# Some remarks on the Poisson-Nijenhuis and the Jacobi structures

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**Abstract**

We study the relationship between two (compatible) Jacobi structures on a manifold $M$, using their associated homogeneous Poisson structures on $\mathbb{R} \times M$, in the case where these Poisson tensors are related by a Nijenhuis tensor.

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## Introduction

All objects considered in this paper (manifolds, maps, differential forms, vector and tensor fields) are assumed to be differentiable of class $C^\infty$.

Let $M$ be a smooth manifold equipped with a *Poisson tensor* $\Lambda$ and a *Nijenhuis tensor* $N$, that is, a tensor field of type $(1,1)$ whose *Nijenhuis torsion* $\tau(N)$ vanishes everywhere. The Nijenhuis torsion $\tau(N)$ of $N$ is given by the following formula, where $X$ and $Y$ are vector fields on $M$,

$$\tau(N)(X,Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X,Y]).$$

We denote by $\Lambda^\# : T^*M \to TM$ the vector bundle map such that, for any $x \in M$, $\alpha, \beta \in T_x^*M$,

$$< \beta, \Lambda^\#(\alpha) >= \Lambda_x(\alpha, \beta).$$

With $\Lambda$ and $N$ we can define a tensor field $\mathcal{R}(\Lambda, N)$ of type $(2,1)$, called the *Magri-Morosi concomitant* [Ma-Mo 84] of $\Lambda$ and $N$, that is defined, for any pair of a 1-form $\alpha$ and a vector field $X$, by

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$$\mathcal{R}(\Lambda, N)(\alpha, X) = (\mathcal{L}_{\Lambda^\#(\alpha)}N)X - \Lambda^\#(\mathcal{L}_X({}^tN\alpha)) + \Lambda^\#(\mathcal{L}_{NX}\alpha), \tag{1}$$

where ${}^tN$ is the transpose of $N : TM \to TM$.

Let $\Lambda$ and $N$ be, respectively, a Poisson tensor and a Nijenhuis tensor on $M$. The triple $(M, \Lambda, N)$ is called a *Poisson-Nijenhuis manifold* [KSch-Ma 90] if

$$N\Lambda^\# = \Lambda^\# \, {}^tN \quad \text{and} \quad \mathcal{R}(\Lambda, N) = 0.$$

If $(M, \Lambda, N)$ is a Poisson-Nijenhuis manifold, there exists a sequence $(\Lambda_k)_{k\in\mathbb{N}}$ of Poisson tensors on $M$, with $\Lambda_k = N^k\Lambda$. Moreover, these Poisson tensors are pairwaise compatible, that is, $[\Lambda_l, \Lambda_k] = 0$, for all $l, k \in \mathbb{N}$.

Besides the Poisson-Nijenhuis structure, we are going to use, in what follows, the notion of Jacobi manifold.

A *Jacobi manifold* [Lich 78] is a triple $(M, C, E)$ where $C$ and $E$ are respectively a bivector and a vector field on $M$, such that

$$[E, C] = 0 \quad \text{and} \quad [C, C] = 2E \wedge C.$$

When $E = 0$, the Jacobi manifold is a Poisson manifold. The *Jacobi bracket* of $f, g \in C^\infty(M, \mathbb{R})$ is given by

$$\{f, g\} = C(df, dg) + f(E.g) - g(E.f),$$

and it defines a *local Lie algebra structure* on $C^\infty(M, \mathbb{R})$.

If $(M, C, E)$ is a Jacobi manifold and $h$ is a nowhere vanishing function on $M$, then the pair $(hC, C^\#(dh) + hE) = (C_h, E_h)$ defines a new Jacobi structure on $M$, which is said to be *conformally equivalent* to $(C, E)$.

With each Jacobi manifold $(M, C, E)$, we may associate a homogeneous Poisson structure $(\Lambda, \frac{\partial}{\partial t})$ on $\mathbb{R} \times M$, with $\Lambda$ given by

$$\Lambda = \exp(-t)(C + \frac{\partial}{\partial t} \wedge E), \tag{2}$$

where $t$ is the canonical coordinate on $\mathbb{R}$.

