ON DECOMPOSITIONS IN GENERALISED LORENTZ-ZYGMUND SPACES

J. S. NEVES

ABSTRACT. Various characterisations are given of the generalised Lorentz-Zygmund (GLZ) spaces $L_{p,q;\boldsymbol{\alpha}}(R)$, with $p, q \in (0, +\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$ and (R, μ) a finite measure space. Given a measure space (R, μ) and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, we obtain equivalent representations for the (quasi-) norm of the GLZ space $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. Moreover, when (R, μ) is a finite measure space and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$, we present an equivalent norm for the space $L_{1,1;\boldsymbol{\alpha}}(R)$ in terms of decompositions. We show how the equivalent norms considered for $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with (R, μ) a finite measure space, and the decomposition norm in $L_{1,1;\boldsymbol{\alpha}}(R)$ can be employed to get simple proofs of some extrapolation results involving these spaces.

1. INTRODUCTION

In [EK], Edmunds and Krbec obtained some decompositions for the exponential Orlicz space $L_{\Phi_1}(\Omega)$, usually denoted by $E_\alpha(\Omega)$, with Young function Φ_1 given by $\Phi_1(t) = \exp t^\alpha$ for large t, where $\alpha > 0$ and Ω is a measurable subset of $\mathbb{R}^n$ with finite n-dimensional Lebesgue measure $|\Omega|_n$. Without loss of generality, it was assumed that $|\Omega|_n = 1$. They showed that considering a suitable decomposition of $(0, 1)$ into a union of disjoint intervals $\{(t_k, t_{k-1})\}_{k\in\mathbb{N}}$ it is enough to control only the blow up of the norms $\|f^*\|_{L_k(t_k,t_{k-1})}$, where f^* is the non-increasing rearrangement of f, by the same power $k^{-1/\alpha}$ to have $L_{\Phi_1}(\Omega)$. The proof was based on the fact that $L_{\Phi_1}(\Omega)$ coincides with the Zygmund space $L^\infty(\log L)^{-1/\alpha}(\Omega)$ (see [BR80, Theorem D] or [BS88, Lemma IV.6.2]). In Section 3, we extend this result to the generalised Lorentz-Zygmund (GLZ) spaces $L_{p,q;\boldsymbol{\alpha}}(R)$, with $p, q \in (0, +\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$, and (R, μ) a finite measure space, cf. Theorem 3.2. The method of the proof is different from, and in our opinion easier than, that used in [EK].

In [Tri93], Triebel gave an equivalent norm for the exponential Orlicz space $L_{\Phi_1}(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$ with finite volume; see also [EGO98]. With this equivalent norm, he proved that the embeddings $id : B^{n/p}_{pp}(\Omega) \to E_\alpha(\Omega)$ and $id : H^{n/p}_p(\Omega) \to E_\alpha(\Omega)$, with $1 < p < +\infty$, $0 < \alpha < p'$ and Ω a bounded C^∞-domain in $\mathbb{R}^n$, are compact and obtained estimates for the approximation and entropy numbers of those embeddings. Let us just mention that $B^{n/p}_{pp}(\Omega)$ and $H^{n/p}_p(\Omega)$ are classical Besov spaces and fractional Sobolev spaces, respectively. We refer to [Tri93] for more details. Equivalent norms for the double exponential Orlicz space $L_{\Phi_2}(\Omega)$, usually denoted by $EE_\alpha(\Omega)$, with Young function Φ_2 given by $\Phi_2(t) = \exp\exp t^\alpha$ for large t, where $\alpha > 0$ and Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, were obtained by Edmunds, Gurka and Opic in [EGO98]. The proof was also based on the fact that $L_{\Phi_2}(\Omega)$ coincides with the GLZ space $L_{\infty,\infty;0,-\frac{1}{\alpha}}(\Omega)$, see [EGO95, Lemma 3.9]. Following the same technique as in [EGO98], we obtain in Section 4 equivalent representations for the (quasi-) norms of the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with (R, μ) a measure space and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, *i.e.*

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$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, $\alpha_1, \dots, \alpha_{m-1} \leq 0$ and $\alpha_m < 0$, cf. Theorem 4.1 and its Corollaries. In particular, when (R, μ) has finite measure we obtain equivalent norms for the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, extending in this way the results in [Tri93] and [EGO98]. Still in Section 4, we give an equivalent norm for the spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with (R, μ) a non-atomic finite measure space and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$, *i.e.* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, $\alpha_1, \dots, \alpha_{m-1} \geq 0$ and $\alpha_m > 0$, in terms of decompositions. This result extends a result obtained by Edmunds and Triebel, cf. [ET96, Theorem 2, p. 72], for the spaces $L^1(\log L)^\alpha(\Omega)$, with $\alpha > 0$ and Ω a measurable subset of $\mathbb{R}^n$ with finite volume. We refer to [FK98, Theorem 3.4] for a different proof of this result.

In Section 5, we show how the equivalent norms obtained in Section 4 for $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, and the decomposition norm in $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_+$, can be employed to get simple proofs of some extrapolation results involving these spaces. Let us remark that we do not follow a general setting in terms of abstract extrapolation methods considered by Jawerth and Milman, cf. [JM91] (see also [Mil94]). We mention that the starting point of the extrapolation theory was the Theorem of Yano [Yan51] which can be described as follows. Suppose that T is a bounded linear operator on $L_p(0,1)$ for $p > 1$ with $\|T\|_{L_p \to L_p} = \mathcal{O}((p-1)^{-\alpha})$ as $p \downarrow 1$, for some $\alpha > 0$; then these estimates can be extrapolated to $L^1(\log L)^\alpha(0,1) \to L_1(0,1)$; see [Zyg59, Theorem XII.4.11 (ii), p. 119] for a more general formulation. We refer to [Tor86, Theorem IV.5.3, p.92] where T was supposed to be sublinear. We also refer to [FK98, Theorem 4.2] where T was supposed to be subadditive. In [Ste70, p. 23] and [ET96, p. 74] the case was considered when T is the Hardy-Littlewood maximal operator. It should be emphasised that the decomposition approach, used in [ET96] and [FK98], skips completely the machinery of weak type inequalities and the Marcinkiewicz interpolation Theorem, since it follows at once from the expression of the norm in $L^1(\log L)^\alpha(\Omega)$, with $\alpha > 0$. There is also a dual statement for operators acting from $L_p(R_0)$ into $L_p(R_1)$, with (R_0, μ_0) and (R_1, μ_1) finite measure spaces, for p close to $+\infty$, such that $\|T\|_{L_p \to L_p} = \mathcal{O}(p^{1/\alpha})$ as $p \to +\infty$, for some $\alpha > 0$; then there exist positive constants λ, K such that $\int_{R_1} \exp(\lambda |Tf|^\alpha)\, d\mu_1 \leq K$ for each f with $|f| \leq 1$; see [Zyg59, Theorem XII.4.11 (i), p. 119]. There is also a version of this result for sublinear operators. We refer to Section 5 for more details.

2. Notation and Preliminaries

As usual, $\mathbb{R}^n$ denotes Euclidean n-dimensional space. Let (R, Σ, μ), usually denoted by (R, μ), be a totally σ-finite measure space and referred in the sequel only as a measure space. A set $E \in \Sigma$ is called an atom of (R, Σ, μ) if $\mu(E) > 0$ and $F \subset E$, $F \in \Sigma$ implies either $\mu(F) = 0$ or $\mu(E \backslash F) = 0$. If there are no atoms, then (R, Σ, μ) is called non-atomic. A measure space (R, μ) is called resonant if it is one of the following two types: (i) non-atomic; (ii) completely atomic, with all atoms having equal measure. We refer to [BS88, pp.45–51] for more details and for a different, but equivalent, definition. When $R = \mathbb{R}^n$ we shall always take μ to be Lebesgue measure μ_n, and shall write $|\Omega|_n = \mu_n(\Omega)$ for any measurable subset Ω of $\mathbb{R}^n$. The family of all extended scalar-valued (real or complex) μ-measurable functions on R will be denoted by $\mathcal{M}(R, \mu)$; $\mathcal{M}_0(R, \mu)$ will stand for the subset of $\mathcal{M}(R, \mu)$ consisting of all those functions which are finite μ-a.e.

Definition 2.1. *Let $f \in \mathcal{M}_0(R, \mu)$. The distribution function μ_f of f is defined by*

$$\mu_f(\lambda) = \mu\{x \in R : |f(x)| > \lambda\}, \quad \text{for all } \lambda \geq 0, \tag{2.1}$$

and the non-increasing rearrangement of f is the function f^* defined on $[0, +\infty)$ by

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad \text{for all } t \geq 0. \tag{2.2}$$

If (R, μ) is a finite measure space, then the distribution function μ_f is bounded by $\mu(R)$ and so $f^*(t) = 0$ for all $t \geq \mu(R)$. In this case we may regard f^* as a function defined on the interval $[0, \mu(R))$; for more details we refer to [BS88].

