$*$-LIE-TYPE MAPS ON ALTERNATIVE $*$-ALGEBRAS

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ABSTRACT. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. In this paper we study the characterization of multiplicative $*$-Lie-type maps. As application, we get a result on alternative W^*-algebras.

Keywords: alternative $*$-algebra, alternative W^*-algebras.

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1. INTRODUCTION AND PRELIMINARIES

The study of additivity of maps have received a fair amount of attention of mathematicians. The first quite surprising result is due to Martindale who established a condition on a ring such that multiplicative bijective maps are all additive [17]. Besides, over the years several works have been published considering different types of associative and non-associative algebras among them we can mention [3, 8, 9, 10, 11, 12, 13]. In order to add new ingredients to the study of additivity of maps, many researches have devoted themselves to the investigation of two new products, presented by Brešar and Fošner in [2, 14], where the definition is as follows: for $a, b \in R$, where R is a $*-$ring, we denote by $\{a, b\}_* = ab + ba^*$ and $[a, b]_* = ab - ba^*$ the $*$-Jordan product and the $*$-Lie product, respectively. In [5], the authors proved that a map φ between two factor von Newmann algebras is a $*$-ring isomorphism if and only if $\varphi(\{a, b\}_*) = \{\varphi(a), \varphi(b)\}_*$. In [7], Ferreira and Costa extended these new products and defined two other types of applications, named multiplicative $*$-Jordan n-map and multiplicative $*$-Lie n-map and used it to impose condition such that a map between C^*-algebras is a $*$-ring isomorphism.

With this picture in mind, in this article we will discuss when a multiplicative $*$-Lie n-map is a $*$-isomorphism in the case of alternative $*$-algebras and, just as it was done in [6], we provide an application on alternative W^*-algebras. Throughout the paper, the ground field is assumed to be the field of complex numbers.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and $\varphi : \mathfrak{A} \to \mathfrak{A}'$ a map. We have the following concepts:

i. φ *preserves product* if $\varphi(ab) = \varphi(a)\varphi(b)$, for all elements $a, b \in \mathfrak{A}$;
ii. φ *preserves Lie product* if $\varphi(ab - ba) = \varphi(a)\varphi(b) - \varphi(b)\varphi(a)$, for any $a, b \in \mathfrak{A}$;
iii. φ is *additive* if $\varphi(a + b) = \varphi(a) + \varphi(b)$, for any $a, b \in \mathfrak{A}$;
iv. φ is *isomorphism* if φ is a bijection additive that preserves products and scalar multiplication;
v. φ is *unital* if $\varphi(1_{\mathfrak{A}}) = 1_{\mathfrak{A}'}$.

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An algebra $\mathfrak{A}$ is called $*$-*algebra* if $\mathfrak{A}$ is endowed with a involution. By involution, we mean a mapping $*:\mathfrak{A}\to\mathfrak{A}$ such that $(x+y)^*=x^*+y^*$, $(x^*)^*=x$ and $(xy)^*=y^*x^*$ for all $x,y\in\mathfrak{A}$. An element $s\in\mathfrak{A}$ satisfying $s^*=s$ is called *symmetric element* of $\mathfrak{A}$.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two $*$-algebras and $\varphi:\mathfrak{A}\to\mathfrak{A}'$ a map. We have the following definitions:

i. φ *preserves involution* if $\varphi(a^*)=\varphi(a)^*$, for all elements $a\in\mathfrak{A}$;
ii. φ is $*$-*isomorphism* if φ is an isomorphism that preserves involution;
iii. φ is $*$-*additive* if it preserves involution and it is additive.

Definition 1.1 (To see [7])**.** Consider a $*$-algebra $\mathfrak{A}$, we denote $[x_1,x_2]_*=x_1x_2-x_2x_1^*$, for all $x_1,x_2\in\mathfrak{A}$ and the sequence of polynomials,

$$p_{1_*}(x)=x \text{ and } p_{n_*}(x_1,x_2,\ldots,x_n)=\left[p_{(n-1)_*}(x_1,x_2,\ldots,x_{n-1}),x_n\right]_*,$$

for all integers $n\geq 2$ and $x_1,\ldots,x_n\in\mathfrak{A}$.

Thus, $p_{2_*}(x_1,x_2)=[x_1,x_2]_*=x_1x_2-x_2x_1^*$, for all $x_1,\,x_2\in\mathfrak{A}$, $p_{3_*}(x_1,x_2,x_3)=[[x_1,x_2]_*,x_3]_*$, for all $x_1,\,x_2,\,x_3\in\mathfrak{A}$, etc. Note that p_{2_*} is the product introduced by Brešar and Fošner [2, 14]. Then, using the nomenclature introduced in [7] we have a new class of maps (not necessarily additive).

Definition 1.2. Consider two $*$-algebras $\mathfrak{A}$ and $\mathfrak{A}'$. A map $\varphi:\mathfrak{A}\longrightarrow\mathfrak{A}'$ is *multiplicative $*$-Lie n-map* if

$$\varphi(p_{n_*}(x_1,x_2,\ldots,x_j,\ldots,x_n))=p_{n_*}(\varphi(x_1),\varphi(x_2),\ldots,\varphi(x_j),\ldots,\varphi(x_n)),$$

for all $x_1,x_2,\ldots,x_n\in\mathfrak{A}$, where $n\geq 2$ is an integer. Multiplicative $*$-Lie 2-map, $*$-Lie 3-map and $*$-Lie n-map are collectively referred to as *multiplicative $*$-Lie-type maps*.

