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AN ALGORITHM FOR THE INERTIA SETS OF TREE SIGN PATTERNS

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ABSTRACT: A matrix whose entries are $+$, $-$ or 0 is said a sign pattern. The inertia set of an n-by-n symmetric sign pattern A is the set of inertias of all real symmetric matrices with the same sign pattern as A. We present an algorithm to compute the inertia set of any symmetric tree (or acyclic) sign pattern. The procedure generalizes some recent results. Some examples are provided.

KEYWORDS: Inertia, sign pattern matrix, tridiagonal pattern, trees, stars, double stars, 2-generalized stars.
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1. Introduction

In the literature, the study of combinatorial and qualitative information based on the signs of the entries of a matrix has attracted much attention. A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern* (*matrix*). For each $n \times n$ sign pattern A there is a natural class of real matrices whose entries have the signs indicated by A, i.e., the *sign pattern class* of a sign pattern A is defined by

$$Q(A) = \{B \mid \operatorname{sign} B = A\} .$$

We are interested in symmetric matrices and in the sign symmetric classes

$$Q_{SYM}(A) = \{B \mid \operatorname{sign} B = A \text{ and } B = B^T\} .$$

Define the inertia of an n-by-n real symmetric matrix H as the triple $\operatorname{In}(H) = (\pi, \nu, \delta)$, where π is the number of positive eigenvalues, ν is the number of negative eigenvalues and $\delta = n - \pi - \nu$ the number of the zero eigenvalues. For a symmetric sign pattern A, we define the *inertia* (*set*) of A to be

$$\operatorname{In}(A) = \{ \operatorname{In}(B) \mid B \in Q_{SYM}(A)\} .$$

We say that the sign pattern A *requires unique inertia* and is *sign nonsingular* if every real matrix in $Q_{SYM}(A)$ has the same inertia and is nonsingular,

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respectively. If two sign patterns A_1 and A_2 are congruent, *i.e.*, if for all $B_1 \in Q_{SYM}(A_1)$ and $B_2 \in Q_{SYM}(A_2)$ there exists a nonsingular real matrix S such that $B_1 = SB_2S^T$, then we say that A_1 and A_2 are *sign congruent* and write $A_1 \approx A_2$.

By Sylvester's law of inertia we may say that two congruent sign patterns have the same inertia set. For example, the symmetric sign pattern

$$\begin{pmatrix} 0 & + & + \\ + & + & 0 \\ + & 0 & 0 \end{pmatrix}$$

is sign congruent to

$$\begin{pmatrix} 0 & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{pmatrix}$$

and, therefore, it requires unique inertia $(2,1,0)$ and, consequently, it is sign nonsingular. On the other hand, the sign pattern

$$\begin{pmatrix} + & + & + \\ + & + & 0 \\ + & 0 & - \end{pmatrix}$$

is sign congruent to

$$\begin{pmatrix} * & 0 & 0 \\ 0 & + & 0 \\ 0 & 0 & - \end{pmatrix},$$

where $*$ is 0, $+$ or $-$, and, therefore, it requires the inertia set $\{(1,1,1), (1,2,0), (2,1,0)\}$.

A diagonal sign pattern where each of whose entries is $+$ or $-$ is called a *signature pattern*. The square of a signature pattern is a signature pattern with all diagonal entries equal to $+$. A sign pattern with exactly one entry in each row and each column equal to $+$ and all other entries equal to 0 is called a *permutation pattern*. Two congruent sign patterns by the way of a signature pattern or of a permutation pattern are called, respectively, *signature congruent* or *permutation congruent* patterns.

An $n \times n$ symmetric sign pattern $A = (a_{ij})$ is associated with the undirected graph G on vertices $1, \dots, n$ having an edge between vertices i and j if and only if $i \neq j$ and $a_{ij} \neq 0$. We often write $A = A(G)$ and we simply say that

G is the graph of the sign pattern A. If the graph of A is a tree, we say A is a *tree* (or *acyclic*) *sign pattern.*

A vertex of G with degree 1 is said *terminal.* If a vertex has degree at least 2 and it is incident in a terminal vertex, then it is called a *foot* of the graph.

Recently the inertia sets of some types of symmetric sign patterns, namely stars, double stars, 2−generalized stars and paths, were considered by different authors, cf. [4, 8, 9, 10, 12, 13, 14]. Using some matrix results, da Fonseca [6] considered several results on tridiagonal matrices. In this paper we present an algorithm that allows us to compute the inertia set of any symmetric sign pattern whose graph is a tree, using mainly tools well known from congruences between matrices (cf., *e.g.*, Cain and Marques de Sá [1, 2] or da Fonseca [5]).