Definition 2.2. *Two functions $f \in \mathcal{M}_0(R,\mu)$ and $g \in \mathcal{M}_0(S,\nu)$ are said to be equimeasurable if they have the same distribution function, i.e., if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.*

Now let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. Let us denote by ϑ_α^m and ω_α^m the real functions defined by

$$\vartheta_\alpha^m(t) = \prod_{i=1}^m l_i^{\alpha_i}(t), \quad \text{for all } \ t \in (0,+\infty), \tag{2.3}$$

and

$$\omega_\alpha^m(t) = \prod_{i=1}^m l_{i-1}^{\alpha_i}(t), \quad \text{for all } \ t \in [1,+\infty), \tag{2.4}$$

where $l_0, l_1, \ldots, l_m$ are non-negative functions defined on $(0,+\infty)$ by

$$l_0(t) = t, \ l_1(t) = 1 + |\log t|, \ l_i(t) = 1 + \log l_{i-1}(t), \quad i \in \{2, \ldots, m\}. \tag{2.5}$$

Definition 2.3 (cf. [EGO97])**.** *Let $p, q \in (0,+\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. The generalised Lorentz-Zygmund (GLZ) space $L_{p,q;\boldsymbol{\alpha}}(R)$ is defined to be the set of all functions $f \in \mathcal{M}_0(R,\mu)$ such that*

$$\|f\|_{p,q;\boldsymbol{\alpha};R} := \|t^{\frac{1}{p}-\frac{1}{q}} \vartheta_\alpha^m(t) \ f^*(t)\|_{q,(0,+\infty)} \tag{2.6}$$

is finite. Here $\|.\|_{q,(0,+\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0,+\infty)$.

We remark that in [EGO97], the space $L_{p,q;\boldsymbol{\alpha}}(R)$ and the quasi-norm $\|.\|_{p,q;\boldsymbol{\alpha};R}$ defined above are denoted by $L_{p,q;\alpha_1,\ldots,\alpha_m}(R)$ and $\|.\|_{p,q;\alpha_1,\ldots,\alpha_m;R}$, respectively. We use the notation in [EGO97] only when we are considering particular cases.

Let us observe that when we consider $\boldsymbol{\alpha} = (0, \ldots, 0)$ in the previous Definition, we get the Lorentz space $L_{p,q}(R)$ endowed with the (quasi-) norm $\|.\|_{p,q;R}$, which is just the Lebesgue space $L_p(R)$ endowed with the (quasi-) norm $\|.\|_{p;R}$ when $p = q$; if $p = q$, $m = 1$ and $(R,\mu) = (\Omega, \mu_n)$, we get the Zygmund space $L^p(\log L)^{\alpha_1}(\Omega)$ endowed with the (quasi-) norm $\|.\|_{p;\alpha_1;\Omega}$.

Let us introduce some more notation, that will be needed in Section 4. Let $m \in \mathbb{N}$ with $m \geq 2$. We define the numbers $exp_0, \ldots, exp_m$ by

$$exp_0 = 1, \ \ exp_i = e^{exp_{i-1}}, \ i \in \{1, \ldots, m\}.$$

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. Let us denote by γ_α^m the non-negative function defined by

$$\gamma_\alpha^m(t) = \prod_{i=1}^m \ell_{i-1}^{\alpha_i}(t), \quad \text{for all } \ t \in [exp_{m-2}, +\infty), \tag{2.7}$$

where $\ell_0, \ldots, \ell_m$ are the non-negative functions defined by

$$\ell_0(t) = t, \ t \geq 1; \ \ell_i(t) = \log \ell_{i-1}(t), \ t \geq exp_{i-1}, \ i \in \{1, \ldots, m\}.$$

We are going to need in Section 3 the following Lemma, which is very easy to prove.

Lemma 2.1. *(i) Let $m, k \in \mathbb{N}$. Then*

$$l_m(e^{-k+1}) = l_{m-1}(k).$$

(ii) Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then

$$l_m(k) \leq l_m(k+1) \leq e \ l_m(k).$$

(iii) Let $\alpha \in \mathbb{R}$ and $m, k \in \mathbb{N}$. Then for each $t \in (e^{-k}, e^{-k+1})$, we have the inequalities

$$\min\{1, e^\alpha\} \ l_{m-1}^\alpha(k) \leq l_m^\alpha(t) \leq \max\{1, e^\alpha\} \ l_{m-1}^\alpha(k).$$

(iv) *Let $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$ and $k \geq 2$. Then the inequalities*

$$\min\{1, e^{-\alpha}\}\, l_{m-1}^{\alpha}(k) \leq l_m^{\alpha}(t) \leq \max\{1, e^{-\alpha}\}\, l_{m-1}^{\alpha}(k)$$

hold for each $t \in (e^{-k+1}, e^{-k+2})$.

The following Lemma, with an obvious proof, will be used later on.

Lemma 2.2. *Let $k \in \mathbb{N}$ and $q_0 > exp_{k-1}$. Then*

(i) $\ell_k(q) \leq l_k(q)$, *for each* $q \in [exp_{k-1}, +\infty)$;
(ii) $l_k(q) \leq e^k \ell_k(q)$, *for each* $q \in [exp_k, +\infty)$;
(iii) $l_k(q) \leq \left(\dfrac{k}{\ell_k(q_0)} + 1\right) \ell_k(q)$, *for each* $q \in [q_0, +\infty)$.

By a Young function Φ we mean a continuous non-negative, strictly increasing and convex function on $[0, +\infty)$ satisfying

$$\lim_{t \to 0^+} \frac{\Phi(t)}{t} = \lim_{t \to +\infty} \frac{t}{\Phi(t)} = 0.$$

Given a Young function Φ and any measurable subset Ω of $\mathbb{R}^n$, $L_\Phi(\Omega)$ will denote the corresponding Orlicz space, *i.e.* the collection of functions $f \in \mathcal{M}_0(\Omega, \mu_n)$ for which there is a $\lambda > 0$ such that $\int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx < +\infty$, equipped with the Luxemburg norm $||.||_{\Phi,\Omega}$ given by

$$||f||_{\Phi,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

We refer to [Ada75, Chapter VIII] and [KJF77, Chapter III] for more details.

Let Φ_1 and Φ_2 be Young functions. Recall that Φ_2 *dominates* Φ_1 *globally* if there is a positive constant κ such that

$$\Phi_1(t) \leq \Phi_2(\kappa t) \tag{2.8}$$

for all $t \geq 0$. Similarly, Φ_2 *dominates* Φ_1 *near infinity* if there are positive constants κ and t_0 such that (2.8) holds for all $t \in [t_0, +\infty)$. Two Young functions are said to be *equivalent globally* (*near infinity*) if each dominates the other globally (near infinity). We have from [Ada75, Theorem 8.12, pp. 234-235] the following result: If Φ_1 and Φ_2 are equivalent globally (or near infinity and $|\Omega|_n < +\infty$), then $L_{\Phi_1}(\Omega) = L_{\Phi_2}(\Omega)$ and the corresponding norms are equivalent.

Lemma 2.3 (cf. [EGO98]). *Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume and let $\alpha > 0$. Then*

(i) *the space $L^\infty(\log L)^{-1/\alpha}(\Omega) = L_{\infty,\infty;-\frac{1}{\alpha}}(\Omega)$ coincides with the Orlicz space $L_{\Phi_1}(\Omega)$, where $\Phi_1(t) = \exp t^\alpha$ for all $t \geq t_0$ with some $t_0 \in (0, +\infty)$, and the corresponding (quasi-) norms are equivalent;*
(ii) *the space $L^\infty(\log\log L)^{-1/\alpha}(\Omega) = L_{\infty,\infty;0,-\frac{1}{\alpha}}(\Omega)$ coincides with the Orlicz space $L_{\Phi_2}(\Omega)$, where $\Phi_2(t) = \exp\exp t^\alpha$ for all $t \geq t_0$ with some $t_0 \in (0, +\infty)$, and the corresponding (quasi-) norms are equivalent.*

We will denote the Orlicz spaces $L_{\Phi_1}(\Omega)$ and $L_{\Phi_2}(\Omega)$, considered in Lemma 2.3, by $E_\alpha(\Omega)$ and $EE_\alpha(\Omega)$, respectively. In view of the same Lemma, we may endow these spaces with the quasi-norms $||.||_{E_\alpha(\Omega)} := ||.||_{\infty,\infty;-\frac{1}{\alpha};\Omega}$ and $||.||_{EE_\alpha(\Omega)} := ||.||_{\infty,\infty;0,-\frac{1}{\alpha};\Omega}$. For more details we refer to [EGO98].

