strengthenings of the notion of refinement of covers (see e.g., [13]). Let U, V be covers. We say that

(1) U is a *star refinement* of V, denoted by $U \leq^* V$, if $U^2 \leq V$;
(2) U is a *barycentric refinement* of V, denoted by $U \leq_1^* V$, if there is a cover W of L with $UW \leq V$;
(3) U is a *connected refinement* of V, denoted by $U \leq_2^* V$, if for all $S \subseteq U$ such that $a \wedge b \neq 0$ for all $a, b \in S$, there is a $v \in V$ with $\bigvee S \leq v$;
(4) U is a *regular refinement*, denoted by $U \leq_3^* V$, of V if for all $a, b \in U$ with $a \wedge b \neq 0$, there is a $v \in V$ with $a \vee b \leq v$.

Note that star refinement is the strongest relation, and regular refinement is the weakest and it implies ordinary refinement. Further, conditions (2) and (3) are generally unrelated, even classically, as displayed in the following diagram.

$$\begin{array}{ccccc} & & U \leq^* V & & \\ & \swarrow & & \searrow & \\ U \leq_1^* V & & & & U \leq_2^* V \\ & \searrow & & \swarrow & \\ & & U \leq_3^* V & & \\ & & \Downarrow & & \\ & & U \leq V & & \end{array}$$

2.2.3. Farness

If U is a cover of L, elements $a, b \in L$ are said to be *U-far* if

$$\forall u \in U, \quad u \wedge a \neq 0 \implies u \wedge b = 0.$$

For a general view of the importance of the farness relation in the uniform context, we refer the reader to [2]. We note that if a and b are U-far and $V \leq U$, then a and b are also V-far. Further, if a and b are U-far and $a' \leq a$ and $b' \leq b$, then a' and b' are also U-far. The following proposition summarizes a number of other useful characterizations:

Proposition 2.5. *Let L be a locale, U a cover of L and $a, b \in L$. Then the following are equivalent*:

(i) *The elements a and b are U-far*;
(ii) *For every $u \in U$, either $u \leq a^*$ or $u \leq b^*$*;
(iii) $Ua \wedge b = 0$;
(iv) $a \wedge Ub = 0$;
(v) *a^{**} and b^{**} are U-far.*

2.2.4. Covering uniformities

From now on we shall always assume that $1 \neq 0$ in L (that is, $|L| \geq 2$). A (covering) *uniformity* on L is a nonempty system $\boldsymbol{\mathcal{U}}$ of covers of L such that

(U1) $U \in \boldsymbol{\mathcal{U}}$ and $U \leq V$ implies $V \in \boldsymbol{\mathcal{U}}$,
(U2) $U, V \in \boldsymbol{\mathcal{U}}$ implies $U \wedge V \in \boldsymbol{\mathcal{U}}$,
(U3) for every $U \in \boldsymbol{\mathcal{U}}$ there is a $V \in \boldsymbol{\mathcal{U}}$ such that $VV \leq U$, and
(U4) for every $a \in L$, $a = \bigvee\{b \mid b \lhd_{\boldsymbol{\mathcal{U}}} a\}$

where we write $b \lhd_{\boldsymbol{\mathcal{U}}} a$ if $Ub \leq a$ for some $U \in \boldsymbol{\mathcal{U}}$.

Without (U4) one speaks of a preuniformity, without (U1) one speaks of a *basis of a* (*pre*)*uniformity* (in the latter case one obtains, of course, a (pre)uniformity adding all the V with $V \geq U \in \boldsymbol{\mathcal{U}}$).

A *uniform frame* (resp. *preuniform frame*) is a pair $(L, \boldsymbol{\mathcal{U}})$ where $\boldsymbol{\mathcal{U}}$ is a uniformity (resp. preuniformity) on L.

2.2.5. The metric uniformity of $\mathfrak{L}(\mathbb{R})$

The frame of reals carries a natural uniformity, its *metric uniformity* [3], generated by covers

$$D_n = \left\{ (r, s) \in \mathfrak{L}(\mathbb{R}) \mid s - r = \tfrac{1}{n} \right\}, \quad n \in \mathbb{N}.$$

We will consider, more generally, the covers

$$D_\delta = \left\{ (r, s) \in \mathfrak{L}(\mathbb{R}) \mid \ s - r = \tfrac{1}{\delta} \right\}, \quad \delta \in \mathbb{Q}^+$$

(where $\mathbb{Q}^+$ denotes the set of positive rational numbers).

3. Prediameters

Let us recall (cf. [24, 1.2] or [19, XI.3.1]) that a *prediameter* on a frame L is a function $f \colon L \to [0, +\infty]$ with the following properties:

(PD1) $f(0) = 0$;
(PD2) $a \leq b$ implies $f(a) \leq f(b)$ for all $a, b \in L$;
(PD3) For all $\epsilon > 0$, the set $\{ a \in L \mid f(a) < \epsilon \}$ is a cover of L.

Consider now the following two properties:

(PD4) If $a, b \in L$ are such that $a \wedge b \neq 0$, then $f(a \vee b) \leq f(a) + f(b)$;

(PD5) If $a, b \in L$ are such that $a \wedge b \neq 0$, then $f(a \vee b) \leq 2\max\{ f(a), f(b) \}$ (and so, in particular, $f(a \vee b) \leq 2f(a) + 2f(b)$).

Clearly, (PD4) implies (PD5). A prediameter satisfying (PD4) is referred to as a *diameter*. Moreover, a prediameter satisfying (PD5) is a *weak diameter* (cf. [24]). The latter should not be confused with the notion of *strong prediameter* (cf. [19]) — i.e., a prediameter which additionally satisfies

(PD6) If $S \subseteq L$ is such that $a \wedge b \neq 0$ for all $a, b \in S$, then $f(\bigvee S) \leq 2 \sup\{ f(s) \mid s \in S \}$.

Clearly, every strong prediameter is a weak diameter. For our purposes we shall be interested only in weak diameters, but in passing we shall also present an application to strong prediameters. The following lemma about weak diameters will be crucial in the proof of our uniform insertion theorem.

Lemma 3.1. *Let f be a weak diameter on a locale L. Let $a_1, \dots, a_k \in L$ with $a_{i-1} \wedge a_i \neq 0$ for all $i = 2, \dots, k$. Then,*

$$f(\bigvee_{j=1}^{k} a_j) \leq 2f(a_1 \vee a_2) + 4\sum_{i=3}^{k-1} f(a_{i-1} \vee a_i) + 2f(a_{k-1} \vee a_k).$$

Proof. Obviously we can assume that every summand in the right hand side is finite (and in that case, by (PD5), the left hand side is also readily seen to be finite, and so each $f(\bigvee_{j=1}^{i} a_j)$ is also finite). We proceed by induction over k. If $k = 1$ or $k = 2$ there is nothing to prove. If $k = 3$, we have $f(a_1 \vee a_2 \vee a_3) \leq 2f(a_1 \vee a_2) + 2f(a_2 \vee a_3)$ by (PD5). Assume now it holds for all sequences of length $< k$ and let $a_1, \dots, a_k \in L$ with $a_{i-1} \wedge a_i \neq 0$ for all $i = 2, \dots, k$. Let

$$A := \{ i \in \{ 1, \dots, k \} \mid f(\bigvee_{j=1}^{k} a_j) \leq 2f(\bigvee_{\ell=1}^{i} a_\ell) \}.$$

One has trivially $k \in A$, so $A \neq \varnothing$, hence there is a well-defined $m = \min A$. If $m = 1$ or $m = 2$, the formula in the statement holds trivially so assume $m > 2$. By minimality (and because $m > 1$) $m - 1 \notin A$ — i.e., $2f(\bigvee_{\ell=1}^{m-1} a_\ell) < f(\bigvee_{j=1}^{k} a_j)$.

Now, by way of contradiction suppose $2f(\bigvee_{\ell=m-1}^{k} a_\ell) < f(\bigvee_{j=1}^{k} a_j)$. Then

$$2\max\{ f(\bigvee_{\ell=1}^{m-1} a_\ell), f(\bigvee_{\ell=m-1}^{k} a_\ell \}) < f(\bigvee_{j=1}^{k} a_j). \tag{3.1}$$

But

$$f(\bigvee_{j=1}^{k} a_j) = f(\bigvee_{\ell=1}^{m-1} a_\ell \vee \bigvee_{\ell=m-1}^{k} a_\ell)$$

and $(\bigvee_{\ell=1}^{m-1} a_\ell) \wedge (\bigvee_{\ell=m-1}^{k} a_\ell) \geq a_{m-1} \neq 0$, so by (PD5) it follows that

$$f(\bigvee_{j=1}^{k} a_j) \leq 2\max\{ f(\bigvee_{\ell=1}^{m-1} a_\ell), f(\bigvee_{\ell=m-1}^{k} a_\ell) \}. \tag{3.2}$$

Combining (3.1) and (3.2) we reach a contradiction. Hence, we have

$$f(\bigvee_{j=1}^{k} a_j) \leq 2f(\bigvee_{\ell=m-1}^{k} a_\ell). \tag{3.3}$$

Now, if $m = k$, from (3.3) we see that the desired formula holds, so we may as well assume $m < k$. Now, we have

$$f(\bigvee_{j=1}^{k} a_j) = \tfrac{1}{2} f(\bigvee_{j=1}^{k} a_j) + \tfrac{1}{2} f(\bigvee_{j=1}^{k} a_j) \leq f(\bigvee_{\ell=1}^{m} a_\ell) + f(\bigvee_{\ell=m-1}^{k} a_\ell) \tag{3.4}$$

