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THE REPRODUCING KERNEL STRUCTURE ASSOCIATED TO FOURIER TYPE SYSTEMS AND THEIR QUANTUM ANALOGUES

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ABSTRACT: We study mapping properties of operators with kernels defined via an abstract formulation of quantum (q-) Fourier type systems. We prove Ismail´s conjecture regarding the existence of a reproducing kernel structure behind these kernels. The results are illustrated with Fourier kernels with ultraspherical and Jacobi weights, their continuous q-extensions and generalizations. As a byproduct of this approach, a new class of sampling theorems is obtained, as well as Neumann type expansions in Bessel and q-Bessel functions.

KEYWORDS: Reproducing kernel, q-Fourier series, orthogonal polynomials, basic hypergeometric functions, sampling theorems.
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1. Introduction

The Gegenbauer expansion of the two variable complex exponential in terms of the ultraspherical polynomials

$$e^{ixt} = \Gamma(\nu)\left(\frac{t}{2}\right)^{-\nu}\sum_{k=0}^{\infty} i^k(\nu+k)J_{\nu+k}(t)C_k^{\nu}(x) \tag{1}$$

has the remarkable feature of being at the same time an expansion in a Neumann series of Bessel functions. The usefulness of this expansion was made very clear in a paper authored by Ismail and Zhang, where it was used to solve the eigenvalue problem for the left inverse of the differential operator, on L^2 spaces with ultraspherical weights [19]. The consideration of the q-analogue of this diagonalization problem led the authors to extend Gegenbauer´s formula to the q-case. This task required the introduction of a new q-analogue of the exponential, a two variable function denoted by $\mathcal{E}_q(x;t)$ which became known in the literature as the curly q-exponential function, bearing the name from its notational convention. Ismail and Zhang´s formula

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is

$$\mathcal{E}_q(x;it)=\frac{t^{-\nu}(q;q)_\infty}{(-qt^2;q^2)_\infty(q^{\nu+1};q)_\infty}\sum_{k=0}^{\infty}i^kq^{k^2/4}\frac{(1-q^{k+\nu})}{(1-q^\nu)}J_{\nu+k}^{(2)}(2t;q)C_k(x;q^\nu|q) \tag{2}$$

The functions involved in this formula will be defined in section 4. Since its introduction, the function $\mathcal{E}_q$ was welcomed as a proper q-analogue of the exponential function, since it was suitable to provide a satisfactory q-analogue of the Fourier theory of integral transformations and series developments. This suitability was made concrete by Bustoz and Suslov in [5], where the authors introduced the subject of q-Fourier series. Some of the subsequent research activity has been already collected in a book [26]. Among recent developments not yet included in this book, we quote the orthogonality relations for sums of curly exponential functions [22], obtained using spectral methods, and the construction of a q-analogue of the Whittaker-Shannon-Kotel´nikov sampling theorem [18]. The designation "Quantum" has appear often in recent literature on q-analysis, as in the monographs [21] and [20]. This designation is very convenient, since q-special functions are intimately connected with representations of quantum groups [6].

An abstract formulation designed to capture the essential properties of q-Fourier type systems was proposed in [14] and we proceed to describe it here. Let $\{p_n(x)\}$ be a complete orthonormal system in $L^2(\mu)$ and assume that $\{r_n(x)\}$ is a discrete orthonormal system whose orthogonality relation is

$$\sum_{j=0}^{\infty}\rho(t_j)r_n(t_j)\overline{r_m(t_j)}=\delta_{mn}$$

and with dual orthogonality

$$\sum_{k=0}^{\infty}r_k(t_n)\overline{r_k(t_m)}=\frac{\delta_{mn}}{\rho(t_n)}$$

Assume also that the the system $\{r_n(x)\}$ is complete in $L^2(\sum\rho(t_j)\delta_{t_j})$. Now define a sequence of functions $\{F_n(x)\}$ by

$$F_n(x)=\sum_{k=0}^{\infty}r_k(t_n)p_k(x)u_k \tag{3}$$

where $\{u_k\}$ is an arbitrary sequence of complex numbers in the unit circle. The following theorem is due to Ismail and comprises in an abstract form the fundamental fact behind the theory of basic analogs of Fourier series on a q-quadratic grid [5].

Theorem A [14] *The system $\{F_n(x)\}$ is orthogonal and complete in $L^2(\mu)$.*

To give an idea of what is involved in this statement, we sketch Ismail´s argument. Since, by (3), $r_k(t_n)$ are the Fourier coeficients of F_n in the basis $\{u_k p_k\}$, the use of Parseval´s formula gives

$$\int F_n(x)\overline{F_n(x)}d\mu(x) = \sum_{k=0}^{\infty} r_k(t_n)\overline{r_k(t_m)} = \frac{\delta_{mn}}{\rho(x_n)}$$

and the orthogonality relation is proved. To show the completeness, choose $f \in L^2(\mu)$ and assume $\int F_n(x)f(x)d\mu(x) = 0$ for all m. Again Parseval´s formula implies $\sum_{k=0}^{\infty} f_k r_k(t_m) = 0$ for all m, where f_k are the Fourier coefficients of f in the basis $\{p_k\}$. Now the completeness of $\{r_k\}$ implies $f_k = 0$. Therefore $f = 0$ almost everywhere in $L^2(\mu)$.

In [15], Ismail posed the problem of studying the mapping properties of operators with kernels defined as above and conjectured that there was a reproducing kernel Hilbert space structure behind these operators. We will show that Ismail´s conjecture is true. Our approach will reveal a reproducing kernel structure reminiscent of the well known structure of the Paley Wiener space of functions bandlimited to a real interval. However, even in the case when the system $\{F_n(x)\}$ is the set of the complex exponentials, we obtain results that, as far as our knowledge goes, seem to be new. When the system $\{F_n(x)\}$ is the set of basis functions of the q-Fourier series constructed with the function $\mathcal{E}_q(x;it)$, we will obtain results that complement the investigations done in [26] and [18]. In particular it will be shown that the sampling theorem derived in [18] lives in a reproducing kernel Hilbert space and that the correspondent q-analogues of the *Sinc* function provide an orthogonal basis for that space.