Let $m \in \mathbb{N}$. We denote by $\mathcal{R}_+^m$ and $\mathcal{R}_-^m$ the following subsets of $\mathbb{R}^m$:

$$\mathcal{R}_+^m = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : \alpha_1, \ldots, \alpha_{m-1} \geq 0 \ \text{ and } \ \alpha_m > 0\}$$

$$\mathcal{R}_-^m = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : \alpha_1, \ldots, \alpha_{m-1} \leq 0 \ \text{ and } \ \alpha_m < 0\}.$$

Given a Banach space X let us denote by X^* its dual space.

Let $j_0 \in \mathbb{N}$ and let $\{A_j\}_{j \geq j_0}$ be a sequence of Banach spaces. We denote by $l_1(A_j)$ the space of all sequences $a = \{a_j\}_{j \geq j_0}$ with $a_j \in A_j$, $j \geq j_0$, such that

$$\|a\|_{l_1(A_j)} = \sum_{j=j_0}^{+\infty} \|a_j\|_{A_j} < +\infty.$$

By $l_\infty(A_j)$ we denote the space of all sequences $a = \{a_j\}_{j \geq j_0}$ with $a_j \in A_j$, $j \geq j_0$, for which $\|a\|_{l_\infty(A_j)} = \sup_{j \geq j_0} \|a_j\|_{A_j}$ is finite. The space $c_0(A_j)$ is the subspace of $l_\infty(A_j)$ consisting of all sequences $a = \{a_j\}_{j \geq j_0}$ such that

$$\lim_{j \to +\infty} \|a_j\|_{A_j} = 0.$$

By Lemma 1.11.1 in [Tri78, pp.68-69], generalised in an obvious way,

$$[c_0(A_j)]^* = l_1(A_j^*), \tag{2.9}$$

with the usual interpretation (not only isomorphic but also isometric); see [Tri78] for more details.

For two non-negative expressions (*i.e.* functions or functionals) $\mathcal{A}$, $\mathcal{B}$ we use the symbol $\mathcal{A} \precsim \mathcal{B}$ to mean that $\mathcal{A} \leq c\,\mathcal{B}$, for some positive constant c independent of the variables in the expressions $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{B} \precsim \mathcal{A}$, we write $\mathcal{A} \approx \mathcal{B}$.

We adopt the convention that $\frac{a}{+\infty} = 0$ and $\frac{a}{0} = +\infty$ for all $a > 0$. If $p \in [1, +\infty]$, the conjugate number p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$.

3. Decompositions

As was said in the Introduction, the following results extend the decompositions considered in [EK] for the exponential Orlicz spaces $E_\alpha(\Omega)$.

Let us assume, in this Section, that (R, μ) is a finite measure space. Without loss of generality we suppose that $\mu(R) = 1$; see Remark 3.1. In the sequel, we shall consider the decomposition of $(0,1)$ into $\{(e^{-k}, e^{-k+1})\}_{k \geq 1}$.

Theorem 3.1. *Let $p, q \in (0, +\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^n$. Then for each $f \in L_{p,q;\boldsymbol{\alpha}}(R)$ we have*

(i) if $0 < q < +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \approx \left[\sum_{k=1}^{+\infty} \left(e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k})\right)^q\right]^{1/q} \tag{3.1}$$

$$\approx \left[\sum_{k=2}^{+\infty} \left(e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k+1})\right)^q\right]^{1/q}; \tag{3.2}$$

(ii) if $q = +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \approx \sup_{k \geq 1} \left\{e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k})\right\} \tag{3.3}$$

$$\approx \sup_{k \geq 2} \left\{e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k+1})\right\}. \tag{3.4}$$

Proof. (i) Let $0 < q < +\infty$ and suppose $f \in L_{p,q;\boldsymbol{\alpha}}(R)$. Then by Lemma 2.1 it follows that

$$\begin{aligned} \|f\|_{p,q;\boldsymbol{\alpha};R}^q &\geq c_1 \sum_{k=2}^{+\infty} \left(e^{-k(\frac{1}{p}-\frac{1}{q})} \vartheta_\alpha^m(e^{-k+1})\; f^*(e^{-k+1})\right)^q\; e^{-k} \\ &\geq c_2 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega_\alpha^m(k)\; f^*(e^{-k})\right)^q. \end{aligned}$$

Conversely, for $f \in L_{p,q;\boldsymbol{\alpha}}(R)$, we have again by Lemma 2.1

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \;\le\; c_3 \sum_{k=2}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega_\alpha^m(k)\; f^*(e^{-k+1})\right)^q \le c_4 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega_\alpha^m(k)\; f^*(e^{-k})\right)^q ,$$

which gives the desired inequalities.

(ii) The proof of the case $q = +\infty$ is similar to the previous one. □

Let Ω be a measurable subset of $\mathbb{R}^n$ such that $|\Omega|_n = 1$. By Theorem 3.1 we conclude that

$$\|f\|_{E_\alpha(\Omega)} \approx \sup_{k\ge 1} \frac{f^*(e^{-k})}{k^{1/\alpha}} \approx \sup_{k\ge 2} \frac{f^*(e^{-k+1})}{k^{1/\alpha}}, \quad \text{for each} \quad f \in E_\alpha(\Omega),$$

and

$$\|f\|_{EE_\alpha(\Omega)} \approx \sup_{k\ge 1} \frac{f^*(e^{-k})}{(1+\log k)^{1/\alpha}} \approx \sup_{k\ge 2} \frac{f^*(e^{-k+1})}{\log^{1/\alpha} k}, \quad \text{for each} \quad f \in EE_\alpha(\Omega).$$

The next Lemma, with an easy proof, will be used to prove the last result of this Section.

Lemma 3.1. *Let $f \in \mathcal{M}_0(R,\mu)$, $J_k = (e^{-k}, e^{-k+1})$, $k \ge 1$. Then*

(i) for each $k \in \mathbb{N}$ we have

$$c_1 f^*(e^{-k+1}) \le \|f^*\|_{k,J_k} \le c_2 f^*(e^{-k}), \tag{3.5}$$

where c_1 and c_2 are positive constants independent of f and k;

(ii) for each $k \ge 2$ we have

$$c_1 f^*(e^{-k+2}) \le \|f^*\|_{k,J_{k-1}} \le c_2 f^*(e^{-k+1}), \tag{3.6}$$

where c_1 and c_2 are positive constants independent of f and k.

Theorem 3.2. *Let $p, q \in (0,+\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^n$. Let $J_k = (e^{-k}, e^{-k+1})$, $k \ge 1$, and $I_k = J_{k-1}$, $k \ge 2$. Then for each $f \in L_{p,q;\boldsymbol{\alpha}}(R)$ we have*

(i) if $0 < q < +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \;\approx\; \left[\sum_{k=1}^{+\infty} \left(e^{-k/p}\, \omega_\alpha^m(k)\, \|f^*\|_{k,J_k}\right)^q\right]^{1/q} \tag{3.7}$$

$$\approx\; \left[\sum_{k=2}^{+\infty} \left(e^{-k/p}\, \omega_\alpha^m(k)\, \|f^*\|_{k,I_k}\right)^q\right]^{1/q}; \tag{3.8}$$

(ii) if $q = +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \;\approx\; \sup_{k\ge 1} \left\{e^{-k/p}\, \omega_\alpha^m(k)\, \|f^*\|_{k,J_k}\right\} \tag{3.9}$$

$$\approx\; \sup_{k\ge 2} \left\{e^{-k/p}\, \omega_\alpha^m(k)\, \|f^*\|_{k,I_k}\right\}. \tag{3.10}$$

Proof. (i) Suppose $0 < q < +\infty$ and let $f \in L_{p,q;\boldsymbol{\alpha}}(R)$. Then by (3.1) and by (3.5), we have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \;\ge\; c_1 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega_\alpha^m\; \|f^*\|_{k,J_k}\right)^q .$$

By (3.2) and by (3.6), we also have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \;\ge\; c_2 \sum_{k=2}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega_\alpha^m(k)\; \|f^*\|_{k,I_k}\right)^q .$$

Conversely, for $f \in L_{p,q;\boldsymbol{\alpha}}(R)$, by (3.2) and by (3.5), we have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \quad \leq \quad c_3 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}} \, \omega^m_\alpha(k) \, \|f^*\|_{k,J_k} \right)^q .$$

By (3.2), by Lemma 2.1 and by (3.6), we have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \quad \leq \quad c_4 \sum_{k=3}^{+\infty} \left(e^{-\frac{k}{p}} \, \omega^m_\alpha(k) \, f^*(e^{-k+2}) \right)^q \leq c_5 \sum_{k=2}^{+\infty} \left(e^{-\frac{k}{p}} \, \omega^m_\alpha(k) \, \|f^*\|_{k,I_k} \right)^q ,$$

which gives the desired inequalities.