This paper is divided into two sections. Section 1 is devoted to the subject of compatible Jacobi manifolds. We show how the compatibility is related with the Lichnerowicz-Jacobi cohomology and we present a way of generating compatible Jacobi structures on a manifold. In section 2, we establish the conditions on the Poisson-Nijenhuis structure of $\mathbb{R} \times M$ to ensure the compatibility of the corresponding Jacobi structures on $M$.

## 1 Compatible Jacobi manifolds

The notion of compatibility of two Jacobi structures on a manifold was introduced in [NdC 98]. We recall that two Jacobi structures $(C_1, E_1)$ and $(C_2, E_2)$ on a manifold $M$ are said to be *compatible* if $(C_1 + C_2, E_1 + E_2)$ is again a Jacobi structure on $M$.

**Proposition 1.1** [NdC 98] *Let $M$ be a manifold endowed with two Jacobi structures $(C_1, E_1)$ and $(C_2, E_2)$. Then, $(C_1, E_1)$ and $(C_2, E_2)$ are compatible if and only if*

$$[E_1, C_2] + [E_2, C_1] = 0 \quad \textit{and} \quad [C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1.$$

There are some equivalent ways of expressing the compatibility of two Jacobi structures on a manifold [NdC 98]. But the study of the compatibility of two Jacobi structures on a manifold, can also be done using the Lichnerowicz-Jacobi cohomology. If $(M, C, E)$ is a Jacobi manifold, let us denote by $A^k(M)$ the space of skew-symmetric contravariant tensor fields of order $k$ ($k$-tensors) on $M$ and define the differential operator

$$\sigma : A^k(M) \to A^{k+1}(M), \quad \sigma(P) = -[C, P] + kE \wedge P. \tag{3}$$

The restriction of $\sigma$ to the subspace

$$A_I^k(M) = \{P \in A^k(M) : \mathcal{L}_E P = 0\}$$

of the invariant $k$-tensors with respect to the vector field $E$ is a *cohomology operator* on the Jacobi manifold and the resultant cohomology is called the *Lichnerowicz-Jacobi cohomology* of $M$ [Le-Ma-Pa 97].

**Proposition 1.2** *Two Jacobi structures $(C_1, E_1)$ and $(C_2, E_2)$ on a manifold $M$ are compatible if and only if*

$$\sigma_1(C_2) = -\sigma_2(C_1) \quad \textit{and} \quad \sigma_1(E_2) = -\sigma_2(E_1),$$

*where $\sigma_i, i = 1, 2$, are the cohomology operators of the Lichnerowicz-Jacobi cohomology of $M$, with respect to both Jacobi structures.*

**Proof.**
A direct computation using Proposition 1.1 and the definition (3) of the cohomology operators $\sigma_i, i = 1, 2$, gives the desired result. □

In [NdC 98] it was proved that two conformally equivalent Jacobi structures on $M$ are compatible. Another way of obtaining compatible Jacobi structures uses the Lie derivative on the direction of some vector field.

**Proposition 1.3** *Let $X$ be a vector field on the Jacobi manifold $(M, C, E)$ such that*

$$\mathcal{L}_X(\mathcal{L}_X C) = 0 \quad \textit{and} \quad \mathcal{L}_X(\mathcal{L}_X E) = 0.$$

*Then the pair $(C_1, E_1) = (\mathcal{L}_X C, \mathcal{L}_X E)$ defines a new Jacobi structure on $M$ which is compatible with $(C, E)$.*

**Proof.**
With $E_1 = \mathcal{L}_X E = [X, E]$, we have $\mathcal{L}_{E_1} C = -\mathcal{L}_E(\mathcal{L}_X C)$, that is

$$\mathcal{L}_{E_1} C + \mathcal{L}_E C_1 = 0. \tag{4}$$

So,

$$\begin{aligned} \mathcal{L}_{E_1} C_1 &= \mathcal{L}_X(\mathcal{L}_{E_1} C) \\ &= -\mathcal{L}_X(\mathcal{L}_E C_1) \\ &= -\mathcal{L}_{[X,E]}(\mathcal{L}_X C) \\ &= -\mathcal{L}_{E_1} C_1, \end{aligned}$$