An algebra $\mathfrak{A}$ (not necessarily associative or commutative) is called *alternative algebra* if it satisfies the identities $a^2b=a(ab)$ and $ba^2=(ba)a$, for all elements $a,b\in\mathfrak{A}$. One easily sees that any associative algebra is an alternative algebra. An alternative algebra $\mathfrak{A}$ is called *prime* if for any elements $a,b\in\mathfrak{A}$ satisfying the condition $a\mathfrak{A}b=0$, then either $a=0$ or $b=0$.

We consider an alternative algebra $\mathfrak{A}$ with identity $1_{\mathfrak{A}}$. Fix a nontrivial idempotent element $e_1\in\mathfrak{A}$ and denote $e_2=1_{\mathfrak{A}}-e_1$. It is easy to see that $(e_ka)e_j=e_k(ae_j)$ $(k,j=1,2)$ for all $a\in\mathfrak{A}$. Then $\mathfrak{A}$ has a Peirce decomposition

$$\mathfrak{A}=\mathfrak{A}_{11}\oplus\mathfrak{A}_{12}\oplus\mathfrak{A}_{21}\oplus\mathfrak{A}_{22},$$

where $\mathfrak{A}_{kj}:=e_k\mathfrak{A}e_j$ $(k,j=1,2)$ (see [15]), satisfying the following multiplicative relations:

(i) $\mathfrak{A}_{kj}\mathfrak{A}_{jl}\subseteq\mathfrak{A}_{kl}$ $(k,j,l=1,2)$;
(ii) $\mathfrak{A}_{kj}\mathfrak{A}_{kj}\subseteq\mathfrak{A}_{jk}$ $(k,j=1,2)$;
(iii) $\mathfrak{A}_{kj}\mathfrak{A}_{ml}=\{0\}$, if $j\neq m$ and $(k,j)\neq(m,l)$, $(k,j,m,l=1,2)$;
(iv) $x_{kj}^2=0$, for all $x_{kj}\in\mathfrak{A}_{kj}$ $(k,j=1,2;\ k\neq j)$.

2. Main theorem

In the following we shall prove a part of the main result of this paper.

Theorem 2.1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2=1_{\mathfrak{A}}-e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. Suppose that $\mathfrak{A}$ satisfies*

(1) $$(e_j\mathfrak{A})x=\{0\} \textit{ for any } j\in\{1,2\} \quad \textit{implies} \quad x=0$$

Suppose also that $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a multiplication bijective unital map which satisfies

$$\varphi(p_{n_*}(a, b, \xi, \ldots, \xi)) = p_{n_*}(\varphi(a), \varphi(b), \varphi(\xi), \ldots, \varphi(\xi)), \tag{2}$$

for all $a, b \in \mathfrak{A}$ and $\xi \in \{e_1, e_2, 1_{\mathfrak{A}}\}$. Then φ is $*$-additive.

The following claims and lemmas have the same hypotheses as the Theorem 2.1 and we need them to prove the $*$-additivity of φ.

Claim 2.1. $*(\mathfrak{A}_{kj}) \subset \mathfrak{A}_{jk}$*, for $j, k \in \{1, 2\}$.*

Proof. If $a_{kj} \in \mathfrak{A}_{kj}$ then

$$a_{kj}^* = (e_k a_{kj} e_j)^* = (e_j)^*(a_{kj})^*(e_k)^* = e_j(a_{kj})^* e_k \in \mathfrak{A}_{jk}.$$

□

It is easy to check the following result (see [6]).

Claim 2.2. *Let x, y, h in $\mathfrak{A}$ such that $\varphi(h) = \varphi(x) + \varphi(y)$. Then, given $z \in \mathfrak{A}$,*

$$\varphi(p_{n_*}(h, z, \xi, \ldots, \xi)) = \varphi(p_{n_*}(x, z, \xi, \ldots, \xi)) + \varphi(p_{n_*}(y, z, \xi, \ldots, \xi))$$

and

$$\varphi(p_{n_*}(z, h, \xi, \ldots, \xi)) = \varphi(p_{n_*}(z, x, \xi, \ldots, \xi)) + \varphi(p_{n_*}(z, y, \xi, \ldots \xi))$$

for $\xi \in \{e_1, e_2, 1_{\mathfrak{A}}\}$.

Claim 2.3. $\varphi(0) = 0$.

Proof. Since φ is surjective, there exists $x \in \mathfrak{A}$ such that $\varphi(x) = 0$. Then,

$$\begin{aligned}\varphi(0) &= \varphi(p_{n_*}(0, x, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}})) = p_{n_*}(\varphi(0), \varphi(x), \varphi(1_{\mathfrak{A}}), \ldots, \varphi(1_{\mathfrak{A}})) \\ &= p_{n_*}(\varphi(0), 0, \varphi(1_{\mathfrak{A}}), \ldots, \varphi(1_{\mathfrak{A}})) = 0.\end{aligned}$$

□

The next results aim to show the additivity of φ.