(because of (3.3) and the fact that $m \in A$). We use induction twice:

$$f(\bigvee_{\ell=1}^{m} a_\ell) \leq 2f(a_1 \vee a_2) + 4 \sum_{i=3}^{m-1} f(a_{i-1} \vee a_i) + 2f(a_{m-1} \vee a_m)$$

and

$$f(\bigvee_{\ell=m-1}^{k} a_\ell) \leq 2f(a_{m-1} \vee a_m) + 4 \sum_{i=m+1}^{k-1} f(a_{i-1} \vee a_i) + 2f(a_{k-1} \vee a_k).$$

This together with (3.4) gives the desired inequality. □

The combination of the previous lemma with (PD5) yields the following

Corollary 3.2. *Let f be a weak diameter on a locale L. Let $a_1, \ldots, a_k \in L$ with $a_i \wedge a_{i-1} \neq 0$ for all $i = 2, \ldots, k$. Then*

$$f(\bigvee_{j=1}^{k} a_j) \leq 4f(a_1) + 12f(a_2) + 16 \sum_{i=3}^{k-2} f(a_i) + 12f(a_{k-1}) + 4f(a_k).$$

Remark 3.3. The last corollary is, in a certain sense, an improvement of [24, Lemma 3.9] (cf. also [19, Lemma XI.3.2.4]), which shows a similar inequality whenever f satisfies a property stronger than (PD5) —too strong for our purposes—, namely:

(3W) If $a, b, c \in L$ are such that $a \wedge b \neq 0 \neq b \wedge c$, then $f(a \vee b \vee c) \leq 2 \max\{ f(a), f(b), f(c) \}$.

Of course, the price one has to pay for considering (PD5) instead of (3W) is that the inequality in Corollary 3.2 is not as sharp as that in [24, Lemma 3.9].

We close this section with an application of Lemma 3.1 to strong prediameters. For that, we first recall the definition of star-additive diameter (see [19, XI.1.2]). It is an important notion, since any such diameter immediately induces a uniformity on L, and it can be satisfactorily approximated by a *metric diameter* (see [19, XI.1.3]). Precisely, a diameter f is said to be *star-additive* if

(DS) If $a \in L$ and $S \subseteq L$ are such that $a \wedge b \neq 0$ for all $b \in S$, then $f(a \vee \bigvee S) \leq f(a) + \sup\{ f(b) + f(c) \mid b, c \in S \}$.

We then have the following (compare with [19, Proposition XI.3.2.5]):

Proposition 3.4. *Let L be a locale and f a strong prediameter on L. Then there is a star-additive diameter d on L such that*

$$\tfrac{1}{32} f \leq d \leq f.$$

We omit the details of the proof, as it is very similar to [19, Proposition XI.3.2.5] (instead of [19, Lemma XI.3.2.4] and property (3W), one uses Lemma 3.1).

In the remainder of this section, we shall specialize the previous results towards proving the uniform insertion theorem.

Lemma 3.5. *Let L be a locale, $\{V_n\}_{n\in\mathbb{Z}}$ a sequence of covers with $V_{n-1} \leq_3^* V_n$ for all $n \in \mathbb{Z}$ and set $f\colon L \to [0,\infty]$ given by*

$$f(a) = \inf\{\, 2^n \mid \exists u \in V_n \text{ with } a \leq u \,\}.$$

Then f is a weak diameter on L.

Proof. Note that if $f(a) \leq 2^n$, then there is a $v \in V_n$ such that $a \leq v$. Properties (PD1) and (PD2) are obvious, and (PD3) follows from the fact that each V_n is a cover and $V_n \subseteq \{\, a \in L \mid f(a) < 2^{n+1} \,\}$. Let us finally show (PD5) so let $a, b \in L$ such that $a \wedge b \neq 0$. If $f(a) = +\infty$ or $f(b) = +\infty$, then there is nothing to prove. Now assume without loss of generality that $f(a) \leq f(b) < +\infty$. If $f(b) = 0$, then $f(a) = 0$ — i.e., for all $n \in \mathbb{Z}$ there are $u_n, v_n \in V_n$ with $a \leq u_n$ and $b \leq v_n$. Now, let $n \in \mathbb{Z}$. Then $u_{n-1} \wedge v_{n-1} \geq a \wedge b \neq 0$, and since $V_{n-1} \leq_3^* V_n$, there is a $v \in V_n$ with $u_{n-1} \vee v_{n-1} \leq v$. Consequently $a \vee b \leq v$ and so $f(a \vee b) = 0$. Assume $f(b) = 2^n$, then there are $u, v \in V_n$ with $a \leq u$ and $b \leq v$. Since $u \wedge v \neq 0$ and $V_n \leq_3^* V_{n+1}$, there is a $w \in V_{n+1}$ such that $u, v \leq w$. Hence, $a \vee b \leq w$ and so $f(a \vee b) \leq 2^{n+1} = 2 \cdot 2^n$, as desired. □

Remarks 3.6.

(1) It is easy to check that the previous lemma also holds when one replaces the relation $\leq_3^*$ by $\leq_2^*$ and the words "weak diameter" by "strong prediameter".

(2) The lemma above can be clearly adapted to a sequence $\{V_n\}_{n\in\mathbb{N}}$ with $V_{n+1} \leq_3^* V_n$ in which case $f\colon L \to [0,\infty]$ is given by $f(a) = \inf\{\, 2^{-n} \mid \exists u \in V_n \text{ with } a \leq u \,\}$.

We also state the following for future reference:

Corollary 3.7. *Let L be a locale, $\{V_n\}_{n\in\mathbb{Z}}$ a sequence of covers with $V_{n-1} \leq_3^* V_n$ for all $n \in \mathbb{Z}$. Let $a_1, \ldots, a_k \in L$ with $a_{i-1} \wedge a_i \neq 0$ for all $i = 2, \ldots, k$, and suppose that $a_i \in V_{n_i}$ for all $i = 1, \ldots, k$. Suppose also that*

$$\sum_{i=1}^{k} 2^{n_i+4} < 2^n.$$

Then there is a $v \in V_{n-1}$ such that $a_1, a_k \leq v$.

Proof. Let f denote the weak diameter given by Lemma 3.5. By the definition of f, we have $f(a_i) \leq 2^{n_i}$ for all $i = 1, \ldots, k$. In particular,

$$4f(a_1) + 12f(a_2) + 16\sum_{i=3}^{k-2} f(a_i) + 12f(a_{k-1}) + 4f(a_k) \leq 16\sum_{i=1}^{k} f(a_i) \leq \sum_{i=1}^{k} 2^{n_i+4} < 2^n.$$

By Corollary 3.2, $f(a_1 \vee a_k) < 2^n$, so it follows by the definition of f that there is a $v \in V_{n-1}$ with $a_1 \vee a_k \leq v$. □

4. Farness for sublocales and characterizations of uniform continuity

4.1. Covers of $\mathsf{S}(L)$

A subset $\mathfrak{U} \subseteq \mathsf{S}(L)$ is a *cover* of $\mathsf{S}(L)$ if $\bigvee \mathfrak{U} = L$. In this context, we shall say that a cover $\mathfrak{U}$ of $\mathsf{S}(L)$ *refines* a cover $\mathfrak{V}$ of $\mathsf{S}(L)$ if for every $S \in \mathfrak{U}$ there is a $T \in \mathfrak{V}$ such that $S \subseteq T$. In that case we shall write $\mathfrak{U} \leq \mathfrak{V}$.

We will be particularly interested in *open covers* of $\mathsf{S}(L)$, that is, covers of the form

$$\mathfrak{o}[U] := \{\mathfrak{o}(u) \mid u \in U\}$$

for a cover U of L. Observe that if U and V are covers of L, then $U \leq V$ in the sense of Subsection 2.2 if and only if $\mathfrak{o}[U] \leq \mathfrak{o}[V]$.

4.2. Farness for general sublocales

Let $\mathfrak{U}$ be a cover of $\mathsf{S}(L)$. Then, sublocales S and T of L are said to be $\mathfrak{U}$*-far* if

$$\forall D \in \mathfrak{U}, \quad D \cap S \neq \mathsf{O} \implies D \cap T = \mathsf{O}.$$

The following observations are trivial:

Remarks 4.1. Let $\mathfrak{U}$ be a cover of $\mathsf{S}(L)$, and let S, T be sublocales of L. Then:

(1) If S and T are $\mathfrak{U}$-far and $S' \subseteq S$ and $T' \subseteq T$, then S' and T' are also $\mathfrak{U}$-far;
(2) If $\mathfrak{U} \leq \mathfrak{V}$ and S and T are $\mathfrak{V}$-far, then S and T are also $\mathfrak{U}$-far.

With only a couple of exceptions, we shall be interested in the case where the cover $\mathfrak{U}$ is open, say $\mathfrak{U} = \mathfrak{o}[U]$ for a cover U of L. In that case, we shall simply say that S and T are U*-far* when they are $\mathfrak{o}[U]$-far. This notion coincides with that of [2] (see Subsection 2.2 above) in the sense that elements a and b of L are U-far if and only if $\mathfrak{o}(a)$ and $\mathfrak{o}(b)$ are U-far.