and

$$\mathcal{L}_{E_1} C_1 = 0. \tag{5}$$

On the other hand,

$$\begin{aligned} \mathcal{L}_X([C,C]) &= [\mathcal{L}_X C, C] + [C, \mathcal{L}_X C] \\ &= 2[C_1, C] \end{aligned}$$

and

$$\mathcal{L}_X(E \wedge C) = E_1 \wedge C + E \wedge C_1;$$

so,

$$[C_1, C] = E_1 \wedge C + E \wedge C_1. \tag{6}$$

Because $\mathcal{L}_X E_1 = \mathcal{L}_X C_1 = 0$, if we take the Lie derivative on the direction of $X$ of both members of (6), we obtain

$$[C_1, C_1] = 2E_1 \wedge C_1. \tag{7}$$

The equalities (5) and (7) prove that $(C_1, E_1)$ is a Jacobi structure, while (4) and (6) show the compatibility. □

We can also prove the following result.

**Proposition 1.4** *Let $X_1$ and $X_2$ be two vector fields on the Jacobi manifold $(M, C, E)$ such that*

$$[X_1, X_2] = 0, \quad [X_1, \mathcal{L}_{X_2} C] = 0, \quad [X_1, \mathcal{L}_{X_2} E] = 0$$

*and*

$$\mathcal{L}_{X_i}(\mathcal{L}_{X_i} C) = 0, \quad \mathcal{L}_{X_i}(\mathcal{L}_{X_i} E) = 0, \quad i = 1, 2.$$

*Then*

$$(C_1, E_1) = (\mathcal{L}_{X_1} C, \mathcal{L}_{X_1} E) \quad \textit{and} \quad (C_2, E_2) = (\mathcal{L}_{X_2} C, \mathcal{L}_{X_2} E)$$

*are compatible Jacobi structures on $M$.*

**Proof.**

We only have to check the compatibility. By Proposition 1.3, we know that the Jacobi structures $(C, E)$ and $(C_2, E_2)$ are compatible. Then,

$$[C, C_2] = E \wedge C_2 + E_2 \wedge C \quad \text{and} \quad [E, C_2] + [E_2, C] = 0. \tag{8}$$

If we take the Lie derivative, on the direction of $X_1$, of both members of equalities (8), we obtain

$$[C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1 \quad \text{and} \quad [E_1, C_2] + [E_2, C_1] = 0. \tag{9}$$

□

## 2 Homogeneous Poisson-Nijenhuis manifolds

Let $M$ be a manifold endowed with two Jacobi structures $(C_1, E_1)$ and $(C_2, E_2)$. Take the corresponding homogeneous Poisson structures on $\mathbb{R} \times M$, $(\Lambda_i = \exp(-t)(C_i + \frac{\partial}{\partial t} \wedge E_i), \frac{\partial}{\partial t})$, $i = 1, 2$.

**Proposition 2.1** [NdC 98] *The Jacobi structures $(C_1, E_1)$ and $(C_2, E_2)$ on $M$ are compatible if and only if $\Lambda_1$ and $\Lambda_2$ are compatible Poisson tensors on $\mathbb{R} \times M$.*

Using this result, we want to study the relationship between two compatible Jacobi structures on $M$ and their associated (compatible) homogeneous Poisson strucutres on $\mathbb{R} \times M$, in the case where these Poisson structures are related by a Nijenhuis tensor.

Let $\bar{N}$ be a $(1,1)$-tensor on $\mathbb{R} \times M$ such that $\mathcal{L}_{\frac{\partial}{\partial t}} \bar{N} = 0$. Then, $\bar{N}$ is given by

$$\bar{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g \frac{\partial}{\partial t} \otimes dt, \tag{10}$$

where $N$ is a $(1,1)$-tensor on $M$, $Y$ is a vector field on $M$, $\gamma$ is a 1-form on $M$ and $g \in C^\infty(M, \mathbb{R})$. Reciprocally, if $\bar{N}$ is given by (10), then $\mathcal{L}_{\frac{\partial}{\partial t}} \bar{N} = 0$.

