On computing real logarithms for matrices in the Lie group of special Euclidean motions in $\mathbb{R}^n$

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Abstract

We show that the diagonal Padé approximants methods, both for computing the principal logarithm of matrices belonging to the Lie group $SE(n, \mathbb{R})$ of special Euclidean motions in $\mathbb{R}^n$ and to compute the matrix exponential of elements in the corresponding Lie algebra $se(n, \mathbb{R})$, are structure preserving. Also, for the particular cases when $n = 2, 3$ we present an alternative closed form to compute the principal logarithm. These low dimensional Lie groups play an important role in the kinematic motion of many mechanical systems and, for that reason, the results presented here have immediate applications in robotics.

Key-words: Lie group of Euclidean motions in $\mathbb{R}^n$, matrix logarithms, matrix exponentials, Padé approximants method.

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1 Introduction

It is well known that under some spectral conditions any invertible real matrix has a real logarithm (Culver, [3]). Lately, there has been an increasing interest in developing computational techniques for real logarithms of real matrices, the most significant work in this area being Kenney and Laub [13], [14] and [15] and Dieci, Morini and Papini [7]. An important focus of recent work is on developing approximating methods for matrix Lie groups that are structure preserving, in the sense that they produce a matrix in the corresponding Lie algebra. For instance, Dieci [6], showed that some of the methods already developed, when applied to an orthogonal matrix, respectively symplectic matrix, always produce a real logarithm that is skewsymmetric, respectively hamiltonian. Cardoso and Silva Leite [4] extended the results of Dieci to a much vaster class of Lie groups. This article continues in the same direction and the main objective is to show that the Padé approximants method is structure preserving for the Lie group of special Euclidean motions in $\mathbb{R}^n$. We also present closed forms for computing the real logarithm for the special cases when $n = 2, 3$. Our motivation comes from the importance that these Lie groups play in applications to Engineering.

We now introduce some notation that will be used throughout the whole paper.

Let $gl(n, \mathbb{R})$ denote the real vector space consisting of all $n \times n$ matrices with real entries. $gl(n, \mathbb{R})$ equipped with the commutator operation $(A, B) \to AB - BA$, forms a Lie algebra, which is the Lie algebra of the Lie group $GL(n, \mathbb{R})$ consisting of all invertible matrices in $gl(n, \mathbb{R})$. The $n \times n$ identity matrix will be denoted by I. Details about the theory of matrix Lie groups and corresponding Lie algebras may be found in Sattinger and Weaver [19]. Now, the *rotation group in* $\mathbb{R}^n$ may be defined as

$$SO(n, \mathbb{R}) = \left\{ X \in GL(n, \mathbb{R}) : X^T X = I \ \wedge \ det(X) = 1 \right\}$$

and the *special group of Euclidean motions in* $\mathbb{R}^n$ by

$$SE(n, \mathbb{R}) = \left\{ \begin{bmatrix} R & 0 \\ v & 1 \end{bmatrix} : \ R \in SO(n, \mathbb{R}) \wedge v \in \mathbb{R}^{1 \times n} \right\}.$$

Both, $SO(n, \mathbb{R})$ and $SE(n, \mathbb{R})$ are Lie groups with corresponding Lie algebras defined respectively by

$$so(n, \mathbb{R}) = \left\{ A \in gl(n, \mathbb{R}) : A^T = -A \right\}$$

and

$$se(n, \mathbb{R}) = \left\{ \begin{bmatrix} S & 0 \\ u & 0 \end{bmatrix} : \ S \in so(n, \mathbb{R}) \wedge u \in \mathbb{R}^{1 \times n} \right\}.$$

One important issue in the control of mechanical systems it that of path planning trajectories. Park and Ravani [17] and Crouch, Kun and Silva Leite [5] generalized the classical De Casteljau algorithm for constructing Bézier curves on Lie groups. For the classical algorithm see, for instance, Farin [9]. It turns out that the implementation of the De Casteljau algorithm on Lie groups depends on sucessive computations of matrix logarithms and exponentials. Since displacements of a rigid body form a Lie group, these interpolation techniques may be an efficient way to path planning. In this context the Lie group of special Euclidean motions in $\mathbb{R}^n$, when $n = 2, 3$, plays an important role. For instance, the motion (at every instant of time) of a unicycle which rolls without slipping on a plane, is described by a matrix

$$X(t) = \begin{bmatrix} A(t) & 0 \\ x(t) & 1 \end{bmatrix} \in SE(2, \mathbb{R}),$$

where $x(t) \in \mathbb{R}^{1\times 2}$ describes the position at time t of the center of mass of the unicycle, with respect to an orthonormal fixed frame in the plane, and $A(t) \in SO(2, \mathbb{R})$ describes its orientation at time t, with respect to the same inertial frame.