The outline of the paper is as follows. The next section contains the main results of the paper, formulated and proved in the general framework described above. An integral transformation between two Hilbert spaces is defined, basis for both spaces are provided, and the formula for the reproducing kernel of the image Hilbert space is deduced. We also prove an abstract sampling theorem in this context, that generalizes the one in [18]. The remaining sections consider three applications of these results, using specific systems of orthogonal polynomials as well as Bessel functions and their generalizations. The first application is associated to formula (1) and systems of complex exponentials. The reproducing kernel in this case is written in terms of Bessel functions. The second application is linked to (2) and to systems of curly q-exponentials, and we write the reproducing kernel as a ${}_2\phi_2$ basic hypergeometric function. These two examples explore the interplay between Lommel polynomials and Bessel functions and the corresponding relations between their q-analogues. In the last section we consider a construction of a general character, designed originally in the papers [19], [16] and [14]. It allows to extend the interplay between Bessel functions and Lommel polynomials to a more general class of functions. Using this construction we will make a brief discussion about the application of our results to spaces weighted by Jacobi weights and their q-analogues and, in the case of the Jacobi weights, we evaluate the reproducing kernel explicitly. In this instance, the Bessel functions from the ultraspherical case will be replaced by confluent hypergeometric functions.

2. The reproducing kernel structure

In this section we will show the existence of a reproducing kernel structure behind the abstract setting of the previous section. The results will follow from the study of the mapping properties of an integral transform whose kernel is obtained from the sequence of functions $\{r_k\}$ and $\{p_k\}$. Our first technical problem comes from the fact that, when $\{r_k\}$ is a discrete system of orthogonal polynomials with a determinate moment problem, then $\{r_k(t)\} \in l^2$ if and only if x is a mass point for the measure of orthogonality. For this reason the series

$$\sum_{k=0}^{\infty} r_k(t)p_k(x)u_k$$

would diverge if t is not such a point (this is pointed out in Section 5 of [14]). Since we want our kernel to be defined for every t, we will assume the existence of an auxiliary system of functions $\{\mathfrak{J}_k(x)\} \in l^2$ for every t real, and such that every function $\mathfrak{J}_k$ interpolates p_k at the mass points $\{x_n\}$ in the sense that

$$\mathfrak{J}_k(x_n) = \lambda_n r_k(\frac{1}{x_n}) \tag{4}$$

for every $k = 0, 1, ...$ and $n = 0, 1, ...$and some constant λ_n independent of k.

Remark 1. *Claiming that such a function $\mathfrak{J}_k$ exists, seems a priori to be a strong assumption. However, as it will be seen in the last section, a general constructive method is available in order to find such a function under very natural requirements on the polynomials r_k.*

Remark 2. *In the abstract formulation it may not be clear why the constant λ_n must be present. Actually the construction would work without it, but for technical reasons that will become evident upon consideration of examples we prefer to use it. Otherwise, careful bookkeeping of the normalization constants would be required in the remaining sections.*

Now we can use the functions $\mathfrak{J}_k(t)$ to define a kernel $K(x,t)$ as

$$K(x,t) = \sum_{k=0}^{\infty} \mathfrak{J}_k(t) p_k(x) u_k \tag{5}$$

Such a kernel is well defined and belongs to $L^2(\mu)$, since it is a sum of basis functions of $L^2(\mu)$. From (3), (4) and (5) we have

$$\begin{aligned} K(x,x_n) &= \sum_{k=0}^{\infty} \mathfrak{J}_k(x_n) p_k(x) u_k \\ &= \lambda_n \sum_{k=0}^{\infty} r_k(\frac{1}{x_n}) p_k(x) u_k \\ &= \lambda_n F_n(x) \end{aligned}$$

and Theorem A with $t_n = \frac{1}{x_n}$ shows that $K(x,x_n)$ is an orthogonal basis for the space $L^2(\mu)$. Now define an integral transformation F by setting

$$(Ff)(t) = \int f(x) K(x,t) d\mu(x)$$

We will study this transform as a map whose domain is the Hilbert space $L^2(\mu)$ and we will define on its range, $F(L^2(\mu))$, the norm

$$\|Ff\|_{F(L^2(\mu))} = \|f\|_{L^2(\mu)}$$

Theorem 1. *The transform F is a Hilbert space isomorphism mapping the space $L^2(\mu)$ into $F(L^2(\mu))$ and the basis $\{\frac{1}{u_n} p_n(x)\}$ into the basis $\{\mathfrak{J}_n(x)\}$. As a consequence, $\{\mathfrak{J}_n(x)\}$ is a basis of the space $F(L^2(\mu))$. Moreover, every* *function f of the form*

$$f(x)=\int u(t)K(t,x)d\mu(t) \tag{6}$$

with $u\in L^2(\mu)$, admits an expansion

$$f(t)=\sum_{n=0}^{\infty}a_n\mathfrak{J}_n(t)$$

where the coefficients a_k are given by

$$a_n=\left\langle u,\frac{1}{u_n}p_n(.)\right\rangle_{L^2(\mu)}$$

Proof: Endowing the range of F with the inner product

$$\langle Ff,Fg\rangle_{F(L^2(\mu))}=\langle f,g\rangle_{L^2(\mu)} \tag{7}$$

then $F(L^2(\mu))$ becomes a Hilbert space isometrically isomorphic to $L^2(\mu)$ under the isomorphism F. We already know by default that $\{p_n(x)\}$ is a basis for $L^2(\mu)$. It remains to prove that $\{\mathfrak{J}_n(t)\}$ is a basis for $F(L^2(\mu))$. Observe that

$$\begin{aligned}(Fp_n)(t) &= \int p_n(x)K(x,t)d\mu(x)\\ &= \sum_{k=0}^{\infty}\mathfrak{J}_k(t)u_k\int p_n(x)p_k(x)d\mu(x)\\ &= \mathfrak{J}_n(t)u_n\end{aligned}$$