(ii) The proof of the case $q = +\infty$ is similar to the previous one. □

Let Ω be a measurable subset of $\mathbb{R}^n$ such that $|\Omega|_n = 1$. By Theorem 3.2 we conclude that for each $f \in E_\alpha(\Omega)$

$$\|f\|_{E_\alpha(\Omega)} \quad \approx \quad \sup_{k \geq 1} \frac{\|f^*\|_{k,J_k}}{k^{1/\alpha}} \approx \sup_{k \geq 2} \frac{\|f^*\|_{k,I_k}}{k^{1/\alpha}}. \tag{3.11}$$

The first estimate in (3.11) is given in [EK] by Corollary 2.3. The counterpart for the spaces $EE_\alpha(\Omega)$ is given by

$$\|f\|_{EE_\alpha(\Omega)} \approx \sup_{k \geq 1} \frac{\|f^*\|_{k,J_k}}{(1+\log k)^{1/\alpha}} \approx \sup_{k \geq 2} \frac{\|f^*\|_{k,I_k}}{\log^{1/\alpha} k}, \quad \text{for all} \quad f \in EE_\alpha(\Omega).$$

Remark 3.1. If (R, μ) is a finite measure space with measure $\mu(R)$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{R}^m$, we have $\vartheta^m_\alpha(s) \approx \vartheta^m_\alpha(s\,\mu(R))$, for all $s \in (0,1)$. This follows from the estimates $e^{-j}\, l_i(s) \leq l_i(s\mu(R)) \leq e^j \, l_i(s)$, for all $s \in (0,1)$ and $i = 1, \ldots, m$ where j is a positive integer such that $e^{j-1} \leq l_1(\mu(R)) \leq e^j$.

With the previous considerations, it is easy to see that the estimates in Theorem 3.1 and Theorem 3.2 still hold, up to constants, if we replace $f^*(e^{-k})$ by $f^*(e^{-k}\,\mu(R))$, for each $k \in \mathbb{N}$, and $J_k = (e^{-k}, e^{-k+1})$ by $J_k = (e^{-k}\,\mu(R), e^{-k+1}\,\mu(R))$, for each $k \in \mathbb{N}$, respectively.

4. Equivalent (Quasi-) Norms for some Generalised Lorentz-Zygmund Spaces

In this Section, we are going to consider in the first part the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, and in the second part the GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_+$.

4.1. The GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$.

First we are going to recall a Lemma.

Lemma 4.1 (cf. [GO98], Lemma 5.1)**.** *Let $m \in \mathbb{N}$ and $\nu > 0$. Then there is a constant $c \in (0, +\infty)$ such that for all $s \in (0,1)$,*

$$\sup_{q \in [1,+\infty)} l^{-\nu}_{m-1}(q) s^{1/q} \leq c\, l^{-\nu}_m(s).$$

With the help of the previous result, it is not difficult to prove the next Lemma.

Lemma 4.2. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Let $t_0 \in (0, +\infty)$. Then there is a positive constant c such that $\omega^m_\alpha(q) s^{1/q} \leq c\, \vartheta^m_\alpha(s)$, for all $s \in (0, t_0)$ and all $q \in [1, +\infty)$.*

The following result generalises Theorem 3.1 in [EGO98].

Theorem 4.1. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Let $t_0 \in (0, +\infty)$.*

(i) Let $p \in (0, +\infty]$. Then for each $f \in L_{p,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{p,\infty;\boldsymbol{\alpha};R} \approx \sup_{q \in [1,+\infty)} \omega^m_\alpha(q) \, \|f^*\|_{\frac{q}{q/p+1},\infty;(0,t_0)} + \sup_{t_0 \leq t < +\infty} \{t^{1/p} \vartheta^m_\alpha(t) f^*(t)\}. \tag{4.1}$$

(ii) Then for each $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)}. \tag{4.2}$$

Proof. We follow the proof of Theorem 3.1 in [EGO98], where the case $p = +\infty$, $m = 2$, $\alpha_1 = 0$, $\alpha_2 < 0$ and $\mu(R) < +\infty$ with $t_0 = \mu(R)$ was considered.

(i) Let $t_0 \in (0,+\infty)$ and $\mathcal{A} := \mathcal{B} + \mathcal{C}$ where

$$\mathcal{B} := \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\, \|f^*\|_{\frac{q}{q/p+1},\infty;(0,t_0)} \quad \text{and} \quad \mathcal{C} := \sup_{t_0\le t<+\infty} \{t^{1/p}\vartheta_\alpha^m(t) f^*(t)\}.$$

Suppose $f \in L_{p,\infty;\boldsymbol{\alpha}}(R)$. By Lemma 4.2 there is a constant $c_1 > 0$ such that for all $q \in [1,+\infty)$,

$$\omega_\alpha^m(q)\, \|f^*\|_{\frac{q}{q/p+1},\infty;(0,t_0)} \quad \le \quad c_1 \sup_{0<s<t_0} \{\vartheta_\alpha^m(s)\; s^{\frac{1}{p}} f^*(s)\}.$$

Passing to the supremum over all $q \in [1,+\infty)$, we get the inequality $\mathcal{B} \le c_1 \|f\|_{p,\infty;\boldsymbol{\alpha};R}$. Hence

$$\mathcal{A} \quad \le 2\max\{1, c_1\} \|f\|_{p,\infty;\boldsymbol{\alpha};R}. \tag{4.3}$$

Conversely, suppose the right hand-side of (4.1) is finite. Fix $s \in (0,t_0)$ and set $q = 1 + |\log s|$. Then $\mathcal{B} \ge \omega_\alpha^m(q)\; s^{\frac{1}{q}+\frac{1}{p}} f^*(s) \ge e^{-1}\vartheta_\alpha^m(s) s^{\frac{1}{p}} f^*(s)$. Taking the supremum over all $s \in (0,t_0)$, we obtain the inequality

$$\mathcal{B} \ge e^{-1} \sup_{0<t<t_0} \{t^{1/p}\vartheta_\alpha^m(t) f^*(t)\}.$$

So $\mathcal{A} \ge e^{-1}\|f\|_{p,\infty;\boldsymbol{\alpha};R}$, which together with (4.3) gives the estimate (4.1).

(ii) Let $t_0 \in (0,+\infty)$. First we prove the following estimate

$$\mathcal{A} + \mathcal{B} \approx f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)}, \tag{4.4}$$

where

$$\mathcal{A} := \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\, \|f^*\|_{q,\infty;(0,t_0)} \quad \text{and} \quad \mathcal{B} := \sup_{t_0\le t<+\infty} \{\vartheta_\alpha^m(t) f^*(t)\}.$$

Suppose the right hand-side of (4.4) is finite. Since $\|f^*\|_{q,\infty;(0,t_0)} \le \|f^*\|_{q;(0,t_0)}$, cf. Proposition IV.4.2 of [BS88], and $\mathcal{B} \le f^*(t_0)$, we immediately obtain the inequality

$$\mathcal{A} + \mathcal{B} \le f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)}.$$

Now we prove the converse inequality. Suppose that $\mathcal{A} + \mathcal{B} < +\infty$. If $1 \le q < q_1$ then

$$\|f^*\|_{q;(0,t_0)} \quad \le \quad \|f^*\|_{q_1,\infty;(0,t_0)} t_0^{\frac{1}{q}-\frac{1}{q_1}} (1 - \frac{q}{q_1})^{-1/q}. \tag{4.5}$$

Let $q \in [1,+\infty)$. Since $l_j(q) \le l_j(2q) \le e l_j(q)$, for all $j \in \mathbb{N}_0$ we have by (4.5), with $q_1 = 2q$, the following inequalities

$$\omega_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)} \quad \le \quad c_1\omega_\alpha^m(2q)\, \|f^*\|_{2q,\infty;(0,t_0)} \le c_1 \sup_{r\in[2,+\infty)} \omega_\alpha^m(r)\, \|f^*\|_{r,\infty;(0,t_0)}.$$

Therefore, passing to the supremum over all $q \in [1,+\infty)$, we get the inequality

$$\sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)} \le c_1\, \mathcal{A}. \tag{4.6}$$

Now it easily follows from (4.6) that

$$f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)} \le \max\{c_1, \vartheta_{-\alpha}^m(t_0)\}\, (\mathcal{A} + \mathcal{B})$$

and (4.4) is proved. The estimate (4.2) follows from (4.1), with $p = +\infty$, and from (4.4). □

When (R,μ) is a finite measure space the previous estimates are much nicer.