Similarly, the kinematic motion of an autonomous underwater vehicle is described by a matrix

$$X(t) = \begin{bmatrix} A(t) & 0 \\ x(t) & 1 \end{bmatrix} \in SE(3, \mathbb{R}),$$

where $x(t) \in \mathbb{R}^{1\times 3}$ describes the position, at time t, of the center of mass of the vehicle in 3-space and $A(t) \in SO(3, \mathbb{R})$ describes its orientation at time t, with respect to an inertial frame.

The organization of the paper is as follows. We start with some basics about logarithms of matrices and show that the principal logarithm of elements in $SE(n, \mathbb{R})$ belong to the corresponding Lie algebra. Section 3 is devoted to show that the Padé approximants methods for computing the principal logarithm is structure preserving for the special Euclidean group. We also prove a dual result for the exponential of elements in the Lie algebra $se(n, \mathbb{R})$. An algorithm to compute the principal logarithm of a matrix in $SE(n, \mathbb{R})$ is included. Finally we present closed forms for the principal logarithm of elements in $SE(2, \mathbb{R})$ and $SE(3, \mathbb{R})$.

2 The principal logarithm in $SE(n, \mathbb{R})$

Consider the matrix equation $e^X = T$, where T is a given matrix belonging to the general linear Lie group $GL(n, \mathbb{R})$. All solutions X of this equation, not necessarily real, are called *logarithms* of T. It turns out however, (see, for instance, Culver [3] or Horn and Johnson [11]), that if the spectrum of T, denoted by $\sigma(T)$, does not intersect $\mathbb{R}^-$, then T has a unique real logarithm whose spectrum lies in the strip $\{z \in \mathbb{C} : -\pi < Im(z) < \pi\}$. This logarithm is called the *principal logarithm* of T and will be denoted by $Log(T)$. It also happens that if $\|I - T\| < 1$, for any matrix norm $\|.\|$, then the power series $\sum_{k=1}^{\infty} \frac{(I-T)^k}{k}$ converges to the principal logarithm of T. So, it makes sense to write

$$Log(T) = \sum_{k=1}^{\infty} \frac{(I - T)^k}{k}, \qquad \|I - T\| < 1. \tag{1}$$

We also recall a result about matrix square roots, namely that if a real matrix T satisfies $\sigma(T) \cap \mathbb{R}_0^- = \emptyset$, then there exists a unique real square root of T having eigenvalues with positive real part (see, for instance, De Prima and Johnson [8]). This square root of T will be the only one used along the paper and will be denoted by $T^{\frac{1}{2}}$, without any further reference. In the particular situation when

$$T = \begin{bmatrix} R & 0 \\ v & 1 \end{bmatrix} \in SE(n, \mathbb{R}),$$

then

$$T^{\frac{1}{2}} = \begin{bmatrix} R^{\frac{1}{2}} & 0 \\ v(R^{\frac{1}{2}} + I)^{-1} & 1 \end{bmatrix}$$

and using the fact that $R^{\frac{1}{2}} \in SO(n, \mathbb{R})$ whenever $R \in SO(n, \mathbb{R})$, we conclude that $T^{\frac{1}{2}} \in SE(n, \mathbb{R})$.

The next result stresses the importance of the principal logarithm of a matrix belonging to the Lie group $SE(n, \mathbb{R})$.