2. Tree sign patterns

In this section, we give an algorithm to compute the inertia set of any symmetric sign pattern whose associated graph is a tree. Notice that given any symmetric tree sign pattern, the inertia set does not depend on the sign of the off-diagonal elements, since two sign patterns under these conditions are signature congruent. Henceforth, we may consider all off diagonal entries as $+$.

We start by considering a symmetric sign pattern whose graph is a star, following the ideas developed in [6].

Set $(*|*,\ldots,*)$ for the main diagonal of a sign pattern A when it contains at least one zero not in the first position.

Theorem 2.1 ([6, 13])**.** *Up to permutation congruence, signature congruence, and negation, a symmetric star sign pattern*

$$A=\begin{pmatrix} * & + & + & \cdots & + \\ + & * & & & \\ + & & * & & \\ \vdots & & & \ddots & \\ + & & & & * \end{pmatrix}_{n\times n}, \tag{2.1}$$

where each diagonal entry is 0*,* $+$ *or* $-$*, requires unique inertia if and only if the main diagonal* A *is*

$$(*|*,\ldots,*) \quad or \quad (\diamond,+,\ldots,+)\,,$$

where $\diamond$ *is* 0 *or* $-$*.*

Proof: With the exception of the $(1,1)$-entry, if one of the diagonal entries is zero, then

$$A \approx \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}_{n-2\times n-2},$$

and therefore A requires unique inertia.

Suppose now that all diagonal entries are nonzero, possibly with the exception of the $(1,1)$-entry. Then, we get

$$A \approx (*) \oplus \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}_{n-1\times n-1}.$$

In this case, A requires unique inertia if and only if all the diagonal entries different from the $(1,1)$-entry have the same sign and the $(1,1)$-entry has a sign different from the other diagonal elements or is equal to 0. ■

From the proof above we get immediately the following result.

Corollary 2.2 ([13]). *Let A be an $n \times n$ symmetric star sign pattern having the form (2.1), and suppose that A has unique inertia.*

(1) *If the diagonal of A has the form $(*|*,\ldots,*)$, then* $\mathrm{In}(A) = (\ell+1, s+1, n-\ell-s-2)$, *where ℓ and s are, respectively, the number of positive and negative entries in the last $n-1$ diagonal positions of A.*
(2) *If the diagonal of A has the form $(\diamond, +, \ldots, +)$, with $\diamond$ equal to 0 or $-$, then* $\mathrm{In}(A) = (n-1, 1, 0)$.

In particular, if a $n \times n$ symmetric star sign pattern A has all diagonal entries equal to zero, then $\mathrm{In}(A) = (1, 1, n-2)$. It also follows from the proof of Theorem 2.1 that if the main diagonal of an $n \times n$ symmetric star sign pattern A, up to permutation congruence, signature congruence, and negation, neither it is $(*|*,\ldots,*)$, nor it is

$$(0, +, \ldots, +) \quad \text{or} \quad (-, +, \ldots, +),$$

then

$$\mathrm{In}(A) = \{(\pi+1, \nu, 0), (\pi, \nu+1, 0), (\pi, \nu, 1)\},$$

where π and ν are, respectively, the number of $+$ and $-$ in the last $n-1$ diagonal entries of A.

We can now state the algorithm which will be used in all results of the paper.

Algorithm For the general case, let us assume that $A(G)$ is a symmetric sign pattern of order n, whose graph G is a tree. Identify the foot of G incident on the largest number of terminal vertices. If there is more than one, one can choose arbitrarily one of them. Reorder, if necessary, the indices of G in such a way that the vertex, say k, is the chosen foot and let that terminal vertices are $k+1, \ldots, n$. Denoting by H subgraph of G, with vertices $k, k+1, \ldots, n$, which is a star, we get

$$A \approx \left(\begin{array}{ccc|cccc} & & & \diamond & & & \\ & A(G \setminus H) & & \vdots & & & \\ & & & \diamond & & & \\ \hline \diamond & \cdots & \diamond & * & + & \cdots & + \\ & & & + & * & & \\ & & & + & & \ddots & \\ & & & + & & & * \end{array}\right), \tag{2.2}$$

where each $*$ is $0, +$ or $-$, and each $\diamond$ is 0 or $+$.

If the diagonal of $A(H)$ has the form $(*|*, \ldots, *)$, we can obtain

$$A \approx A(G \setminus H) \oplus A(H),$$

and, therefore,

$$\mathrm{In}(A) = \mathrm{In}(A(G \setminus H)) + \mathrm{In}(A(H)),$$

repeating now the procedure for $A(G \setminus H)$.