**Remarks**

1. The image of a vector field on $\mathbb{R} \times M$ of the form $\bar{X} = \exp(-t)(f \frac{\partial}{\partial t} + X)$, where $X$ is a vector field on $M$ and $f \in C^\infty(M, \mathbb{R})$, by the $(1,1)$-tensor given by (10) is a vector field on $\mathbb{R} \times M$ of the same type of $\bar{X}$.
2. If $\Lambda$ is the homogeneous Poisson tensor on $\mathbb{R} \times M$, given by (2), then $\bar{N}\Lambda$ is a homogeneous bivector on $\mathbb{R} \times M$: $\mathcal{L}_{\frac{\partial}{\partial t}}(\bar{N}\Lambda) = -\bar{N}\Lambda$

**Proposition 2.2** *Let $\bar{N}$ be a $(1,1)$-tensor on $\mathbb{R} \times M$ given by (10). Then $\bar{N}$ is a Nijenhuis tensor on $\mathbb{R} \times M$ if and only if*

$$\begin{array}{rl} i) & \tau(N) = Y \otimes d\gamma; \\ ii) & \mathcal{L}_N \gamma = g d\gamma; \\ iii) & \mathcal{L}_Y N = -Y \otimes dg; \\ iv) & {}^t N(dg) = \mathcal{L}_Y \gamma + g dg. \end{array}$$

**Proof.**

First, remark that if $X$ is a vector field on $M$, then $\bar{N}X = NX + < \gamma, X > \frac{\partial}{\partial t}$ and that $\bar{N}(\frac{\partial}{\partial t}) = Y + g\frac{\partial}{\partial t}$. If $X_1$ and $X_2$ are vector fields on $M$, then

$$\tau(\bar{N})(X_1, X_2) \;=\; \tau(N)(X_1, X_2) + (X_2. < \gamma, X_1 > - X_1. < \gamma, X_2 >$$

$$
\begin{aligned}
&+<\gamma,[X_1,X_2]>)\bar{N}(\frac{\partial}{\partial t})+((NX_1).<\gamma,X_2> -(NX_2).<\gamma,X_1> \\
&-<\gamma,[NX_1,X_2]>-<\gamma,[X_1,NX_2]>-<\gamma,N[X_1,X_2]>)\frac{\partial}{\partial t} \\
=\ &\tau(N)(X_1,X_2)-d\gamma(X_1,X_2)Y-(gd\gamma(X_1,X_2)-\mathcal{L}_N\gamma(X_1,X_2))\frac{\partial}{\partial t}.
\end{aligned}
$$

So, $\tau(\bar{N})(X_1,X_2)=0$ if and only if

$$
\tau(N)(X_1,X_2)=(Y\otimes d\gamma)(X_1,X_2) \tag{11}
$$

and

$$
\mathcal{L}_N\gamma(X_1,X_2)=gd\gamma(X_1,X_2). \tag{12}
$$

Let $X$ be a vector field on $M$. Then,

$$
\begin{aligned}
\tau(\bar{N})(X,\frac{\partial}{\partial t}) &= [NX,Y]+((NX).g)\frac{\partial}{\partial t}-(Y.<\gamma,X>)\frac{\partial}{\partial t} \\
&\quad -N[X,Y]-<\gamma,[X,Y]>\frac{\partial}{\partial t}-(X.g)(Y+g\frac{\partial}{\partial t}) \\
&= -(\mathcal{L}_Y N)(X)-(Y\otimes dg)(X) \\
&\quad +(<{}^tNdg,X>-<\mathcal{L}_Y\gamma,X>-<gdg,X>)\frac{\partial}{\partial t}.
\end{aligned}
$$

So, $\tau(\bar{N})(X,\frac{\partial}{\partial t})=0$ if and only if

$$
(\mathcal{L}_Y N)(X)=-(Y\otimes dg)(X) \tag{13}
$$

and

$$
<{}^tNdg-\mathcal{L}_Y\gamma-gdg,X>=0. \tag{14}
$$

Equalities (11), (12), (13) and (14) end the proof. □

Let us take the homogeneous Poisson tensor $\Lambda=\exp(-t)(C+\frac{\partial}{\partial t}\wedge E)$ on $(\mathbb{R}\times M)$.