Theorem 2.1 *If $T \in SE(n, \mathbb{R})$ and $\sigma(T) \cap \mathbb{R}^- = \emptyset$ then $Log(T) \in se(n, \mathbb{R})$.*

Proof - We first note that, as proved in Dieci [6], the analogue of this theorem, for the case when the Lie group $SE(n, \mathbb{R})$ is replaced by the rotation group $SO(n, \mathbb{R})$, is also true. (For a generalization to other Lie groups see also Cardoso and Silva Leite [4]). If $\|I - T\| < 1$ the theorem follows by applying this result after having used the series expansion of $Log(T)$. On the other hand, if $\|I - T\| \geq 1$, there exists a positive integer k such that $\|I - T^{\frac{1}{2^k}}\| < 1$, where $T^{\frac{1}{2^k}}$ is obtained from T after k sucessive square roots. So, $Log(T^{\frac{1}{2^k}}) \in se(n, \mathbb{R})$. Now, since $Log(T) = 2^k Log(T^{\frac{1}{2^k}})$, (see Kenney and Laub [14]), it follows that $Log(T) \in se(n, \mathbb{R})$.

$\square$

3 Padé approximants method

We refer to Baker [1] and Baker and Graves-Morris [2] for more details concerning to general theory of Padé approximants. Here we recall how to obtain the diagonal Padé approximants of a scalar function.

Assume that $f(x) = \sum_{i=1}^{\infty} c_i x^i$ is the MacLaurin series of f and that $R_{mm}(x) = \frac{P_m(x)}{Q_m(x)}$, with $Q_m(0) = 1$, is the (m, m) diagonal Padé approximant of f. Then, one may write

$$R_{mm}(x) = \frac{P_m(x)}{Q_m(x)} = \frac{a_0 + a_1 x + \cdots + a_m x^m}{1 + b_1 x + \cdots + b_m x^m}, \tag{2}$$

where the constants b_i, $i = 1, \cdots, m$, are uniquely determined by solving the system of linear algebraic equations $A_1 X = B_1$, with

$$A_1 = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_m \\ c_2 & c_3 & c_4 & \cdots & c_{m+1} \\ c_3 & c_4 & c_5 & \cdots & c_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_m & c_{m+1} & c_{m+2} & \cdots & c_{2m-1} \end{bmatrix}, \; X = \begin{bmatrix} b_m \\ b_{m-1} \\ b_{m-2} \\ \vdots \\ b_1 \end{bmatrix}, \; B_1 = -\begin{bmatrix} c_{m+1} \\ c_{m+2} \\ c_{m+3} \\ \vdots \\ c_{2m} \end{bmatrix}$$

and the a_i's, $i = 1, \cdots, m$, are then obtained through the following formulas:

$$\begin{array}{rcl} a_0 & = & c_0, \\ a_i & = & c_i + \sum_{k=1}^{i} b_k c_{i-k}, \quad i = 1, \cdots, m. \end{array} \tag{3}$$

The following result will be essencial to prove the main result in this section.

Lemma 3.1 *If f is an odd function, then the coefficients of the diagonal Padé approximants of f satisfy: $a_{2i} = b_{2i+1} = 0, \ \forall i.$*

Proof - If f is odd, then the even coefficients c_{2i} in its MacLaurin series vanish. In this situation, a simple manipulation with properties of determinants will show that any matrix obtained from A_1 above, by replacing each of its even columns by column B_1, will have zero determinant. It then follows from applying Cramer's rule to the system $A_1 X = B_1$ that $b_{2i+1} = 0, \forall i$. Replacing this in (3) we also get $a_{2i} = 0, \ \forall i$.

□

It is well known that square Padé approximants $R_{mm}(A)$ of the matrix function $f(A) = Log(I - A)$ may be used to approximate the principal logarithm of any matrix $T = I - A$ with $\|I - T\| < 1$. It turns out that some important simplifications take place if instead of using the (m, m) diagonal Padé approximant of $f(A) = Log(I - A)$ one uses the (m, m) diagonal Padé approximant of $g(B) = Log[(I + B)(I - B)^{-1}]$, where $B = A(A - 2I)^{-1}$. We denote this approximant by $S_{mm}(B)$. These two Padé approximants are related through the identity

$$R_{mm}(A) = S_{mm}[A(A - 2I)^{-1}]. \tag{4}$$

The following result shows the advantage of working with $S_{mm}[A(A - 2I)^{-1}]$ instead of $R_{mm}(A)$ as an approximation for $Log(I - T)$.