Given a cover U of L and a sublocale $S \subseteq L$, we set

$$U * S := \bigvee \{\, \mathfrak{o}(u) \mid u \in U,\ \mathfrak{o}(u) \cap S \neq \mathsf{O} \,\}$$

(see also [13] or [11,20] for this concept in the more general context of nearness structures). Notice that $U * S$ is an open sublocale of L, and that $S \subseteq U * S$ (the latter follows easily because U is a cover and because families of open sublocales are *distributive* — i.e., $S \cap \bigvee_{i \in I} \mathfrak{o}(a_i) = \bigvee_{i \in I} S \cap \mathfrak{o}(a_i)$ for every $\{a_i\}_{i \in I} \subseteq L$, cf. [21]). Note also that for every $a \in L$ one has $U * \mathfrak{o}(a) = \mathfrak{o}(Ua)$. Moreover, if $S \subseteq T$, then $U * S \subseteq U * T$.

In the case of open covers, we can give a few more characterizations of farness:

Proposition 4.2. *Let L be a locale and U a cover of L. For sublocales S and T of L, the following conditions are equivalent:*

(i) *S and T are U-far;*
(ii) $(U * S) \cap T = \mathsf{O}$;
(iii) $T \subseteq (U * S)^{\#}$;
(iv) *$\overline{S}$ and $\overline{T}$ are U-far.*

Moreover, if S and T are U-far, then $\overline{S}\cap\overline{T}=\mathsf{O}$.

Proof. (i) $\Longleftrightarrow$ (ii): Since families of open covers are distributive, $(U*S)\cap T=\bigvee\{\,\mathfrak{o}(u)\cap T\mid u\in U,\ \mathfrak{o}(u)\cap S\neq\mathsf{O}\,\}$. Then, $(U*S)\cap T=\mathsf{O}$ iff for each $u\in U$, $\mathfrak{o}(u)\cap S\neq\mathsf{O}$ implies $\mathfrak{o}(u)\cap T=\mathsf{O}$ — i.e., iff S and T are U-far.

(ii) $\Longleftrightarrow$ (iii): This equivalence follows because $U*S$ is open and hence complemented.

(i) $\Longleftrightarrow$ (iv): Assume that S and T are U-far; equivalently one has $T\subseteq(U*S)^{\#}$ and since $(U*S)^{\#}$ is closed, it follows that $\overline{T}\subseteq(U*S)^{\#}$. The latter is in turn equivalent to $\overline{T}$ and S being U-far. Now, (iv) follows repeating the argument with S and $\overline{T}$. The reverse implication is trivial by Remark 4.1 (1).

For the last assertion, if S and T are U-far, then so are $\overline{S}$ and $\overline{T}$ and by (ii) it follows that $\overline{S}\cap\overline{T}\subseteq(U*\overline{S})\cap\overline{T}=\mathsf{O}$. □

Since being U-far is a symmetric relation, we may exchange the roles of S and T in the conditions of the previous proposition. We also have the following:

Corollary 4.3. *Let L be a locale and U a cover of L. For sublocales S and T of L, the following conditions are equivalent*:

(i) $U*S\subseteq T$;
(ii) $U*\overline{S}\subseteq\operatorname{int}T$;
(iii) *S and $T^{\#}$ are U-far.*

Proof. The equivalence between (i) and (ii) follows since

$$\begin{aligned} U*S\subseteq T &\iff U*S\subseteq\operatorname{int}T && \text{because } U*S \text{ is open,}\\ &\iff U*S\cap(\operatorname{int}T)^{\#}=\mathsf{O} && \text{because } \operatorname{int}T \text{ is complemented,}\\ &\iff U*\overline{S}\cap(\operatorname{int}T)^{\#}=\mathsf{O} && \text{because of Proposition 4.2,}\\ &\iff U*\overline{S}\subseteq\operatorname{int}T && \text{because } \operatorname{int}T \text{ is complemented.}\end{aligned}$$

Now, $U*\overline{S}\subseteq\operatorname{int}T$ if and only if $\overline{S}$ and $(\operatorname{int}T)^{\#}=\overline{T^{\#}}$ are U-far (see [7, Eq. 4.2] for the equality), which by Proposition 4.2 holds if and only if S and $T^{\#}$ are U-far. Thus the equivalence between (ii) and (iii) follows. □

4.3. Arbitrary real functions and uniform homomorphisms

Let L and M be frames and let $\mathcal{U}$ (resp. $\mathcal{V}$) be a basis for a (pre)uniformity on L (resp. M). Recall that a frame homomorphism $f\colon L\to M$ is a *uniform homomorphism* $(L,\mathcal{U})\to(M,\mathcal{V})$ if for every $U\in\mathcal{U}$ there is some $V\in\mathcal{V}$ such that $V\leq h[U]$. We are particularly interested in the uniform homomorphisms $\mathfrak{L}(\mathbb{R})\to(L,\mathcal{U})$ where $\mathfrak{L}(\mathbb{R})$ is endowed with its natural metric uniformity whose basis consists of the covers

$$D_\delta=\left\{\,(r,s)\in\mathfrak{L}(\mathbb{R})\mid s-r=\tfrac{1}{\delta}\,\right\}$$

for $\delta\in\mathbb{Q}^+$ (cf. Subsection 2.2.5). In other words, a real-valued function $f\in\mathcal{R}(L)$ is a uniform homomorphism if for every $n\in\mathbb{N}$ there is a $U\in\mathcal{U}$ such that $U\leq f[D_n]=\{\,f(r,s)\mid(r,s)\in D_n\,\}$. In view of this, we introduce the following terminology:

Definition 4.4. Let $(L, \mathcal{U})$ be a preuniform frame. An $f \in \mathsf{F}(L)$ is *uniformly continuous* if for every $n \in \mathbb{N}$ there is a $U \in \mathcal{U}$ such that $\mathfrak{o}[U] \leq \{\, f(r,s)^{\#} \mid (r,s) \in D_n \,\}$.

Remarks 4.5.

(1) The set $\{\, f(r,s)^{\#} \mid (r,s) \in D_n \,\}$ is a cover of $\mathsf{S}(L)$. Indeed, more generally let $f, g \in \mathsf{F}(L)$ with $f \geq g$ and let $\delta \in \mathbb{Q}^+$. Consider the following subset of $\mathsf{S}(L)$:

$$D_\delta^{f,g} := \{\, (f(r,—) \vee g(—,s))^{\#} \mid (r,s) \in D_\delta \,\}.$$

If $f = g$ we simply denote $D_\delta^f := D_\delta^{f,f} = \{\, f(r,s)^{\#} \mid (r,s) \in D_n \,\}$. Notice that, since $f \geq g$ and D_δ is a cover of $\mathfrak{L}(\mathbb{R})$, we have

$$\begin{aligned} \bigvee D_\delta^{f,g} &= \bigvee \{\, (f(r,—) \vee g(—,s))^{\#} \mid (r,s) \in D_\delta \,\} \supseteq \bigvee \{\, (g(r,—) \vee g(—,s))^{\#} \mid (r,s) \in D_\delta \,\} \\ &= \bigvee \{\, g(r,s)^{\#} \mid (r,s) \in D_\delta \,\} = (\bigcap \{\, g(r,s) \mid (r,s) \in D_\delta \,\})^{\#} = L \end{aligned}$$

— i.e., $D_\delta^{f,g}$ is a cover of $\mathsf{S}(L)$.

(2) If $f \in \mathsf{C}(L)$ (i.e., $f \colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ is of the form $f = \mathfrak{c} \circ g$ for a frame homomorphism $g \colon \mathfrak{L}(\mathbb{R}) \to L$), it is clear that f is uniformly continuous (in the sense just defined) if and only if g is a uniform homomorphism.

(3) Actually, it is not necessary to require f to be continuous in order to recover the usual notion of uniform continuity. Indeed, we shall show in Proposition 4.8 below that uniform continuity (via Definition 4.4) implies continuity. Hence, by virtue of (2), uniformly continuous maps in $\mathsf{F}(L)$ correspond precisely to uniform homomorphisms $\mathfrak{L}(\mathbb{R}) \to (L, \mathcal{U})$, thus ensuring that this is the right notion of uniform continuity for maps in $\mathsf{F}(L)$.

Before proving Proposition 4.8, we need a couple of lemmas.

Lemma 4.6. *Let $(L, \mathcal{U})$ be a preuniform frame and $f, g \in \mathsf{F}(L)$ with $f \geq g$. For every $\delta \in \mathbb{Q}^+$ and every $r, s \in \mathbb{Q}$ with $s - r > \frac{1}{\delta}$ the sublocales $f(r,—)$ and $g(—,s)$ are $D_\delta^{f,g}$-far.*

Proof. Let $\delta \in \mathbb{Q}^+$. Suppose by contradiction that there are $r, s \in \mathbb{Q}$ with $s - r > \frac{1}{\delta}$ such that $f(r,—)$ and $g(—,s)$ are not $D_\delta^{f,g}$-far. Then there is an $(r', s') \in D_\delta^{f,g}$ such that

$$f(r,—) \cap (f(r',—) \vee g(—,s'))^{\#} \neq \mathsf{O} \quad \text{and} \quad g(—,s) \cap (f(r',—) \vee g(—,s'))^{\#} \neq \mathsf{O}.$$

Now, by Remarks 2.1 one has that $(f(r',—) \vee g(—,s'))^{\#} \subseteq f(—,r') \cap g(s',—)$ and so

$$f(r,—) \cap f(—,r') \cap g(s',—) \neq \mathsf{O} \quad \text{and} \quad g(—,s) \cap f(—,r') \cap g(s',—) \neq \mathsf{O}.$$

In particular, we have

$$f(r,—) \cap f(—,r') \neq \mathsf{O} \quad \text{and} \quad g(—,s) \cap g(s',—) \neq \mathsf{O}.$$

Hence, $r' \leq r$ and $s \leq s'$. This means that $\frac{1}{\delta} = s' - r' \geq s - r > \frac{1}{\delta}$, a contradiction. □