Lemma 2.1. *For any $a_{11} \in \mathfrak{A}_{11}$ and $b_{22} \in \mathfrak{A}_{22}$, we have*

$$\varphi(a_{11} + b_{22}) = \varphi(a_{11}) + \varphi(b_{22}).$$

Proof. Since φ is surjective, given $\varphi(a_{11}) + \varphi(b_{22}) \in \mathfrak{A}'$ there exists $h \in \mathfrak{A}$ such that $\varphi(h) = \varphi(a_{11}) + \varphi(b_{22})$. We may write $h = h_{11} + h_{12} + h_{21} + h_{22}$, with $h_{jk} \in \mathfrak{A}_{jk}$ $(k, j = 1, 2)$. Besides, by Claims 2.2 and 2.3

$$\varphi(p_{n_*}(e_1, h, e_1, \ldots, e_1)) = \varphi(p_{n_*}(e_1, a_{11}, e_1, \ldots, e_1)) + \varphi(p_{n_*}(e_1, b_{22}, e_1, \ldots, e_1)),$$

that is,

$$\varphi(-h_{21} + h_{21}^*) = \varphi(0) + \varphi(0) = 0.$$

Then, by injectivity of φ, $-h_{21} + h_{21}^* = 0$. Thus $h_{21} = 0$. Moreover,

$$\varphi(p_{n_*}(e_2, h, e_2, \ldots, e_2)) = \varphi(p_{n_*}(e_2, a_{11}, e_2, \ldots, e_2)) + \varphi(p_{n_*}(e_2, b_{22}, e_2, \ldots, e_2)),$$

that is,

$$\varphi(-h_{12} + h_{12}^*) = 0.$$

Again, by injectivity of φ we conclude that $h_{12} = 0$.

Furthermore, given $d_{21} \in \mathfrak{A}_{21}$,

$$\varphi(p_{n_*}(d_{21}, h, e_1, \ldots, e_1)) = \varphi(p_{n_*}(d_{21}, a_{11}, e_1, \ldots, e_1)) + \varphi(p_{n_*}(d_{21}, b_{22}, e_1, \ldots, e_1)),$$

that is,

$$\varphi(d_{21}h_{11} - (d_{21}h_{11})^*) = \varphi(d_{21}a_{11} - (d_{21}a_{11})^*).$$

Then we conclude, by injectivity of φ, that $d_{21}h_{11}-(d_{21}h_{11})^*=d_{21}a_{11}-(d_{21}a_{11})^*$, that is, $d_{21}(h_{11}-a_{11})=0$. Even more, $(e_2\mathfrak{A})(h_{11}-a_{11})=0$, which implies that $h_{11}=a_{11}$ by Condition (1) of Theorem 2.1.

Finally, given $d_{12}\in\mathfrak{A}_{12}$, a similar calculation gives us $h_{22}=b_{22}$. Therefore $h=a_{11}+b_{22}$. □

Lemma 2.2. *For any $a_{12}\in\mathfrak{A}_{12}$ and $b_{21}\in\mathfrak{A}_{21}$, we have $\varphi(a_{12}+b_{21})=\varphi(a_{12})+\varphi(b_{21})$.*

Proof. Since φ is surjective, given $\varphi(a_{12})+\varphi(b_{21})\in\mathfrak{A}'$ there exists $h\in\mathfrak{A}$ such that $\varphi(h)=\varphi(a_{12})+\varphi(b_{21})$. We may write $h=h_{11}+h_{12}+h_{21}+h_{22}$, with $h_{jk}\in\mathfrak{A}_{jk}$ $(k,j=1,2)$. Now, by Claims 2.2 and 2.3

$$\varphi(p_{n_*}(e_1,h,e_1,\ldots,e_1))=\varphi(p_{n_*}(e_1,a_{12},e_1,\ldots,e_1))+\varphi(p_{n_*}(e_1,b_{21},e_1,\ldots,e_1)),$$

that is,

$$\varphi(-h_{21}+h_{21}^*)=\varphi(-b_{21}+b_{21}^*).$$

Then, by injectivity of φ, $-h_{21}+h_{21}^*=-b_{21}+b_{21}^*$. Thus $h_{21}=b_{21}$. Moreover,

$$\varphi(p_{n_*}(e_2,h,e_2,\ldots,e_2))=\varphi(p_{n_*}(e_2,a_{12},e_2,\ldots,e_2))+\varphi(p_{n_*}(e_2,b_{21},e_2,\ldots,e_2)),$$

that is,

$$\varphi(-h_{12}+h_{12}^*)=\varphi(-a_{12}+a_{12}^*).$$

Again, by injectivity of φ we conclude that $h_{12}=a_{12}$.

Furthermore, given $d_{21}\in\mathfrak{A}_{21}$,

$$\begin{aligned}\varphi(d_{21}h_{11}-(d_{21}h_{11})^*)&=\varphi(p_{n_*}(d_{21},h,e_1,\ldots,e_1))\\&=\varphi(p_{n_*}(d_{21},a_{12},e_1,\ldots,e_1))+\varphi(p_{n_*}(d_{21},b_{21},e_1,\ldots,e_1))=0.\end{aligned}$$

Then we conclude, by injectivity of φ, that $d_{21}h_{11}-(d_{21}h_{11})^*=0$, that is, $d_{21}h_{11}=0$. Even more, $(e_2\mathfrak{A})h_{11}=0$, which implies that $h_{11}=0$ by Condition (1) of Theorem 2.1.

