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COMPREHENSIVE FACTORIZATION AND UNIVERSAL I-CENTRAL EXTENSIONS IN THE MAL'CEV CONTEXT

DOMINIQUE BOURN AND DIANA RODELO

Abstract: We show that, under suitable left exact conditions on a reflection functor I, the construction of the associated universal I-central extension is reduced to the comprehensive factorization of a specific internal functor. This observation produces some existence conditions which hold in particular for any reflection from a Mal'cev variety to any Birkhoff subvariety.

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Introduction

Two well-known and, apparently, independent subjects are brought together in this work: central extensions and factorization of functors. The aim is to make explicit the unexpected relationship between them in order to provide the construction of the universal central extension.

The categorical Galois theory developed by G. Janelidze in [17] and [18] gave a final generalized interpretation of the classical Galois theory, completing the work of other authors (see references in [17] and [18]). Several examples were investigated and pointed out, eventually, a strong analogy between the notion of "covering" and the notion of "central" morphism. This led to the introduction of the definition of a *central extension* with respect to a reflection $I : \mathbb{D} \to \mathbb{C}$ associated with the inclusion of a full replete admissible subcategory of an exact category [1], $j : \mathbb{C} \rightarrowtail \mathbb{D}$, see [19]. Correlatively, it emphasized the question on the existence of the universal I-central extension associated to any extension. A positive answer to this question is given here, based on a special factorization of functors.

The *comprehensive factorization* of a functor in the set theoretical context gives a factorization into a final functor and a discrete fibration [24]. The generalization of such a factorization into an exact context was first done in [5]. Such a factorization represented an important tool in the development of the general non-abelian cohomology theory where internal n-groupoids

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played the role of chain complexes [4]. It turns out that the factorization of specific internal functors still holds in the more general context of *efficiently regular* categories [6], an intermediate notion between regular [1] and exact categories. So, the range of examples can be widened to include many topological situations.

The answer to our initial question (Theorem 2.1) is given for an efficiently regular category $\mathbb{D}$ such that the functor I satisfies suitable left exact conditions: given any extension $f : X \twoheadrightarrow Y$, the associated universal I-central extension $\bar{f}$ is reduced to the comprehensive factorization of the upper internal functor $\underline{\eta}_1 f : R[f] \to IR[f]$, of the following diagram where $R[f]$ denotes the kernel equivalence relation of f, and $\bar{f}$ the quotient of the domain R of the discrete fibration involved in this factorization:

$$\begin{tikzcd}
R[f] \arrow[r, "R(\hat{\eta})"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R \arrow[r, two heads, "\overline{\eta}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X \arrow[r, "\hat{\eta}"'] \arrow[d, "f"'] & \bar{X} \arrow[r, two heads, "\overline{\eta}_0"'] \arrow[d, "\bar{f}"] & IX \arrow[d, "If"] \\
Y \arrow[r, equal] & Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

The relatively strong left exact suitable conditions needed in the general situation become much simpler in the Mal'cev context on which we focus our attention and which deals with many subtle variations about the left exactness of the functor I. These conditions hold for any inclusion $j : \mathbb{C} \rightarrowtail \mathbb{D}$ where $\mathbb{D}$ is a Mal'cev variety and $\mathbb{C}$ is any Birkhoff subvariety. Given any finitely cocomplete efficiently regular Mal'cev category $\mathbb{D}$, they hold also when $\mathbb{C}$ is the subcategory $M\mathbb{D}$ of *commutative* objects of $\mathbb{D}$.

The article is organized along the following lines: the first section refreshes the results of [4] about the comprehensive factorization; Section 2 deals with the above specified result in the efficiently regular context; Section 3 modulates the assumptions to the regular Mal'cev context, and finally the last section is devoted to the related, but more specific question, of the preservation by I of products.

1. The comprehensive factorization

We shall suppose all our categories $\mathbb{E}$ finitely complete. Given the following right hand side commutative square, we denote the kernel equivalence relation of f by $R[f]$ and the induced map between the kernel equivalences by $R(x)$:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "R(x)"'] & X \arrow[l, "s_0"' description] \arrow[r, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\
R[f'] \arrow[r, shift left=2] \arrow[r, shift right=2] & X' \arrow[l] \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

1.1. The shifting functor *Dec***.** An internal groupoid $\underline{X}_1$ in $\mathbb{E}$ will be presented (see [4]) as a reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ endowed with an operation d_2:

$$\begin{tikzcd}
R[d_0]^2 \arrow[r, shift left=3, "d_2"] \arrow[r, "d_1" description] \arrow[r, shift right=3, "d_0"'] \arrow[r, bend left=60, "R(d_2)"] & R[d_0] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[r, bend left=60, "d_2"] & X_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & X_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

making the previous diagram satisfy all the simplicial identities, including the degeneracies. In the set theoretical context, this operation d_2 associates the composite $\psi.\phi^{-1}$ with any pair (ϕ, ψ) of arrows of $\underline{X}_1$ with same domain. Any equivalence relation $R \rightrightarrows X$ on an object X in $\mathbb{E}$ provides an internal groupoid:

$$\begin{tikzcd}
R[p_0]^2 \arrow[r, shift left=3, "p_2"] \arrow[r, "p_1" description] \arrow[r, shift right=3, "p_0"'] \arrow[r, bend left=60, "p_3"] & R[p_0] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[r, bend left=60, "p_2"] & R \arrow[r, shift left=3, "p_1"] \arrow[r, shift right=3, "p_0"'] & X \arrow[l, "s_0" description]
\end{tikzcd}$$

which, in some formal circumstances, will be denoted by $\underline{R}_1$.
Let $Grd\mathbb{E}$ denote the category of internal groupoids and internal functors in $\mathbb{E}$, and $()_0 : Grd\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with an internal groupoid $\underline{X}_1$ its "object of objects" X_0. This functor is a left exact fibration. Any fibre (above an object X) has a terminal object $\underline{\nabla}_1(X)$ which is the undiscrete equivalence relation on the object X:

$$\begin{tikzcd}
X \times X \arrow[r, shift left=3, "p_0"] \arrow[r, shift right=3, "p_1"'] & X \arrow[l, "s_0" description]
\end{tikzcd}$$

and an initial object $\underline{\Delta}_1(X)$ which is the discrete equivalence relation on X:

$$\begin{tikzcd}
X \arrow[r, shift left=3, "1_X"] \arrow[r, shift right=3, "1_X"'] & X \arrow[l, "1_X"']
\end{tikzcd}$$