Since $\{p_n(x)\}$ is a basis for $L^2(\mu)$ and F is an isomorphism between $L^2(\mu)$ and $F(L^2(\mu))$, then $\{\mathfrak{J}_n(x)\}$ is a basis for $F(L^2(\mu))$. To prove the last assertion of the theorem, observe that function f defined by (6) belongs to $F(L^2(\mu))$ and therefore can be expanded in the basis $\{\frac{1}{u_n}p_n(x)\}$. The Fourier coefficients of this expansion are

$$a_n=\langle f,\mathfrak{J}_n(.)\rangle_{F(L^2(\mu))}=\left\langle Fu,F(\frac{1}{u_n}p_n(.))\right\rangle_{F(L^2(\mu))}=\left\langle u,\frac{1}{u_n}p_n(.)\right\rangle_{L^2(\mu)}$$

where we have used (7) in the last identity. ■

Let H be a class of complex valued functions, defined in a set $X\subset\mathbf{C}$, such that H is a Hilbert space with the norm of $L^2(X,\mu)$. The function $R(s,x)$ is a *reproducing kernel* of H if

i) $R(.,x) \in H$ for every $x \in X$;
ii) $f(x) = \langle f(.), R(.,x) \rangle$ for every $f \in H$, $x \in X$.

The space H is said to be a Hilbert space with reproducing kernel if it contains a reproducing kernel. It is easy to see that the space H has a reproducing kernel if and only if point evaluations in H are bounded in H. This is indeed the case of the image space of the transform F, as will be seen in the next proposition.

Theorem 2. *The space $F(L^2(\mu))$ is a Hilbert space with reproducing kernel given by*

$$R(t,s) = \int K(x,t)\overline{K(x,s)}d\mu(x) \tag{8}$$

Proof: An application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} \|(Ff)(t)\| &= \left|\int f(x)K(x,t)d\mu(x)\right| \leq \|f\|_{L^2(\mu)} \|K\|_{L^2(\mu)} && (9) \\ &= \|Ff\|_{F(L^2(\mu))} \|K\|_{L^2(\mu)} && (10) \end{aligned}$$

Therefore, point evaluations in $F(L^2(\mu))$ are bounded and $F(L^2(\mu))$ is a Hilbert space with reproducing kernel. To evaluate the reproducing kernel, observe that

$$R(t,s) = F(K(.,s))(t)$$

Therefore, writing an arbitrary $g \in F(L^2(\mu))$ in the form $g = F(f)$ with $f \in L^2(\mu)$, we have, by the isometric property of F,

$$\langle g, R(.,s)\rangle_{F(L^2(\mu))} = \langle F(f), F(K(.,s))\rangle_{L^2(\mu)} = f(s)$$

This proves that $R(t,s)$ is the reproducing kernel of $F(L^2(\mu))$. ■

Given the existence of a reproducing kernel structure, it is natural to look for a sampling theorem valid for functions in $F(L^2(\mu))$. This is the content of our next result.

Theorem 3. *Every function of the form*

$$f(x) = \int u(t)K(t,x)d\mu(t)$$

with $u \in L^2(\mu)$ can be written as the sampling expansion

$$f(x) = \sum f(t_n)\frac{R(x,t_n)}{R(t_n,t_n)}. \tag{11}$$

The sum in (11) converges absolutely. Furthermore, it converges uniformly in every set such that $\|K(.,t)\|_{L^2(\mu)}$ is finite.

Proof: As we have seen before, $K(x,x_n)$ is an orthogonal basis for the space $L^2(\mu)$. Since F is an isometry between the spaces $L^2(\mu)$ and $F(L^2(\mu))$ then the functions

$$\frac{R(t,t_n)}{\sqrt{R(t_n,t_n)}} = \frac{\int K(x,t_n)K(x,s)d\mu(x)}{\sqrt{\int |K(x,t_n)|^2 d\mu(x)}}$$

form an orthonormal basis for the space $F(L^2(\mu))$. The Fourier coefficients of a function $f \in F(L^2(\mu))$ in such a basis are

$$\left\langle f(.), \frac{R(.,t_n)}{\sqrt{R(t_n,t_n)}} \right\rangle = \frac{f(t_n)}{\sqrt{R(t_n,t_n)}}$$

This gives (11). This expansion is convergent in norm and, due to inequality (9), convergence in norm implies uniform convergence in every set such that $\|K(.,t)\|_{L^2(\mu)}$ is finite. To prove the absolute convergence, apply Schwarz inequality for sums to (11)

$$\left[\sum f(t_n)\frac{R(x,t_n)}{R(t_n,t_n)}\right]^2 \leq \sum \left|\frac{f(t_n)}{\sqrt{R(t_n,t_n)}}\right|^2 \sum \left|\frac{R(x,t_n)}{\sqrt{R(t_n,t_n)}}\right|^2$$

We have seen that $\frac{f(t_n)}{\sqrt{R(t_n,t_n)}}$ are the Fourier coefficients of the function f in the basis $\{\frac{R(.,t_n)}{\sqrt{R(t_n,t_n)}}\}$. On the other side, $R(x,t_n)/\sqrt{R(t_n,t_n)}$ are the Fourier coefficients of the function $K(x,t)$ in the basis $K(x,t_n)/\|K(.,t_n)\|_{L^2(\mu)}$. Therefore both sequences are in l^2 and the theorem is proved. ■