Finally, given $d_{12}\in\mathfrak{A}_{12}$, a similar calculation gives us $h_{22}=0$. Therefore, we conclude that $h=a_{12}+b_{21}$. □

Lemma 2.3. *For any $a_{11}\in\mathfrak{A}_{11}$, $b_{12}\in\mathfrak{A}_{12}$, $c_{21}\in\mathfrak{A}_{21}$ and $d_{22}\in\mathfrak{A}_{22}$ we have*

$$\varphi(a_{11}+b_{12}+c_{21}+d_{22})=\varphi(a_{11})+\varphi(b_{12})+\varphi(c_{21})+\varphi(d_{22}).$$

Proof. Since φ is surjective, given $\varphi(a_{11})+\varphi(b_{12})+\varphi(c_{21})+\varphi(d_{22})\in\mathfrak{A}'$ there exists $h\in\mathfrak{A}$ such that $\varphi(h)=\varphi(a_{11})+\varphi(b_{12})+\varphi(c_{21})+\varphi(d_{22})$. We may write $h=h_{11}+h_{12}+h_{21}+h_{22}$, with $h_{jk}\in\mathfrak{A}_{jk}$ $(k,j=1,2)$. Applying Lemmas 2.1 and 2.2 we have

$$\varphi(h)=\varphi(a_{11})+\varphi(b_{12})+\varphi(c_{21})+\varphi(d_{22})=\varphi(a_{11}+d_{22})+\varphi(b_{12}+c_{21}).$$

Now, observing that $p_{n_*}(e_1,a_{11}+d_{22},e_1,\ldots,e_1)=0=p_{n_*}(e_1,b_{12},e_1,\ldots,e_1)$ and by Claims 2.2 and 2.3 we obtain

$$\begin{aligned}&\varphi(p_{n_*}(e_1,h,e_1,\ldots,e_1))\\&=\varphi(p_{n_*}(e_1,a_{11}+d_{22},e_1,\ldots,e_1))+\varphi(p_{n_*}(e_1,b_{12}+c_{21},e_1,\ldots,e_1))\\&=\varphi(p_{n_*}(e_1,c_{21},e_1,\ldots,e_1)),\end{aligned}$$

that is,

$$\varphi(-h_{21}+h_{21}^*)=\varphi(-c_{21}+c_{21}^*).$$

Then, by injectivity of φ, $-h_{21}+h_{21}^*=-c_{21}+c_{21}^*$. Thus $h_{21}=c_{21}$.

In a similar way, using e_2 rather than e_1 in the previous calculation, we conclude that $h_{12} = b_{12}$. Also, given $x_{21} \in \mathfrak{A}_{21}$,

$$\begin{aligned}&\varphi(p_{n_*}(x_{21}, h, e_1, \dots, e_1))\\&= \varphi(p_{n_*}(x_{21}, a_{11} + d_{22}, \dots, e_1)) + \varphi(p_{n_*}(x_{21}, b_{12} + c_{21}, e_1, \dots, e_1))\\&= \varphi(p_{n_*}(x_{21}, a_{11}, e_1, \dots, e_1)),\end{aligned}$$

since $p_{n_*}(x_{21}, b_{12} + c_{21}, e_1, \dots, e_1) = 0 = p_{n_*}(x_{21}, d_{22}, \dots, e_1)$. Again, by injectivity of φ we conclude, by following the same strategy as in the proof of Lemma 2.1, that $h_{11} = a_{11}$. Now, using e_2 rather than e_1 and x_{12} rather than x_{21} in the previous calculation we obtain $h_{22} = d_{22}$. Therefore, $h = a_{11} + b_{12} + c_{21} + d_{22}$. □

Lemma 2.4. *For any $a_{jk}, b_{jk} \in \mathfrak{A}_{jk}$, with $j \neq k$, we have $\varphi(a_{jk} + b_{jk}) = \varphi(a_{jk}) + \varphi(b_{jk})$.*

Proof. We shall prove the case $j = 1$ and $k = 2$. The other case is done in a similar way. Since φ is surjective, given $\varphi(a_{12}) + \varphi(b_{12}) \in \mathfrak{A}'$ and $\varphi(-a_{12}^*) + \varphi(-b_{12}^*)$ there exist $h \in \mathfrak{A}$ and $t \in \mathfrak{A}$ such that $\varphi(h) = \varphi(a_{12}) + \varphi(b_{12})$ and $\varphi(t) = \varphi(-a_{12}^*) + \varphi(-b_{12}^*)$. We may write $h = h_{11} + h_{12} + h_{21} + h_{22}$ and $t = t_{11} + t_{12} + t_{21} + t_{22}$, with $h_{jk}, t_{jk} \in \mathfrak{A}_{jk}$ $(k, j = 1, 2)$.