Lemma 4.7. *Let $(L, \mathcal{U})$ be a preuniform frame and $f \in \mathsf{F}(L)$ be such that for every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathcal{U}$ such that the sublocales $f(r,—)$ and $f(—,s)$ are U-far whenever $s - r > \frac{1}{\delta}$. Then f is continuous (i.e., $f \in \mathsf{C}(L)$).*

Proof. For each $\delta \in \mathbb{Q}^+$, let U_δ denote the uniform cover such that $f(r,\text{—})$ and $f(\text{—},s)$ are U_δ-far whenever $s-r>\frac{1}{\delta}$. To show that f is continuous we have to prove that for every $r,s\in\mathbb{Q}$, the sublocales $f(r,\text{—})$ and $f(\text{—},s)$ are closed. For each $r\in\mathbb{Q}$, by Proposition 4.2 we obtain

$$f(r,\text{—}) \subseteq \left(U_\delta * f(\text{—}, r+\tfrac{2}{\delta})\right)^{\#}$$

for every $\delta\in\mathbb{Q}^+$. Thus,

$$f(r,\text{—}) \subseteq \bigcap_{\delta\in\mathbb{Q}^+} \left(U_\delta * f(\text{—}, r+\tfrac{2}{\delta})\right)^{\#}.$$

From Remarks 2.1 and the fact that $S\subseteq U*S$ for any sublocale $S\in\mathsf{S}(L)$, we obtain

$$\begin{aligned}\bigcap_{r<t} f(\text{—},t)^{\#} \subseteq \bigcap_{r<t} f(t,\text{—}) = f(r,\text{—}) \subseteq \bigcap_{\delta\in\mathbb{Q}^+}\left(U_\delta * f(\text{—}, r+\tfrac{2}{\delta})\right)^{\#} \subseteq \bigcap_{\delta\in\mathbb{Q}^+} f(\text{—}, r+\tfrac{2}{\delta})^{\#} = \\ = \bigcap_{t>r} f(\text{—},t)^{\#}.\end{aligned}$$

Since $\bigcap_{\delta\in\mathbb{Q}^+}\left(U_\delta * f(\text{—}, r+\frac{2}{\delta})\right)^{\#}$ is closed, $f(r,\text{—})$ is a closed sublocale for every $r\in\mathbb{Q}$. Similarly, we can conclude that $f(\text{—},s)$ is closed for every $s\in\mathbb{Q}$. □

Proposition 4.8. *Let $(L,\mathcal{U})$ be a preuniform frame and $f\in\mathsf{F}(L)$ be uniformly continuous. Then f is continuous (i.e., $f\in\mathsf{C}(L)$).*

Proof. Let us check that the assumption of Lemma 4.7 is satisfied. Let $\delta\in\mathbb{Q}^+$ and select an $n\in\mathbb{N}$ with $\frac{1}{n}\le\frac{1}{\delta}$. By uniform continuity, there is a uniform cover $U\in\mathcal{U}$ of L such that $\mathfrak{o}[U]\le D^f_n$. By Lemma 4.6 it follows that $f(r,\text{—})$ and $f(\text{—},s)$ are D^f_n-far whenever $s-r>\frac{1}{n}$. Therefore, by Remark 4.1 (2), one has that $f(r,\text{—})$ and $f(\text{—},s)$ are U-far whenever $s-r>\frac{1}{\delta}$. □

We are interested in giving a few more characterizations of uniform continuity via farness. Recall that the map $\mathfrak{c}\colon L\to\mathsf{S}(L)^{op}$ denotes the canonical frame homomorphism that sends $a\in L$ to $\mathfrak{c}(a)$. We first need the following.

Lemma 4.9. *Let $(L,\mathcal{U})$ be a preuniform frame and let U be a cover of L. If $\delta\in\mathbb{Q}^+$ and $f,g\in\mathcal{R}(L)$ are such that $f\ge g$, then the following are equivalent:*

(i) *the elements $f(\text{—},r)$ and $g(s,\text{—})$ are U-far whenever $s-r>\frac{1}{\delta}$;*
(ii) *the sublocales $\mathfrak{c}f(r,\text{—})$ and $\mathfrak{c}g(\text{—},s)$ are U-far whenever $s-r>\frac{1}{\delta}$.*

Proof. First, notice for any $r,s\in\mathbb{Q}$ we have the following equivalences:

$$\begin{aligned}\mathfrak{c}f(r,\text{—}) \text{ and } \mathfrak{c}g(\text{—},s) \text{ are } U\text{-far} &\iff \forall u\in U,\quad \mathfrak{c}f(r,\text{—})\cap\mathfrak{o}(u)=\mathsf{O} \\ &\qquad\text{or } \mathfrak{c}g(\text{—},s)\cap\mathfrak{o}(u)=\mathsf{O} \\ &\iff \forall u\in U,\quad \mathfrak{c}f(r,\text{—})\subseteq\mathfrak{c}(u) \text{ or } \mathfrak{c}g(\text{—},s)\subseteq\mathfrak{c}(u) \\ &\iff \forall u\in U,\quad u\le f(r,\text{—}) \text{ or } u\le g(\text{—},s).\end{aligned}$$

Now, assume (i) holds and let $r,s\in\mathbb{Q}$ with $s-r>\frac{1}{\delta}$. Select $p,q\in\mathbb{Q}$ such that $r<p<q<s$ with $q-p>\frac{1}{\delta}$. Then, $f(\text{—},p)$ and $g(q,\text{—})$ are U-far — i.e., for all $u\in U$, one has $u\le f(\text{—},p)^*$ or $u\le g(q,\text{—})^*$ (cf. Proposition 2.5). By Remarks 2.1, it follows that for all $u\in U$ either $u\le f(r,\text{—})$ or $u\le g(\text{—},s)$. By the equivalences above, $\mathfrak{c}f(r,\text{—})$ and $\mathfrak{c}g(\text{—},s)$ are U-far. The converse is even easier (it follows at once from the equivalences above and Remarks 2.1). □

Proposition 4.10. *Let $(L,\boldsymbol{\mathcal{U}})$ be a preuniform frame and $f \in \mathcal{R}(L)$. Then the following are equivalent:*

(i) *$\mathfrak{c}f$ is uniformly continuous;*
(ii) *f is a uniform homomorphism;*
(iii) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that the elements $f(\text{—},r)$ and $f(s,\text{—})$ are U-far whenever $s - r > \frac{1}{\delta}$;*
(iv) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that for every $n \in \mathbb{N}$ the elements $f(\text{—},r)$ and $f(s,\text{—})$ are U^n-far whenever $s - r > \frac{n}{\delta}$;*
(v) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that the sublocales $\mathfrak{c}f(r,\text{—})$ and $\mathfrak{c}f(\text{—},s)$ are U-far whenever $s - r > \frac{1}{\delta}$;*
(vi) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that for every $n \in \mathbb{N}$ the sublocales $\mathfrak{c}f(r,\text{—})$ and $\mathfrak{c}f(\text{—},s)$ are U^n-far whenever $s - r > \frac{n}{\delta}$.*

Proof. The equivalence between (ii), (iii) and (iv) is proved in [2, Theorem 3.1]. Moreover, (iii) and (v) are equivalent by Lemma 4.9, and so are (iv) and (vi). Finally, (i) and (ii) are equivalent as observed in Remark 4.5 (2). □

As a consequence, we also have the following characterization of uniform continuity in terms of farness:

Corollary 4.11. *Let $(L,\boldsymbol{\mathcal{U}})$ be a preuniform frame. Then $f \in \mathsf{F}(L)$ is uniformly continuous if and only if for every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that the sublocales $f(r,\text{—})$ and $f(\text{—},s)$ are U-far whenever $s - r > \frac{1}{\delta}$.*

Proof. The "only if" implication follows from Proposition 4.8 and Proposition 4.10 (v) whereas the "if" holds because of Lemma 4.7 and Proposition 4.10 (v). □

We end up this section with a useful lemma which deals with uniformly continuous functions in terms of scales.

Lemma 4.12. *Let $(L,\boldsymbol{\mathcal{U}})$ be a preuniform frame. If a family $\{S_r\}_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ satisfies the following conditions:*

(1) $\bigcap_{r\in\mathbb{Q}} S_r = \mathsf{O} = \bigcap_{r\in\mathbb{Q}} S_r^{\#}$, *and*
(2) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that $U * S_r \subseteq S_s$ (resp. $U * S_s \subseteq S_r$) whenever $s - r > \frac{1}{\delta}$,*

then the formulas

$$h(p,\text{—}) = \bigcap_{r>p} S_r \quad \text{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^{\#}$$

(*resp.*

$$h(p,\text{—}) = \bigcap_{r>p} S_r^{\#} \quad \text{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s)$$

define a uniformly continuous $h \in \mathsf{F}(L)$.

Proof. Let $\{S_r\}_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ be a family of sublocales such that (1) holds and for every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that $U * S_r \subseteq S_s$ whenever $s - r > \frac{1}{\delta}$. First, we claim $\{S_r\}_{r\in\mathbb{Q}}$ is a descending scale in $\mathsf{S}(L)^{op}$.