Otherwise, we get

$$A \approx A((G \setminus H) \cup \{k\}) \oplus A(H \setminus \{k\}),$$

and

$$\mathrm{In}(A) = \mathrm{In}(A((G \setminus H) \cup \{k\})) + \mathrm{In}(A(H \setminus \{k\})),$$

where $A(H \setminus \{k\})$ is a diagonal sign pattern, repeating now the procedure for $A((G \setminus H) \cup \{k\})$. Notice that the sign of the diagonal entry in the position k may vary from the original sign.

Iterating this process a sufficient number of times, we eventually obtain

$$A \approx A_1 \oplus \cdots \oplus A_r,$$

where each A_i is either a star sign pattern or a diagonal sign pattern. Henceforth,

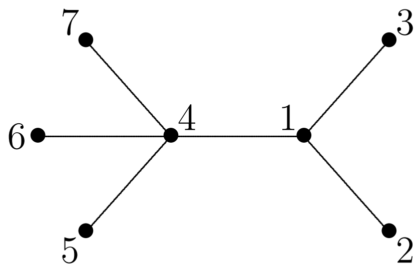
$$\mathrm{In}(A) = \mathrm{In}(A_1) + \cdots + \mathrm{In}(A_r).$$

Notice that since an $n \times n$ symmetric star sign pattern B with all the diagonal elements equal to zero has inertia $\mathrm{In}(B) = (1, 1, n-2)$, it follows from the algorithm above that the inertia of a tree sign pattern whose diagonal elements are all zero is $\mathrm{In}(A) = (\ell, \ell, n-2\ell)$, where ℓ is the number of iterated steps in the algorithm above.

For example, consider the following symmetric sign pattern

$$A = \left(\begin{array}{ccc|cccc} + & + & + & + & & & \\ + & - & & & & & \\ + & & + & & & & \\ \hline + & & & - & + & + & + \\ & & & + & + & & \\ & & & + & & 0 & \\ & & & + & & & - \end{array} \right),$$

whose graph is



Following the algorithm described above, we get

$$A \approx \begin{pmatrix} + & + & + \\ + & - & \\ + & & + \end{pmatrix} \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix},$$

and, therefore, the inertia set of A is

$$\mathrm{In}(A) = \mathrm{In} \begin{pmatrix} + & + & + \\ + & - & \\ + & & + \end{pmatrix} + (2, 2, 0) .$$

Finally, by Theorem 2.1, we get

$$\mathrm{In} \begin{pmatrix} + & + & + \\ + & - & \\ + & & + \end{pmatrix} = \{(1, 1, 1), (2, 1, 0), (1, 2, 0)\} ,$$

and

$$\mathrm{In}(A) = \{(3,3,1),(4,3,0),(3,4,0)\}\ .$$

Still with the same graph, considering the sign pattern

$$B = \left(\begin{array}{ccc|cccc} + & + & + & + & & & \\ + & - & & & & & \\ + & & + & & & & \\ \hline + & & & - & + & + & + \\ & & & + & + & & \\ & & & + & & - & \\ & & & + & & & - \end{array}\right),$$

we get

$$B \approx \begin{pmatrix} + & + & + & + \\ + & - & & \\ + & & + & \\ + & & & * \end{pmatrix} \oplus \begin{pmatrix} + & & \\ & - & \\ & & - \end{pmatrix},$$

and since

$$\mathrm{In}\begin{pmatrix} + & + & + & + \\ + & - & & \\ + & & + & \\ + & & & * \end{pmatrix} = \{(1,2,1),(1,3,0),(2,1,1),(2,2,0),(3,1,0)\}\ ,$$

we reach the set of inertias:

$$\mathrm{In}(B) = \{(2,4,1),(2,5,0),(3,3,1),(3,4,0),(4,3,0)\}\ .$$

Finally, considering the symmetric sign pattern

$$C = \left(\begin{array}{ccc|cccc} 0 & + & + & * & & & \\ + & 0 & & & & & \\ + & & 0 & & & & \\ \hline * & & & 0 & + & + & + \\ & & & + & 0 & & \\ & & & + & & 0 & \\ & & & + & & & 0 \end{array}\right),$$

whose diagonal elements are all zero, and following the algorithm, we get

$$\mathrm{In}(C) = (1,1,1) + (1,1,2) = (2,2,3)\,.$$

In the next sections, we will show how the algorithm described here can be used to give straightforward proofs of the main results provided in [3, 4, 9, 10].

3. Symmetric n-star sign patterns

Let us define the symmetric n-star sign pattern

$$A_n = \begin{pmatrix} B_1 & J_1 & & & \\ J_1^t & B_2 & J_2 & & \\ & J_2^t & \ddots & \ddots & \\ & & \ddots & \ddots & J_{n-1} \\ & & & J_{n-1}^t & B_n \end{pmatrix}, \tag{3.1}$$

where each B_i is an $\ell_i \times \ell_i$ symmetric star sign pattern with $\ell_i > 1$, and, for $i = 1, \ldots, n-1$, J_i is the $\ell_i \times \ell_{i+1}$ sign pattern having the sign $+$ is position $(1,1)$, and 0 elsewhere. For each $i = 1, \ldots, n$, d_i denotes the diagonal of B_i.