First we show that $h \in \mathfrak{A}_{12}$. By Claim 2.2 we get

$$\begin{aligned}\varphi(-h_{21} + h_{21}^*) &= \varphi(p_{n_*}(e_1, h, e_1, \dots, e_1))\\&= \varphi(p_{n_*}(e_1, a_{12}, e_1, \dots, e_1)) + \varphi(p_{n_*}(e_1, b_{12}, e_1, \dots, e_1)) = 0.\end{aligned}$$

Then, by injectivity of φ we obtain $h_{21} = 0$. Also, given $d_{12} \in \mathfrak{A}_{12}$,

$$\begin{aligned}\varphi(d_{12}h_{22} - (d_{12}h_{22})^*) &= \varphi(p_{n_*}(d_{12}, h, e_2, \dots, e_2))\\&= \varphi(p_{n_*}(d_{12}, a_{12}, e_2, \dots, e_2)) + \varphi(p_{n_*}(d_{12}, b_{12}, e_2, \dots, e_2)) = 0,\end{aligned}$$

that is, $d_{12}h_{22} = 0$, which implies that $h_{22} = 0$ by Condition (1) of Theorem 2.1. Now, using $d_{21} \in \mathfrak{A}_{21}$ rather than d_{12} in the previous calculation, we conclude that $h_{11} = 0$. Therefore, $h = h_{12} \in \mathfrak{A}_{12}$.

In a similar way, we obtain $t = t_{21} \in \mathfrak{A}_{21}$. Finally, by Lemma 2.3

$$\begin{aligned}\varphi(a_{12} + b_{12} - a_{12}^* - b_{12}^*) &= \varphi(p_{n_*}(e_1 + a_{12}, e_2 + b_{12}, e_2, \dots, e_2))\\&= p_{n_*}(\varphi(e_1 + a_{12}), \varphi(e_2 + b_{12}), \varphi(e_2), \dots, \varphi(e_2))\\&= p_{n_*}(\varphi(e_1), \varphi(e_2), \varphi(e_2), \dots, \varphi(e_2))\\&+ p_{n_*}(\varphi(e_1), \varphi(b_{12}), \varphi(e_2), \dots, \varphi(e_2))\\&+ p_{n_*}(\varphi(a_{12}), \varphi(e_2), \varphi(e_2), \dots, \varphi(e_2))\\&+ p_{n_*}(\varphi(a_{12}), \varphi(b_{12}), \varphi(e_2), \dots, \varphi(e_2))\\&= \varphi(p_{n_*}(e_1, e_2, e_2, \dots, e_2))\\&+ \varphi(p_{n_*}(e_1, b_{12}, e_2, \dots, e_2))\\&+ \varphi(p_{n_*}(a_{12}, e_2, e_2, \dots, e_2))\\&+ \varphi(p_{n_*}(a_{12}, b_{12}, e_2, \dots, e_2))\\&= \varphi(a_{12} - a_{12}^*) + \varphi(b_{12} - b_{12}^*)\\&= \varphi(a_{12}) + \varphi(b_{12}) + \varphi(-a_{12}^*) + \varphi(-b_{12}^*)\\&= \varphi(h_{12}) + \varphi(t_{21}) = \varphi(h_{12} + t_{21}).\end{aligned}$$

Since φ is injective, we have $a_{12} + b_{12} - a_{12}^* - b_{12}^* = h_{12} + t_{21}$, this is, $h = h_{12} = a_{12} + b_{12}$. □

Lemma 2.5. *For any $a_{jj}, b_{jj} \in \mathfrak{A}_{jj}$, with $j \in \{1, 2\}$, we have $\varphi(a_{jj} + b_{jj}) = \varphi(a_{jj}) + \varphi(b_{jj})$.*

Proof. We shall prove the case $j = 1$, since the other case is done in a similar way. Since φ is surjective, given $\varphi(a_{11}) + \varphi(b_{11}) \in \mathfrak{A}'$ there exists $h \in \mathfrak{A}$ such that $\varphi(h) = \varphi(a_{11}) + \varphi(b_{11})$. We may write $h = h_{11} + h_{12} + h_{21} + h_{22}$, with $h_{jk} \in \mathfrak{A}_{jk}$ $(k, j = 1, 2)$. Now, by Claim 2.2

$$\begin{aligned}\varphi(-h_{21} + h_{21}^*) &= \varphi(p_{n_*}(e_1, h, e_1, \ldots, e_1)) \\ &= \varphi(p_{n_*}(e_1, a_{11}, e_1, \ldots, e_1)) + \varphi(p_{n_*}(e_1, b_{11}, e_1, \ldots, e_1)) = 0.\end{aligned}$$

Then, by injectivity of φ we obtain $h_{21} = 0$. Also,

$$\begin{aligned}\varphi(-h_{12} + h_{12}^*) &= \varphi(p_{n_*}(e_2, h, e_2, \ldots, e_2)) \\ &= \varphi(p_{n_*}(e_2, a_{11}, e_2, \ldots, e_2)) + \varphi(p_{n_*}(e_2, b_{11}, e_2, \ldots, e_2)) = 0,\end{aligned}$$

that is, $h_{12} = 0$ by injectivity of φ. Moreover, given $d_{12} \in \mathfrak{A}_{12}$,

$$\begin{aligned}\varphi(d_{12}h_{22} - (d_{12}h_{22})^*) &= \varphi(p_{n_*}(d_{12}, h, e_2, \ldots, e_2)) \\ &= \varphi(p_{n_*}(d_{12}, a_{11}, e_2, \ldots, e_2)) + \varphi(p_{n_*}(d_{12}, b_{11}, e_2, \ldots, e_2)) \\ &= 0.\end{aligned}$$