**Lemma 2.1** *With the notations of Proposition 2.2,* $\bar{N}\Lambda=\Lambda\,{}^t\bar{N}$ *if and only if*

$$
\begin{aligned}
i)&\quad NE=C^{\#}(\gamma)+gE; \\
ii)&\quad NC-C\,{}^tN=E\otimes Y+Y\otimes E; \\
iii)&\quad <\gamma,E>=0.
\end{aligned}
$$

**Proof.**
Let $\alpha$ be any 1-form on $M$. Then,

$$
\begin{aligned}
\bar{N}(\Lambda^{\#}(\alpha)) &= \exp(-t)(\bar{N}(C^{\#}(\alpha)-<\alpha,E>\frac{\partial}{\partial t})) \\
&= \exp(-t)(N(C^{\#}(\alpha))-<\alpha,E>Y-(<\alpha,C^{\#}(\gamma)>+<\alpha,E>g)\frac{\partial}{\partial t})
\end{aligned}
$$

and, taking account that

$$^{t}\bar{N}(\alpha) = {}^{t}N\alpha + <\alpha, Y> dt,$$

we compute

$$\Lambda^{\#}({}^{t}\bar{N}\alpha) = \exp(-t)(C^{\#}({}^{t}N\alpha) + <\alpha, Y> E - <\alpha, NE> \frac{\partial}{\partial t})$$

and we conclude that $\bar{N}\Lambda^{\#}(\alpha) = \Lambda^{\#}({}^{t}N\alpha)$ if and only if

$$<\alpha, NE> = <\alpha, C^{\#}(\gamma) + gE> \qquad (15)$$

and

$$(NC^{\#} - Y \otimes E)\alpha = (C^{\#}\,{}^{t}N + E \otimes Y)\alpha. \qquad (16)$$

Equalities (15) and (16) give conditions *i)* and *ii)*.

On the other hand,

$$\bar{N}(\Lambda^{\#}(dt)) = \exp(-t)(NE + <\gamma, E> \frac{\partial}{\partial t}) \qquad (17)$$

and, because $^{t}\bar{N}(dt) = \gamma + gdt$,

$$\Lambda^{\#}({}^{t}\bar{N}(dt)) = \exp(-t)(C^{\#}(\gamma) + gE). \qquad (18)$$

Once the right members of (17) and (18) are equal, we obtain condition *iii)*.

□

Let us take the Magri-Morosi concomitant $\mathcal{R}(\Lambda, \bar{N})$ of $\Lambda$ and $\bar{N}$, (1).

**Lemma 2.2** *With the notations of Proposition 2.2, $\mathcal{R}(\Lambda, \bar{N}) = 0$ if and only if*

*i)* $NE + \mathcal{L}_Y E = gE - C^{\#}(dg)$

*ii)* $\mathcal{L}_Y C = gC - NC + Y \otimes E$

*iii)* $(\mathcal{L}_E N)X + <\gamma, X> E = C^{\#}(i_X d\gamma) + (X.g)E$

*iv)* $\mathcal{R}(C, N)(df, X) = (X.(Y.f))E - (X.(E.f))Y - <\gamma, X> C^{\#}(df)$,

*where $X$ is any vector field on $M$ and $f$ is any function of $C^{\infty}(M, \mathrm{I\!R})$.*

**Proof.**

Let $f$ be any function of $C^{\infty}(M, \mathrm{I\!R})$. The component of the vector field $\mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t})$ on $\frac{\partial}{\partial t}$ is

$$\exp(-t)(< df, C^{\#}(dg) + \mathcal{L}_Y E + NE - gE >). \qquad (19)$$

The other components (without $\frac{\partial}{\partial t}$) of $\mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t})$, obtained from the computation of $< dh, \mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t}) >$, for any $h \in C^{\infty}(M, \mathrm{I\!R})$, are

$$\exp(-t)(< dh, (-\mathcal{L}_Y C)^{\#}(df) + gC^{\#}(df) - N(C^{\#}(df)) + (E.f)Y >). \qquad (20)$$

From (19) and (20), we conclude that $\mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t}) = 0$ gives conditions $i)$ and $ii)$ of the Lemma.