They produce respectively a right adjoint and a left adjoint of the forgetful functor $()_0$. An internal functor $\underline{f}_1 : \underline{X}_1 \rightarrow \underline{Y}_1$ is $()_0$-cartesian if and only if the following square is a pullback in $\mathbb{E}$, in other words if and only if it is internally fully faithful:

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

Accordingly any internal functor $\underline{f}_1$ induces the following decomposition, where the lower quadrangle is a pullback:

$$\begin{tikzcd}
X_1 \arrow[rr, "f_1"] \arrow[dr, "\gamma_1"] \arrow[dd, "{(d_0,d_1)}"'] & & Y_1 \arrow[dd, "{(d_0,d_1)}"] \\
& Z_1 \arrow[ur, "\phi_1"] \arrow[dl] & \\
X_0 \times X_0 \arrow[rr, "f_0 \times f_0"'] & & Y_0 \times Y_0
\end{tikzcd}$$

by a fully faithful functor $\underline{\phi}_1$ and a bijective on objects functor $\underline{\gamma}_1$. We shall need the following pieces of definition:

Definition 1.1. *An internal functor* $\underline{f}_1$ *is said to be* $()_0$*-faithful when the previous factorization* γ_1 *is a monomorphism. It is said to be* $()_0$*-full when this same map* γ_1 *is a strong epimorphism. It is said to be a discrete fibration when the following square is a pullback:*

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, shift right=3, "d_1"'] \arrow[d, dotted, shift left=3, "d_0"] & Y_1 \arrow[d, shift right=3, "d_1"'] \arrow[d, dotted, shift left=3, "d_0"] \\
X_0 \arrow[r, "f_0"'] \arrow[u, dotted] & Y_0 \arrow[u, dotted]
\end{tikzcd}$$

The codomain $\underline{Y}_1$ being an internal groupoid, the square with d_0 is a pullback as well. It is easy to check that, when $\underline{f}_1$ is a discrete fibration and its codomain $\underline{Y}_1$ is an equivalence relation, then the same holds for its domain $\underline{X}_1$.

Lemma 1.1. *Suppose $\underline{f}_1$ is a discrete fibration. Then the object Z_1 in the previous decomposition is $R[f_1]$ and the map γ_1 is $s_0 : X_1 \rightarrowtail R[f_1]$, so that any discrete fibration is $()_0$-faithful. A discrete fibration is $()_0$-cartesian if and only if it is monomorphic.*

Proof: Thanks to the Yoneda Lemma, it is sufficient to prove the first assertion in *Set*. Then the map f_1 is a monomorphism if and only if $\gamma_1 = s_0 : X_1 \rightarrowtail R[f_1]$ is an isomorphism. The internal functor $\underline{f}_1$ being a discrete fibration, f_1 is a monomorphism if and only if f_0 is a monomorphism. ■

The class $Disf$ of discrete fibrations contains the isomorphisms, is stable under composition and such that when $\underline{g}_1.\underline{f}_1$ and $\underline{g}_1$ are in $Disf$, then $\underline{f}_1$ is in $Disf$. The discrete fibrations are stable under pullbacks.

Given an internal groupoid $\underline{X}_1$, we define $Dec\underline{X}_1$ as the following internal groupoid obtained by shifting the indexation:

$$\begin{tikzcd}
R[d_0]^3 \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1" description] \arrow[r, shift right=2, "p_0"'] \arrow[r, bend left=50, "p_3"] & R[d_0]^2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[r, bend left=50, "p_2"] & R[d_0] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & X_1 \arrow[l, "s_0" description]
\end{tikzcd}$$

It is the kernel equivalence relation of the map $d_0 : X_1 \to X_0$. We denote by $\underline{\epsilon}_1\underline{X}_1 : Dec\underline{X}_1 \to \underline{X}_1$ the following internal functor:

$$\begin{tikzcd}
R[d_0] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "d_2"'] & X_1 \arrow[l] \arrow[d, "d_1"] \\
X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l]
\end{tikzcd}$$

which is a discrete fibration and a strong epimorphism in $Grd\mathbb{E}$, since it is levelwise split. It is clear that this shifting construction Dec is functorial and left exact; and that the internal functors $\underline{\epsilon}_1$ determine a natural transformation $Dec \Rightarrow Id$ (which is actually underlying a comonad, see [4]). Moreover the following diagram, in the category $Grd\mathbb{E}$, is a kernel equivalence relation with its quotient:

$$\begin{tikzcd} Dec^2\underline{X}_1 \arrow[r, shift left=1.5, "\epsilon_1 Dec\underline{X}_1"] \arrow[r, shift right=1.5, "Dec\epsilon_1\underline{X}_1"'] & Dec\underline{X}_1 \arrow[r, two heads, "\epsilon_1\underline{X}_1"] & \underline{X}_1 \end{tikzcd} \quad (1)$$

Notice that an internal functor $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is now a discrete fibration if and only if the following diagram is a pullback in $Grd\mathbb{E}$:

$$\begin{tikzcd} Dec\underline{X}_1 \arrow[r, two heads, "\epsilon_1\underline{X}_1"] \arrow[d, "Dec\underline{f}_1"'] & \underline{X}_1 \arrow[d, "\underline{f}_1"] \\ Dec\underline{Y}_1 \arrow[r, two heads, "\epsilon_1\underline{Y}_1"'] & \underline{Y}_1 \end{tikzcd}$$

1.2. The regular context. Recall that a category $\mathbb{E}$ is regular [1] when the regular epimorphisms are stable under pullbacks and any effective equivalence relation (i.e. which is the kernel equivalence relation of some map) admits a quotient. Then the strong epimorphisms coincide with the regular epimorphisms, and the (regular epimorphism, monomorphism) factorization system is pullback stable. Let us begin by recalling the well-known:

Theorem 1.1. [Barr-Kock] *Let $\mathbb{E}$ be a regular category. Given any commutative diagram:*

$$\begin{tikzcd} R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "R(x)"'] & X \arrow[l, "s_0"' description] \arrow[r, two heads, "f"] \arrow[d, "x"] & Y \arrow[d, "y"] \\ R[f'] \arrow[r, shift left=2] \arrow[r, shift right=2] & X' \arrow[l] \arrow[r, "f'"'] & Y' \end{tikzcd}$$

where f is a regular epimorphism, then the right hand side square is a pullback if and only if the internal functor $R(x) : R[f] \to R[f']$ is a discrete fibration.