Lemma 3.2 *The Padé approximant $S_{mm}(X)$ of the matrix function $g(X) = Log[(I + X)(I - X)^{-1}]$ is of the form $g(X) = \alpha(X)[\beta(X)]^{-1}$, where α is a polynomial function of the odd powers of X and β is a polynomial function of the even powers of X, both of degree $\leq m$.*

Proof - The result is an immediate consequence of the lemma 3.1, applied to the odd function $g(x) = Log\frac{1+x}{1-x}$.

□

We are now ready to present the main result.

Theorem 3.3 *If $T \in SE(n, \mathbb{R})$ and $\|I - T\| < 1$, then $R_{mm}(I - T) \in se(n, \mathbb{R})$.*

Proof - Let $A = I - T$ and $B = A(A - 2I)^{-1}$. Due to the relation (4), we just have to prove that $S_{mm}(B)$ belongs to $se(n, \mathbb{R})$. We first show that if $T \in SE(n, \mathbb{R})$, then $B \in se(n, \mathbb{R})$. We then proceed to show that $se(n, \mathbb{R})$ is invariant under S_{mm}, which will conclude the proof.

A simple calculation shows that if

$$T = \begin{bmatrix} R & 0 \\ v & 1 \end{bmatrix},$$

where $R \in SO(n, \mathbb{R})$ and $v \in \mathbb{R}^{1\times n}$, then

$$B = \begin{bmatrix} -(I-R)(R+I)^{-1} & 0 \\ v(R+I)^{-1} & 0 \end{bmatrix}.$$

(Note that, since we are assuming that $\|I-T\| < 1$, this implies $\rho(I-T) < 1$, where $\rho(X)$ denotes the spectrum radius of X, and so $R+I$ is always invertible). It turns out however that, since $R^T R = I$, $(I-R)(R+I)^{-1} \in so(n, \mathbb{R})$ and, as a consequence, $B \in se(n, \mathbb{R})$. Now, let us prove that if $B \in se(n, \mathbb{R})$ so does $S_{mm}(B)$. Assume that

$$B = \begin{bmatrix} S & 0 \\ u & 0 \end{bmatrix},$$

for some $n \times n$ skewsymmetric matrix S and some $u \in \mathbb{R}^{1\times n}$. Now, applying the last lemma together with the fact that all odd powers of a skewsymmetric matrix are still skewsymmetric, while its even powers are symmetric, it is an easy exercise to check that $\alpha(B) \in se(n, \mathbb{R})$, say

$$\alpha(B) = \begin{bmatrix} C & 0 \\ x & 0 \end{bmatrix}, \tag{5}$$

with $C^T = -C$ and $x \in \mathbb{R}^{1\times n}$, and that

$$\beta(B) = \begin{bmatrix} V & 0 \\ w & c \end{bmatrix},$$

where V is symmetric and invertible, $w \in \mathbb{R}^{1\times n}$ and c is a nonzero real number. Since

$$[\beta(B)]^{-1} = \begin{bmatrix} V^{-1} & 0 \\ -\frac{1}{c}wV^{-1} & \frac{1}{c} \end{bmatrix}, \tag{6}$$

it follows from (5) and (6) that

$$S_{mm}(B) = \alpha(B)[\beta(B)]^{-1} = \begin{bmatrix} CV^{-1} & 0 \\ xV^{-1} & 0 \end{bmatrix}. \tag{7}$$

Now, since C and V are polynomials in S, we have $CV = VC$, $CV^{-1} = V^{-1}C$ and also, using the skewsymmetry of C and the symmetry of V^{-1}, it follows that $CV^{-1} \in so(n, \mathbb{R})$ and therefore $S_{mm}(B) \in se(n, \mathbb{R})$, as required.

□

Using this theorem and the explanation during its proof, together with the theorem in the last section and the fact that $R_{mm}(I-T) \approx Log(T)$, it is clear that when T satisfies the norm condition $\|I-T\| < 1$, an efficient way to find the principal logarithm of $T \in SE(n, \mathbb{R})$, is by computing $S_{mm}(B)$, where $B = (I-T)(-I-T)^{-1}$. In the case when $\|I-T\| \geq 1$, and $\sigma(T) \cap \mathbb{R}_0^- = \emptyset$, one combines, in the usual way, the previous method with the so called *inverse squaring and scaling technique*. For that find an nonnegative integer k such that the matrix $T^{\frac{1}{2^k}}$ satisfies the norm assumptions of the last theorem, then apply the Padé approximants method to obtain $Log(T^{\frac{1}{2^k}})$ and finally recover $Log(T)$ using the identity $Log(T) = 2^k Log(T^{\frac{1}{2^k}})$.