Corollary 4.1. *Suppose (R,μ) is a measure space such that $\mu(R)<+\infty$. Let $m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}^m_-$.*

(i) Let $p\in(0,+\infty]$. Then for each $f\in L_{p,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{p,\infty;\boldsymbol{\alpha};R}\approx\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,\|f\|_{\frac{q}{q/p+1},\infty;R}. \tag{4.7}$$

(ii) Then for each $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,\|f\|_{q;R}. \tag{4.8}$$

Proof. The results follow from the theorem with $t_0=\mu(R)$ and from the fact that $f^*(t)=0$, $t\geq\mu(R)$. For the part (ii) we use also Proposition II.1.8 in [BS88]; see also Theorem 1.8.5 in [Zie89]. ☐

From (ii) of Corollary 4.1 we recover the results of Theorem 3.1 in [EGO98] for the spaces $E_\alpha(\Omega)$ and $EE_\alpha(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n<+\infty$.

Corollary 4.2. *Let $m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}^m_-$. Let $t_0\in(0,+\infty)$. If $j_0\in\mathbb{N}$ and $q_0\geq1$ then for all $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx f^*(t_0)+\sup_{j\in\mathbb{N},j\geq j_0}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)} \tag{4.9}$$

$$\approx f^*(t_0)+\sup_{q\in[q_0,+\infty)}\omega^m_\alpha(q)\,\|f^*\|_{q;(0,t_0)}. \tag{4.10}$$

Proof. We follow the proof of Corollary 3.2 in [EGO98], where the case $m=2$, $\alpha_1=0$, $\alpha_2<0$ and $\mu(R)<+\infty$ with $t_0=\mu(R)$ was proved. For $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, $j_0\in\mathbb{N}$ and $q_0\geq1$ we denote

$$\begin{aligned}
S_1(f)&=f^*(t_0)+\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,\|f^*\|_{q;(0,t_0)},\\
S_2(f)&=f^*(t_0)+\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,t_0^{-1/q}\,\|f^*\|_{q;(0,t_0)},\\
S_3(f)&=f^*(t_0)+\sup_{q\in[q_0,+\infty)}\omega^m_\alpha(q)\,\|f^*\|_{q;(0,t_0)},\\
\sigma_1(f)&=f^*(t_0)+\sup_{j\in\mathbb{N},j\geq j_0}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)},\\
\sigma_2(f)&=f^*(t_0)+\sup_{j\in\mathbb{N},j\geq j_0}\omega^m_\alpha(j)\,t_0^{-1/j}\,\|f^*\|_{j;(0,t_0)},\\
\sigma_3(f)&=f^*(t_0)+\sup_{j\in\mathbb{N},j\geq[q_0]+1}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)},
\end{aligned}$$

where $[q_0]$ denotes the integer part of q_0.

(i) Let $t_0\in(0,+\infty)$, $j_0\in\mathbb{N}$ and $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. First we prove that

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx f^*(t_0)+\sup_{j\in\mathbb{N},j\geq j_0}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)}.$$

If $q\in[1,+\infty)$, we put $j=\max\{j_0,[q]+1\}$ and choose $n\in\mathbb{N}$ such that $e^{n-1}\geq j_0$. Then

$$j\leq j_0([q]+1)<j_0q\leq e^{n-1}(q+1)\leq e^{n-1}2q\leq e^nq$$

and hence

$$l_{k-1}(j)\leq e^n l_{k-1}(q),\quad k=2,\ldots,m.$$

Therefore

$$e^{n(\alpha_1+\cdots+\alpha_m)}\omega^m_\alpha(q)\leq\omega^m_\alpha(j). \tag{4.11}$$

Since $j \geq [q]+1 > q$, we get by Hölder's inequality together with (4.11) the inequality

$$\omega_\alpha^m(q) t_0^{-1/q} \|f^*\|_{q;(0,t_0)} \leq c\, \omega_\alpha^m(j) t_0^{-1/j} \|f^*\|_{j;(0,t_0)},$$

where $c = e^{-n(\alpha_1+\cdots+\alpha_m)} > 1$, and hence

$$S_2(f) \leq c\, \sigma_2(f). \tag{4.12}$$

It is easy to see that $S_1(f) \approx S_2(f)$, $\sigma_1(f) \approx \sigma_2(f)$, and since $\sigma_1(f) \leq S_1(f)$ we have, together with (4.12), the estimates

$$\sigma_1(f) \leq S_1(f) \approx S_2(f) \leq c\, \sigma_2(f) \approx \sigma_1(f). \tag{4.13}$$

So (4.9) it follows from (4.2) and (4.13).

(ii) Let $t_0 \in (0,+\infty)$, $q_0 \geq 1$ and $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. From (4.2) it follows that

$$S_3(f) \leq S_1(f) \approx \|f\|_{\infty,\infty;\boldsymbol{\alpha};R}. \tag{4.14}$$

Since $\sigma_3(f) = \sigma_1(f)$ if $j_0 = [q_0]+1$, we have by (4.9)

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx \sigma_3(f) \leq S_3(f). \tag{4.15}$$

Therefore, by (4.14) and (4.15) we get (4.10) and the proof is finished. □

When (R,μ) is a measure space of finite measure we obtain simple equivalent norms.

Corollary 4.3. *Suppose (R,μ) is a measure space such that $\mu(R) < +\infty$. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. If $j_0 \in \mathbb{N}$ and $q_0 \geq 1$ then for all $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx \sup_{j\in\mathbb{N}, j\geq j_0} \omega_\alpha^m(j)\, \|f\|_{j;R} \tag{4.16}$$

$$\approx \sup_{q\in[q_0,+\infty)} \omega_\alpha^m(q)\, \|f\|_{q;R}. \tag{4.17}$$

Proof. The results follow from Corollary 4.2 with $t_0 = \mu(R)$ and Proposition II.1.8 in [BS88]; see also Theorem 1.8.5 in [Zie89]. □

If we consider $m=1$, $\alpha_1 < 0$ and Ω a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n < +\infty$ in the above Corollary we recover part (i) of Corollary 3.2 in [EGO98].

Corollary 4.4. *Let $m \in \mathbb{N}$, $m \geq 2$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. Let $t_0 \in (0,+\infty)$. If $j_0 \in \mathbb{N}$, $j_0 \geq [exp_{m-2}]+1$ and $q_0 > exp_{m-2}$ then for all $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx f^*(t_0) + \sup_{j\in\mathbb{N}, j\geq j_0} \gamma_\alpha^m(j)\, \|f^*\|_{j;(0,t_0)} \tag{4.18}$$

$$\approx f^*(t_0) + \sup_{q\in[q_0,+\infty)} \gamma_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)}. \tag{4.19}$$

Proof. (i) Let $j_0 \in \mathbb{N}$, $j_0 \geq [exp_{m-2}]+1$. Since $j_0 > exp_{m-2}$, it follows from (i) and (iii) of Lemma 2.2 that, for each $k \in \{1,\ldots,m-1\}$, $\ell_k(j) \approx l_k(j)$, for all $j \geq j_0$. Therefore, the estimate (4.18) follows from (4.9).

(ii) Let $q_0 > exp_{m-2}$. Then for $k = 1,\ldots,m-1$, the estimate $\ell_k(q) \approx l_k(q)$, for all $q \geq q_0$, follows from (i) and (iii) of Lemma 2.2. Therefore, the estimate (4.19) follows from (4.10). □

Corollary 4.5. *Suppose (R,μ) is a measure space such that $\mu(R) < +\infty$. Let $m \in \mathbb{N}$, $m \geq 2$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. If $j_0 \in \mathbb{N}$, $j_0 \geq [exp_{m-2}]+1$ and $q_0 > exp_{m-2}$ then for all $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx \sup_{j\in\mathbb{N}, j\geq j_0} \gamma_\alpha^m(j)\, \|f\|_{j;R} \tag{4.20}$$

$$\approx \sup_{q\in[q_0,+\infty)} \gamma_\alpha^m(q)\, \|f\|_{q;R}. \tag{4.21}$$

Proof. The results follow from Corollary 4.4 with $t_0 = \mu(R)$ and Proposition II.1.8 in [BS88]; see also Theorem 1.8.5 in [Zie89]. □

If we consider $m = 2$, $\alpha_1 = 0$, $\alpha_2 < 0$, and Ω a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n < +\infty$ in the above Corollary we recover part (ii) of Corollary 3.2 in [EGO98].

