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ON THE EXPONENTIAL FUNCTION TAIL

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ABSTRACT: In this note we show that for every positive ϵ there exists an interval $[T, +\infty)$ in which every term $x^n/n!$ of the exponential series expansion bounds from below the scaled exponential function ϵe^x. This, in particular, implies that the sequence of functions $\{e^{-x}x^n/n!\}$ converges uniformly to the function zero everywhere.

1. Introduction

The value of $x^n/n!$, for every $n = 0, 1, \ldots$, bounds from below the value of the exponential function e^x for every real x. For every fixed $\epsilon > 0$, the fact that

$$\lim_{x\to+\infty} \frac{x^n/n!}{\epsilon e^x} = 0$$

holds for any n suggests that a similar statement holds for the scaled exponential function ϵe^x. In words, for every "x large enough", the value $x^n/n!$ bounds from below the value ϵe^x. We claim that 'x large enough' is independent of n, *i.e.*, for every $\epsilon > 0$, there exists T such that

$$\max_{n=0,1,\ldots} \left\{ \frac{e^{-x}x^n}{n!} \right\} \leq \epsilon, \text{ for every } x \geq T. \quad (1)$$

As we shall see, (1) in particular implies the uniform convergence of the sequence of functions $\{f_n(x) \equiv e^{-x}x^n/n!\}$ to the function zero everywhere.

In this paper we prove the claim and present some of its consequences. The paper is structured as follows. In Section 2 we prove a few preliminary lemmas. In Section 3 we prove the claim. In Section 4 we present some of its consequences and we also mention some key aspects that must hold in order to be able to generalize this type of claim to other functions.

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2. Preliminary results

A naive analysis leads to a partial result, formally presented in Proposition 1 below. Essentially, this is the result that motivated our claim.

Proposition 1. *For every $\epsilon \geq \sqrt{2}-1 \approx 0.414$, there exists T such that*

$$\max_{n=0,1,\dots} \left\{ \frac{e^{-x}x^n}{n!} \right\} \leq \epsilon, \qquad \textit{for every } x \geq T.$$

Proof: Recall the power series of the exponential function,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \text{for every } x \in \mathbb{R}. \tag{2}$$

When $\epsilon \geq 1$, the result follows trivially with, *e.g.*, $T = 0$. Consider the case $\epsilon \in [\sqrt{2}-1, 1)$. For any $n \geq 1$,

$$\frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \leq e^x, \qquad \text{for every } x \in \mathbb{R}.$$

Thus, for any $n \geq 1$,

$$\frac{x^n}{n!} + \frac{x^{n-1}}{n!} \underbrace{\left(\epsilon n + (\epsilon - 1)x + \frac{\epsilon}{n+1}x^2 \right)}_{P_{\epsilon,n}(x)} \leq \epsilon e^x, \qquad \text{for every } x \in \mathbb{R}.$$

Now, we show that the quadratic function $P_{\epsilon,n}$ is always nonnegative. Since $n/(n+1) \geq 1/2$, the discriminant of $P_{\epsilon,n}$ is

$$(\epsilon-1)^2 - 4\,(\epsilon n)\,\frac{\epsilon}{n+1} \leq (\epsilon-1)^2 - 2\epsilon^2 = 1 - 2\epsilon - \epsilon^2 \leq 0,$$

for every $\epsilon \in [\sqrt{2}-1, 1)$. Thus, $P_{\epsilon,n}(x) \geq 0$, for every $x \in \mathbb{R}$, which in particular implies that

$$\max_{n=1,2,\dots} \left\{ \frac{x^n}{n!} \right\} \leq \epsilon e^x, \qquad \text{for every } x \in \mathbb{R}.$$

The desired follows because the missing term $x^0/0! = 1 \leq \epsilon e^x$, for every $x \geq T \equiv \ln(1/\epsilon)$. ∎

The following lemma provides a closed-form formula for the maximum over n in (1). It will allow to put our claim in a limit sense.

Lemma 1. *For every $x > 0$,*

$$\max_{n=0,1,\dots} \left\{ \frac{x^n}{n!} \right\} = \frac{x^{\lfloor x \rfloor}}{\lfloor x \rfloor !}$$

attained at $\{x-1, x\}$ when x is integer, and at $\{\lfloor x \rfloor\}$ when x is not integer.

