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ON THE FISCHER INEQUALITY

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ABSTRACT. A simple proof of Fischer's inequality concerning positive definite matrices is given. It is based on a theorem by M. Fiedler giving an upper bound for the determinant of the sum of two positive semidefinite matrices.

Let G be an $n \times n$ positive definite hermitian matrix, partitioned in the form

$$G = \begin{bmatrix} H & X \\ X^* & K \end{bmatrix}$$

where H and K are square. A well known result of E. Fischer [2] (which generalizes the famous Hadamard determinant theorem) states that

$$\det G \leqslant \det H . \det K$$

with equality if and only if $X = 0$.

This has been proved and generalized in a variety of ways. We offer here still another proof. It is based on the following result of M. Fiedler [1]: if A and B are $n \times n$ positive semidefinite hermitian matrices, with eigenvalues $\alpha_1 \geqslant \dots \geqslant \alpha_n$ and $\beta_1 \geqslant \dots \geqslant \beta_n$, respectively, then

$$\det (A + B) \leqslant \prod_{i=1}^{n} (\alpha_i + \beta_{n-i+1}). \tag{1}$$

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We apply a trick of H. Wielandt [4, p. 120], which has been used to deduce inequalities for eigenvalues of complementary principal submatrices of hermitian matrices from inequalities for eigenvalues of sums of hermitian matrices [3], [4]. The idea is as follows: Since G is positive definite, we have $G = T^*T$ for some nonsingular T. Suppose H is $r \times r$ and write $T = [T_1 T_2]$ where T_1 is $n \times r$. Then $H = T_1^* T_1$, $K = T_2^* T_2$ and $TT^* = T_1 T_1^* + T_2 T_2^*$. Denote by $\lambda_1 \geqslant \ldots \geqslant \lambda_r$ (resp. $\mu_1 \geqslant \ldots \geqslant \mu_{n-r}$) the eigenvalues of H (resp. K) which, apart from zeros, coincide with those of $T_1 T_1^*$ (resp. $T_2 T_2^*$).

We then have, using (1),

$$\det G = \det TT^* = \det (T_1 T_1^* + T_2 T_2^*) \leqslant$$

$$\leqslant \lambda_1 \ldots \lambda_r \, \mu_{n-r} \ldots \mu_1 = \det H . \det K.$$

Now suppose $\det G = \det H . \det K$. Following Fiedler's proof of (1), we see that there is equality there if and only if $A + B$ has eigenvalues $\alpha_1 + \beta_n, \ldots, \alpha_n + \beta_1$. Hence, in our situation, there exists a unitary U such that $UT_1 T_1^* U^* = \mathrm{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$ and $UT_2 T_2^* U^* = \mathrm{diag}(0, \ldots 0, \mu_{n-r}, \ldots, \mu_1)$. Therefore we have $UT_1 T_1^* \, T_2 T_2^* U^* = 0$. Since T_1 and T_2^* are full-rank matrices, it follows that $X = T_1^* T_2 = 0$, as desired.

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