For every $r < s$ we have that there is a $U \in \mathcal{U}$ such that $U * S_r \subseteq S_s$. From Corollary 4.3 we get $\overline{S_r} \subseteq \operatorname{int} S_s$ which implies

$$S_r \cap S_s^{\#} \subseteq \overline{S_r} \cap (\operatorname{int} S_s)^{\#} = \mathsf{O}.$$

— i.e., $\{S_r\}_{r\in\mathbb{Q}}$ is a descending scale in $\mathsf{S}(L)^{op}$. Hence, the formulas

$$h(p,\text{—}) = \bigcap_{r>p} S_r \quad \text{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^{\#}$$

determine an $h \in \mathsf{F}(L)$. Now, let $\delta \in \mathbb{Q}^+$ and take the $U \in \mathcal{U}$ given by (2). Let $p, q \in \mathbb{Q}$ such that $q - p > \frac{1}{\delta}$. Select $r', s' \in \mathbb{Q}$ such that $p < r' < s' < q$ and $s' - r' > \frac{1}{\delta}$, then $U * S_{r'} \subseteq S_{s'}$. By Corollary 4.3, we have that $\overline{S_{r'}}$ and $(\operatorname{int} S_{s'})^{\#}$ are U-far. Now,

$$h(p,\text{—}) = \bigcap_{r>p} S_r \subseteq S_{r'} \subseteq \overline{S_{r'}} \quad \text{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^{\#} \subseteq S_{s'}^{\#} \subseteq (\operatorname{int} S_{s'})^{\#}$$

so, by Remark 4.1 (1), $h(p,\text{—})$ and $h(\text{—},q)$ are U-far. Thus h is uniformly continuous by Corollary 4.11. Similarly, one can prove the statement inside parentheses. □

5. Insertion theorem for uniform locales

Lemma 5.1. *Let $(L, \mathcal{U})$ be a preuniform frame and let $f, g \in \mathsf{F}(L)$ with $f \geq g$. Assume that for every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathcal{U}$ such that for every $n \in \mathbb{N}$, the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are U^n-far whenever $s-r > \frac{n}{\delta}$. Then, there is a sequence $\{V_n\}_{n\in\mathbb{Z}} \subseteq \mathcal{U}$ such that for every $n \in \mathbb{Z}$ the following properties are satisfied:*

(1) $V_n \leq_1^* V_{n+1}$;
(2) *For every $r, s \in \mathbb{Q}$ such that $s - r > 2^n$, the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are V_n-far.*

Proof. Let V_0 be the cover given by the assumption by choosing $\delta = 1$. Moreover, for $n \geq 1$, set $V_n := (V_0)^{2^n}$. Clearly, property (2) is satisfied when $n \geq 0$. Now, for $n \geq 0$, condition (1) is also satisfied. Indeed, by an application of Lemma 2.4 (3), we have that

$$V_n V_0 = (V_0)^{2^n} V_0 = (V_0)^{2^{n+1}} = V_{n+1},$$

hence $V_n \leq_1^* V_{n+1}$. Now we recursively define V_n for $n < 0$. First, for each $n < 0$, let U_n denote the cover given by the assumption for $\delta = \frac{1}{2^n}$. For $n = -1$, pick a $V_{-1} \in \mathcal{U}$ such that $V_{-1}^2 \leq V_0 \wedge U_{-1}$ (recall the axiom (U3)). Clearly, conditions (1) and (2) are satisfied (the refinement $V_{-1} \leq V_0$ is a star-refinement, so *a fortiori* it is barycentric). Suppose now that for an $n < 0$ we have constructed $V_n, V_{n+1}, \ldots, V_{-1}$ satisfying (1) and (2). Then we choose a $V_{n-1} \in \mathcal{U}$ such that $V_{n-1}^2 \leq V_n \wedge U_{n-1}$. The sequence $\{V_n\}_{n\in\mathbb{Z}}$ clearly satisfies the required conditions. □

We are now ready to prove the main result of this paper.

Theorem 5.2 *(Uniform insertion theorem). Let $(L, \mathcal{U})$ be a preuniform frame and $f, g \in \mathsf{F}(L)$ with $f \geq g$. Then the following are equivalent:*

(i) *There exists a uniformly continuous $h \in \mathsf{F}(L)$ such that $f \geq h \geq g$;*
(ii) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathcal{U}$ such that for every $n \in \mathbb{N}$ the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are U^n-far whenever $s - r > \frac{n}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii): This implication follows at once from Proposition 4.10 (vi), the definition of the partial order in $\mathsf{F}(L)$ (recall (2.3)) and Remark 4.1 (1).

(ii) $\Longrightarrow$ (i): Let $\{V_n\}_{n\in\mathbb{Z}} \subseteq \boldsymbol{\mathcal{U}}$ denote the sequence of uniform covers given by Lemma 5.1. We define a family $\{b_r\}_{r\in\mathbb{Q}} \subseteq L$ as follows

$$b_r := \bigvee_{n\in\mathbb{Z}} \bigvee A_r^n,$$

where

$$\begin{aligned} A_r^n := \big\{ a \in V_n \mid \exists k \in \mathbb{N},\ \exists n_1, \ldots, n_k \in \mathbb{Z}, \\ \exists a_i \in V_{n_i} \text{ for all } i = 1, \ldots, k \text{ such that } a_1 = a,\ n_1 = n, \\ a_{i-1} \wedge a_i \neq 0\ (i = 2, \ldots, k), \text{ and } \mathfrak{o}(a_k) \cap f(r - \sum_{i=1}^{k} 2^{n_i+5}, —) \neq \mathsf{O} \big\}. \end{aligned}$$

Set also $B_r := \mathfrak{o}(b_r)$ for every $r \in \mathbb{Q}$. Clearly, $B_r = \bigvee\{ \mathfrak{o}(a) \mid \exists n \in \mathbb{Z} \text{ with } a \in A_r^n \}$.

First we will show that

$$f(—, r)^{\#} \subseteq B_r \subseteq g(—, r)^{\#} \tag{5.1}$$

for every $r \in \mathbb{Q}$. For the first inclusion, note that for each $n \in \mathbb{Z}$ one clearly has $V_n * f(r - 2^{n+5}, —) = \bigvee\{ \mathfrak{o}(a) \mid a \in V_n,\ \mathfrak{o}(a) \cap f(r - 2^{n+5}, —) \neq \mathsf{O} \} \subseteq B_r$. Consequently,

$$\begin{aligned} B_r &\supseteq \bigvee_{n\in\mathbb{Z}} V_n * f(r - 2^{n+5}, —) \supseteq \bigvee_{n\in\mathbb{Z}} f(r - 2^{n+5}, —) = \bigvee_{s<r} f(s, —) \supseteq \bigvee_{s<r} f(—, s)^{\#} \\ &= \big(\bigcap_{s<r} f(—, s)\big)^{\#} = f(—, r)^{\#}. \end{aligned}$$

Let us now show the inclusion $B_r \subseteq g(—, r)^{\#}$. Let $a \in A_r^n$; our goal is to show that $\mathfrak{o}(a) \subseteq g(—, r)^{\#}$. Since $a \in A_r^n$, there is a $k \in \mathbb{N}$ and there are $n_i \in \mathbb{Z}$ and $a_i \in V_{n_i}$ for all $i = 1, \ldots, k$ satisfying $n_1 = n$, $a_1 = a$, $a_{i-1} \wedge a_i \neq 0$ for every $i = 2, \ldots, k$, and $\mathfrak{o}(a_k) \cap f(r - \sum_{i=1}^{k} 2^{n_i+5}, —) \neq \mathsf{O}$. Take an $m \in \mathbb{Z}$ such that

$$2^{m-1} \leq \sum_{i=1}^{k} 2^{n_i+4} < 2^m.$$

By Corollary 3.7 (recall that barycentric refinement implies regular refinement), there is a $v \in V_{m-1}$ such that $a_1, a_k \leq v$. We have that

$$r - (r - \sum_{i=1}^{k} 2^{n_i+5}) = \sum_{i=1}^{k} 2^{n_i+5} > \sum_{i=1}^{k} 2^{n_i+4} \geq 2^{m-1}$$

so by Lemma 5.1 (2), $f(r - \sum_{i=1}^{k} 2^{n_i+5}, —)$ and $g(—, r)$ are V_{m-1}-far. Consequently,

$$\mathfrak{o}(a) = \mathfrak{o}(a_1) \subseteq \mathfrak{o}(v) \subseteq V_{m-1} * f(r - \sum_{i=1}^{k} 2^{n_i+5}, —) \subseteq g(—, r)^{\#}$$

where the second inclusion holds because $v \in V_{m-1}$ and

$$\mathsf{O} \neq \mathfrak{o}(a_k) \cap f(r - \sum_{i=1}^{k} 2^{n_i+5}, —) \subseteq \mathfrak{o}(v) \cap f(r - \sum_{i=1}^{k} 2^{n_i+5}, —).$$

Hence, (5.1) holds. Now, we will show that the conditions of Lemma 4.12 hold for the family $\{B_r\}_{r\in\mathbb{Q}}$. First, notice that by (5.1) one has

$$\bigcap_{r\in\mathbb{Q}} B_r \subseteq \bigcap_{r\in\mathbb{Q}} g(-,r)^{\#} \subseteq \bigcap_{r\in\mathbb{Q}} g(r,-) = \mathsf{O}$$

and similarly

$$\bigcap_{r\in\mathbb{Q}} B_r^{\#} \subseteq \bigcap_{r\in\mathbb{Q}} f(-,r)^{\#\#} \subseteq \bigcap_{r\in\mathbb{Q}} f(-,r) = \mathsf{O}.$$

Let $\delta \in \mathbb{Q}^+$ and select an $n \in \mathbb{Z}$ such that $\frac{1}{\delta} > 2^{n+5}$. Let $s - r > \frac{1}{\delta}$; we will show that

$$V_n b_r \leq b_s, \tag{5.2}$$

which is clearly equivalent to $V_n * B_r \subseteq B_s$. Now, since $b_r = \bigvee_{m\in\mathbb{Z}} \bigvee A_r^m$, by virtue of Proposition 2.3 (7), proving (5.2) is further equivalent to show that if $a \in A_r^m$ and $v \in V_n$ is such that $v \wedge a \neq 0$, then $v \leq b_s$. If $a \in A_r^m$, there is a $k \in \mathbb{N}$ such that for every $i = 1, \ldots, k$ there is an $a_i \in V_{n_i}$ with $a_1 = a$, $n_1 = m$, $a_{i-1} \wedge a_i \neq 0$ for every $i = 2, \ldots, k$ and $\mathfrak{o}(a_k) \cap f(r - \sum_{i=1}^k 2^{n_i+5}, -) \neq \mathsf{O}$. But since $s - 2^{n+5} > r$ it follows that $f(r - \sum_{i=1}^k 2^{n_i+5}, -) \subseteq f(s - 2^{n+5} - \sum_{i=1}^k 2^{n_i+5}, -)$ and so

$$\mathfrak{o}(a_k) \cap f(s - 2^{n+5} - \sum_{i=1}^k 2^{n_i+5}, -) \neq \mathsf{O}.$$

Hence, if $v \in V_n$ is such that $v \wedge a \neq 0$, it follows that $v \in A_s^n$, which yields $v \leq b_s$, as required.