4.2. **The GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$.**

Let us assume, in this Subsection, that (R,μ) is a finite measure. Again, without loss of generality we suppose that $\mu(R) = 1$. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$. Let us consider the spaces $L_{1,1;\boldsymbol{\alpha}}(R)$ and $L_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ endowed with $\|.\|_{1,1;\boldsymbol{\alpha};R}$ and $\|.\|_{\infty,\infty;-\boldsymbol{\alpha};R}$, respectively.

The triangle inequality for $\|.\|_{1,1;\boldsymbol{\alpha};R}$ follows immediately by the property,

$$\int_0^t \varphi(s)(f+g)^*(s)ds \le \int_0^t \varphi(s)f^*(s)ds + \int_0^t \varphi(s)g^*(s)ds, \quad 0 < t < 1,$$

whenever φ is a non-negative decreasing function on $(0,1)$, cf. [Lor51, p. 38] or [BR80, p.23].

Let us introduce the functional $\|f\|_{(\infty,\infty;-\boldsymbol{\alpha};R)} = \sup\limits_{0<t<1} \vartheta^m_{-\alpha}(t) f^{**}(t)$. Then by Lemma 3.2 in [EGO97], we have

$$\|f\|_{\infty,\infty;-\boldsymbol{\alpha};R} \le \|f\|_{(\infty,\infty;-\boldsymbol{\alpha};R)} \precsim \|f\|_{\infty,\infty;-\boldsymbol{\alpha};R},$$

for all $f \in L_{\infty,\infty;-\boldsymbol{\alpha}}(R)$. The triangle inequality for $\|.\|_{(\infty,\infty;-\boldsymbol{\alpha};R)}$ it follows from the sub-additivity of $f \mapsto f^{**}$, cf. Theorem II.3.4 in [BS88].

Before we give the next result, we refer to [BS88] for the definitions of mutually associate and rearrangement-invariant Banach function spaces. Given a Banach function space X let us denote by X' its associate space.

Lemma 4.3. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$. If (R,μ) is a resonant measure space, then*

$$X = (L_{1,1;\boldsymbol{\alpha}}(R), \|.\|_{1,1;\boldsymbol{\alpha};R})$$

and

$$Y = (L_{\infty,\infty;-\boldsymbol{\alpha}}(R), \|.\|_{(\infty,\infty;-\boldsymbol{\alpha};R)})$$

are rearrangement-invariant Banach function spaces and they are mutually associate (up to equivalence of norms).

Proof. There is no difficulty in verifying that X and Y are Banach function spaces and the rearrangement invariance is obvious, since two equimeasurable functions have the same non-increasing rearrangement.

Now we are going to prove that X and Y are mutually associate. We follow the proof of Theorem IV.6.5 in [BS88] and the proof of Lemma 3.4 in [EGO97].

Suppose $g \in Y$. Then for any $f \in X$ with $\|f\|_X \le 1$, we have by the Hardy-Littlewood inequality, cf. Theorem II.2.2 in [BS88],

$$\int_R |fg|\, d\mu \quad \le \quad \int_0^1 f^*(t)g^*(t)\, dt \le \sup_{0<t<1} \{g^{**}(t)\vartheta^m_{-\alpha}(t)\}\|f\|_X = \|g\|_Y\, \|f\|_X.$$

Hence taking the supremum over all $f \in X$ with $\|f\|_X \le 1$, we get

$$\|g\|_{X'} = \sup\{\int_R |fg|\, d\mu : f \in X, \|f\|_X \le 1\} \le \|g\|_Y. \tag{4.22}$$

To establish an inequality reverse to (4.22), it is sufficient by the Luxemburg representation Theorem (see [BS88, Theorem II.4.10, p. 62]) to do so for the measure space $(\mathbb{R}^+, \mu_1)$ and functions g in $\mathbb{R}^+$ for which $g = g^*$. Suppose g belongs to the associate space X' of X, and also under the previous conditions, then by Hölder's inequality, cf. Corollary II.4.5 in [BS88], for $0<t<1$,

$$tg^{**}(t)=\int_0^1 \chi_{[0,t]}(s)g^*(s)\,ds \le \|\chi_{[0,t]}\|_X\|g\|_{X'}.$$

Since

$$\|\chi_{[0,t]}\|_X=\int_0^1 \chi_{[0,t]}(s)\vartheta_\alpha^m(s)\,ds=\int_0^t \vartheta_\alpha^m(s)\,ds \quad\approx\quad t\,\vartheta_\alpha^m(t),$$

we get

$$\|g\|_Y \precsim \|g\|_{X'}. \tag{4.23}$$

The estimates (4.22) and (4.23) together show that Y is equivalent to the associate of X and hence, by the Lorentz-Luxemburg Theorem, cf. Theorem I.2.7 in [BS88], the spaces X and Y are mutually associate. □

Proposition 4.1. *Suppose (R,μ) is a non-atomic measure space. Let $m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}_-^m$. Then, up to equivalence of norms,*

$$(L^0_{\infty,\infty;\boldsymbol{\alpha}}(R))^*=L_{1,1;-\boldsymbol{\alpha}}(R), \tag{4.24}$$

where $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$ is the completion of $L_\infty(R)$ in $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$.

Proof. We apply Theorem II.5.5 in [BS88, pp. 67-68] to the space $X=L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. It is easy to see that $\lim_{t\to 0^+}\varphi_X(t)=0$, where φ_X is the fundamental function of X, see [BS88, p. 65]. Therefore, by Theorem II.5.5 in [BS88], $(X_b)^*=X'$. But by Lemma 4.3, X' coincides with $L_{1,1;-\boldsymbol{\alpha}}(R)$ up to equivalence of norms and, by Proposition I.3.10 in [BS88, p. 17], X_b coincides with the space $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$. □

Let $j_0,m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}_-^m$. We denote by $c_0^s(L_j(R))$ the subspace of $c_0(L_j(R))$ which consists of all elements $\{F_j\}_{j\ge j_0}$ of $c_0(L_j(R))$ with $F_j=\omega_\alpha^m(j)f$, for all $j\ge j_0$, where $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. In what follows, and according to Corollary 4.3, we consider the space $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$ endowed with the norm

$$\|\cdot\|^d_{\infty,\infty;\boldsymbol{\alpha};R}=\sup_{j\in\mathbb{N},j\ge j_0}\omega_\alpha^m(j)\ \|\cdot\|_{j;R}.$$

Proposition 4.2. *Let $j_0,m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}_-^m$. Then*

$$L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)=\{f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R):\lim_{j\to+\infty}\omega_\alpha^m(j)\ \|f\|_{j;R}=0\}$$

and $(L^0_{\infty,\infty;\boldsymbol{\alpha}}(R),\|.\|^d)$ is isometric to $(c_0^s(L_j(R)),\|.\|_{l_\infty(L_j(R))})$.

Proof. If $f\in L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$, the results follow easily.

Conversely, suppose $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$ with $\lim_{j\to+\infty}\omega_\alpha^m(j)\|f\|_{j;R}=0$. Let $\epsilon>0$. Then there is $j_1\in\mathbb{N}$, with $j_1\ge j_0$, such that for all $j\ge j_1$ we have the inequality

$$\omega_\alpha^m(j)\|f\|_{j;R}<\frac{\epsilon}{2}. \tag{4.25}$$

Since $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, f is finite $\mu-a.e.$ For each $n\in\mathbb{N}$ let us consider the set $R_n=\{x\in R:|f(x)|>n\}$. Now we introduce a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L_\infty(R)$ by $f_n(x)=f(x)$ if $x\in R\backslash R_n$, and $f_n(x)=0$ otherwise. Then, for each $n\in\mathbb{N}$, we have by (4.25)

$$\|f-f_n\|^d_{\infty,\infty;\boldsymbol{\alpha};R}\le\max_{j\in\mathbb{N},j_0\le j\le j_1}\omega_\alpha^m(j)\|f\|_{j;R_n}+\frac{\epsilon}{2}=\omega_\alpha^m(k)\|f\|_{k;R_n}+\frac{\epsilon}{2}. \tag{4.26}$$

Now

$$\|\omega_\alpha^m(k)f\|^k_{k;R_n}=\|\left(\omega_\alpha^m(k)f\right)^k\chi_{R_n}\|_{1;R}.$$

Let us consider, for each $n\in\mathbb{N}$, a function defined $\mu-a.e.$ on R by

$$g_n=\left(\omega_\alpha^m(k)|f|\right)^k\chi_{R_n}.$$

We note that for all $n \in \mathbb{N}$, $|g_n| \le h$, $\mu - a.e$ on R, where $h = (\omega_\alpha^m(k)|f|)^k$, $\mu - a.e$ on R, is a function in $L_1(R)$. Since $\lim_{n\to+\infty} \chi_{R_n} = 0$, $\mu - a.e.$ it follows from the Lebesgue dominated convergence Theorem that $\lim_{n\to+\infty} \|\omega_\alpha^m(k)f\|_{k;R_n}^k = 0$. Hence, there is $n_0 \in \mathbb{N}$ such that

$$\omega_\alpha^m(k)\|f\|_{k;R_n} < \frac{\epsilon}{2}, \quad \text{for each} \quad n \ge n_0. \tag{4.27}$$

Therefore, from (4.26) and (4.27), we get $\lim_{n\to+\infty} \|f - f_n\|_{\infty,\infty;\boldsymbol{\alpha};R}^d = 0$, which shows that $f \in L_{\infty,\infty;\boldsymbol{\alpha}}^0(R)$.