Remark 3. *The construction of this section is reminiscent of the reproducing kernel structure of the Paley-Wiener space. In the classical situations generalizing this structure, there is an integral transform whose kernel is defined as*

$$K(x,t) = \sum S_k(t)e_k(x) \tag{12}$$

where $e_k(x)$ is an orthogonal basis for the domain Hilbert space and $S_k(t)$ is a sequence of functions such that there exists a sequence $\{t_n\}$ satisfying the sampling property

$$S_k(t_n) = a_n\delta_{n,k} \tag{13}$$

As an instance, take $S_k(t) = \frac{\sin \pi(t-k)}{\pi(t-k)}$ *and* $e_k(x) = e^{ikx}$*. Then (12) is*

$$e^{itx} = \sum_{k=-\infty}^{\infty} \frac{\sin \pi(t-k)}{\pi(t-k)} e^{ikx}$$

and $K(x,t)$ *is the kernel of the Fourier transform. The corresponding reproducing kernel Hilbert space is the Paley-Wiener space. In our construction we made a modification of this classical setting: Instead of the sequence of functions* S_k*, with the sampling property (13), we considered a sequence of functions* $\{\mathfrak{J}_k\}$*, interpolating an orthogonal system* $\{r_k\}$ *in the sense of (4). And we have seen that the essential properties of classical reproducing kernel settings are kept. However, this modification allows to recognize a class of reproducing kernel Hilbert spaces that were obscured until now. This will became clear in the next section. For an account of Hilbert spaces defined by transforms with kernels as (12), see* [12], [23] *and* [8] *, with many historical notes and references. The root of these ideas is in Hardy´s groundbreaking paper* [11]*. For an application of this classical set up to Jackson q-integral transforms and the third Jackson q-Bessel function, see* [1]*.*

3. The Fourier system with ultraspherical weights

The *nth* ultraspherical (or Gegenbauer) polynomial of order ν is denoted by $C_n^\nu(x)$. These polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} C_n^\nu(x) C_m^\nu(x)(1-x^2)^{\nu-1/2} dx = \frac{(2\nu)_n \sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{n!(\nu+n)\Gamma(\nu)} \delta_{m,n}$$

and form a complete sequence in the Hilbert space $L^2[(-1,1),(1-x^2)^{\nu-1/2}]$. For typographical convenience we will introduce the following notation for this Hilbert space:

$$H^\nu = L^2[(-1,1),(1-x^2)^{\nu-1/2}]$$

The Bessel function of order ν, $J_\nu(x)$, is defined by the power series expansion

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n} \tag{14}$$

The *nth* Lommel polynomial of order ν, denoted by $h_{n,\nu}(x)$, is related to the Bessel functions by the relation

$$J_{\nu+k}(x) = h_{k,\nu}(\frac{1}{x}) J_\nu(x) - h_{k-1,\nu-1}(\frac{1}{x}) J_{\nu-1}(x). \tag{15}$$

The Lommel polynomials satisfy the discrete orthogonality relation

$$\sum_{k=0}^{\infty}\frac{1}{(j_{\nu,k})^2}h_{n,\nu+1}(\pm\frac{1}{j_{\nu,k}})h_{m,\nu+1}(\pm\frac{1}{j_{\nu,k}})=\delta_{nm}$$

and the dual orthogonality

$$\sum_{k=0}^{\infty}h_{k,\nu+1}(\pm\frac{1}{j_{\nu,n}})h_{k,\nu+1}(\pm\frac{1}{j_{\nu,m}})=(j_{\nu,k})^2\delta_{nm}$$

They form a complete orthogonal system in the l^2 space weighted by the discrete measure with respect to which they are orthogonal. We will use these two complete orthogonal systems in our first illustration of the general results. Set

$$p_k(x)=\Gamma(\nu)(\nu+k)C_k^{\nu}(x)$$

and

$$r_k(t)=h_{k,\nu-1}(t)$$

Consider also

$$\mathfrak{J}_k(t)=J_{\nu+k}(t).$$

Denote by $j_{\nu,k}$ the kth zero of the Bessel function of order ν. Substituting $x=j_{\nu,n}$ in (15), the following interpolating property is obtained

$$h_{k,\nu-1}(\frac{1}{j_{\nu,n}})=-\frac{J_{\nu+k}(j_{\nu,n})}{J_{\nu-1}(j_{\nu,n})} \tag{16}$$

The interpolating property (16) will play the role of (4) with $\lambda_n=-\frac{1}{J_{\nu-1}(j_{\nu,n})}$. Consider also the sequence of complex numbers $\{u_n\}$ defined as

$$u_k=i^k$$

and set

$$K(x,t)=\Gamma(\nu)\sum_{k=0}^{\infty}i^k(\nu+k)J_{\nu+k}(t)C_k^{\nu}(x)$$

Using (1) we have

$$K(x,t)=\left(\frac{t}{2}\right)^{\nu}e^{ixt}$$

and

$$K(x,j_{\nu,n-1})=\left(\frac{j_{\nu,n-1}}{2}\right)^{\nu}e^{ixj_{\nu,n-1}}$$

leading us to conclude that $\{e^{ij_{\nu,n-1}x}\}$ is an orthogonal basis for the space H^ν (observe that setting $\nu = 1/2$ the orthogonality and completeness of the complex exponentials $\{e^{i\pi nx}\}$ in $L^2(-1,1)$ is obtained as a special case). The transformation F is defined, for every $f \in H^\nu$, as

$$(Ff)(t) = \left(\frac{t}{2}\right)^\nu \int_{-1}^{1} f(x)e^{ixt}(1-x^2)^{\nu-1/2}dx$$

Since $\{i^{-n}\Gamma(\nu)(\nu+n)C_n^\nu(t)\}$ forms a basis of the space H^ν, it follows from Theorem 1 that $\{J_{\nu+n}(x)\} = F\{i^{-n}\Gamma(\nu)(\nu+n)C_n^\nu(t)\}$ is a basis of the space $F(H^\nu)$. Furthermore, the application of Theorem 1 to this setting gives our first expansion result.