Theorem 3.1. *The symmetric n-star sign pattern (3.1) requires unique inertia if and only if, up to permutation congruence, signature congruence and negation, for each $k = n, \ldots, 1$, $d_k = (*|*, \ldots, *)$ or $d_k = (\diamond, +, \ldots, +)$ and, in this last case, $d_{k-1} = (*|*, \ldots, *)$ or $d_{k-1} = (\diamond', -, \ldots, -)$, where $\diamond$ is 0 or $-$, and $\diamond'$ is 0 or $+$.*

Proof: $(\Leftarrow)$ The proof will be handle by induction over $n \geq 1$. When $n = 1$ the result was proved in Theorem 2.1. So, let $n \geq 2$ and consider the diagonal d_n. If $d_n = (*|*, \ldots, *)$, then by our algorithm we get

$$A_n \approx A_{n-1} \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \begin{pmatrix} *_2 & & \\ & \ddots & \\ & & *_s \end{pmatrix},$$

where A_{n-1} is the symmetric $(n-1)$-star sign pattern with diagonal blocks $B_1, \ldots, B_{t-1}$. By induction, A_{n-1} has unique inertia and, thus, also A_n has unique inertia. If $d_n = (\diamond, +, \ldots, +)$, again by the algorithm, we get

$$A_n \approx \begin{pmatrix} B_1 & J_1 & & \\ J_1^t & \ddots & \ddots & \\ & \ddots & \ddots & J'_{n-2} \\ & & J'^t_{n-2} & B'_{n-1} \end{pmatrix} \oplus \begin{pmatrix} + & & \\ & \ddots & \\ & & + \end{pmatrix},$$

where B'_{n-1} is the symmetric star sign pattern obtained from B_{n-1} by adding the sign $-$ in the last diagonal position. Notice that if $d_{n-1} = (*|*, \ldots, *)$, then the main diagonal of B'_{n-1} is also of the same form, and if $d_{n-1} = (\diamond', -, \ldots, -)$, then the diagonal of B'_{n-1} still have this pattern. Thus, by induction we find that A_n has unique inertia.

($\Rightarrow$) Without loss of generality, assume that up to permutation congruence, signature congruence and negation, neither $d_n = (*|*, \ldots, *)$ nor is $d_n = (\diamond, +, \ldots, +)$. Then, by the algorithm,

$$A_n \approx \begin{pmatrix} B_1 & J_1 & & \\ J_1^t & \ddots & \ddots & \\ & \ddots & \ddots & J'_{n-2} \\ & & J'^t_{n-2} & B'_{n-1} \end{pmatrix} \oplus \begin{pmatrix} *_2 & & \\ & \ddots & \\ & & *_s \end{pmatrix}, \tag{3.2}$$

with B'_{n-1} the symmetric star sign pattern obtained from B_{n-1} by adding $*$ in the last diagonal position, where $*$ may be $0, +$ or $-$. If $d'_{n-1} = (*|*, \ldots, *)$, we get

$$A_n \approx \begin{pmatrix} B_1 & J_1 & & \\ J_1^t & \ddots & \ddots & \\ & \ddots & \ddots & J_{n-3} \\ & & J^t_{n-3} & B_{n-2} \end{pmatrix} \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \begin{pmatrix} *'_2 & & & \\ & \ddots & & \\ & & *'_r & \\ & & & * \end{pmatrix} \oplus \begin{pmatrix} *_2 & & \\ & \ddots & \\ & & *_s \end{pmatrix},$$

and therefore, A_n does not require unique inertia. In any other case, choosing appropriately the sign of $*$, the same reasoning also shows that A_n does not require unique inertia. Finally, note that a similar situation occurs if $d_n = (\diamond, +, \ldots, +)$ and neither $d_{n-1} = (*|*, \ldots, *)$ nor $d_{n-1} = (\diamond', -, \ldots, -)$, where $\diamond$ is 0 or $-$, and $\diamond'$ is 0 or $+$. ∎

The characterization of 2-star sign patterns requiring unique inertia, established by Yanling in [14], is now a particular case of the previous theorem.

Corollary 3.2 ([14]). *The symmetric 2-star sign pattern*

$$A_2 = \begin{pmatrix} B_1 & J_1 \\ J_1^t & B_2 \end{pmatrix}, \tag{3.3}$$

requires unique inertia if and only if, up to permutation congruence, signature congruence and negation, the main diagonal has one of the following forms:

$$d_1 = (*|*, \ldots, *);\ d_2 = (*|*, \ldots, *)\ , \qquad (\diamond, +, \ldots, +; 0, -, \ldots, -)\ ,$$

$$d_1 = (\diamond, +, \ldots, +);\ d_2 = (*|*, \ldots, *)\ , \qquad (\diamond, +, \ldots, +; +, -, \ldots, -)\ ,$$

where $*$ can be $0, +$ or $-$, and $\diamond$ is 0 or $-$.