Corollary 1.1. *Let $\mathbb{E}$ be a regular category. Suppose the following whole rectangle and the left hand side square are pullbacks:*

$$\begin{tikzcd} X' \arrow[r, two heads, "x"] \arrow[d, "f'"'] & X \arrow[r, "u"] \arrow[d, "f"'] & U \arrow[d, "\phi"] \\ Y' \arrow[r, two heads, "y"'] & Y \arrow[r, "v"'] & V \end{tikzcd}$$

If y is regular epimorphism, then the right hand square is a pullback.

When $\mathbb{E}$ is regular, a regular epimorphism in a fibre of the fibration $()_0 : Grd\mathbb{E} \to E$ is an internal functor:

$$\begin{tikzcd}
X_1 \arrow[r, two heads, "f_1"] \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "d_0"] & Y_1 \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "d_0"] \\
X_0 \arrow[u] \arrow[r, equal] & X_0 \arrow[u]
\end{tikzcd}$$

where f_1 is a regular epimorphism. Accordingly any of theses fibres is regular, and any change of base functor is left exact and preserves the regular epimorphisms. The canonical (regular epimorphism, monomorphism) decomposition of the terminal map $\underline{X}_1 \to \underline{\nabla}_1 X_0$ in the fibre:

$$\underline{X}_1 \twoheadrightarrow \underline{\Sigma}_1 \underline{X}_1 \rightarrowtail \underline{\nabla}_1 X_0$$

gives an equivalence relation $\underline{\Sigma}_1 \underline{X}_1$ called the *support* of the internal groupoid. Clearly the construction of the support extends to a functor $\underline{\Sigma}_1 : Grd\mathbb{E} \to Req\mathbb{E}$, where $Req\mathbb{E}$ denotes the category of equivalence relations in $\mathbb{E}$; it is a reflection of the inclusion $Req\mathbb{E} \rightarrowtail Grd\mathbb{E}$ of the equivalence relations and, up to equivalence, a fibration, i.e. a fibred reflection in the sense of [4].

Definition 1.2. *An internal functor $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ will be said to be a $\underline{\Sigma}_1$-discrete fibration, when it is a discrete fibration such that $\underline{\Sigma}_1 \underline{f}_1$ is a discrete fibration.*

Immediately we get:

Lemma 1.2. *A discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ is $\underline{\Sigma}_1$-discrete if and only if the following square in $Grd\mathbb{E}$ is a pullback:*

$$\begin{tikzcd}
\underline{X}_1 \arrow[r, two heads] \arrow[d, "\underline{f}_1"'] & \underline{\Sigma}_1 \underline{X}_1 \arrow[d, "\Sigma_1 \underline{f}_1"] \\
\underline{Y}_1 \arrow[r, two heads] & \underline{\Sigma}_1 \underline{Y}_1
\end{tikzcd}$$

Proof: This is a straightforward consequence of the previous corollary. ■

Any discrete fibration between equivalence relations is $\underline{\Sigma}_1$-discrete. The class $\underline{\Sigma}_1$-*dis* of $\underline{\Sigma}_1$-discrete fibrations contains the isomorphisms, is stable under composition and such that when $\underline{g}_1 . \underline{f}_1$ and $\underline{g}_1$ are in $\underline{\Sigma}_1$-*dis*, then $\underline{f}_1$ is in $\underline{\Sigma}_1$-*dis*. The $\underline{\Sigma}_1$-discrete fibrations are stable under pullbacks and $\underline{\Sigma}_1$ preserves these pullbacks. Any monomorphic discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ being $()_0$-cartesian, the previous lemma shows that it is a $\underline{\Sigma}_1$-discrete fibration. Given an internal groupoid $\underline{X}_1$, the discrete fibration $\underline{\epsilon}_1\underline{X}_1$ is $\underline{\Sigma}_1$-discrete if and only if $\underline{X}_1$ is actually an equivalence relation (the map $d_1 : X_1 \to X_0$ being split).

Definition 1.3. *An internal groupoid $\underline{X}_1$ is said to have an effective support when the equivalence relation $\underline{\Sigma}_1\underline{X}_1$ is effective. We shall denote by $Gref\mathbb{E}$ the full subcategory of $Grd\mathbb{E}$ whose objects are the internal groupoids with effective support.*

The internal groupoids with effective support are stable under products. When $\mathbb{E}$ is exact [1] (i.e. when moreover any equivalence relation is effective), any internal groupoid has an effective support and we have $Gref\mathbb{E} = Grd\mathbb{E}$. We denote by $\pi_0 : Gref\mathbb{E} \to \mathbb{E}$ the functor which associates with an internal groupoid $\underline{X}_1$ the quotient of the effective equivalence relation $\underline{\Sigma}_1\underline{X}_1$; it is a left adjoint to the inclusion $\underline{\Delta}_1 : \mathbb{E} \to Gref\mathbb{E}$ and consequently is right exact. When an internal groupoid $\underline{X}_1$ has an effective support, then the image by π_0 of the above kernel equivalence relation with quotient (1) in $Grd\mathbb{E}$ is the following coequalizer diagram in $\mathbb{E}$:

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[r, two heads, "q\underline{X}_1"] & \pi_0\underline{X}_1
\end{tikzcd}$$

Proposition 1.1. *Any discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ in $Gref\mathbb{E}$ is $\underline{\Sigma}_1$-discrete if and only if the following square is a pullback in $\mathbb{E}$:*

$$\begin{tikzcd}
X_0 \arrow[r, two heads, "q\underline{X}_1"] \arrow[d, "f_0"'] & \pi_0\underline{X}_1 \arrow[d, "\pi_0\underline{f}_1"] \\
Y_0 \arrow[r, two heads, "q\underline{Y}_1"'] & \pi_0\underline{Y}_1
\end{tikzcd}$$

The functor π_0 preserves pullbacks, when they exist in $Gref\mathbb{E}$, of $\underline{\Sigma}_1$-discrete fibrations along any map.