Note that when we apply the inverse squaring and scaling technique to matrices in $SE(n, \mathbb{R})$ no structure is lost. In fact, as we have already pointed out at the end of the last section, $T^{\frac{1}{2}}$ is always in $SE(n, \mathbb{R})$ whenever T is. So, theorem 3.3 guarantees that the approximating value of $Log(T^{\frac{1}{2}})$ belongs to $se(n, \mathbb{R})$ and, since this is a vector space, rescaling is not going to change the structure of the final result.

We now summarize the main steps for computing the principal logarithm of $T \in SE(n, \mathbb{R})$. Although the accuracy of the Padé approximation increases with m, it has been shown by Kenney and Laub [13], that $R_{88}(I - T)$ is already within 10^{-18} of $Log(T)$, whenever $\|I - T\| < 0.25$. So, having in mind implementations of this algorithm, we work bellow with the $(8, 8)$ diagonal Padé approximant.

According to the discussion presented before the statement of the lemma 3.2, we use the Padé approximant S_{88} instead of R_{88} in the next algorithm. Using the Derive program to compute S_{88}, one obtains

$$S_{88}(A) = \alpha(A)[\beta(A)]^{-1},$$

where

$$\begin{aligned} \alpha(A) &= 2(225225A - 345345A^3 + 147455A^5 - 15159A^7) \\ \beta(A) &= 35(6435I - 12012A^2 + 6930A^4 - 1260A^6 + 35A^8). \end{aligned} \tag{8}$$

Algorithm

Suppose that $T \in SE(n, \mathbb{R})$ and $\sigma(T) \cap \mathbb{R}^- = \emptyset$.

1. Compute k successive square roots of T until $\|I - T^{\frac{1}{2^k}}\| < 0.25$;
2. Take $A := I - T^{\frac{1}{2^k}}$ and $B := A(A - 2I)^{-1}$;
3. Compute $S_{88}(B) = \alpha(B)[\beta(B)]^{-1}$ where α and β are given by (8).
4. Approximate $Log(T)$ using the following relation

$$Log(T) \approx 2^k S_{88}(B) \in se(n, \mathbb{R}).$$

For the sake of completeness we include here a result which is the dual of theorem 3.3, in the sense that it provides a stable method to compute the exponential of a matrix in the Lie algebra $se(n, \mathbb{R})$.

One the most effective ways to compute the exponential of a matrix (see, for instance, Moler and Van Loan [16]), is to use the method of Padé approximants together with scaling and squaring techniques. This consists in approximating the exponential, e^A, of a matrix A, by

$$e^A \approx [R_{mm}(\frac{A}{2^j})]^{2^j}, \tag{9}$$

where $R_{mm}(X)$ is the (m,m) diagonal Padé approximant of e^X and j is a nonnegative integer such that $\|\frac{A}{2^j}\| < 1$. It happens that the exponential of a matrix in $se(n,\mathbb{R})$ belongs to the corresponding Lie group $SE(n,\mathbb{R})$ and so it is important to be able to guarantee that the procedure just described, to approximate the exponential, always produces a matrix in that Lie group, no matter what order of the Padé approximant is taken. The next theorem ensures that is always the case, and the proof is based on a similar result for the orthogonal group.

Theorem 3.4 *If $A \in se(n,\mathbb{R})$, then $R_{mm}(A) \in SE(n,\mathbb{R})$.*

Proof - The (m,m) diagonal Padé approximant of e^A is given by

$$R_{mm}(A) = Q_m(A)[Q_m(-A)]^{-1},$$

where $Q_m(A)$ is a polynomial of degree m in A, the coefficient of A^k being

$$c_k = \frac{(2m-k)!m!}{(2m)!k!(m-k)!}, \; k = 0, \cdots m.$$

Now, since

$$A = \begin{bmatrix} S & 0 \\ u & 0 \end{bmatrix},$$

for some $S \in so(n,\mathbb{R})$ and $u \in \mathbb{R}^{1\times n}$, we obtain after some algebraic computations