Now we can define a linear mapping H from $L_{\infty,\infty;\boldsymbol{\alpha}}^0(R)$ onto $c_0^s(L_j(R))$ by

$$H(f) = \{\omega_\alpha^m(j)f\}_{j\ge j_0}, \qquad \text{for all} \quad f \in L_{\infty,\infty;\boldsymbol{\alpha}}^0(R).$$

We also have $\|H(f)\|_{c_0^s(L_j(R))} = \|f\|_{\infty,\infty;\boldsymbol{\alpha};R}^d$, for all $f \in L_{\infty,\infty;\boldsymbol{\alpha}}^0(R)$, and the proof is finished. □

The next result gives an equivalent norm for the GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}_+^m$, in terms of decompositions.

Theorem 4.2. *Suppose (R,μ) is a non-atomic measure space. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_+^m$. Let $j_0 \in \mathbb{N}$ with $j_0 \ge 2$. Then $L_{1,1;\boldsymbol{\alpha}}(R)$ is the set of all measurable functions $g : R \to \mathbb{C}$ which can be represented as*

$$g = \sum_{j=j_0}^{+\infty} g_j, \tag{4.28}$$

with g_j a measurable function on R that belongs to $L_{j'}(R)$, for each $j \ge j_0$, such that

$$\sum_{j=j_0}^{+\infty} \omega_\alpha^m(j)\|g_j\|_{j';R} < +\infty. \tag{4.29}$$

The infimum of the expression (4.29) taken over all admissible representations (4.28) is an equivalent norm on $L_{1,1;\boldsymbol{\alpha}}(R)$

Proof. Let $j_0 \in \mathbb{N}$. Let us consider a measurable function $h : R \to \mathbb{C}$ that can be represented as

$$h = \sum_{j=j_0}^{+\infty} g_j, \tag{4.30}$$

with g_j a measurable function on R that belongs to $L_{j'}(R)$, for each $j \ge j_0$, such that

$$\sum_{j=j_0}^{+\infty} \omega_\alpha^m(j)\|g_j\|_{j';R} < +\infty$$

and let us define

$$\Phi_h(f) = \int_R hf\, d\mu, \quad \text{for all} \quad f \in L_{\infty,\infty;-\boldsymbol{\alpha}}^0(R). \tag{4.31}$$

Then $\Phi_h \in (L_{\infty,\infty;-\boldsymbol{\alpha}}^0(R))^*$ and

$$\|\Phi_h|(L_{\infty,\infty;-\boldsymbol{\alpha}}^0(R))^*\| \le \inf \sum_{j=j_0}^{+\infty} \omega_\alpha^m(j)\|g_j\|_{j';R}, \tag{4.32}$$

where the infimum is taken over all admissible representations (4.30). In fact, for all $f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$, we have by Theorem 1.27 in [Rud86, p.22] and by Hölder's inequality, the following

$$|\Phi_h(f)| \leq \sum_{j=j_0}^{+\infty} \|g_j\|_{j';R}\|f\|_{j;R} \leq \|f\|^d_{\infty,\infty;-\boldsymbol{\alpha};R} \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j';R}.$$

Thus, Φ_h is a bounded linear functional on $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$ (the linearity of Φ_h is obvious) such that

$$\|\Phi_h|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \leq \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j';R}$$

and we get (4.32).

Now we follow the reasoning in the proof of Theorem 2.6.2/2 in [ET96, pp. 72-74]. Let $G \in (L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*$. Since $L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ is isometric to $c^s_0(L_j(R))$, cf. Proposition 4.2, $G \circ H^{-1} \in (c^s_0(L_j(R)))^*$, where H is the isometry considered in the referred proposition. By Hahn-Banach theorem, there exists a bounded linear functional $\widetilde{G \circ H^{-1}}$ on $c_0(L_j(R))$, which is an extension of $G \circ H^{-1}$ to $c_0(L_j(R))$ and has the same norm

$$\|\widetilde{G \circ H^{-1}}|(c_0(L_j(R)))^*\| = \|G \circ H^{-1}|(c^s_0(L_j(R)))^*\|.$$

But by (2.9), $\widetilde{G \circ H^{-1}}$ can be identified with an element $\{\tilde{G}_j\}_{j \geq j_0} \in l_1((L_j(R))^*)$ such that

$$\|G \circ H^{-1}|(c^s_0(L_j(R)))^*\| = \|\widetilde{G \circ H^{-1}}|(c_0(L_j(R)))^*\| = \sum_{j=j_0}^{+\infty} \|\tilde{G}_j|(L_j(R))^*\|. \tag{4.33}$$

Since each $\tilde{G}_j$ can be identified with a $\tilde{g}_j \in L_{j'}(R)$ by

$$\tilde{G}_j(f) = \int_R \tilde{g}_j\, f\, d\mu, \quad \text{for all } \ f \in L_j(R),$$

with $\|\tilde{G}_j|(L_j(R))^*\| = \|\tilde{g}_j\|_{j',R}$, it follows from (4.33) that

$$\|G|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| = \|G \circ H^{-1}|(c^s_0(L_j(R)))^*\| = \sum_{j=j_0}^{+\infty} \|\tilde{g}_j\|_{j',R}. \tag{4.34}$$

Using Theorem 1.38 in [Rud86, p. 29] we get

$$G(f) = \sum_{j=j_0}^{+\infty} \tilde{G}_j(\omega^m_{-\alpha}(j)f) = \int_R h\, f\, d\mu, \quad \text{for all } \ f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$h = \sum_{j=j_0}^{+\infty} g_j \quad \text{and} \quad g_j = \tilde{g}_j\, \omega^m_{-\alpha}(j), \ j \geq j_0,$$

because, for each $f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$,

$$\sum_{j=j_0}^{+\infty} \int_R |f\, \omega^m_{-\alpha}(j)\tilde{g}_j|\, d\mu \leq \|f\|^d_{\infty,\infty;-\boldsymbol{\alpha};R} \sum_{j=j_0}^{+\infty} \|\tilde{g}_j\|_{j';R} < +\infty.$$

From (4.34), we get

$$\|G|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \geq \inf \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j',R}, \tag{4.35}$$

where the infimum is taken over all admissible representations of h that satisfy (4.29). But since $G = \Phi_h$, we have from (4.32) and (4.35) that

$$\|G|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \quad = \quad \inf \sum_{j=j_0}^{+\infty} \omega_\alpha^m(j)\|g_j\|_{j',R},$$

where the infimum is taken over all admissible representations of h that satisfy (4.29).

Now given a function h represented as (4.28) and satisfying (4.29), we infer by (4.24) that there is a $g \in L_{1,1;\boldsymbol{\alpha}}(R)$ such that

$$\Phi_h(f) = \int_R f\, g\, d\mu, \quad \text{for all} \;\; f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx \|\Phi_h|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\|.$$

Then it follows, by Theorem 1.39 in [Rud86, p. 30], that $g = h\ \mu - a.e.$, because it is easy to see that $g, h \in L_1(R)$.

Conversely, let $g \in L_{1,1;\boldsymbol{\alpha}}(R)$. By (4.24), g defines a linear functional Λ_g on $L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ such that

$$\Lambda_g(f) = \int_R f\, g\, d\mu, \quad \text{for all} \;\; f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx \|\Lambda_g|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\|.$$

Since there is a function h that can be represented as (4.28) and satisfying (4.29) for which $\Lambda_g = \Phi_h$, it follows as above that $g = h\ \mu - a.e.$ □

5. Applications

As was referred in the Introduction, there is a version of the extrapolation result in [Zyg59, Theorem XII.4.11 (i), p. 119] for sublinear operators. Therefore we start this section by defining sublinear operator and by recalling that extrapolation result; see [Tor86, Theorem V.3.3, p.124] or [FK98, Theorem 4.1] for instance.