By Lemma 4.12, the function $h \in \mathsf{F}(L)$ given by

$$h(p,-) = \bigcap_{r>p} B_r \quad \text{and} \quad h(-,q) = \bigcap_{s<q} B_s^{\#}$$

is uniformly continuous. Finally, $f \geq h \geq g$ because, from (5.1) and Remarks 2.1, we have

$$h(p,-) = \bigcap_{r>p} B_r \subseteq \bigcap_{r>p} g(-,r)^{\#} \subseteq \bigcap_{r>p} g(r,-) = g(p,-)$$

and

$$h(-,q) = \bigcap_{s<q} B_s^{\#} \subseteq \bigcap_{s<q} f(-,s)^{\#\#} \subseteq \bigcap_{s<q} f(-,s) = f(-,q)$$

for every $p, q \in \mathbb{Q}$ (see (2.3)). □

5.1. The bounded case

Specializing Theorem 5.2 one can easily obtain the Uniform Insertion Theorem for bounded functions — i.e., what is stated in Theorem 5.4 below. However, in this subsection, we present an alternative (and easier) proof of this special case by using a different technique; namely the so-called Katětov's Lemma (see [15] for the original formulation for power sets). For that purpose, we recall that a binary relation $\Subset$ on a lattice L is a *Katětov relation* if it satisfies the following conditions for all $a, b, a', b' \in L$:

(K1) $a \Subset b \implies a \leq b$;
(K2) $a' \leq a \Subset b \leq b' \implies a' \Subset b'$;

(K3) $a \Subset b$ and $a' \Subset b \implies (a \vee a') \Subset b$;
(K4) $a \Subset b$ and $a \Subset b' \implies a \Subset (b \wedge b')$;
(K5) $a \Subset b \implies \exists c \in L,\ a \Subset c \Subset b$.

The following extends the original idea of Katětov from power sets to complete lattices (cf. [16,17]).

Lemma 5.3 *(Katětov's Lemma). Let L be a complete lattice, $\Subset$ a Katětov relation on L and $\lhd$ a transitive and irreflexive relation on a countable set D. Further, let $\{a_d\}_{d\in D}$ and $\{b_d\}_{d\in D}$ be two families of elements of L such that*

$$d_1 \lhd d_2 \quad \textit{implies} \quad a_{d_2} \leq a_{d_1}, \quad b_{d_2} \leq b_{d_1} \quad \textit{and} \quad a_{d_2} \Subset b_{d_1}.$$

Then there exists a family $\{c_d\}_{d\in D} \subseteq L$ such that

$$d_1 \lhd d_2 \quad \textit{implies} \quad c_{d_2} \Subset c_{d_1}, \quad a_{d_2} \Subset c_{d_1} \quad \textit{and} \quad c_{d_2} \Subset b_{d_1}.$$

Let now $(L, \boldsymbol{\mathcal{U}})$ be a (pre)uniform frame. Then it is readily verified that the relation $\lhd_{\mathcal{U}}$ in $\mathsf{S}(L)$ defined by

$$S \lhd_{\mathcal{U}} T \iff \text{there is a } U \in \boldsymbol{\mathcal{U}} \text{ such that } U * S \subseteq T$$

is a Katětov relation on $\mathsf{S}(L)$.

Theorem 5.4 *(Uniform insertion theorem for bounded functions). Let $(L, \boldsymbol{\mathcal{U}})$ be a preuniform frame and let $f, g \in \mathsf{F}(L)$ be bounded functions with $f \geq g$. Then the following are equivalent*:

(i) *There exists a uniformly continuous $h \in \mathsf{F}(L)$ such that $f \geq h \geq g$;*
(ii) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \boldsymbol{\mathcal{U}}$ such that the sublocales $f(r, \text{—})$ and $g(\text{—}, s)$ are U-far whenever $s - r > \frac{1}{\delta}$.*

Proof. (i) $\implies$ (ii): This implication follows at once from Proposition 4.10 (v), the definition of the partial order in $\mathsf{F}(L)$ (recall (2.3)) and Remark 4.1 (1).

(ii) $\implies$ (i): Since f and g are bounded, by (2.5) take $\alpha, \beta \in \mathbb{Q}$ with $\alpha < \beta$ such that

$$f(\beta, \text{—}) = L, \quad f(\text{—}, \alpha) = L, \quad g(\beta, \text{—}) = L \quad \text{and} \quad g(\text{—}, \alpha) = L. \tag{5.3}$$

By assumption, one has in particular that $g(\text{—}, s) \lhd_{\mathcal{U}} f(r, \text{—})^{\#}$ for every $s > r$. Since $\lhd_{\mathcal{U}}$ is a Katětov relation, by Lemma 5.3 there is a family $\{C_p\}_{p\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ such that

$$g(\text{—}, s) \lhd_{\mathcal{U}} C_q \lhd_{\mathcal{U}} C_p \lhd_{\mathcal{U}} f(r, \text{—})^{\#} \tag{5.4}$$

whenever $r < p < q < s$. We will use Lemma 4.12 to show that $\{C_p\}_{p\in\mathbb{Q}}$ determines a uniformly continuous function. First, from (5.4) it is easy to see that $\bigcap_{p\in\mathbb{Q}} C_p = \mathsf{O} = \bigcap_{p\in\mathbb{Q}} C_p^{\#}$. We only have to show that

$$\forall \delta \in \mathbb{Q}^+, \text{ there is some } U \in \boldsymbol{\mathcal{U}} \text{ such that } U * C_s \subseteq C_r \text{ for every } s - r > \tfrac{1}{\delta}. \tag{5.5}$$

Let $\delta \in \mathbb{Q}^+$. Notice that if $\beta < s$ or $\alpha > r$, from (5.3) and (5.4) one obtains $C_s \subseteq f(\beta, \text{—})^{\#} = \mathsf{O}$ or $L = g(\text{—}, \alpha) \subseteq C_r$ which clearly yields $U * C_s \subseteq C_r$ for any $U \in \boldsymbol{\mathcal{U}}$. Thus, it suffices to show (5.5) for every $s - r > \frac{1}{\delta}$ with $\alpha \leq r < s \leq \beta$. Select an $n \in \mathbb{N}$ and $t_0, t_1, \ldots, t_{n+1} \in \mathbb{Q}$ such that they satisfy

$$t_0 = \alpha < t_1 < t_2 < \cdots < t_n < \beta = t_{n+1}$$

and $t_{k+1} - t_k < \frac{1}{2\delta}$ for all $k = 0, \ldots, n$. Set $U := U_0 \wedge U_1 \cdots \wedge U_n$, where U_k is the cover that witnesses the relation $C_{t_{k+1}} \lhd_{\mathcal{U}} C_{t_k}$ for $k = 0, \ldots, n$. Thus $U * C_{t_{k+1}} \subseteq C_{t_k}$ for every $k = 0, \ldots, n$. Let $s - r > \frac{1}{\delta}$ with $\alpha \le r < s \le \beta$ and pick a $k \in \{0, \ldots, n\}$ such that $r \le t_k < t_{k+1} \le s$. Hence,

$$U * C_s \subseteq U * C_{t_{k+1}} \subseteq C_{t_k} \subseteq C_r$$

as required. In conclusion, $\{C_p\}_{p\in\mathbb{Q}}$ determines a uniformly continuous $h \in \mathsf{F}(L)$ given by $h(r, —) = \bigcap_{r<p} C_p^{\#}$ and $h(—, s) = \bigcap_{q<s} C_q$. Furthermore, by (5.4) one may easily check that $g \le h \le f$. $\square$

Condition (ii) in Theorem 5.2 is formally stronger than condition (ii) in Theorem 5.4. The following proposition and the remark afterwards explain the reason behind this discrepancy:

Proposition 5.5. *Let $(L, \mathcal{U})$ be a preuniform frame and $f, g \in \mathsf{F}(L)$ with $f \ge g$. Fix a $\delta_0 \in \mathbb{Q}^+$. Then the following are equivalent:*

(i) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathcal{U}$ such that for every $n \in \mathbb{N}$ the sublocales $f(r, —)$ and $g(—, s)$ are U^n-far whenever $s - r > \frac{n}{\delta}$;*
(ii) *The following two conditions hold:*