$$\begin{aligned} R_{mm}(A) &= \begin{bmatrix} Q_m(S) & 0 \\ w_1 & 1 \end{bmatrix} \begin{bmatrix} Q_m(-S) & 0 \\ w_2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} Q_m(S) & 0 \\ w_1 & 1 \end{bmatrix} \begin{bmatrix} [Q_m(-S)]^{-1} & 0 \\ -w_2[Q_m(-S)]^{-1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{mm}(S) & 0 \\ w & 1 \end{bmatrix}, \end{aligned}$$

for some $w_1, w_2 \in \mathbb{R}^{1\times n}$ and $w = (w_1 + w_2)[Q_m(-S)]^{-1} \in \mathbb{R}^{1\times n}$. Now use the fact that S is skew-symmetric and a result in Cardoso and Silva Leite [4], to conclude that $R_{mm}(S)$ is orthogonal. Therefore $R_{mm}(A) \in SE(n,\mathbb{R})$.

□

Since $se(n,\mathbb{R})$ is a Lie algebra, and so closed under scalar multiplication, and $SE(n,\mathbb{R})$ is closed under matrix multiplication, it follows from the previous theorem that (9) gives an approximation to e^A which belongs to the right Lie group, $SE(n,\mathbb{R})$.

4 Closed forms for logarithms in $SE(2,\mathbb{R})$ and $SE(3,\mathbb{R})$

In this section, we derive formulas that provide an alternative way to compute the principal matrix logarithm for the most important Euclidean groups in applications, $SE(2,\mathbb{R})$ and $SE(3,\mathbb{R})$.

If $T = \begin{bmatrix} R & 0 \\ v & 1 \end{bmatrix} \in SE(2,\mathbb{R})$ and $\sigma(T) \cap \mathbb{R}^- = \emptyset$, then $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\in SO(2,\mathbb{R})$, for some $\theta \in]-\pi,\pi[$. In this case $Log(R) = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$ and if we assume further that $\theta \neq 0$, then $I - R$ is invertible and it results from applying the series expansion (1) that

$$
\begin{aligned}
Log(T) &= \begin{bmatrix} Log(R) & 0 \\ v(I-R)^{-1}Log(R) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ v_1 \dfrac{\theta\sin\theta}{2(1-\cos\theta)} - v_2 & v_1 + v_2\dfrac{\theta\sin\theta}{2(1-\cos\theta)} & 0 \end{bmatrix},
\end{aligned} \tag{10}
$$

with $v = (v_1, v_2)$.

Although, as stated in section 2, the convergence of the series is only guaranteed when $\|I - T\| < 1$, it turns out however that the eigenvalues of the matrix in (10) satisfy $-\pi < Im(z) < \pi$ and its exponential is equal to T. So, we have the guarantee that the formula (10) always gives the principal logarithm of $T \in SE(n,\mathbb{R})$.

Now, suppose that

$$T = \begin{bmatrix} R & 0 \\ v & 1 \end{bmatrix} \quad \in SE(3,\mathbb{R}),$$

so that $R \in SO(3,\mathbb{R})$ and $v \in \mathbb{R}^{1\times 3}$. It happens that 1 is always an eigenvalue of $R \in SO(3,\mathbb{R})$ and, consequently, the matrix $(I-R)$ is never invertible. So the technique used for the case $n = 2$ can not be applied. However, assuming the convergence of the series defining the logarithm we may write

$$Log(T) = Log \begin{bmatrix} R & 0 \\ v & 1 \end{bmatrix} = \begin{bmatrix} Log(R) & 0 \\ vV & 0 \end{bmatrix},$$

where $V := -\sum_{k=1}^{\infty} \dfrac{(I-R)^{k-1}}{k}$.

In order to deal with this series we use the Schur decomposition of R to reduce to the situation of the previous example. Since the eigenvalues of R are $\{1, e^{\pm i\theta}\}$,

$$R = Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} Q^T,$$

is the real Schur decomposition of R, where Q is an orthogonal matrix and $\theta = \arccos(\frac{trace(R)-1}{2})$ is assumed nonzero. So,

$$Log(R) = Q \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta \\ 0 & \theta & 0 \end{bmatrix} Q^T \tag{11}$$

and

$$I - R = Q \begin{bmatrix} 0 & 0 \\ 0 & I - R(\theta) \end{bmatrix} Q^T,$$

where $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Since $I - R(\theta)$ is invertible, we may now apply the techniques used for the case $n = 2$. A simple calculation shows that

$$\begin{aligned} V &= Q \begin{bmatrix} 1 & 0 \\ 0 & (I - R(\theta))^{-1} Log(R(\theta)) \end{bmatrix} Q^T \\ &= Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\theta \sin\theta}{2(1-\cos\theta)} & \frac{\theta}{2} \\ 0 & -\frac{\theta}{2} & \frac{\theta \sin\theta}{2(1-\cos\theta)} \end{bmatrix} Q^T. \end{aligned} \tag{12}$$