Proof: It is a straightforward consequence of Corollary 1.1. ∎

1.3. The efficiently regular context. We recall here from [6] an intermediate notion between regular and exact categories which allows us to integrate many topological situations.

Definition 1.4. *A regular category $\mathbb{E}$ is said to be* efficiently regular *when any equivalence relation T on an object X which is a subobject $j : T \rightarrowtail R[f]$*

of an effective equivalence relation $R[f]$ on X by an effective monomorphism in $\mathbb{E}$ (which means that j is the equalizer of some pair of maps in $\mathbb{E}$) is itself effective.

Any exact category is always efficiently regular. The category $GpTop$ (resp. $AbTop$) of topological (resp. abelian) groups is efficiently regular, but not exact. More generally any category $Top^{\mathbb{T}}$ of topological protomodular algebras (where $\mathbb{T}$ is a protomodular theory) is efficiently regular: it is a regular category according to [3], and clearly an equivalence relation T on X is effective if and only if the object T is endowed with the topology induced by the topological product, which is the case when $j : T \rightarrowtail R[f]$ is an effective monomorphism. When $\mathbb{E}$ is efficiently regular, so is any slice category $\mathbb{E}/Y$, and any fibre of the fibration $()_0 : Grd\mathbb{E} \to \mathbb{E}$.

An important fact in an efficiently regular category $\mathbb{E}$ is that any discrete fibration above an effective equivalence relation $R[f]$:

$$\begin{tikzcd}
S \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "\bar{f}"'] & U \arrow[r, dotted, two heads, "q'"] \arrow[d, "f"'] & Q \arrow[d, dotted, "\phi"] \\
R[f] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

makes its domain S an effective equivalence relation on U. Accordingly we can complete the diagram with its quotient Q which makes the right hand side a pullback (by the Barr-Kock theorem). So, when $\mathbb{E}$ is efficiently regular, any $\underline{\Sigma}_1$-discrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ having its codomain $\underline{Y}_1$ with effective support has its domain $\underline{X}_1$ with effective support. Consequently, in this context, the category $Gref\mathbb{E}$ admits pullbacks of $\underline{\Sigma}_1$-discrete fibrations along any map, and the functor $\pi_0 : Gref\mathbb{E} \to \mathbb{E}$ preserves them (Proposition 1.1).

Now suppose we have a discrete fibration $\underline{f}_1 : R[q_0] \to \underline{Y}_1$ with q_0 a regular epimorphism:

$$\begin{tikzcd}
R[f_1] \arrow[r, shift left, "R(p_0)"] \arrow[r, shift right, "R(p_1)"'] \arrow[d, shift right] \arrow[d, shift left] & R[f_0] \arrow[r, dotted, two heads, "q_1"] \arrow[d, shift right] \arrow[d, shift left] & T_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
R[q_0] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "f_1"'] & X \arrow[r, two heads, "q_0"'] \arrow[d, "f_0"] & T_0 \\
Y_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Y_0 &
\end{tikzcd}$$

Then completing the diagram by the vertical kernel equivalence relations makes the upper left hand side horizontal diagram an effective equivalence relation, which produces, by its quotient T_1, an internal groupoid $\underline{T}_1$ and a discrete fibration $\underline{q}_1 : R[f_0] \to \underline{T}_1$. The following result enlarges a previous version only asserted in an efficiently regular Mal'cev context [8]:

Theorem 1.2. *Let $\mathbb{E}$ be an efficiently regular category. Then for the above construction:*
1) if $\underline{Y}_1$ is an equivalence relation, then $\underline{T}_1$ is an equivalence relation; the converse is true when f_0 is a regular epimorphism
2) if $\underline{Y}_1$ has an effective support, then $\underline{T}_1$ has an effective support; the converse is true when f_0 is a regular epimorphism.

Proof: 1) According to Lemma 1.1 in [8], when $\underline{Y}_1$ is an equivalence relation, then we have $R[f_0] \cap R[q_0] = \Delta_X$, and since q_0 is a regular epimorphism, $\underline{T}_1$ is an equivalence relation. The converse is true when f_0 is a regular epimorphism, since, then, the role of $\underline{Y}_1$ and $\underline{T}_1$ are totally symmetric.
2) Suppose $\underline{Y}_1$ has an effective support and let $q_{\underline{Y}_1}$ be the quotient of the

effective equivalence relation $\underline{\Sigma}_1 Y_1$. Then consider the following diagram:

$$\begin{tikzcd}
R[f_1] \arrow[r, shift left, "R(p_0)"] \arrow[r, shift right, "R(p_1)"'] \arrow[d, shift left] \arrow[d, shift right] & R[f_0] \arrow[r, two heads, "q_1"'] \arrow[d, shift left] \arrow[d, shift right] \arrow[rr, bend left, "R(q_0)"] & T_1 \arrow[r, "\psi"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & R[q] \arrow[dl, shift right, "p_0"'] \arrow[dl, shift left, "p_1"] \\
R[q_0] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "f_1"'] \arrow[dd, bend right=60, "R(f_0)"'] & X \arrow[r, two heads, "q_0"] \arrow[d, "f_0"] & T_0 \arrow[d, dotted, "q"] & \\
Y_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, two heads, "\sigma_1"'] & Y_0 \arrow[r, two heads, "q_{\underline{Y}_1}"'] & Q & \\
\Sigma_1 Y_1 \arrow[ur, shift left, "d_0"] \arrow[ur, shift right, "d_1"'] & & &
\end{tikzcd}$$