Then, by injectivity of φ, $d_{12}h_{22} = 0$, which implies that $h_{22} = 0$ by Condition (1) of Theorem 2.1. Finally, given $d_{21} \in \mathfrak{A}_{21}$, by Lemmas 2.3 and 2.4 we have

$$\begin{aligned}\varphi(d_{21}h_{11} - (d_{21}h_{11})^*) &= \varphi(p_{n_*}(d_{21}, h, e_1, \ldots, e_1)) \\ &= \varphi(p_{n_*}(d_{21}, a_{11}, e_1, \ldots, e_1)) + \varphi(p_{n_*}(d_{21}, b_{11}, e_1, \ldots, e_1)) \\ &= \varphi(d_{21}a_{11} - (d_{21}a_{11})^*) + \varphi(d_{21}b_{11} - (d_{21}b_{11})^*) \\ &= \varphi(d_{21}a_{11}) + \varphi(-(d_{21}a_{11})^*) + \varphi(d_{21}b_{11}) + \varphi(-(d_{21}b_{11})^*) \\ &= \varphi(d_{21}a_{11} + d_{21}b_{11}) + \varphi(-(d_{21}a_{11})^* - (d_{21}b_{11})^*) \\ &= \varphi(d_{21}(a_{11} + b_{11}) - (a_{11}^* + b_{11}^*)d_{21}^*),\end{aligned}$$

that is, $d_{21}h_{11} - (d_{21}h_{11})^* = d_{21}(a_{11} + b_{11}) - (a_{11}^* + b_{11}^*)d_{21}^*$, by injectivity of φ. Thus, $d_{21}(h_{11} - (a_{11} + b_{11})) = 0$, which implies that $h_{11} = a_{11} + b_{11}$ by Condition (1) of Theorem 2.1. □

Proof of Theorem 2.1. Now using Lemmas 2.3, 2.4 and 2.5 is easy see that φ is additive. Besides, using additivity of φ and since φ is unital, we have for $a \in \mathfrak{A}$,

$$\begin{aligned}2^{n-2}(\varphi(a) - \varphi(a)^*) &= p_{n_*}(\varphi(a), 1_{\mathfrak{A}'}, \ldots, 1_{\mathfrak{A}'}) = p_{n_*}(\varphi(a), \varphi(1_{\mathfrak{A}}), \ldots, \varphi(1_{\mathfrak{A}})) \\ &= \varphi(p_{n_*}(a, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}})) = \varphi(2^{n-2}(a - a^*)) \\ &= 2^{n-2}\varphi(a - a^*) = 2^{n-2}(\varphi(a) - \varphi(a^*)),\end{aligned}$$

then $\varphi(a^*) = \varphi(a)^*$ and we conclude that φ preserves involution. □

Remark 2.1. *Observe that the Theorem 2.1 holds for any field of characteristic different of* 2*. In the proof the Theorem 2.1 we established the additivity of φ without using the unital assumption of φ.*

Theorem 2.2. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. Let $\varphi : \mathfrak{A} \to \mathfrak{A}'$ be a complex scalar multiplication bijective unital map. Suppose that $\mathfrak{A}$ satisfies the conditions of the Theorem 2.1, namely,*

$$(e_j\mathfrak{A})x = \{0\} \text{ for any } j \in \{1,2\} \quad \text{implies} \quad x = 0,$$

$$\varphi(p_{n_*}(a, b, \xi, \ldots, \xi)) = p_{n_*}(\varphi(a), \varphi(b), \varphi(\xi), \ldots, \varphi(\xi)),$$

for all $a, b \in \mathfrak{A}$ and $\xi \in \{e_1, e_2, 1_{\mathfrak{A}}\}$.

Even more, if $\mathfrak{A}'$ satisfies the condition

$$(\varphi(e_j)\mathfrak{A}')y = \{0\} \text{ for any } j \in \{1,2\} \quad \text{implies} \quad y = 0, \tag{3}$$

then φ is $*$-isomorphism.

With this hypothesis and Theorem 2.1 we have already proved that φ is $*$-additive. It remains for us to show that φ preserves product. In order to do that we will prove some more lemmas. Firstly, we observe that,

Claim 2.4. *$q_j = \varphi(e_j)$ is an idempotent in $\mathfrak{A}'$, for $j \in \{1,2\}$.*

Proof. Since φ is a complex scalar multiplication, it follows that

$$\begin{aligned} 2^{n-1}i\, q_j = 2^{n-1}i\varphi(e_j) &= \varphi(2^{n-1}ie_j) = \varphi(p_{n_*}(ie_j, e_j, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}})) \\ &= p_{n_*}(i\varphi(e_j), \varphi(e_j), \varphi(1_{\mathfrak{A}}), \ldots, \varphi(1_{\mathfrak{A}})) \\ &= p_{n_*}(i\varphi(e_j), \varphi(e_j), 1_{\mathfrak{A}'}, \ldots, 1_{\mathfrak{A}'})) = 2^{n-1}i\varphi(e_j)^2 = 2^{n-1}i\, {q_j}^2. \end{aligned}$$