Proof: Let $u_n \equiv x^n/n!$, for some fixed $x > 0$, so that, for every $n \geq 1$, $u_n = (x/n)u_{n-1}$. If x is integer then $u_0 < u_1 < \cdots < u_{x-1} = u_x$ and $u_x > u_{x+1} > \dots$. If x is not integer then $u_0 < u_1 < \cdots < u_{\lfloor x \rfloor}$ and $u_{\lfloor x \rfloor} > u_{\lfloor x \rfloor + 1} > \dots$. The desired result follows. ■

From Lemma 1, the proposed claim, see (1), is true if and only if

$$\lim_{x \to +\infty} \frac{x^{\lfloor x \rfloor}}{\lfloor x \rfloor ! e^x} \tag{3}$$

exists and it is zero. The fact that Lemma 1 requires $x > 0$ causes no difficulty because we can always assume without loss of generality that we are looking for some $T > 0$.

When restricted to integer sequences, (3) is easily shown to be zero. The proof uses Stirling's formula: for all n sufficiently large,

$$\sqrt{2n\pi} < \frac{n!e^n}{n^n} < \sqrt{2n\pi}\left(1 + \frac{1}{12n-1}\right).$$

Thus, $n^n/(n!e^n)$ goes to zero as n goes to infinity. The proof that (3) exists and it is zero turns out to be harder.

A related limit is easy to compute. Let Γ denote the Gamma function which is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt \quad (x > 0).$$

This function extends the factorial function over the positive reals. Stirling's asymptotic formula — see [1, page 105] — states that

$$\ln \Gamma(x) \sim \ln\left(\sqrt{\frac{2\pi}{x}}\frac{x^x}{e^x}\right) + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \cdots$$

where the '$\sim$' sign means that the alternating series indicated is divergent for all x but, when $x > 0$, it has the property that any partial sum differs from $\ln\Gamma(x)$ by an amount which in absolute value is less than the last term of the partial sum. Thus, for all $x > 0$,

$$\left|\ln\left(\frac{\Gamma(x)e^x}{x^x}\sqrt{\frac{x}{2\pi}}\right) - \frac{1}{12x}\right| \leq \frac{1}{12x}$$

which implies

$$\frac{\Gamma(x)e^x}{x^x}\sqrt{\frac{x}{2\pi}} \geq 1,$$

or, by using the fact that $\Gamma(x+1) = x\Gamma(x)$,

$$\frac{\Gamma(x+1)e^x}{x^x} \geq \sqrt{2\pi x}.$$

Thus,

$$\lim_{x\to+\infty} \frac{x^x}{\Gamma(x+1)e^x} = 0. \tag{4}$$

3. The main result

The limiting functions in (3) and (4) are related in the following way,

$$\frac{x^{\lfloor x\rfloor}}{\lfloor x\rfloor! e^x} = \left(\frac{x^x}{\Gamma(x+1)e^x}\right)\left(\frac{\Gamma(x+1)}{\Gamma(\lfloor x\rfloor+1)}\right)\left(\frac{x^{\lfloor x\rfloor}}{x^x}\right), \quad \text{for every } x > 0. \tag{5}$$

In Lemma 2 below we show that the product of the last two factors in (5) is bounded.

Lemma 2. *There exists $T > 0$ such that for every $x > T$,*

$$\left(\frac{\Gamma(x+1)}{\Gamma(\lfloor x\rfloor+1)}\right)\left(\frac{x^{\lfloor x\rfloor}}{x^x}\right) \leq e^{\gamma+1}, \tag{6}$$

where γ denotes the Euler's constant ($\gamma = 0.55721\ldots$).

Proof: Since $\Gamma(x+1) = x\Gamma(x)$ for every $x > 0$,

$$\frac{\Gamma(x+1)}{\Gamma(\lfloor x\rfloor+1)} = \left(\prod_{j=0}^{\lfloor x\rfloor-1}\left(1 + \frac{x-\lfloor x\rfloor}{\lfloor x\rfloor - j}\right)\right)\left(\frac{\Gamma(x-\lfloor x\rfloor+1)}{\Gamma(1)}\right).$$

The last factor is smaller than $\Gamma(2)/\Gamma(1) = 1$. Now, we need to show that

$$\left(\prod_{j=0}^{\lfloor x\rfloor-1}\left(1 + \frac{x-\lfloor x\rfloor}{\lfloor x\rfloor - j}\right)\right)\left(\frac{x^{\lfloor x\rfloor}}{x^x}\right) \leq e^{\gamma+1}, \tag{7}$$

for all x large enough. From the arithmetic-geometric mean inequality, the left-hand-side of (7) is less than or equal to

$$\left(\frac{1}{\lfloor x\rfloor}\sum_{j=0}^{\lfloor x\rfloor-1}\left(1+\frac{x-\lfloor x\rfloor}{\lfloor x\rfloor-j}\right)\right)^{\lfloor x\rfloor}\left(\frac{x^{\lfloor x\rfloor}}{x^x}\right)=$$

$$=\left(\frac{1+\left(\frac{x-\lfloor x\rfloor}{\lfloor x\rfloor}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\lfloor x\rfloor}\right)}{x^{(x-\lfloor x\rfloor)/\lfloor x\rfloor}}\right)^{\lfloor x\rfloor}. \tag{8}$$