Theorem 4. *Let f be a function of the form*

$$f(t) = \left(\frac{t}{2}\right)^\nu \int_{-1}^{1} u(x)e^{ixt}(1-x^2)^{\nu-1/2}dx \tag{17}$$

where $u \in H^\nu$. Then f can be written as

$$f(t) = \sum_{n=0}^{\infty} a_n J_{\nu+n}(t) \tag{18}$$

with the coefficients a_n given by

$$a_n = i^{-n}\Gamma(\nu)(\nu+n)\int_{-1}^{1} u(x)C_n^\nu(x)(1-x^2)^{\nu-1/2}dx \tag{19}$$

Remark 4. *Expansions of the type (18) are known as Neumann series of Bessel functions (see chapter 16 of* [24]*).*

In the next result we obtain the explicit formula for the reproducing kernel of $F(H^\nu)$.

Theorem 5. *The space $F(H^\nu)$ is a Hilbert space with reproducing kernel $R^\nu(t,s)$ given by*

$$R^\nu(t,s) = \Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\left(\frac{ts}{t-s}\right)^\nu J_\nu(t-s) \tag{20}$$

Proof: By Theorem 2 we know that $F(H^\nu)$ is a Hilbert space with reproducing kernel. By (8) we have

$$R^\nu(t,s) = \left(\frac{ts}{2}\right)^\nu \int_{-1}^{1} e^{ix(t-s)}(1-x^2)^{\nu-\frac{1}{2}}dx$$

Using the Poisson integral in the form (see [24, pag. 50])

$$J_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{itx}(1-x^2)^{\nu-\frac{1}{2}}dx$$

we obtain (20). ■

Observe that Graf´s addition formula [24, pag. 145] can be used to give yet another form to this kernel:

$$R^\nu(t,s) = \Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\left(\frac{ts}{t-s}\right)\sum_{n=-\infty}^{\infty} J_{\nu+m}(t)J_m(s)$$

The sampling theorem resulting from this construction seems to have been hitherto unnoticed, although it deals with very classical mathematics.

Theorem 6. *Let f be a function of the form (17), where $u \in H^\nu$. Then f can be represented as the following absolutely convergent sampling series*

$$f(t) = \frac{1}{\Gamma(\nu+1)}\sum_{n=0}^{\infty} f(j_{\nu,n})\left[\frac{1}{j_{\nu,n}(t-j_{\nu,n})}\right]^\nu J_\nu(t-j_{\nu,n}) \qquad (21)$$

The convergence is uniform in compact subsets of $\mathbf{R}$.

Proof: In order to apply apply Theorem 3 we must evaluate the quotient $R^\nu(t,j_{\nu,n})/R^\nu(j_{\nu,n},j_{\nu,n})$. Substituting in (20) and using (14) yields, after some simplification,

$$\frac{R^\nu(t,j_{\nu,n})}{R^\nu(j_{\nu,n},j_{\nu,n})} = \frac{1}{\Gamma(\nu+1)}\left[\frac{1}{j_{\nu,n}(t-j_{\nu,n})}\right]^\nu J_\nu(t-j_{\nu,n})$$

■

Remark 5. *When $\nu = 1/2$ we have*

$$R^{\frac{1}{2}}(t,s) = \sqrt{2}(ts)^{1/2}\frac{\sin(t-s)}{(t-s)}$$

and the sampling theorem states that every function of the form

$$f(t)=\left(\frac{t}{2}\right)^{1/2}\int_{-1}^{1}u(x)e^{ixt}dx$$

with $u\in L^2[(-1,1)]$ *can be represented as.*

$$f(t)=\sqrt{2}\sum_{n=0}^{\infty}f(n)\frac{\sin(t-n)}{n^{1/2}(t-n)}$$

4. The q-Fourier system with q-ultraspherical weights

We proceed to describe the q-analogue of the previous situation. Choose a number q such that $0<q<1$. The notational conventions from [9]

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1}),$$

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n,\quad (a_1,...,a_m;q)_n=\prod_{l=1}^{m}(a_l;q)_n,\quad |q|<1,$$

where $n=1,2,\ldots,$ will be used. The symbol ${}_{r+1}\phi_r$ stands for the function

$${}_{r+1}\phi_r\left(\begin{array}{c}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{array}\middle|\,q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_{r+1};q)_n}{(q,b_1,\ldots,b_r;q)_n}z^n.$$

The q-exponential function that we talked about in the introduction is defined in terms of basic hypergeometric series as

$$\mathcal{E}_q(x;t)=\frac{(-t;q^{\frac{1}{2}})_\infty}{(qt^2;q^2)_\infty}{}_{r+1}\phi_r\left(\begin{array}{c}q^{\frac{1}{3}}e^{i\theta},q^{\frac{1}{3}}e^{-i\theta}\\ q^{\frac{1}{2}}\end{array}\middle|\,q^{\frac{1}{2}},-t\right)$$

where $x=\cos\theta$. The continuous q-ultraspherical polynomials of order ν are denoted by $C_n^\nu(x;q^\nu|q)$ and satisfy the orthogonality

$$\int_{-1}^{1}C_n^\nu(x;q^\nu|q)C_m^\nu(x;q^\nu|q)w(x;q^\nu\mid q)dx=\frac{(2\pi q^\nu,q^{\nu+1};q)_\infty}{(q,q^{2\nu};q)_\infty}\frac{(1-q^\nu)(q^{2\nu};q)_n}{(1-q^{n+\nu})(q;q)_n}\delta_{m,n}$$

where the weight function $w(x;\beta\mid q)$ is

$$w(\cos\theta;\beta|q)=\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{\sin\theta(\beta e^{2i\theta},\beta e^{-2i\theta};q))_\infty},(0<\theta<\pi)$$