From the proof of Theorem 3.1 for $n = 2$ and Corollary 2.2, it is now easy to give the inertia set of a symmetric 2-star sign pattern that requires unique inertia.

Corollary 3.3 ([14]). *Let A be an $n \times n$ symmetric 2-star sign pattern having the form (3.3), and suppose that A requires unique inertia. Let $\ell = |\{i : a_i = + \text{ and } 2 \leq i \leq \ell_1\}|$, $\ell' = |\{i : a_i = + \text{ and } \ell_1 + 2 \leq i \leq n\}|$, $s = |\{i : a_i = - \text{ and } 2 \leq i \leq \ell_1\}|$, and $s' = |\{i : a_i = - \text{ and } \ell_1 + 2 \leq i \leq n\}|$. If the diagonal of A has the form:*

(1) $d_1 = (*|*, \ldots, *)$ *and* $d_2 = (*|*, \ldots, *)$, *then* $\mathrm{In}(A) = (\ell + \ell' + 2, s + s' + 2, n - \ell - \ell' - s - s' - 4)$;
(2) $d_1 = (\diamond, +, \ldots, +)$ *and* $d_2 = (*|*, \ldots, *)$, *then* $\mathrm{In}(A) = (\ell_1 + \ell', s' + 2, \ell_2 - \ell' - s' - 2)$;
(3) $(\diamond, +, \ldots, +; 0, -, \ldots, -)$, *then* $\mathrm{In}(A) = (\ell_1, \ell_2, 0)$;
(4) $(\diamond, +, \ldots, +; +, -, \ldots, -)$, *then* $\mathrm{In}(A) = (\ell_1, \ell_2, 0)$.

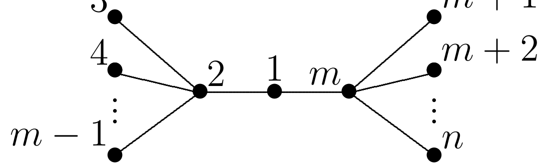
Next we will give a new example which can be easily generalized.

4. An example of generalized star sign pattern

Let

$$A = \begin{pmatrix} a_1 & + & & & & + & & & \\ + & a_2 & + & \cdots & + & & & & \\ & + & a_3 & & & & & & \\ & \vdots & & \ddots & & & & & \\ & + & & & a_{m-1} & & & & \\ + & & & & & a_m & + & \cdots & + \\ & & & & & + & a_{m+1} & & \\ & & & & & \vdots & & \ddots & \\ & & & & & + & & & a_n \end{pmatrix}, \tag{4.1}$$

be an $n \times n$ symmetric generalized star sign pattern, with $n \geq 3$, where $n - m + 1 \geq m - 1$, $m > 4$ and each a_i is 0, $+$ or $-$, for $i = 1, \ldots, n$. The graph of A is:



(4.2)

Setting $S_2 = G(A)\setminus\{1, m, m+1, \ldots, n\}$ and $S_m = G(A)\setminus\{1, 2, \ldots, m-1\}$, we define d_2 and d_m the main diagonals of the star sign patterns $A(S_2)$ and $A(S_m)$, respectively.

Theorem 4.1. *The symmetric generalized star sign pattern (4.1) requires unique inertia if and only if, up to permutation congruence, signature congruence and negation, the main diagonal has one of the following forms:*

(1) $d_2 = (*|*, \ldots, *)$ *and* $d_m = (*|*, \ldots, *)$*;*
(2) $a_1 = +$*,* $d_2 = (\diamond, + \ldots, +)$ *and* $d_m = (*|*, \ldots, *)$*;*
(3) $a_1 = 0$ *and* $d_m = (*|*, \ldots, *)$*;*
(4) $(\diamond'; \diamond, + \ldots, +; \diamond, +, \ldots, +)$*;*
(5) $a_1 = \diamond'$*,* $d_2 = (*|*, \ldots, *)$ *and* $d_m = (\diamond, +, \ldots, +)$*;*
(6) $a_1 = 0$*,* $d_2 = (*|*, \ldots, *)$ *and* $d_m = (*, \ldots, *, +, -)$*,*

where each $*$ *is* $0, +$ *or* $-$*,* $\diamond$ *is* 0 *or* $-$*, and* $\diamond'$ *is* 0 *or* $+$*.*

Proof: If $d_m = (*|*, \ldots, *)$, then by the algorithm,

$$A \approx A' \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \begin{pmatrix} a_{m+1} & & \\ & \ddots & \\ & & a_{n-1} \end{pmatrix},$$

where A' is a symmetric star sign pattern whose graph is the star, subgraph of (4.2), with central vertex 2 and terminal vertices $1, 3, \ldots, m-1$. Thus, A requires unique inertia if and only if (1), (2) and (3) are verified.