We are going to show that $R(q_0)$ is a regular epimorphism, which will imply that the factorization ψ is a regular epimorphism and the effective equivalence relation $R[q]$ is the support of $\underline{T}_1$. This will come from the fact that σ_1 is a regular epimorphism. For that, first consider the following diagram, where the right hand side square is a pullback:

$$\begin{tikzcd}
R[q_0] \arrow[r, "\hat{\sigma}_1"] \arrow[d, "f_1"'] \arrow[rr, bend left, "p_0"] & P_1 \arrow[r, "\delta_0"] \arrow[d, "\phi_1"'] & X \arrow[d, "f_0"] \\
Y_1 \arrow[r, two heads, "\sigma_1"'] \arrow[rr, bend right, "d_0"'] & \Sigma_1 Y_1 \arrow[r, "d_0"'] & Y_0
\end{tikzcd}$$

Since $\underline{f}_1$ is a discrete fibration, the left hand side square is a pullback, and the factorization $\hat{\sigma}_1$ is a regular epimorphism. Then consider the following diagram where the right hand side quadrangle is a pullback and the map $\hat{\pi}$ is the factorization induced by the right hand side square:

$$\begin{tikzcd}
R[q_0] \arrow[rr, shift left, "p_0"] \arrow[rr, shift right, "p_1"'] \arrow[dr, two heads, "\hat{\sigma}_1"'] \arrow[dd, "R(f_0)"'] & & X \arrow[rr, two heads, "q_0"] \arrow[dr, "\hat{\pi}"] \arrow[dd, "f_0"] & & T_0 \arrow[dd, "q"] \\
& P_1 \arrow[rr, "\delta_0", "\delta_1"'] \arrow[ur, dotted] \arrow[dl, dotted, "\phi_1"'] & & P_0 \arrow[ur, dotted, two heads, "\bar{q}"'] \arrow[dl, dotted, "\phi_0"] & \\
\Sigma_1 Y_1 \arrow[rr, shift left, "d_0"] \arrow[rr, shift right, "d_1"'] & & Y_0 \arrow[rr, two heads, "q_{\underline{Y}_1}"'] & & Q
\end{tikzcd}$$

The maps q_0 and d_1 produce the factorization $\delta_1 : P_1 \to P_0$ which makes the two quadrangles with dotted parallel edges commute. Then the upward quadrangle is a pullback and makes δ_1 a regular epimorphism, since so is q_0. It is easy to check that $\hat{\pi}.p_1 = \delta_1.\hat{\sigma}_1$. This composite is a regular epimorphism since so are δ_1 and $\hat{\sigma}_1$. Accordingly the map $\hat{\pi}$ is a regular epimorphism, and also the map $R(q_0) = R(\bar{q}).R(\hat{\pi})$ as a composite of regular epimorphisms:

$$\begin{tikzcd}
R[f_0] \arrow[r, two heads, "R(\hat{\pi})"'] \arrow[rr, shift left=3ex, "R(q_0)"] \arrow[d, shift right=1.5ex] \arrow[d, shift left=1.5ex] & R[\phi_0] \arrow[r, two heads, "R(\bar{q})"'] \arrow[d, shift right=1.5ex] \arrow[d, shift left=1.5ex] & R[q] \arrow[d, shift right=1.5ex] \arrow[d, shift left=1.5ex] \\
X \arrow[u] \arrow[r, two heads, "\hat{\pi}"'] \arrow[d, "f_0"'] & P_0 \arrow[u] \arrow[r, two heads, "\bar{q}"'] \arrow[d, "\phi_0"'] & T_0 \arrow[u] \arrow[d, "q"] \\
Y_0 \arrow[r, equal] & Y_0 \arrow[r, two heads, "q_{\underline{Y}_1}"'] & Q
\end{tikzcd}$$

the map $R(\bar{q})$ because the two right hand side squares are pullbacks, the map $R(\hat{\pi})$ because it is the product of the regular epimorphism $\hat{\pi}$ by itself in the regular category $\mathbb{E}/Y_0$. Again the converse is true when f_0 is a regular epimorphism, since, then, the role of $\underline{Y}_1$ and $\underline{T}_1$ are totally symmetric. ■

1.4. The comprehensive factorization. We shall suppose from now on that $\mathbb{E}$ is an efficiently regular category. Let $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ be any internal functor. Then consider the following diagram in $Grd\mathbb{E}$ with the right hand side pullbacks:

$$\begin{tikzcd}
Dec^2\underline{X}_1 \arrow[r, "\underline{\psi}^2_1"'] \arrow[rr, shift left=3ex, "Dec^2\underline{f}_1"] \arrow[d, shift right=1.5ex, "\underline{\epsilon}_1 Dec\underline{X}_1"'] \arrow[d, shift left=1.5ex, "Dec\underline{\epsilon}_1\underline{X}_1"] & \underline{V}_1 \arrow[r, "\underline{\phi}^2_1"'] \arrow[d, shift right=1.5ex] \arrow[d, shift left=1.5ex] & Dec^2\underline{Y}_1 \arrow[d, shift right=1.5ex, "\underline{\epsilon}_1 Dec\underline{Y}_1"'] \arrow[d, shift left=1.5ex, "Dec\underline{\epsilon}_1\underline{Y}_1"] \\
Dec\underline{X}_1 \arrow[u] \arrow[r, "\underline{\psi}_1"'] \arrow[d, two heads, "\underline{\epsilon}_1\underline{X}_1"'] & \underline{U}_1 \arrow[u] \arrow[r, "\underline{\phi}_1"'] \arrow[d, two heads, "\underline{e}_1"'] & Dec\underline{Y}_1 \arrow[u] \arrow[d, two heads, "\underline{\epsilon}_1\underline{Y}_1"] \\
\underline{X}_1 \arrow[r, equal] & \underline{X}_1 \arrow[r, "\underline{f}_1"'] & \underline{Y}_1
\end{tikzcd}$$

The middle vertical diagram is then a kernel equivalence relation with quotient, since so is the right hand side vertical one. Since the two upper right hand side vertical arrows are $\underline{\Sigma}_1$-discrete fibrations, so are the two unlabeled projections $\underline{V}_1 \rightrightarrows \underline{U}_1$.