Thus, for computing the principal logarithm of a matrix $T = \begin{bmatrix} R & 0 \\ v & 1 \end{bmatrix} \in SE(3, \mathrm{IR})$, we suggest the following formula:

$$Log(T) = \begin{bmatrix} Log(R) & 0 \\ vV & 0 \end{bmatrix}, \tag{13}$$

where $Log(R)$ and V are given respectively by (11) and (12).

Similarly to the case $n = 2$, the eigenvalues $\{0, \pm i\theta\}$ of this matrix are always in the range $-\pi < Im(z) < \pi$, its exponential is the matrix T, so (13) always gives the principal logarithm of $T \in SE(3, \mathrm{IR})$.

We may also derive closed forms for the principal logarithm using the Lagrange-Hermite interpolation formula. We start with $SE(2, \mathrm{IR})$, just for the sake of completeness.

The principal matrix logarithm is a primary matrix function and, as a consequence, given a matrix T such that $\sigma(T) \cap \mathrm{IR}_0^- = \emptyset$, there exists a scalar polynomial $p(z)$ in the complex variable z such that $Log(T) = p(T)$. This polynomial interpolates the scalar logarithm $Log(z)$ and its derivatives at the eigenvalues of T, that is, $Log^{(k)}(\lambda_i) = p^{(k)}(\lambda_i)$, for $k = 0, 1, \cdots, r_{i-1}$, where r_i is the multiplicity of the eigenvalue λ_i, and may be computed through the Lagrange-Hermite formula, (Horn and Johnson [11]). In general, computing $p(z)$ is hard. However, when the size is small it is possible to find the expression of $p(z)$ after some algebraic manipulations. This is now illustrated for the case when $T \in SE(n, \mathrm{IR})$ with $n = 2, 3$.

Assume that $T \in SE(2, \mathrm{IR})$ satisfies the condition $\sigma(T) \cap \mathrm{IR}^- = \emptyset$ and the eigenvalues of T are 1 and $\cos\theta \pm i \sin\theta$, with $\theta = \arccos(\frac{trace(T)-1}{2}) \neq 0$. By applying the Lagrange-Hermite formula, we obtain the following polynomial representation for $Log(T)$, $T \in SE(2, \mathrm{IR})$:

$$Log(T) = h(\theta) I - (h(\theta) + g(\theta)) T + g(\theta) T^2,$$

where

$$g(\theta) = -\frac{\theta}{2\sin\theta},$$

$$h(\theta) = -\frac{\theta}{2}(\cot\theta + \frac{\sin\theta}{1-\cos\theta}).$$

If $T \in SE(3,\mathbb{R})$ satisfies $\sigma(T) \cap \mathbb{R}_0^- = \emptyset$, then T has a double eigenvalue equal to 1 and a pair $\cos\theta \pm i\sin\theta$, where $\theta = \arccos(\frac{trace(T)-2}{2})$ is assumed to be nonzero. Now, the polynomial representation for $Log(T)$, $T \in SE(3,\mathbb{R})$, obtained from the Lagrange-Hermite formula, is given by

$$Log(T) = (h(\theta)-1)I + (1+g(\theta)-2h(\theta))T + (h(\theta)-2g(\theta))T^2 + g(\theta)T^3,$$

where

$$g(\theta) = \frac{\theta}{4\sin\theta} + \frac{1}{2(1-\cos\theta)} - \frac{\theta\sin\theta}{4(1-\cos\theta)^2},$$

$$h(\theta) = \frac{1}{2} - \frac{\theta\sin\theta + \cos\theta}{2(1-\cos\theta)} - \frac{\theta}{4}\cot\theta + \frac{\theta\sin 2\theta}{8(1-\cos\theta)^2}.$$

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