The polynomials $\{C_n^\nu(x;q^\nu|q)\}$ form a basis of the Hilbert space H_q^ν defined as

$$H_q^\nu = L^2[(-1,1), w(x;q^\nu \mid q)]$$

The second Jackson q-Bessel function of order ν is defined by the power series

$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}(-1)^n \frac{(x/2)^{\nu+2n}}{(q;q)_n(q^{\nu+1};q)_n} q^{n(\nu+n)}$$

Since this is the only q-Bessel function to be used in the text, we will drop the superscript for shortness of the notation and write $J_\nu(x;q) = J_\nu^{(2)}(x;q)$. The q-Lommel polynomials associated to the Jackson q-Bessel function of order ν are denoted by $h_{n,\nu-1}(x;q)$. These polynomials were defined in [13] by means of the relation

$$q^{n\nu+n(n-1)/2}J_{\nu+n}(x;q) = h_{n,\nu}(\frac{1}{x};q)J_\nu(x;q) - h_{n-1,\nu-1}(\frac{1}{x};q)J_{\nu-1}(x;q) \quad (22)$$

The q-Lommel polynomials satisfy the orthogonality relation

$$\sum_{k=1}^{\infty}\frac{A_k(\nu+1)}{(j_{\nu,n}(q))^2}h_{n,\nu+1}(\pm\frac{1}{j_{\nu,n}(q)};q)h_{m,\nu+1}(\pm\frac{1}{j_{\nu,n}(q)};q) = \frac{q^{n\nu+n(n+1)/2}}{1-q^{n+\nu+1}}\delta_{nm}$$

and the dual orthogonality

$$\sum_{k=1}^{\infty}\frac{(1-q^{n+\nu+1})}{q^{n\nu+n(n+1)/2}}h_{n,\nu+1}(\pm\frac{1}{j_{\nu,n}(q)};q)h_{m,\nu+1}(\pm\frac{1}{j_{\nu,n}(q)};q) = \frac{(j_{\nu,n}(q))^2}{A_k(\nu+1)}\delta_{nm}$$

Consider

$$p_k(x) = \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty}\frac{(1-q^{k+\nu})}{(1-q^\nu)}C_k(x;q^\nu|q)$$

$$r_k(t) = h_{k,\nu-1}(2t;q)$$

and

$$\mathfrak{J}_k(t) = q^{k\nu+\binom{k}{2}}J_{\nu+k}(2t;q)$$

The parameters u_k will be given by

$$u_k = q^{k^2/4}i^k$$

Denote by $j_{\nu,k}(q)$ the kth zero of $J_\nu(x;q)$. Setting $t = j_{\nu,k}(q)$ in (22) we have the interpolating property

$$h_{k,\nu-1}(\frac{1}{j_{\nu,n}(q)};q) = -q^{k\nu+\binom{k}{2}}\frac{J_{\nu+k}(j_{\nu,n}(q);q)}{J_{\nu-1}(j_{\nu,n}(q);q)} \quad (23)$$

This means that in (4) we must take $\lambda_n = -\frac{1}{J_{\nu-1}(j_{\nu,n}(q);q)}$. In this context, the kernel $K(x,t)$ is given as

$$K(x,t) = (q;q)_\infty \sum_{k=0}^{\infty} i^k q^{k^2/4} \frac{(1-q^{k+\nu})}{(1-q^\nu)} J_{\nu+k}(2t;q) C_k(x;q^\nu|q)$$

and the use of (2) gives

$$K(x,t) = t^\nu(-qt^2;q^2)_\infty \mathcal{E}_q(x;it)$$

The basis functions of the domain space are

$$F_n(x) = K(x, j_{\nu,k}(q)) = (j_{\nu,k}(q))^\nu (-qj^2_{\nu,k}(q);q^2)_\infty \mathcal{E}_q(x; ij_{\nu,k}(q))$$

Our imediate conclusion is that $\{\mathcal{E}_q(x; ij_{\nu,n-1})\}$ is orthogonal and complete in H_q^ν. Now define the transform

$$(F_q^\nu f)(t) = t^\nu(-qt^2;q^2)_\infty \int_{-1}^{1} f(x)\mathcal{E}_q(x;it) w(x;q^\nu \mid q)dx \tag{24}$$

for every $f \in H_q^\nu$. Use of Theorem 1 shows that

$$q^{n\nu+\binom{n}{2}} J_{\nu+n}(t;q) = F_q^\nu \left(i^{-n} \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty} \frac{(1-q^{k+\nu})}{(1-q^\nu)} C_n(x;q^\nu|q) \right)$$

and $\{q^{n\nu+\binom{n}{2}} J_{\nu+n}(t;q)\}_{n=0}^\infty$ is a basis of the space $F_q^\nu\left(H_q^\nu\right)$. We can also state the q-analogue of the expansion in Theorem 5 and obtain a q-Neumann expansion theorem in q-Bessel functions.

Theorem 7. *Let f be a function of the form*

$$f(t) = t^\nu(-qt^2;q^2)_\infty \int_{-1}^{1} u(x)\mathcal{E}_q(x;it) w(x;q^\nu \mid q)dx \tag{25}$$

where $u \in H_q^\nu$. Then f can be written as

$$f(t) = \sum_{n=0}^{\infty} a_n J_{\nu+n}(t;q)$$

with the coefficients a_n given by

$$a_n = q^{n\nu+\binom{n}{2}-\frac{n^2}{4}} i^{-n} \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty} \frac{(1-q^{k+\nu})}{(1-q^\nu)} \int_{-1}^{1} u(x) C_n(x;q^\nu|q) w(x;q^\nu \mid q)dx$$

Once more we can evaluate the reproducing kernel in an explicit form using the following integral, evaluated, by a fortunate coincidence, in [17]:

$$\int_0^\pi \mathcal{E}_q(\cos\theta;\alpha)\mathcal{E}_q(\cos\theta;\beta)\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\gamma e^{2i\theta},\gamma e^{-2i\theta};q)_\infty}\,d\theta \tag{26}$$
$$= \frac{2\pi(\gamma,q\gamma,-\alpha\beta q^{1/2};q)_\infty}{(q,\gamma^2;q)_\infty(q\alpha^2,q\beta^2;q^2)_\infty}{}_2\phi_2\left(\begin{array}{c} -q^{1/2}\alpha/\beta,-q^{1/2}\beta/\alpha \\ q\gamma,,-\alpha\beta\gamma q^{1/2}\end{array}\middle| q,-\alpha\beta\gamma q^{1/2}\right)$$