When $d_m = (\diamond, +, \ldots, +)$, the application of the algorithm provides

$$A \approx A' \oplus \begin{pmatrix} + & & \\ & \ddots & \\ & & + \end{pmatrix},$$

where

$$A' = \begin{pmatrix} a_1 & + & & & & + \\ + & a_2 & + & \cdots & + & \\ & + & a_3 & & & \\ & \vdots & & \ddots & & \\ & + & & & a_{m-1} & \\ + & & & & & - \end{pmatrix} \approx \begin{pmatrix} a_1 & + & + & & & \\ + & - & & & & \\ + & & a_2 & + & \cdots & + \\ & & + & a_3 & & \\ & & \vdots & & \ddots & \\ & & + & & & a_{m-1} \end{pmatrix} \tag{4.3}$$

is a symmetric 2-star sign pattern whose graph is the subgraph of (4.2) with vertices $1, \ldots, m$. By Theorem 3.1, A requires unique inertia if and only if $d_2 = (*|*, \ldots, *)$ or is equal to $(\diamond, +, \ldots, +)$ and $a_1 = 0, +$.

Finally, when $d_3 = (*, \dots, *, +, -)$, we may write

$$A \approx A'' \oplus \begin{pmatrix} a_{m+1} & & \\ & \ddots & \\ & & a_{n-1} \end{pmatrix},$$

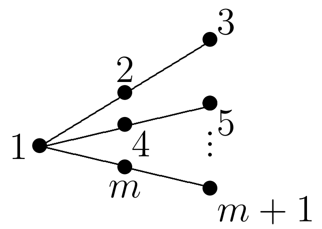
where A'' differs from matrix A' in (4.3) only in the last diagonal position, which can be $0, +$ or $-$. Thus, again by Theorem 3.1, we conclude that A has unique inertia if and only if $d_2 = (*|*, \dots, *)$ and $a_1 = 0$. ∎

5. Symmetric 2-generalized star sign patterns

We turn now our attention to symmetric 2-generalized star sign patterns which have the following form

$$A = \begin{pmatrix} a_1 & + & 0 & + & 0 & \cdots & + & 0 \\ + & a_2 & + & 0 & 0 & \cdots & 0 & 0 \\ 0 & + & a_3 & 0 & 0 & \cdots & 0 & 0 \\ + & 0 & 0 & a_4 & + & \cdots & 0 & 0 \\ 0 & 0 & 0 & + & a_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ + & 0 & 0 & 0 & 0 & \cdots & a_m & + \\ 0 & 0 & 0 & 0 & 0 & \cdots & + & a_{m+1} \end{pmatrix}, \tag{5.1}$$

for some odd integer $m + 1 \geq 3$. The graph of such sign pattern is



(5.2)

Let $H = \{a_{2i+1} : i = 0, 1, \dots\}$ denote the set of all odd diagonal entries of A.

Using our algorithm, we an give a straightforward proof of the main result in [8].

Theorem 5.1 ([8]). *Let A be a symmetric 2-generalized star sign pattern having the form (5.1). Then A requires unique inertia if and only if H has at most one nonzero element, or H has at least two nonzero elements, all having the same sign, and $a_{2i}a_{2i+1} \leq 0$, for all i.*

Proof: We start by noticing that if $a_{m+1} = 0$, then, following our main procedure, we have

$$A \approx A' \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix},$$

where A' is a symmetric 2-generalized star sign pattern whose graph is the subgraph of (5.2) with vertices $1, \ldots, m-1$. Thus, if all elements of H are zero, or if the only nonzero element of H is a_1, then A requires unique inertia. The same is true if H has one and only one nonzero element, say $+ \neq a_1$, since then we get

$$A \approx \begin{pmatrix} 0 & + & 0 \\ + & a_2 & + \\ 0 & + & + \end{pmatrix} \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}.$$

Therefore, without loss of generality, let us assume that all elements of H are $+$, and $a_{2i}a_{2i+1} \leq 0$, for all i. Following the algorithm, we have

$$A \approx A' \oplus (+),$$

where A' is associated with the subgraph G' of (5.2) of by the vertices $1, \ldots, m$, and whose (m, m) diagonal element is $-$. Reordering the vertices of G', we get

$$A' \approx \begin{pmatrix} a_1 & + & 0 & 0 & \cdots & + & 0 \\ + & - & 0 & 0 & \cdots & 0 & 0 \\ + & 0 & a_2 & + & \cdots & 0 & 0 \\ 0 & 0 & + & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ + & 0 & 0 & 0 & \cdots & a_{m-2} & + \\ 0 & 0 & 0 & 0 & \cdots & + & a_{m-1} \end{pmatrix}.$$

Repeating the process above, we eventually obtain

$$A \approx B \oplus (+) \oplus \cdots \oplus (+)\ ,$$

where B is a $(\frac{m}{2}+1) \times (\frac{m}{2}+1)$ symmetric star sign pattern with diagonal $(+, -, \ldots, -)$. By Theorem 2.1, A has unique inertia.