(a) *There is a $U_0 \in \mathcal{U}$ such that for every $n \in \mathbb{N}$ the sublocales $f(r, —)$ and $g(—, s)$ are U_0^n-far whenever $s - r > \frac{n}{\delta_0}$;*
(b) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathcal{U}$ such that the sublocales $f(r, —)$ and $g(—, s)$ are U-far whenever $s - r > \frac{1}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (i): Let $\delta \in \mathbb{Q}^+$ and select an $m \in \mathbb{N}$ such that $\delta < \delta_0 2^m$. For each $n \in \{1, \ldots, 2^m - 1\}$ let U_n be the cover given by (b) by choosing the rational $\frac{\delta_0 2^m}{n} \in \mathbb{Q}^+$. Then

$$f(r, —) \text{ and } g(—, s) \text{ are } U_n\text{-far whenever } s - r > \tfrac{n}{\delta_0 2^m} \tag{5.6}$$

for each $n \in \{1, \ldots, 2^m - 1\}$. Now (recall the axiom (U3)) choose a cover W with the property that

$$W^{2^{m+1}} \le U_0 \wedge \bigwedge_{n=1}^{2^m-1} U_n.$$

We claim that for any $n \in \mathbb{N}$, the sublocales

$$f(r, —) \text{ and } g(—, s) \text{ are } W^n\text{-far whenever } s - r > \tfrac{n}{\delta_0 2^m}. \tag{5.7}$$

Indeed, let $n \in \mathbb{N}$ and $s - r > \frac{n}{\delta_0 2^m}$. We distinguish two cases:

(1) If $n \in \{1, \ldots, 2^m\}$, then $f(r, —)$ and $g(—, s)$ are U_n-far if $n < 2^m$ (by (5.6)) and U_0-far if $n = 2^m$ (by (a)). In either case they are $W^{2^{m+1}}$-far by Remark 4.1 (2). But $n \le 2^m \le 2^{m+1}$ and so $W^n \le W^{2^{m+1}}$, hence they are also W^n-far.
(2) If $n > 2^m$. Since $n2^{-m} > 1$, select an $\ell \in \mathbb{N}$ with $\ell < n2^{-m} \le \ell + 1$. Then one can write $n = \ell 2^m + j$ for a suitable $j \in \{1, \ldots, 2^m\}$, namely $j = n - \ell 2^m$. Since $s - r > \frac{n}{\delta_0 2^m} = \frac{\ell 2^m + j}{\delta_0 2^m} > \frac{\ell}{\delta_0}$, it follows from (a) that $f(r, —)$ and $g(—, s)$ are U_0^ℓ-far. By Lemma 2.4 (4) we conclude that

$$W^n = W^{\ell 2^m + j} \leq W^{\ell 2^m + 2^m} \leq W^{\ell 2^{m+1}} \leq (W^{2^{m+1}})^\ell \leq U_0^\ell,$$

so $f(r,—)$ and $g(—,s)$ are W^n-far, as required.

Hence, (5.7) is proved. Finally, if $s - r > \frac{n}{\delta}$, by the choice of m one has $s - r > \frac{n}{\delta_0 2^m}$ so $f(r,—)$ and $g(—,s)$ are W^n-far. □

Remark 5.6. Let $\boldsymbol{\alpha} \leq g \leq f \leq \boldsymbol{\beta}$ be bounded. Then by choosing $\delta_0 = \frac{1}{\beta-\alpha}$, property (a) in the last proposition is trivially satisfied. Indeed, if $s - r > n(\beta - \alpha)$, then $s - r > \beta - \alpha$ and so either $r < \alpha$ or $s > \beta$. By (2.4), one has $f(r,—) = \mathsf{O}$ or $g(—,s) = \mathsf{O}$, thus $f(r,—)$ and $g(—,s)$ are U-far for any cover U. This explains why condition (ii) in Theorem 5.4 is precisely (b).

6. Uniform separation and extension theorems

As usual, a Katětov-type insertion theorem yields the corresponding Urysohn-type separation result and Tietze-type extension result as simple corollaries. In this final section, we prove the uniform versions of these theorems.

Theorem 6.1 *(Uniform separation theorem). Let $(L, \mathcal{U})$ be a preuniform frame, and let S and T be sublocales of L. Then the following are equivalent*:

(i) *S and T are U-far for some $U \in \mathcal{U}$;*
(ii) *There is a uniformly continuous $h \in \mathsf{F}(L)$ with $\mathbf{0} \leq h \leq \mathbf{1}$ such that $T \subseteq h(0,—)$ and $S \subseteq h(—,1)$.*

Proof. (i) $\Longrightarrow$ (ii): Assume that S and T are U-far for some $U \in \mathcal{U}$. By Proposition 4.2 we have that $\overline{S}$ and $\overline{T}$ are U-far. Consider the characteristic functions of $\overline{S}$ and $\overline{T}^\#$ from Example 2.2 (2), namely the maps $\chi_{\overline{S}}, \chi_{\overline{T}^\#} \in \mathsf{F}(L)$ given by

$$\chi_{\overline{S}}(p,—) = \begin{cases} \mathsf{O} & \text{if } p < 0, \\ \overline{S}^\# & \text{if } 0 \leq p < 1, \\ L & \text{if } p \geq 1, \end{cases} \quad \chi_{\overline{S}}(—,q) = \begin{cases} L & \text{if } q \leq 0, \\ \overline{S} & \text{if } 0 < q \leq 1, \\ \mathsf{O} & \text{if } q > 1, \end{cases}$$

and

$$\chi_{\overline{T}^\#}(p,—) = \begin{cases} \mathsf{O} & \text{if } p < 0, \\ \overline{T} & \text{if } 0 \leq p < 1, \\ L & \text{if } p \geq 1, \end{cases} \quad \chi_{\overline{T}^\#}(—,q) = \begin{cases} L & \text{if } q \leq 0, \\ \overline{T}^\# & \text{if } 0 < q \leq 1, \\ \mathsf{O} & \text{if } q > 1. \end{cases}$$

Note that, since $\overline{S}$ and $\overline{T}$ are U-far, one has $\overline{S} \subseteq \overline{T}^\#$, and therefore it follows that $\mathbf{0} \leq \chi_{\overline{S}} \leq \chi_{\overline{T}^\#} \leq \mathbf{1}$. Furthermore, we claim that for every $\delta \in \mathbb{Q}^+$ the sublocales $\chi_{\overline{T}^\#}(r,—)$ and $\chi_{\overline{S}}(—,s)$ are U-far whenever $s - r > \frac{1}{\delta}$. Indeed, if $r < 0$ or $1 < s$, one clearly has that $\chi_{\overline{T}^\#}(r,—)$ and $\chi_{\overline{S}}(—,s)$ are U-far. If $0 \leq r < s \leq 1$, then $\chi_{\overline{T}^\#}(r,—) = \overline{T}$ and $\chi_{\overline{S}}(—,s) = \overline{S}$ which by assumption are U-far. Consequently, by Theorem 5.4, there is a uniformly continuous $h \in \mathsf{F}(L)$ such that $\mathbf{0} \leq \chi_{\overline{S}} \leq h \leq \chi_{\overline{T}^\#} \leq \mathbf{1}$. Moreover, (recall (2.3)), we have

$$S \subseteq \overline{S} = \chi_{\overline{S}}(—,1) \subseteq h(—,1) \quad \text{and} \quad T \subseteq \overline{T} = \chi_{\overline{T}^\#}(0,—) \subseteq h(0,—)$$

as required.
(ii) $\Longrightarrow$ (i): Since h is uniformly continuous, by Corollary 4.11 there is a $U \in \mathcal{U}$ such that $h(0,—)$ and $h(—,1)$ are U-far. In particular, S and T are U-far. □

Let $(L, \mathcal{U})$ be a (pre)uniform frame and S a sublocale of L with $j_S \colon S \hookrightarrow L$ the localic embedding of S in L. We denote by j_S^* the associated frame surjection. It is shown in [4, Lemma 2.2] that the system

$$\mathcal{U}_S^L := \{j_S^*[U] \mid U \in \mathcal{U}\}$$

is a (pre)uniformity in S.

Remark 6.2. Let S be a sublocale of L and T be a sublocale of S.
It is then easy to see that $\mathcal{U}_T^L = \left(\mathcal{U}_S^L\right)_T^S$.

Let $h \in \mathsf{F}(S)$ be uniformly continuous with respect to $\mathcal{U}_S^L$. We say that an $\overline{h} \in \mathsf{F}(L)$ is a *uniformly continuous extension* of h if it is uniformly continuous with respect to $\mathcal{U}$ and the diagram

$$\begin{array}{ccc} \mathfrak{L}(\mathbb{R}) & \xrightarrow{\overline{h}} & \mathsf{S}(L)^{op} \\ & {\scriptstyle h} \searrow & \downarrow {\scriptstyle (j_S)_{-1}[-]} \\ & & \mathsf{S}(S)^{op} \end{array}$$

commutes, where $(j_S)_{-1}[T] = T \cap S$ for each $T \in \mathsf{S}(L)$ (for more information about the localic preimage we refer to [19, III.4.2]).

Now, we can prove the uniform extension theorem as a corollary of Theorem 5.4 and of the extension result for dense sublocales proved in [2]:

Theorem 6.3 *(Uniform extension theorem). Let $(L, \mathcal{U})$ be a preuniform frame and S a sublocale of L. Then every bounded uniformly continuous $h \in \mathsf{F}(S)$ (with respect to $\mathcal{U}_S^L$) has a bounded uniformly continuous extension $\overline{h} \in \mathsf{F}(L)$ (with respect to $\mathcal{U}$).*

Proof. First, it was shown in [2, Theorem 7.3] (cf. also the remark after its proof) that every bounded uniformly continuous function on a dense sublocale of L has a bounded uniformly continuous extension to L. Since every sublocale is dense in its closure (cf. [19, Proposition III.8.5]), by Remark 6.2 it suffices to show the statement for closed sublocales. More generally, we shall show it for complemented sublocales.