Theorem 8. *The space $F_q^\nu\left(H_q^\nu\right)$ is a space with reproducing kernel $R_q^\nu(t,s)$, given by*

$$R_q^\nu(t,s)=\frac{2\pi(ts)^\nu(q^\nu,q^{\nu+1},-tsq^{1/2};q)_\infty}{(q,q^{2\nu};q)_\infty}{}_2\phi_2\left(\begin{array}{c} -q^{1/2}t/s,-q^{1/2}s/t \\ q^{\nu+1},-tsq^{1/2+\nu}\end{array}\middle| q,-tsq^{\nu+1/2}\right) \tag{27}$$

Proof: Applying Theorem 2 we know that $R_q^\nu(t,s)$ is given by

$$R_q^\nu(t,s)=(ts)^\nu(-qt^2,-qs^2;q^2)_\infty\int_{-1}^1 \mathcal{E}_q(x;it)\mathcal{E}_q(x;-is)w(x;q^\nu\mid q)dx$$

Make the substitutions $x=\cos\theta$, $it=\alpha$, $is=\beta$, and $q^\nu=\gamma$ in (26). Then (27) follows. ■

Of course, as in the preceding sections we can formulate a sampling theorem, although no major simplification seems to occur after performing the required substitutions.

Theorem 9. *Every function of the form (25) admits the expansion*

$$f(x)=\sum_{k=0}^\infty f(t_k)\frac{R^\nu(x,t_k)}{R^\nu(t_k,t_k)} \tag{28}$$

where $t_k=\frac{j_{\nu,k}(q)}{2}$

Remark 6. *When $\nu=\frac{1}{2}$ the orthogonality and completeness of the complex exponentials $\{\mathcal{E}_q(x;i\frac{j_{\frac{1}{2},n-1}(q)}{2}\}$ in $H_q^{\frac{1}{2}}$is obtained. This is the case of the q-Fourier series studied in [26]. The functions $R^{1/2}(x,t_k)/R^{1/2}(t_k,t_k)$ above turn out to be the same as the $Sinc_q(t,k)$ in [18], where is shown that, in this special case, a remarkable simplification occurs and the resulting sampling theorem is an interpolating formula of the Lagrange type. The above discussion adds information that was not available in previous work: The* *function space where these sampling theorems live, are reproducing kernel Hilbert spaces, and the corresponding reproducing kernels are known explicitly as basic hypergeometric functions.*

Remark 7. *By the proof of Theorem 3, the functions* $\frac{R^\nu(x,t_k)}{R^\nu(t_k,t_k)}$ *are orthogonal. In the case* $\nu = \frac{1}{2}$, *this shows that the functions* $Sinc_q(t,n)$ *are orthogonal in the space* $F_q^{\frac{1}{2}}(H_q^{\frac{1}{2}})$. *This result is new and it is a q-analogue of the important fact, proved by Hardy in* [11], *that the functions* $\frac{\sin\pi(t-k)}{\pi(t-k)}$ *are orthogonal in the Paley-Wiener space.*

Remark 8. *Important information concerning the zeros of the second Jackson q-Bessel function, that appear as sampling nodes in the expansion (28), was obtained very recently by Walter Hayman in* [10] *using a method due to Bergweiler and Haymann* [4]. *He proved the asymptotic expansion*

$$j_{\nu,k}^2(q) = 4q^{1-2n-\nu}\{1 + \sum_{\nu=1}^{n} b_\nu q^{k\nu} + O\left|q^{(n+1)k}\right|\}$$

as $k \to \infty$, *with the constants* b_ν *depending on a and q. Therefore, for big k, the sampling nodes are exponentially separated in a similar way to what was observed in* [1] *and* [2]. *In the case where* $\nu = \pm\frac{1}{2}$, *the zeros were studied by Suslov* [25].

5. A generalization

We begin this last section describing a formal approach generalizing the situations studied in the two previous sections. This formal approach was initiated in [19] and [16] with the purpose of finding functions to play the role of the Lommel polynomials in more general situations, and was studied further in [14]. In the context studied in this paper, it will be of particular relevance, since it gives a constructive method to find the functions $\mathfrak{J}_k$ satisfying (4). Let $\{f_{n,\nu}\}$ be a sequence of polynomials defined recursively by $f_{0,\nu}(x) = 1$, $f_{1,\nu}(x) = xB_\nu$ and

$$f_{n+1,\nu}(x) = [xB_{n+\nu}]f_{n,\nu}(x) - C_{n+\nu-1}f_{n-1,\nu}(x)$$

Assuming the positivity condition $B_{n+\nu}B_{n+\nu+1}C_{n+\nu} > 0$ $(n \geq 0)$ and the convergence of the series $\sum_{n=0}^{\infty} \frac{C_{n+\nu}}{B_{n+\nu}B_{n+\nu+1}}$, it can be shown, using facts from the general theory of orthogonal polynomials, that the polynomials $f_{n,\nu}$ are orthogonal with respect to a compact supported discrete measure and that the support points of this measure are $\frac{1}{x_{n,\nu}}$, where the $x_{n,\nu}$ are the zeros of an entire function $\mathcal{J}$ satisfying

$$C_\nu ... C_{\nu+n-1}\mathcal{J}(x;\nu+n) = \mathcal{J}(x;\nu)f_{n,\nu}(\frac{1}{x}) - \mathcal{J}(x;\nu-1)f_{n-1,\nu+1}(\frac{1}{x}) \quad (29)$$