Reciprocally, it is clear from the description above that if H has two nonzero elements with different signs, or has a nonzero element a_{2i+1} such that $a_{2i}a_{2i+1} > 0$, then A does not require unique inertia. ■

This proof allows us to identify the inertia set of a 2-generalized star sign pattern that requires unique inertia.

Corollary 5.2 ([8]). *Let A be a $(m+1)\times(m+1)$ symmetric 2-generalized star sign pattern having the form (5.1), and requiring unique inertia.*

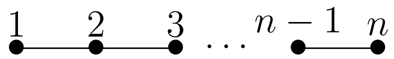
(1) *If H has no nonzero elements, then* $\mathrm{In}(A) = (\frac{m}{2}, \frac{m}{2}, 1)$.
(2) *If H has exactly one $+$ (resp., $-$), then* $\mathrm{In}(A) = (\frac{m}{2}+1, \frac{m}{2}, 0)$ *(resp.,* $\mathrm{In}(A) = (\frac{m}{2}, \frac{m}{2}+1, 0)$*).*
(3) *If H has at least two $+$'s (resp., $-$), and $a_{2i}a_{2i+1} \leq 0$ for all i, then* $\mathrm{In}(A) = (\frac{m}{2}+1, \frac{m}{2}, 0)$ *(resp.,* $\mathrm{In}(A) = (\frac{m}{2}, \frac{m}{2}+1, 0)$*).*

6. Symmetric tridiagonal sign patterns

Let us consider the $n \times n$ $(n \geq 3)$ symmetric tridiagonal sign pattern

$$A_* = \begin{pmatrix} * & + & & & \\ + & * & + & & \\ & + & \ddots & \ddots & \\ & & \ddots & \ddots & + \\ & & & + & * \end{pmatrix},$$

where each diagonal entry is 0, $+$ or $-$. The graph of a tridiagonal sign pattern is the path



Write $A_* = (a_{ij})$. If i is odd (even), we say that the diagonal entry a_{ii} is in a *odd* (*even*) *diagonal position*. If $i < j$, we say that a_{ii} and a_{jj} are in *ascending positions*. Moreover, if i is odd and j is even, we say that a_{ii} and a_{jj} are in *odd-even ascending positions*. If $i < j < k$, i and k are odds and j is even, we say that a_{ii}, a_{jj} and a_{kk} are in *odd-even-odd ascending positions*.

The following result, proved in [6, 10, 13], can be easily obtained using our algorithm.

Theorem 6.1 ([6, 10, 13]). *Let $A_* = (a_{ij})$ be an $n \times n$ symmetric tridiagonal sign pattern. Then,*

(1) *if n is even, A_* has unique inertia if and only if there are no two $+$ nor two $-$ diagonal entries in A_* in odd-even ascending positions. In this case,* $\mathrm{In}(A) = (\frac{n}{2}, \frac{n}{2}, 0)$*;*
(2) *if n is odd, A_* has unique inertia if and only if there are no simultaneous $+$ and $-$ in odd diagonal positions, and neither three $+$ nor three $-$ diagonal entries are in odd-even-odd ascending positions, respectively. In this case* $\mathrm{In}(A_*) = (\frac{n+1}{2}, \frac{n-1}{2}, 0)$ *if there are $+$ in odd*

positions, $\mathrm{In}(A_*) = (\frac{n-1}{2}, \frac{n+1}{2}, 0)$ *if there are* $-$ *in odd positions, and* $\mathrm{In}(A_*) = (\frac{n-1}{2}, \frac{n-1}{2}, 1)$ *if there are neither* $+$ *nor* $-$ *in odd positions.*

Proof: (1) If a_{nn} is zero then, following the procedure described in the previous section, we have

$$A_* \approx A'_* \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix},$$

where $A'_* = (a_{ij})$, for all $i, j = 1, \ldots, n-2$. By induction, we find

$$\mathrm{In}(A_*) = \mathrm{In}(A'_*) + (1,1,0) = \left(\frac{n}{2}, \frac{n}{2}, 0\right).$$