Let S be a complemented sublocale of L, denote by $j_S \colon S \hookrightarrow L$ its localic embedding and let $h \in \mathsf{F}(S)$ be bounded and uniformly continuous with respect to $\mathcal{U}_S^L$. Select $\alpha, \beta \in \mathbb{Q}$ such that $\boldsymbol{\alpha} \le h \le \boldsymbol{\beta}$ and for each $r \in \mathbb{Q}$ set

$$S_r := \begin{cases} \mathsf{O} & \text{if } r < \alpha, \\ h(r, \text{—}) & \text{if } \alpha \le r < \beta, \\ L & \text{if } r \ge \beta, \end{cases} \quad \text{and} \quad T_r := \begin{cases} L & \text{if } r \le \alpha, \\ h(\text{—}, r) & \text{if } \alpha < r \le \beta, \\ \mathsf{O} & \text{if } r > \beta. \end{cases}$$

For each $r < s$ one has $S_s^{\#} \cap S_r = \mathsf{O}$. Indeed, if $r < \alpha$ or $s \ge \beta$ it is trivial because either $S_s^{\#} = \mathsf{O}$ or $S_r = \mathsf{O}$. If $\alpha \le r < s < \beta$ then

$$\begin{aligned} S_s^{\#} \cap S_r = h(s, \text{—})^{\#} \cap h(r, \text{—}) &= h(s, \text{—})^{\#} \cap S \cap h(r, \text{—}) \\ &= h(s, \text{—})^{\#_S} \cap h(r, \text{—}) \subseteq h(\text{—}, s) \cap h(r, \text{—}) = \mathsf{O} \end{aligned}$$

by (2.1). Hence $\{S_r\}_{r \in \mathbb{Q}}$ is a descending scale in $\mathsf{S}(L)^{op}$ and similarly $\{T_r\}_{r \in \mathbb{Q}}$ is an ascending scale in $\mathsf{S}(L)^{op}$. Let $f, g \in \mathsf{F}(L)$ be the functions they generate. From the equalities

$$f(-,r) = \bigcap_{p<r} S_p^{\#} \quad \text{and} \quad g(-,r) = \bigcap_{q<r} T_q,$$

it follows that for each $r \in \mathbb{Q}$ one has $g(-,r) \subseteq f(-,r)$ — i.e., $f \geq g$. Indeed, let $r \in \mathbb{Q}$ and $p < r$. We have to check that $\bigcap_{q<r} T_q \subseteq S_p^{\#}$. If $p < \alpha$ or $r > \beta$ one has either $S_p^{\#} = L$ or $\bigcap_{q<r} T_q = \mathsf{O}$, so the inclusion follows. Suppose now that $\alpha \leq p < r \leq \beta$ and pick a $q' \in \mathbb{Q}$ with $p < q' < r$. Then $\bigcap_{q<r} T_q \subseteq T_{q'} = h(-,q') \subseteq h(p,-)^{\#} = S_p^{\#}$, as desired.

Further, the maps f and g satisfy condition (ii) in Theorem 5.4. Indeed, let $\delta \in \mathbb{Q}^+$. Since h is uniformly continuous there is a $U \in \mathcal{U}$ such that $h(r,-)$ and $h(-,s)$ are $j_S^*[U]$-far (as sublocales of S) whenever $s - r > \frac{1}{\delta}$. Since $\mathfrak{o}_S(j_S^*(u)) = S \cap \mathfrak{o}_L(u)$ for any $u \in L$, then $h(r,-)$ and $h(-,s)$ are U-far (as sublocales of L). We claim that $f(r,-)$ and $g(-,s)$ are U-far whenever $s - r > \frac{1}{\delta}$. Clearly it suffices to show the case where $\alpha \leq r < s \leq \beta$ (as otherwise $f(r,-) = \mathsf{O}$ or $g(-,s) = \mathsf{O}$). Pick $r', s' \in \mathbb{Q}$ with $r < r' < s' < s$ and $s' - r' > \frac{1}{\delta}$. Then $f(r,-) = \bigcap_{r<p} S_p \subseteq S_{r'} = h(r',-)$ and $g(-,s) \subseteq T_{s'} = h(-,s')$. The claim thus follows from Remark 4.1 (1).

Moreover, f and g are bounded by (2.4). By Theorem 5.4 there is a uniformly continuous $\overline{h} \in \mathsf{F}(L)$ with $f \geq \overline{h} \geq g$. Now it follows trivially from (2.4) and (2.5) that $S_r \cap S = h(r,-)$ and $T_r \cap S = h(-,r)$ for each $r \in \mathbb{Q}$. Hence $(j_S)_{-1}[-] \circ f = h = (j_s)_{-1}[-] \circ g$, and so $h \geq (j_S)_{-1}[-] \circ \overline{h} \geq h$ — i.e., $\overline{h}$ is the desired extension of h. □

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References

[1] I. Arrieta, J. Gutiérrez García, J. Picado, Frame presentations of compact hedgehogs and their properties, Quaest. Math. (2023), https://doi.org/10.2989/16073606.2021.2013971, in press.
[2] A.B. Avilez, J. Picado, Uniform continuity of pointfree real functions via farness and related Galois connections, Algebra Univers. 83 (2022) 1–27, art. no. 39.
[3] B. Banaschewski, The Real Numbers in Pointfree Topology, Textos de Matemática, vol. 12, University of Coimbra, 1997.
[4] B. Banaschewski, Uniform completion in pointfree topology, in: Topological and Algebraic Structures in Fuzzy Sets, in: Trends Log. Stud. Log. Libr., vol. 20, Kluwer Acad. Publ., Dordrecht, 2003, pp. 19–56.
[5] J. Blatter, G.L. Seever, Interposition of semicontinuous functions by continuous functions, in: L. Nachbin (Ed.), Analyse Fonctionelle et Applications, Comptes Rendus du Colloque d'Analyse, Rio de Janeiro, 1972, Hermann, Paris, 1975, pp. 27–51.
[6] M.J. Ferreira, J. Gutiérrez García, J. Picado, Completely normal frames and real-valued functions, Topol. Appl. 156 (2009) 2932–2941.
[7] M.J. Ferreira, J. Picado, S. Pinto, Remainders in pointfree topology, Topol. Appl. 245 (2018) 21–45.
[8] J. Gutiérrez García, T. Kubiak, General insertion and extension theorems for localic real functions, J. Pure Appl. Algebra 215 (2011) 1198–1204.
[9] J. Gutiérrez García, T. Kubiak, J. Picado, Localic real functions: a general setting, J. Pure Appl. Algebra 213 (2009) 1064–1074.
[10] J. Gutiérrez García, J. Picado, Insertion and extension results for pointfree complete regularity, Bull. Belg. Math. Soc. Simon Stevin 20 (2013) 675–687.
[11] H. Herrlich, A. Pultr, Nearness, subfitness and sequential regularity, Appl. Categ. Struct. 8 (2000) 67–80.
[12] J. Hoffmann-Jørgensen, A general "in-between" theorem, Math. Scand. 50 (1982) 55–65.
[13] P.T. Johnstone, A constructive theory of uniform locales, I: uniform covers, in: S.J. Andima, R. Kopperman, P.R. Misra, J.Z. Reichman, A.R. Todd (Eds.), General Topology and Applications, in: Lecture Notes in Pure and Appl. Math., vol. 134, Dekker, New York, 1991, pp. 179–193.
[14] T. Kaiser, A sufficient condition of full normality, Comment. Math. Univ. Carol. 37 (1996) 381–389.
[15] M. Katětov, On real valued functions in topological spaces, Fundam. Math. 38 (1951) 85–91; Fundam. Math. 40 (1953) 139–142 (Correction).
[16] T. Kubiak, On Fuzzy Topologies, Ph.D. Thesis, UAM, Poznań, 1985.
[17] T. Kubiak, Separation axioms: extension of mappings and embedding of spaces, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory, in: The Handbooks of Fuzzy Sets Series, vol. 3, Kluwer, Dordrecht, 1999, pp. 433–479.
[18] E.P. Lane, Insertion of a continuous function, Topol. Proc. 4 (1979) 463–478.

[19] J. Picado, A. Pultr, Frames and Locales: Topology Without Points, Frontiers in Mathematics, vol. 28, Springer, Basel, 2012.
[20] J. Picado, A. Pultr, (Sub)fit biframes and non-symmetric nearness, Topol. Appl. 168 (2014) 66–81.
[21] T. Plewe, Sublocale lattices, J. Pure Appl. Algebra 168 (2002) 309–326.
[22] D. Preiss, J. Vilimovský, In-between theorems in uniform spaces, Trans. Am. Math. Soc. 261 (1980) 483–501.
[23] A. Pultr, Pointless uniformities I. Complete regularity, Comment. Math. Univ. Carol. 25 (1984) 91–104.
[24] A. Pultr, Pointless uniformities II. (Dia)metrization, Comment. Math. Univ. Carol. 25 (1984) 105–120.
[25] M.H. Stone, Boundedness properties in function lattices, Can. J. Math. 1 (1949) 176–186.
[26] H. Tong, Some characterizations of normal and perfectly normal spaces, Duke Math. J. 19 (1952) 289–292.