Proposition 1.2. *Suppose $\mathbb{E}$ is efficiently regular and $\underline{U}_1$ of the previous construction has an effective support. Then the image by π_0 of the upper part of the previous diagram:*

$$\begin{tikzcd}
X_1 \arrow[r, "\hat{f}_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T_1 \arrow[r, "\overline{f}_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X_0 \arrow[u] \arrow[r, "\hat{f}_0"'] \arrow[d, dotted, two heads, "q_{\underline{X}_1}"] & T_0 \arrow[u] \arrow[r, "\overline{f}_0"'] \arrow[d, dotted, two heads, "q_{\underline{T}_1}=\pi_0(\underline{e}_1)"] & Y_0 \arrow[u] \\
\pi_0(\underline{X}_1) \arrow[r, equal] & \pi_0(\underline{X}_1) &
\end{tikzcd}$$

produces the universal decomposition of $\underline{f}_1 = \overline{\underline{f}}_1.\hat{\underline{f}}_1 : \underline{X}_1 \to \underline{T}_1 \to \underline{Y}_1$ through the discrete fibration $\overline{\underline{f}}_1$. If, moreover, the internal groupoid $\underline{X}_1$ has an effective support, then we can extend the previous diagram with their (dotted) coequalizers.

Proof: The upper right hand side pullbacks, in the diagram in $Grd\mathbb{E}$, having $\underline{\Sigma}_1$-discrete fibrations as vertical edges are preserved by π_0 (Proposition 1.1) and produces the discrete fibration $\overline{\underline{f}}_1$. Suppose now that $\underline{f}_1 = \underline{g}_1.\underline{h}_1$ with $\underline{g}_1$ a discrete fibration. Then consider the following diagram in $Grd\mathbb{E}$:

$$\begin{tikzcd}
Dec^2\underline{X}_1 \arrow[rr, shift left=4, "Dec^2\underline{f}_1"] \arrow[r, "Dec^2\underline{h}_1"'] \arrow[d, shift right=2, "\underline{\epsilon}_1 Dec\underline{X}_1"'] \arrow[d, shift left=2, "Dec\underline{\epsilon}_1\underline{X}_1"] & Dec^2\underline{Z}_1 \arrow[r, "Dec^2\underline{g}_1"'] \arrow[d, shift right=2] \arrow[d, shift left=2] & Dec^2\underline{Y}_1 \arrow[d, shift right=2, "\underline{\epsilon}_1 Dec\underline{Y}_1"'] \arrow[d, shift left=2, "Dec\underline{\epsilon}_1\underline{Y}_1"] \\
Dec\underline{X}_1 \arrow[u] \arrow[r, "Dec\underline{h}_1"'] \arrow[d, "\underline{\epsilon}_1\underline{X}_1"'] & Dec\underline{Z}_1 \arrow[u] \arrow[r, "Dec\underline{g}_1"'] \arrow[d, "\underline{\epsilon}_1\underline{Z}_1"'] & Dec\underline{Y}_1 \arrow[u] \arrow[d, "\underline{\epsilon}_1\underline{Y}_1"] \\
\underline{X}_1 \arrow[r, "\underline{h}_1"] \arrow[rr, shift right=4, "\underline{f}_1"'] & \underline{Z}_1 \arrow[r, "\underline{g}_1"] & \underline{Y}_1
\end{tikzcd}$$

The right hand side squares are pullbacks since $\underline{g}_1$ is a discrete fibration. Whence the following factorizations with $\underline{\phi}_1 = Dec\underline{g}_1.\underline{\bar{h}}_1$:

$$\begin{tikzcd}
\underline{V}_1 \arrow[r, "\underline{\bar{h}}_1^2"'] \arrow[rr, bend left, "\underline{\phi}_1^2"] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & Dec^2\underline{Z}_1 \arrow[r, "Dec^2\underline{g}_1"'] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & Dec^2\underline{Y}_1 \arrow[d, shift right=2, "\underline{\epsilon}_1 Dec\underline{Y}_1"'] \arrow[d, leftarrow] \arrow[d, shift left=2, "Dec\underline{\epsilon}_1\underline{Y}_1"] \\
\underline{U}_1 \arrow[r, "\underline{\bar{h}}_1"'] \arrow[d, "\underline{e}_1"'] & Dec\underline{Z}_1 \arrow[r, "Dec\underline{g}_1"'] \arrow[d, "\underline{\epsilon}_1\underline{Z}_1"'] & Dec\underline{Y}_1 \arrow[d, "\underline{\epsilon}_1\underline{Y}_1"] \\
\underline{X}_1 \arrow[r, "\underline{h}_1"] \arrow[rr, bend right, "\underline{f}_1"'] & \underline{Z}_1 \arrow[r, "\underline{g}_1"] & \underline{Y}_1
\end{tikzcd}$$

which make the left hand side squares pullbacks. The image by π_0 of the upper pullbacks:

$$\begin{tikzcd}
T_1 \arrow[r, "\tau_1"'] \arrow[rr, bend left, "\bar{f}_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, leftarrow] \arrow[d, shift left=2, "d_1"] & Z_1 \arrow[r, "g_1"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, leftarrow] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, leftarrow] \arrow[d, shift left=2, "d_1"] \\
T_0 \arrow[r, "\tau_0"] \arrow[rr, bend right, "\bar{f}_0"'] & Z_0 \arrow[r, "g_0"] & Y_0
\end{tikzcd}$$

produces the factorization $\underline{\tau}_1 : \underline{T}_1 \to \underline{Z}_1$ such that $\underline{\bar{f}}_1 = \underline{g}_1.\underline{\tau}_1$ we were looking for. It is easy to check that $\underline{\tau}_1.\underline{\hat{f}}_1 = \underline{h}_1$. Finally, when $\underline{X}_1$ has an effective support, we can apply π_0 to the left hand side of our initial diagram: both vertical coequalizers are preserved and coincide. ■