Definition 5.1. *Let (R_0, μ_0) and (R_1, μ_1) be measure spaces. Let T be an operator whose domain is some linear subspace of $\mathcal{M}_0(R_0, \mu_0)$ and whose range is contained in $\mathcal{M}(R_1, \mu_1)$. Then T is said to be sublinear if the relations*

$$|T(f+g)| \le |Tf| + |Tg| \quad \textit{and} \quad |T(\lambda f)| = |\lambda|\,|Tf|$$

hold $\mu_1 - a.e$ on R_1 for all f and g in the domain of T and for all scalars λ.

Theorem 5.1. *Suppose Ω is a measurable subset of $\mathbb{R}^n$ with finite volume. Let $\alpha > 0$ and $q_0 \in [1, +\infty)$. If A is a bounded sublinear operator in $L_q(\Omega)$, $q_0 \le q < +\infty$, such that*

$$\|Af\|_q \le c\, q^{1/\alpha}\|f\|_q, \quad q \ge q_0 \ge 1,$$

then

$$\|Af\|_{E_\alpha(\Omega)} \le c\|f\|_\infty, \quad \textit{for all} \;\; f \in L_\infty(\Omega).$$

Now, by the results of Section 4, the following Theorem is an obvious generalisation of the previous one.

Theorem 5.2. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Suppose (R_0, μ_0) and (R_1, μ_1) are finite measure spaces.*

(i) *Suppose A is a bounded sublinear operator from $L_q(R_0)$ into $L_q(R_1)$ such that either*
$$\|Af\|_{q;R_1} \leq c\,\omega^m_{-\alpha}(q)\|f\|_{q;R_0}, \quad \textit{for all} \;\; f \in L_q(R_0),$$
for each $q \in [q_0, +\infty)$ with $q_0 \geq 1$, or
$$\|Af\|_{q;R_1} \leq c\,\gamma^m_{-\alpha}(q)\|f\|_{q;R_0}, \quad \textit{for all} \;\; f \in L_q(R_0),$$
for each $q \in [q_0, +\infty)$ with $q_0 > exp_{m-2}$ and $m \geq 2$. Then
$$A : L_\infty(R_0) \longrightarrow L_{\infty,\infty;\boldsymbol{\alpha}}(R_1),$$
and
$$\|Af\|_{\infty,\infty;\boldsymbol{\alpha};R_1} \leq c\|f\|_{\infty;R_0}, \quad \textit{for all} \;\; f \in L_\infty(R_0)$$

(ii) *Suppose A is a bounded sublinear operator from $L_q(R_0)$ into $L_q(R_1)$ such that either*
$$\|Af\|_{q;R_1} \leq c\,\omega^m_{\alpha}(q)\|f\|_{q;R_0}, \quad \textit{for all} \;\; f \in L_q(R_0),$$
for each $q \in [q_0, +\infty)$ with $q_0 \geq 1$, or
$$\|Af\|_{q;R_1} \leq c\,\gamma^m_{\alpha}(q)\|f\|_{q;R_0}, \quad \textit{for all} \;\; f \in L_q(R_0),$$
for each $q \in [q_0, +\infty)$ with $q_0 > exp_{m-2}$ and $m \geq 2$. Then
$$A : L_{\infty,\infty;\boldsymbol{\alpha}}(R_0) \longrightarrow L_\infty(R_1),$$
and
$$\|Af\|_{\infty;R_1} \leq c\|f\|_{\infty,\infty;\boldsymbol{\alpha};R_0}, \quad \textit{for all} \;\; f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R_0).$$

Proof. The proof is a consequence of Corollaries 4.3, 4.5 and [KJF77, Theorem 2.11.4, p. 84]. □

If we take $m = 1$, $\boldsymbol{\alpha} = -1/\alpha$, with $\alpha > 0$ in part (i) of the previous Theorem, we recover Theorem 5.1.

Now we present an extrapolation result involving the GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_+$.

Theorem 5.3. *Let (R_0, μ_0) and (R_1, μ_1) be non-atomic finite measure spaces. Let $m \in \mathbb{N}$, $j_0 \geq 2$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{R}^m_+$. Suppose A is an operator whose domain is $\mathcal{M}_0(R_0, \mu_0)$ and whose range is contained in $\mathcal{M}(R_1, \mu_1)$ such that:*

(i) *for every possible representation of $f \in \mathcal{M}_0(R_0, \mu_0)$ by $f = \sum_{j=j_0}^{+\infty} f_j$ (convergent μ_0-a.e. on R_0), with $\{f_j\}_j \subset \mathcal{M}_0(R_0, \mu_0)$, we have $\sum_{j=j_0}^{+\infty} Af_j$ convergent $\mu_1 - a.e.$ on R_1 and the inequality*

$$|Af| \leq |\sum_{j=j_0}^{+\infty} Af_j| \quad \mu_1 - a.e. \text{ on } R_1; \tag{5.1}$$

(ii) *for all $p \in (1, +\infty)$ and all $f \in L_p(R_0)$,*

$$\|Af\|_{p;R_1} \leq c\,\omega^m_{\beta}(\frac{1}{p-1})\,\|f\|_{p;R_0}, \tag{5.2}$$

where c is independent of f, p and $\boldsymbol{\beta}$.

Then

$$\|Af\|_{1,1;\boldsymbol{\alpha};R_1} \leq c'\|f\|_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta};R_0}, \tag{5.3}$$

for all $f \in L_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta}}(R_0)$, for some constant c' independent of f, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Proof. Let $j_0 \geq 2$. Fix $f \in L_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta}}(R_0)$ and $f = \sum_{j=j_0}^{+\infty} f_j$, with

$$\sum_{j=j_0}^{+\infty} \omega_{\alpha+\beta}^m(j) \|f_j\|_{j',R_0} < +\infty. \tag{5.4}$$

We remark that $\sum_{j=j_0}^{+\infty} Af_j$ converges $\mu_1 - a.e.$ on R_1, because by Hölder's inequality and (5.2) we get

$$\sum_{j=j_0}^{+\infty} \int_{R_1} |Af_j|\, d\mu_1 \quad \leq c \sum_{j=j_0}^{+\infty} \omega_{\alpha+\beta}^m(j) \|f_j\|_{j',R_0},$$

and the rest it follows from (5.4) and from Theorem 1.38 in [Rud86, p. 29].

Now, by (5.1), (5.2) and Theorem 4.2,

$$\begin{aligned}
\|Af\|_{1,1;\boldsymbol{\alpha};R_1} &\leq \|\sum_{j=j_0}^{+\infty} Af_j\|_{1,1;\boldsymbol{\alpha};R_1} \leq c_1 \sum_{j=j_0}^{+\infty} \omega_{\alpha}^m(j) \|Af_j\|_{j',R_1} \\
&\leq c_2 \sum_{j=j_0}^{+\infty} \omega_{\alpha}^m(j)\ \omega_{\beta}^m(\frac{1}{j'-1}) \|f_j\|_{j',R_0} \\
&\leq c_2 \sum_{j=j_0}^{+\infty} \omega_{\alpha+\beta}^m(j) \|f_j\|_{j',R_0}. \qquad (5.5)
\end{aligned}$$

Taking the infimum over all the decompositions of f we get (5.3). □

Remark 5.1. *In the Theorem above we only need the condition (5.2) be satisfied for all* p *such that* $1 < p \leq p_0$, *for some* $p_0 \in (1, +\infty)$, *because in that case we can consider* j_0 *large enough. We could also replace (5.2) by the condition*

$$\|Af\|_{p;R_1} \leq c\, \omega_{\beta}^m(\frac{p}{p-1}) \|f\|_{p;R_0},$$

for all $p \in (1, +\infty)$ *(or* $p \in (1, p_0]$*) and for all* $f \in L_p(R_0)$, *where* c *is independent of* f, p *and* $\boldsymbol{\beta}$.

Since the Hardy-Littlewood maximal operator satisfies part (i) of the previous Theorem trivially and condition (5.2) with $m = 1$ and $\boldsymbol{\beta} = 1$, we recover the result already known for the maximal operator, *i.e.*

$$M : L^1(\log L)^{a+1}(\Omega) \longrightarrow L^1(\log L)^a(\Omega),$$

and

$$\|Mf|L^1(\log L)^a(\Omega)\| \leq c_2 \|f|L^1(\log L)^{a+1}(\Omega)\|,$$

for all $f \in L^1(\log L)^{a+1}(\Omega)$, where $a > 0$; see the literature mentioned in the Introduction.

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Júlio S. Neves, Departamento de Matemática da FCT, Universidade de Coimbra, Apartado 3008, 3000 Coimbra, Portugal

Current address: Júlio S. Neves, P/G Pigeonholes, School of Mathematical Sciences, University of Sussex, Falmer, Brighton, BN1 9QH, UK

E-mail address: `jsn@mat.uc.pt or J.S.Neves@sussex.ac.uk`