Then we can conclude that $q_j = {q_j}^2$. Moreover, since e_j is a idempotent in $\mathfrak{A}$ we have that $p_{n_*}(e_j, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}}) = 0$. Besides,

$$0 = \varphi(0) = \varphi(p_{n_*}(e_j, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}})) = p_{n_*}(q_j, 1_{\mathfrak{A}'}, \ldots, 1_{\mathfrak{A}'}).$$

Thus, $q_j - {q_j}^* = 0$, that is, $q_j = {q_j}^*$. □

Lemma 2.6. *For any $a \in \mathfrak{A}$, $\varphi(e_ja) = \varphi(e_j)\varphi(a)$ and $\varphi(ae_j) = \varphi(a)\varphi(e_j)$, with $j \in \{1,2\}$.*

Proof. Firstly, observe that

$$p_{n_*}(ia, e_j, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}}) = 2^{n-2}i(ae_j + e_ja^*)$$

and

$$p_{n_*}(a, e_j, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}}) = 2^{n-2}(ae_j - e_ja^*).$$

Still, by Condition (2) of Theorem 2.1 and $*$-additivity of φ,

$$\begin{aligned} \varphi(2^{n-2}i(ae_j + e_ja^*)) &= \varphi(p_{n_*}(ia, e_j, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}})) = p_{n_*}(\varphi(ia), \varphi(e_j), 1_{\mathfrak{A}'}, \ldots, 1_{\mathfrak{A}'}) \\ &= 2^{n-2}i(\varphi(a)\varphi(e_j) + \varphi(e_j)\varphi(a)^*) \end{aligned}$$

and

$$\begin{aligned} \varphi(2^{n-2}(ae_j - e_ja^*)) &= \varphi(p_{n_*}(a, e_j, 1_{\mathfrak{A}}, \ldots, 1_{\mathfrak{A}})) = p_{n_*}(\varphi(a), \varphi(e_j), 1_{\mathfrak{A}'}, \ldots, 1_{\mathfrak{A}'}) \\ &= 2^{n-2}(\varphi(a)\varphi(e_j) - \varphi(e_j)\varphi(a)^*). \end{aligned}$$

Now, since φ is $*$-additive, multiplying the second equality by i and adding these two equations we obtain $\varphi(ae_j) = \varphi(a)\varphi(e_j)$. The second statement is obtained in a similar way. □

Consider the Peirce decomposition of $\mathfrak{A}'$ with respect to idempotents $q_j = \varphi(e_j)$ of $\mathfrak{A}'$ (with $j \in \{1,2\}$) given by $\mathfrak{A}' = \mathfrak{A}'_{11} \oplus \mathfrak{A}'_{12} \oplus \mathfrak{A}'_{21} \oplus \mathfrak{A}'_{22}$, where $\mathfrak{A}'_{kj} := q_k \mathfrak{A}' q_j$ for $k, j \in \{1,2\}$.

Lemma 2.7. *$\varphi(\mathfrak{A}_{jk}) \subset \mathfrak{A}'_{jk}$, for $j, k \in \{1,2\}$.*

Proof. Given $x \in \mathfrak{A}_{jk}$, we have $x = e_j x e_k$ and then, by Lemma 2.6,

$$\varphi(x) = \varphi(e_j)\varphi(xe_k) = \varphi(e_j)\varphi(x)\varphi(e_k) \in \mathfrak{A}'_{jk}.$$

□

Lemma 2.8. *For $j \neq k$, we have:*

- *If $a_{jk} \in \mathfrak{A}_{jk}$ and $b_{kk} \in \mathfrak{A}_{kk}$ then $\varphi(a_{jk}b_{kk}) = \varphi(a_{jk})\varphi(b_{kk})$;*
- *If $a_{jk} \in \mathfrak{A}_{jk}$ and $b_{jk} \in \mathfrak{A}_{jk}$ then $\varphi(a_{jk}b_{jk}) = \varphi(a_{jk})\varphi(b_{jk})$;*
- *If $a_{jj} \in \mathfrak{A}_{jj}$ and $b_{jk} \in \mathfrak{A}_{jk}$ then $\varphi(a_{jj}b_{jk}) = \varphi(a_{jj})\varphi(b_{jk})$;*
- *If $a_{jk} \in \mathfrak{A}_{jk}$ and $b_{kj} \in \mathfrak{A}_{kj}$ then $\varphi(a_{jk}b_{kj}) = \varphi(a_{jk})\varphi(b_{kj})$.*

Proof. In order to prove the first statement, on the one hand, by Lemma 2.7

$$\begin{aligned}\varphi(a_{jk}b_{kk}) - \varphi(a_{jk}b_{kk})^* &= \varphi(a_{jk}b_{kk} - (a_{jk}b_{kk})^*) = \varphi(p_{n_*}(a_{jk}, b_{kk}, e_k, \ldots, e_k)) \\ &= p_{n_*}(\varphi(a_{jk}), \varphi(b_{kk}), q_k, \ldots, q_k) \\ &= \varphi(a_{jk})\varphi(b_{kk}) - (\varphi(a_{jk})\varphi(b_{kk}))^*\end{aligned}$$

and then $\varphi(a_{jk}b_{kk}) = \varphi(a_{jk})\varphi(b_{kk})$.