The discrete fibrations are stable under composition and pullbacks. Then necessarily the factorization $\underline{\hat{f}}_1$ through the associated universal discrete fibration $\underline{\bar{f}}_1$ belongs to the class of morphisms which is *orthogonal* to the class $Disf$ of discrete fibrations, see Theorem 1.8 in [13], namely the class of those internal functors $\underline{\phi}_1 : \underline{M}_1 \to \underline{N}_1$ such that any commutative square with $\underline{k}_1$ a discrete fibration produces a unique diagonal factorization:

$$\begin{tikzcd}
\underline{M}_1 \arrow[r] \arrow[d, "\underline{\phi}_1"'] & \underline{A}_1 \arrow[d, "\underline{k}_1"] \\
\underline{N}_1 \arrow[r] \arrow[ur, dotted] & \underline{B}_1
\end{tikzcd}$$

So according to the terminology introduced in [24] when $\mathbb{E}$ is *Set*, and extended to any exact context in [4], we shall call this orthogonal class the class of *final* internal functors and the previous decomposition the *comprehensive factorization* of the internal functor $\underline{f}_1$. Clearly, when $\mathbb{E}$ is exact, any internal functor $\underline{f}_1$ admits a comprehensive factorization. This was first shown in [4].

Proposition 1.3. *Suppose $\mathbb{E}$ is an efficiently regular category. Let $\underline{f}_1 : R[q] \to \underline{Y}_1$ be an internal functor. Then the comprehensive factorization of $\underline{f}_1$ does exist. Moreover the groupoid $\underline{T}_1$ is itself an effective equivalence relation which has the same quotient as $R[q]$.*

Proof: The internal functor $\underline{e}_1 : \underline{U}_1 \to \underline{R}[q]$ being a discrete fibration and $\mathbb{E}$ efficiently regular, the internal groupoid $\underline{U}_1$ in the previous construction is an effective equivalence relation. Accordingly the comprehensive factorization holds. Now the following diagram satisfies the condition of Theorem 1.2:

$$\begin{tikzcd}
V_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, shift left] \arrow[d, shift right] & V_0 \arrow[r, two heads, "q_{\underline{V}_1}"'] \arrow[d, shift left] \arrow[d, shift right] & T_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
U_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "e_1"'] & U_0 \arrow[r, two heads, "q_{\underline{U}_1}"] \arrow[d, "e_0"] & T_0 \arrow[d, dotted, "\bar{q}"] \\
R[q] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[r, two heads, "q"'] & Q
\end{tikzcd}$$

And the internal groupoid $\underline{T}_1$ is the effective equivalence relation $R[\bar{q}]$. Not only it has same quotient Q as $R[q]$, but according to Proposition 1.2, the internal functor $\hat{\underline{f}}_1 : R[q] \to R[\bar{q}]$ induces the identity on Q. ■

From that we get:

Corollary 1.2. *Let $\mathbb{E}$ be an efficiently regular category and $\underline{f}_1 : \nabla X \to \underline{Y}_1$ an internal functor. Its comprehensive factorization exists and is of the following form:*

$$\nabla X \to \nabla T \to \underline{Y}_1$$

Proof: The indiscrete equivalence relation ∇X is effective and its quotient is a subobject W of the terminal object 1. According to the previous proposition, the internal groupoid $\underline{T}_1$ is then an effective equivalence relation with quotient W, namely an indiscrete equivalence relation ∇T. ■

In the same way, we get the following proposition which makes the last assertion of Proposition 1.2 more precise:

Proposition 1.4. *Suppose* $\mathbb{E}$ *is an efficiently regular category,* $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ *an internal functor and the internal groupoid* $\underline{U}_1$ *of the previous construction has an effective support. If moreover* $\underline{X}_1$ *has an effective support, then the internal groupoid* $\underline{T}_1$ *given by the comprehensive factorization has an effective support, and the same* π_0 *as* $\underline{X}_1$*.*

Proof: First consider the diagram:

$$\begin{tikzcd}
V_1 \arrow[r, two heads] \arrow[d, shift left] \arrow[d, shift right] & \Sigma_1 V_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, shift left] \arrow[d, shift right] & V_0 \arrow[r, two heads, "q_{\underline{V}_1}"] \arrow[d, shift left] \arrow[d, shift right] & T_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
U_1 \arrow[r, two heads] \arrow[d, two heads, "e_1"'] & \Sigma_1 U_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, two heads, "\Sigma_1 e_1"'] & U_0 \arrow[r, two heads, "q_{\underline{U}_1}"] \arrow[d, two heads, "e_0"] & T_0 \arrow[d, two heads, dotted, "q"] \\
X_1 \arrow[r, two heads] & \Sigma_1 X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[r, two heads, "q_{\underline{X}_1}"'] & Q
\end{tikzcd}$$

We know that the pair $\underline{V}_1 \rightrightarrows \underline{U}_1$ of projections are $\underline{\Sigma}_1$-discrete fibrations which means that the central upper squares are pullbacks. Accordingly, on one hand, the right hand side upper squares are pullbacks by the Barr-Kock theorem and determine a discrete fibration $R[e_0] \to \underline{T}_1$; on the other hand the equivalence relation $\Sigma_1 V_1 \rightrightarrows \Sigma_1 U_1$ is effective. Let us denote by Ξ_1 its quotient which determines the following diagram and produces an internal groupoid $\underline{\Xi}_1$:

$$\begin{tikzcd}
\Sigma_1 V_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, shift left] \arrow[d, shift right] & V_0 \arrow[r, two heads, "q_{\underline{V}_1}"'] \arrow[d, shift left] \arrow[d, shift right] & T_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
\Sigma_1 U_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, two heads, "\xi"'] \arrow[dd, bend right=60, two heads, "\Sigma_1 e_1"'] & U_0 \arrow[r, two heads, "q_{\underline{U}_1}"] \arrow[d, two heads, "e_0"] & T_0 \arrow[d, two heads, dotted, "q"] \\
\Xi_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\bar{\xi}"'] & X_0 \arrow[r, two heads, "q_{\underline{X}_1}"'] & Q \\
\Sigma_1 X_1 \arrow[ur, shift left, "d_0"] \arrow[ur, shift right, "d_1"'] & &
\end{tikzcd}$$

Since $\Sigma_1 e_1 : \Sigma_1 U_1 \twoheadrightarrow \Sigma_1 X_1$ is a regular epimorphism, so is the factorization $\bar{\xi} : \Xi \twoheadrightarrow \Sigma_1 X_1$. Accordingly the internal groupoids $\underline{X}_1$ and $\underline{\Xi}_1$ have the same effective support $\underline{\Sigma}_1\underline{X}_1$. Now, according to Theorem 1.2, since $\underline{\Xi}_1$ has an effective support, this is also the case for $\underline{T}_1$. ■

2. Regular reflections

Now let $j : \mathbb{C} \rightarrowtail \mathbb{D}$ be a full replete inclusion.