The dual orthogonality relation of the polynomials $f_{n,\nu}(x)$ is

$$\sum_{n=0}^{\infty} \frac{B_{\nu+1}}{2\lambda_n(\nu+1)} f_{n,\nu+1}(\frac{1}{x_{\nu,k}}) f_{n,\nu+1}(\frac{1}{x_{\nu,j}}) = \frac{x_{\nu,j}^2}{A_j(\nu+1)}\delta_{j,k}$$

for some constants $A_j(\nu+1)$ and $\lambda_n(\nu+1)$. (for the evaluation of these constants, as well as other parts of the argument missed in this brief sketch, we recommend the reading of section 4 of [14]). From (29) and the above analysis we can obtain the interpolation property

$$\mathcal{J}(x_{n,\nu};\nu+k) = \frac{-\mathcal{J}(x_{n,\nu};\nu-1)}{C_\nu ... C_{\nu+n-1}} f_{k-1,\nu+1}(\frac{1}{x_{n,\nu}})$$

Therefore, in the language of the second section we can set

$$\mathfrak{J}_k(t) = \sqrt{\lambda_k(\nu)}\frac{B_{k+\nu}}{B_\nu}\mathcal{J}(t,k)$$

$$r_k(t) = \sqrt{\lambda_k(\nu)}\frac{B_{k+\nu}}{B_\nu} f_{n,\nu}(t)$$

$$\lambda_n = \frac{-\mathcal{J}(x_{n,\nu};\nu-1)}{C_\nu ... C_{\nu+n-1}}$$

and define the kernel

$$K(x,t) = \sum_{k=0}^{\infty} u_k \sqrt{\lambda_k(\nu)}\frac{B_{k+\nu}}{B_\nu}\mathcal{J}(t,k)p_k(x)$$

where $|u_k| = 1$ and $\{p_n(x)\}$ is an arbitrary complete orthonormal system in $L^2(\mu)$. As before, the kernel $K(x,t)$ can be used to define an integral transformation between two Hilbert spaces. We could now apply the machinery of section 2 and provide a reproducing kernel structure and a sampling theorem by means of an integral transform with the above kernel. However, no simplification would occur on the absence of proper addition formulas for the kernel $K(x,t)$. Choosing families of orthogonal polynomials $f_{n,\nu}(t)$ and $p_n(x)$ in a way that such addition formulas exist is the topic of the second problem in [15].

Operators weighted by the Jacobi measure can be studied with kernels defined by the following generalization of (1) [7, formula 10.20.4]

$$e^{ixt} = e^{-it} \sum_{k=0}^{\infty} \frac{(\alpha+\beta+1)_k}{(\alpha+\beta+1)_{2k}} (2it)^k {}_1F_1 \left(\begin{array}{c} k+\beta+1 \\ 2k+\alpha+\beta+2 \end{array} \middle| 2it \right) P_k^{(\alpha,\beta)}(x) \tag{30}$$

where $\{P_k^{(\alpha,\beta)}(x)\}$ stands for the Jacobi polynomials, orthogonal in the interval $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. Reasoning as before, we can use this formula to generalize the results in the third section to Fourier systems with Jacobi weights. The analogues of the Lommel polynomials can be constructed as the functions $f_{n,\nu}$ described above. These functions preserve the formal properties of the Lommel polynomials and were used in [19, section 4]. Results very similar to those of section 3 would follow, with an extra parameter. Expansions in series of ${}_1F_1$ replace the Neumann expansions and sampling theorems with sampling points located at the zeros of these ${}_1F_1$ can also be derived. To avoid tedious duplication we omit the statement of these results, but we find useful to compute explicitly the corresponding reproducing kernel of the resulting image Hilbert space. Set

$$H^{\alpha,\beta} = L^2[(-1,1),(1-x)^\alpha(1+x)^\beta]$$

and define an integral transform by

$$(Ff)(t) = e^{it} \int_{-1}^{1} f(x) e^{ixt} (1-x)^\alpha (1+x)^\beta dx$$

for every $f \in H^{\alpha,\beta}$. Following section 2 and denoting the reproducing kernel of $F(H^{\alpha,\beta})$ by $R^{\alpha,\beta}(t,s)$, we have

$$R^{\alpha,\beta}(t,s) = e^{i(t-s)} \int_{-1}^{1} e^{ix(t-s)} (1-x)^\alpha (1+x)^\beta dx \tag{31}$$

Now, since the expansion (30) is a Fourier-Jacobi series, the following integral follows at once for every $k = 0, 1, \ldots$

$$\begin{aligned} &\int_{-1}^{1} e^{ixt} P_k^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta dx \\ = \;& e^{-it} \frac{(\alpha+\beta+1)_k}{(\alpha+\beta+1)_{2k}} (2it)^k {}_1F_1 \left(\begin{array}{c} n+\beta+1 \\ 2n+\alpha+\beta+2 \end{array} \middle| 2it \right) \end{aligned}$$

setting $k=0$ gives

$$\int_{-1}^{1} e^{ixt}(1-x)^{\alpha}(1+x)^{\beta}dx = e^{-it}{}_1F_1\left(\begin{array}{c} \beta+1 \\ \alpha+\beta+2 \end{array} \middle| 2it \right) \tag{32}$$

this allows to evaluate the integral in (31) and the result is

$$R^{\alpha,\beta}(t,s) = {}_1F_1\left(\begin{array}{c} \beta+1 \\ \alpha+\beta+2 \end{array} \middle| 2i(t-s) \right)$$

In section 6 of [16], formula (6.13) is a q-analogue of (30) generalizing (2). This formula involves continuous q-analogues of the Jacobi polynomials defined via the Askey Wilson polynomials [3] and a q-exponential function with an extra variable. To evaluate the kernel we would need an extension of formula (26) to this more general and complicated situation. To our knowledge, such a formula has not yet be written and a more detailed analysis of this situation deserves attention in a future discussion.

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