Now to prove the second statement, we have

$$\begin{aligned}&\varphi(a_{jk}b_{jk}) - \varphi(a_{jk}b_{jk})^* - 2^{n-3}\varphi(b_{jk}a_{jk})^* + 2^{n-3}\varphi(b_{jk}a_{jk}^*)^* \\ &\quad = \varphi(a_{jk}b_{jk}) - (a_{jk}b_{jk})^* - 2^{n-3}(b_{jk}a_{jk})^* + 2^{n-3}(b_{jk}a_{jk}^*)^*) \\ &\quad = \varphi(p_{n_*}(a_{jk}, b_{jk}, e_j, \ldots, e_j)) = p_{n_*}(\varphi(a_{jk}), \varphi(b_{jk}), q_j, \ldots, q_j) \\ &\quad = \varphi(a_{jk})\varphi(b_{jk}) - \varphi(a_{jk})^*\varphi(b_{jk})^* \\ &\quad - 2^{n-3}\varphi(b_{jk})^*\varphi(a_{jk})^* + 2^{n-3}\varphi(b_{jk})^*\varphi(a_{jk}^*)^*\end{aligned}$$

and then $\varphi(a_{jk}b_{jk}) = \varphi(a_{jk})\varphi(b_{jk})$.

The others statements are proved in a similar way. □

Since alternative algebras are flexible, we have

$$(x_{kj}, a_{jj}, b_{jj}) + (b_{jj}, a_{jj}, x_{kj}) = 0,$$

for all $x_{kj} \in \mathfrak{A}_{kj}$, $a_{jj}, b_{jj} \in \mathfrak{A}_{jj}$, for $k, j \in \{1,2\}$.

Lemma 2.9. *If $a_{jj}, b_{jj} \in \mathfrak{A}_{jj}$, with $j \in \{1,2\}$, then $\varphi(a_{jj}b_{jj}) = \varphi(a_{jj})\varphi(b_{jj})$.*

Proof. Let x_{kj} be an element of $\mathfrak{A}_{kj}$, with $j \neq k$. Using Lemma 2.8 we obtain

$$\begin{aligned}\varphi(x_{kj})\varphi(a_{jj}b_{jj}) =& \varphi(x_{kj}a_{jj}b_{jj}) = \varphi((x_{kj}a_{jj})b_{jj}) \\ =& (\varphi(x_{kj})\varphi(a_{jj}))\varphi(b_{jj}) = \varphi(x_{kj})(\varphi(a_{jj})\varphi(b_{jj}))\end{aligned}$$

that is,

$$\varphi(x_{kj})(\varphi(a_{jj}b_{jj}) - \varphi(a_{jj})\varphi(b_{jj})) = 0.$$

Now, by Lemma 2.7, $\varphi(x_{kj}) \in \mathfrak{A}'_{kj}$ as well as $\varphi(a_{jj}b_{jj})$ and $\varphi(a_{jj})\varphi(b_{jj}) \in \mathfrak{A}'_{jj}$. Then, $(\varphi(e_k)\mathfrak{A}')(\varphi(a_{jj}b_{jj}) - \varphi(a_{jj})\varphi(b_{jj})) = 0$, which implies that $\varphi(a_{jj}b_{jj}) = \varphi(a_{jj})\varphi(b_{jj})$ by Condition (3) of Theorem 2.2. □

Proof of Theorem 2.2. By additivity of φ and Lemmas 2.8 and 2.9, it follows that $\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in \mathfrak{A}$, this is, φ preserves product as required. $\square$

3. Corollaries

Now we present some consequences of our main results.

Corollary 3.1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. Let $\varphi : \mathfrak{A} \to \mathfrak{A}'$ be a complex scalar multiplication bijective unital map. Suppose that $\mathfrak{A}$ satisfies*

$$(e_j\mathfrak{A})x = \{0\} \text{ for any } j \in \{1,2\} \quad \text{implies} \quad x = 0.$$

Even more, suppose that $\mathfrak{A}'$ satisfies

$$(\varphi(e_j)\mathfrak{A}')y = \{0\} \text{ for any } j \in \{1,2\} \quad \text{implies} \quad y = 0.$$

In this conditions, $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a multiplicative $*$-Lie n-map if and only if φ is $*$-isomorphism.

It is easy to see that any prime alternative algebra satisfy Conditions (1) and (3), so we have the following result:

Corollary 3.2. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two prime alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. In this condition, a complex scalar multiplication $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective unital multiplicative $*$-Lie n-map if and only if φ is $*$-isomorphism.*

To finish we will give an application of the Corollary 3.2. A complete normed alternative complex $*$-algebra A is called an *alternative C^*-algebra* if it satisfies the condition: $\|a^*a\| = \|a\|^2$, for all elements $a \in A$. Alternative C^*-algebras are non-associative generalizations of C^*-algebras and appear in various areas in Mathematics (see more details in the references [18] and [19]). An alternative C^*-algebra A is called an *alternative W^*-algebra* if it is a dual Banach space and a prime alternative W^*-algebra is called *alternative W^*-factor*. It is well known that non-zero alternative W^*-algebras are unital.

Corollary 3.3. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative W^*-factors. In this condition, a complex scalar multiplication $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective unital multiplicative $*$-Lie n-map if and only if φ is $*$-isomorphism.*

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