Otherwise, assume that $a_{nn} = +$ (the proof is analogous if $a_{nn} = -$), and notice that this means that all odd diagonal entries of A_* are 0 or $-$. Applying our algorithm, we get $A_* \approx A'_* \oplus [+]$, where $A'_* = (a'_{ij})$ with $a'_{ij} = a_{ij}$, for all $i, j = 1, \ldots, n-1$, except for the diagonal entry $n-1$, where we have $a'_{n-1,n-1} = -$. Now, if $a'_{n-2,n-2} = -$, then all odd diagonal entries of A'_* must be zero. In this case, reordering the indices of $G \setminus \{n\}$ and applying algorithm to the corresponding matrix, we get

$$A'_* \approx \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus (-),$$

and therefore, $\mathrm{In}(A_*) = \mathrm{In}(A'_*) + (1,0,0) = (\frac{n}{2}, \frac{n}{2}, 0)$. Finally, if a'_{n-2n-2} is 0 or $+$, the algorithm gives $A'_* \approx A''_* \oplus (-)$, where $A''_* = (a''_{ij})$, with $a''_{ij} = a_{ij}$, for all $i, j = 1, \ldots, n-2$. By induction, we have $\mathrm{In}(A_*) = \mathrm{In}(A''_*) + (0,1,0) + (1,0,0) = (\frac{n}{2}, \frac{n}{2}, 0)$.

Conversely assuming the existence of two $+$ diagonal entries in A_* in odd-even ascending positions, say $2k+1$ and $2k'$, then, by algorithm 1, we may write $A_* \approx A'_* \oplus B_1 \oplus \cdots \oplus B_r$, where $A'_* = (a'_{ij})$ is the tridiagonal sign pattern of order $2k+2$, with $a'_{ij} = a_{ij}$ for $1 \leq i, j \leq 2k+1$, and $a'_{2k+2,2k+2} = +$, and each $B_i = (+) \oplus (-)$ or $B_i = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}$. By the main algorithm, we get

$$A'_* \approx \begin{pmatrix} \ddots & \ddots & & \\ \ddots & a_{2k,2k} & \pm & \\ & \pm & * & 0 \\ & & 0 & + \end{pmatrix},$$

where $*$ may be $0, +$ or $-$. Choosing appropriately the sign of $*$, we eventually get

$$A'_* \approx \begin{pmatrix} * & 0 \\ 0 & + \end{pmatrix} \oplus B'_1 \oplus \cdots \oplus B'_s,$$

where each B'_i is a diagonal sign pattern of order 2. Thus, A_* has not unique inertia.

(2) The *only if* condition is similar to the only if condition in (1).

Reciprocally, we start noticing that if all diagonal odd positions are 0, then by our algorithm, we get

$$A_* \approx (0) \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix},$$

and thus $\mathrm{In}(A_*) = (\frac{n-1}{2}, \frac{n-1}{2}, 1)$. Otherwise, assume that there is at least one $+$ or $-$ diagonal entry in an odd position, but not both in odd positions, and neither three $+$ nor three $-$ diagonal entries are in odd-even-odd ascending positions, respectively. The result is clear for $n = 3$. So, let $n > 3$, and consider $a_{nn} = 0$. In this case, following the algorithm, we get

$$A_* \approx A'_* \oplus \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix},$$

where $A'_* = (a_{ij})_{1 \le i,j \le n-2}$, and the result follows by induction. Next, suppose $a_{nn} = +$ (the proof is analogous if $a_{n,n} = -$). Notice that if $a_{n-1,n-1} = +$, then all odd diagonal entries in A_* must be equal to 0. In this case, we get $A_* \approx A'_* \oplus (+)$, where $A'_* = (a'_{ij})$ is a tridiagonal sign pattern of order $n-1$, satisfying $a'_{ij} = a_{ij}$ for all $1 \le i, j \le n-2$, and $a_{n-1,n-1}$ may be $0, +$ or $-$. Therefore, by (a) we get

$$\mathrm{In}(A_*) = \mathrm{In}(A'_*) + (1,0,0) = \left(\frac{n+1}{2}, \frac{n-1}{2}, 0\right).$$

Finally, if $a_{n-1,n-1}$ is 0 or $-$, then we get $A_* \approx A'_* \oplus (+)$, where $A'_* = (a'_{ij})$ is a tridiagonal sign pattern of order $n-1$, satisfying $a'_{ij} = a_{ij}$ for all $1 \le i, j \le n-2$, and $a_{n-1,n-1} = -$. Noticing that $a_{ii} \ne -$, for all odd $1 \le i \le n-1$, again by (a) we find that

$$\mathrm{In}(A_*) = \mathrm{In}(A'_*) + (1,0,0) = \left(\frac{n+1}{2}, \frac{n-1}{2}, 0\right).$$

■

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