On the other hand, if $X$ is any vector field on $M$, the component of the vector field $\mathcal{R}(\Lambda, \bar{N})(df, X)$ on $\frac{\partial}{\partial t}$ is

$$\exp(-t)(< df, -(\mathcal{L}_E N)X - < \gamma, X > E + C^{\#}(i_X d\gamma) + (X.g)E >, \tag{21}$$

while the components without $\frac{\partial}{\partial t}$, obtained from the computation of $< dh, \mathcal{R}(\Lambda, \bar{N})(df, X) >$, for any $h \in C^\infty(M, \mathbb{R})$, are

$$\exp(-t)(< dh, \mathcal{R}(C, N)(df, X) + < \gamma, X > C^{\#}(df) + (X.(E.f))Y - (X.(Y.f))E > . \tag{22}$$

From (21) and (22), we conclude that $\mathcal{R}(\Lambda, \bar{N})(df, X) = 0$ gives conditions $iii)$ and $iv)$ of the Lemma. □

**Theorem 2.1** *Let* $\Lambda = \exp(-t)(C + \frac{\partial}{\partial t} \wedge E)$ *and* $\bar{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g\frac{\partial}{\partial t} \otimes dt$, *be respectively the homogeneous Poisson tensor and the Nijenhuis tensor on* $\mathbb{R} \times M$. *Then, the triple* $(\mathbb{R} \times M, \Lambda, \bar{N})$ *is a Poisson-Nijenhuis manifold if and only if conditions* $i)$, $ii)$ *and* $iii)$ *of Lemma (2.1) and* $i)$ - $iv)$ *of Lemma (2.2) hold. Moreover, the homogeneous Poisson tensor* $\bar{N}\Lambda$ *is given by* $\exp(-t)(C_1 + \frac{\partial}{\partial t} \wedge E_1)$, *where*

$$(C_1, E_1) = (gC - \mathcal{L}_Y C, gE - C^{\#}(dg) - \mathcal{L}_Y E) \tag{23}$$

*is a Jacobi structure on* $M$, *compatible with* $(C, E)$.

**Proof.**

We only have to find the expressions of $C_1$ and $E_1$. For any pair $(f, h)$ of functions on $M$,

$$< dh, (\bar{N}\Lambda)^{\#}(df) >= \exp(-t) < dh, NC^{\#}(df) - (Y \otimes E)(df) > \tag{24}$$

and

$$< dh, (\Lambda^{\#}({}^t\bar{N})(df) >= \exp(-t) < dh, C^{\#}({}^tN(df)) + (E \otimes Y)(df) > . \tag{25}$$

Since $\bar{N}\Lambda = \frac{1}{2}(\bar{N}\Lambda + \Lambda\, {}^t\bar{N})$, we obtain from (24) and (25), using conditions $ii)$ of Lemma 2.1 and $ii)$ of Lemma 2.2,

$$\begin{aligned}(\bar{N}\Lambda)(df, dh) &= \exp(-t)(NC - Y \otimes E)(df, dh) \\ &= \exp(-t)(gC - \mathcal{L}_Y C)(df, dh).\end{aligned} \tag{26}$$

Also,

$$< dh, \bar{N}(\Lambda^{\#}(dt)) >= \exp(-t) < dh, NE > \tag{27}$$

and

$$< dh, \Lambda^{\#}({}^t\bar{N}(dt)) >= \exp(-t) < dh, C^{\#}(\gamma) + gE > . \tag{28}$$

From (27) and (28), and taking account conditions $i)$ of Lemma 2.1 and $i)$ of Lemma 2.2, we conclude that

$$\begin{aligned}(\bar{N}\Lambda)(dt, dh) &= \exp(-t)NE(dt, dh) \\ &= \exp(-t)(gE - C^{\#}(dg) - \mathcal{L}_Y E)(dt, dh). \end{aligned} \tag{29}$$

□

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