Now, we recall that $\lim_{n\to\infty}(1+1/2+1/3+\cdots+1/n-\ln n)=\gamma$. Thus, for all x large enough, (8) is less than

$$\left(\frac{1+\left(\frac{x-\lfloor x\rfloor}{\lfloor x\rfloor}\right)(\ln(\lfloor x\rfloor)+(\gamma+1))}{\lfloor x\rfloor^{(x-\lfloor x\rfloor)/\lfloor x\rfloor}}\right)^{\lfloor x\rfloor}. \tag{9}$$

From the power series of the exponential function, (9) is always less than

$$\left(\frac{e^{\left(\frac{x-\lfloor x\rfloor}{\lfloor x\rfloor}\right)(\ln(\lfloor x\rfloor)+\gamma+1)}}{e^{\left(\frac{x-\lfloor x\rfloor}{\lfloor x\rfloor}\right)(\ln(\lfloor x\rfloor))}}\right)^{\lfloor x\rfloor}=e^{(x-\lfloor x\rfloor)(\gamma+1)}$$

The desired result follows. ■

The upper bound (6) is clearly not tight. Lemma 2 would still hold with $e^{\gamma+\delta}$, where δ is an arbitrary positive real, instead of $e^{\gamma+1}$. We are now ready to prove the claim.

Proposition 2. *For every $\epsilon>0$, there exists T such that*

$$\max_{n=0,1,\ldots}\left\{\frac{e^{-x}x^n}{n!}\right\}\leq\epsilon,\ \textit{for every } x\geq T. \tag{10}$$

Proof: From Lemma 2, the product of last two factors in (5) is bounded for all x large enough. Thus, from (4) and (5) we conclude that (3) exists and it is zero. The desired result follows. ■

4. Consequences

Now, we present a number of consequences from Proposition 2. The first result can also be derived through uniform convergence arguments. The other three results require Proposition 2 to the best of our knowledge.

Corollary 1. *For every $\epsilon > 0$, there exists T such that*

$$\sum_{n=0}^{\infty} \frac{e^{-x}x^n}{n!} p_n \le \epsilon, \ \textit{for every } x \ge T, \tag{11}$$

holds for any sequence $\{p_n\}$ of nonnegative real numbers such that $\sum_{n=0}^{\infty} p_n < +\infty$.

Proof: Clearly, for every $n \ge 0$ and $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \frac{e^{-x}x^n}{n!} p_n \le \sum_{n=0}^{\infty} \left(\max_{k=0,1,\ldots} \frac{e^{-x}x^k}{k!} \right) p_n = \max_{n=0,1,\ldots} \frac{e^{-x}x^n}{n!}. \tag{12}$$

The desired result follows from Proposition 2. ■

A proof of Corollary 1 that uses uniform convergence arguments would be as follows. The sequence of functions $\{f_n(x)\}$, defined by

$$f_n(x) \equiv \frac{e^{-x}x^n}{n!} p_n \qquad (n = 0, 1, \ldots)$$

is such that the sequence of functions $\{\sum_{j=0}^{n} f_j\}$ converges uniformly in $\mathbb{R}$. This follows from well-known Weirstrass test, taking into account that $|f_n(x)| \le p_n$, for every $x \in \mathbb{R}$ and every n, and $\sum_{n=0}^{\infty} p_n < +\infty$. Thus, a generalization of [2, Theorem 7.11,p.149]) shows that

$$\lim_{x \to \infty} \sum_{n=0}^{\infty} \frac{e^{-x}x^n}{n!} p_n = 0.$$

which is another way of expressing (11). We remark that the proof of Corollary 1 didn't make any explicit use of uniform convergence.

Corollary 2. *The sequence of functions $\{e^{-x}x^n/n!\}$ converges uniformly to the function zero everywhere.*

Proof: We need to show that, for every $\epsilon > 0$, there exists an integer N such that

$$\max_{x \in \mathbb{R}} \left\{ \frac{e^{-x}x^n}{n!} \right\} \le \epsilon, \ \text{for every } n \ge N. \tag{13}$$

From Proposition 2, there exists T such that (10) holds. On the other hand,

$$\frac{e^{-x}x^n}{n!} \le \frac{e^{-T}T^n}{n!}, \ \text{for every } x \le T.$$

The sequence $\{e^{-T}T^n/n!\}$ converges to zero as n goes to infinity. Let N be such that $e^{-T}T^n/n! \leq \epsilon$, for all $n \geq N$. For this N, (13) follows. ∎

Corollary 3. *For every $\epsilon > 0$, there exists T such that*

$$\max_{n=0,1,\ldots} \left\{ \sum_{j=0}^{n} \frac{e^{-x}x^j}{j!} p_j^n \right\} \leq \epsilon, \text{ for every } x \geq T,$$

holds for any sequence $\{p^n\}$ such that $p^n \equiv (p_1^n, p_2^n, \ldots, p_n^n) \in \mathbb{R}^{n+1}$ and satisfies $\sum_{j=0}^{n} p_j^n = 1$ and $p_j^n \geq 0$, for every $j = 0, 1, \ldots, n$.

Proof: Clearly, for every $n \geq 0$,

$$\sum_{j=0}^{n} \frac{e^{-x}x^j}{j!} p_j^n \leq \sum_{j=0}^{n} \left(\max_{k=0,1,\ldots} \frac{e^{-x}x^k}{k!} \right) p_j^n = \max_{n=0,1,\ldots} \frac{e^{-x}x^n}{n!}.$$

The desired result follows from Proposition 2. ∎

Corollary 4. *For every $\epsilon > 0$, there exists T such that*

$$\max_{n=0,1,\ldots} \left\{ \frac{1}{n+1} \int_x^{+\infty} \frac{e^{-t}t^n}{n!}\, dt \right\} \leq \epsilon, \text{ for every } x \geq T. \tag{14}$$

Proof: From calculus,

$$\frac{1}{n+1} \int_x^{+\infty} \frac{e^{-t}t^n}{n!}\, dt = \frac{1}{n+1} \sum_{j=0}^{n} \frac{e^{-x}x^j}{j!}.$$

The desired result follows from Corollary 3. ∎

Note the factor $1/(n+1)$ in (14). Had this factor not been there and we would have zero as limit when x goes to $+\infty$ (for a fixed n) and one as limit when n goes to ∞ (for a fixed x). For the type of property studied in this paper to hold in other settings we must have zero in both limits.

It is interesting to observe the type of property studied in this paper with other functions. Here is an example where the property holds with a minor adjustment. The value of x^n, for every $n = 0, 1, \ldots$, bounds from below the value of the function $1/(1-x)$ for every real $x \in [0, 1)$. For every fixed $\epsilon > 0$, we have that

$$\lim_{x \to 1^-} \frac{x^n}{\frac{\epsilon}{1-x}} = \lim_{x \to 1^-} \frac{(1-x)x^n}{\epsilon} = 0 = \lim_{n \to \infty} \frac{(1-x)x^n}{\epsilon}.$$

Hence, the necessary condition stated in the previous paragraph holds. It is not hard to show that, for every $\epsilon > 0$, there exist $N \in \{0, 1, \ldots\}$, such that

$$x^N = \max_{n=N,N+1,\ldots} \{x^n\} \leq \frac{\epsilon}{1-x}, \text{ for every } x \in [0,1). \tag{15}$$

Thus, the sequence of functions $\{x^n(1-x), x \in [0,1)\}$ converges uniformly to the function zero everywhere. Note that statement (15) can also be understood as an adjusted version of (1) with $T = 0$.

It seems that the property studied in this paper, which is shared by the functions e^x and $1/(1-x)$, may happen to be shared by other functions. It would be interesting to analyze this property for the class of real analytic functions f, say

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \qquad \text{for every } |x| < R_f \leq +\infty,$$

for which there exists b, with $0 < b \leq R_f$, such that the following properties hold:

(1) $f^{(n)}(0)x^n/n!$ bounds from below the value of $f(x)$, for every $x \in [0, b)$ and all $n = 0, 1, \ldots$;
(2) $f(x) > 0$ for all $x \in [0, b)$;
(3) $\lim_{x \to b^-} 1/f(x) = 0$.

For every function f in such class we would like to know if the following is true: for every $\epsilon > 0$, there exists $T \in [0, b)$ and an integer N such that

$$\max_{n=N,N+1,\ldots} \left\{ \frac{f^{(n)}(0)x^n}{n! f(x)} \right\} \leq \epsilon, \qquad \text{for every } x \in [T, b).$$

This property implies the uniform convergence of the sequence of functions $\left\{ (f^{(n)}(0)x^n)/(n! f(x)), x \in [0, b) \right\}$ to the function zero everywhere.

References

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