2.1. The reg-epi reflections.

Recall the following:

Definition 2.1. *A reflection $I : \mathbb{D} \to \mathbb{C}$ of the inclusion j is said to be a reg-epi reflection when any projection $\eta_X : X \twoheadrightarrow IX$ is a strong epimorphism.*

This last point is equivalent to saying that $\mathbb{C}$ is stable under subobjects. Also, it is straightforward that the strong epimorphism $1 \twoheadrightarrow I1$, having a retraction, is an isomorphism, and that 1 is in $\mathbb{C}$. In the same way, given a pair (A, B) of objects in $\mathbb{C}$, the strong epimorphism $\eta_{A\times B} : A \times B \twoheadrightarrow I(A \times B)$ has $(Ip_0, Ip_1) : I(A \times B) \to A \times B$ as a retraction, and thus is an isomorphism. Accordingly $\mathbb{C}$ is stable under products. Being also stable under monomorphism, $\mathbb{C}$ is stable under finite limits.

Following [19], we shall now be interested in certain classes of maps with respect to the reflection I:

Definition 2.2. *Given a reg-epi reflection I, a map $f : X \to Y$ in $\mathbb{D}$ is said to be I-trivial when the following square is a pullback:*

$$\begin{tikzcd}
X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & IX \arrow[d, "If"] \\
Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

Clearly the isomorphisms are I-trivial, the I-trivial maps are stable under composition and such that, when $g.f$ and g are I-trivial, then f is I-trivial. Clearly also I-trivial maps are stable under the pullbacks which are preserved by the reflection I. This last point emphasizes the importance of those pullbacks in $\mathbb{D}$ which are preserved by I. An I-trivial map f is certainly I-cartesian, namely universal among the maps above If.

When $\mathbb{D}$ is a regular category, a reflection I is a reg-epi reflection if and only if any η_X is a regular epimorphism. In this case, a map in $\mathbb{C}$ is a regular epimorphism in $\mathbb{C}$ if and only if it is a regular epimorphism in $\mathbb{D}$ and $\mathbb{C}$ is also a regular category.

Example 2.1. *Suppose $\mathbb{D}$ is a regular category. The category $Grd\mathbb{D}$ is not necessarily regular, but the reflection $\underline{\Sigma}_1 : Grd\mathbb{D} \to Req\mathbb{D}$ is a reg-epi reflection. In this context, any $()_0$-cartesian functor $\underline{f}_1$ is $\underline{\Sigma}_1$-trivial. According to Lemma 1.2, a discrete fibration is $\underline{\Sigma}_1$-discrete if and only if it is $\underline{\Sigma}_1$-trivial.*

Again, following [19], we have the following:

Definition 2.3. *Given a regular category $\mathbb{D}$ and a reg-epi reflection I, the reflection I is said to be admissible when any pullback of a regular epimorphism ϕ in $\mathbb{C}$ of the form:*

$$\begin{tikzcd}
X \arrow[r, two heads, "g"] \arrow[d, two heads, "f"'] & A \arrow[d, two heads, "\phi"] \\
Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

is I-trivial.

Lemma 2.1. *Let $\mathbb{D}$ be a regular category and I a reg-epi reflection. Then I is admissible if and only if the pullback along any map of a regular epimorphism in $\mathbb{C}$ is I-trivial. Then I preserves such pullbacks. Accordingly a reg-epi reflection I is admissible if and only if the I-trivial extensions are stable under pullbacks.*

Proof: The first point is a straightforward consequence of Corollary 1.1. And the second one a straightforward consequence of the first one. ■

In the conditions of the previous lemma, when I is admissible, the I-trivial extensions, being also stable under composition, are also stable under products.

Definition 2.4. *Given a regular category $\mathbb{D}$ and a reg-epi reflection I, a map $f : X \to Y$ is called I-central when it is I-trivial up to a regular epimorphism, namely such that there exists a regular epimorphism along which this map f is pulled back onto an I-trivial map. A map f is called I-normal, when its projection $p_0 : R[f] \to X$ (or p_1) is I-trivial.*

The class of I-central morphisms contains the I-trivial morphisms and thus the isomorphisms. It is not stable under composition, nor under pullbacks, in general; however, according to the previous lemma, when I is admissible, the I-central extensions are stable under pullbacks. Any I-normal extension f is I-central (pullback f along itself). Notice that, although any I-trivial extension is I-central, it is not necessarily I-normal. However it is clear that when I-central extensions and I-normal extensions coincide, any I-trivial extension is I-normal.

Proposition 2.1. *Let $\mathbb{D}$ be a regular category and I a reg-epi reflection. Suppose the map $f : X \rightarrow Y$ is I-trivial. Then the two following conditions are equivalent:*
1) f is I-normal
2) $IR[f] \simeq R[If]$
Suppose now $f : X \twoheadrightarrow Y$ is a regular epimorphism. Consider the following conditions:
1) f is an I-trivial extension
2) f is an I-normal extension
3) $IR[f] \simeq R[If]$
If any two conditions are satisfied, the third one holds. Accordingly any I-trivial extension is I-normal if and only if the kernel equivalence relations of I-trivial regular epimorphisms are preserved by I.

Proof: It is a straightforward consequence of Corollary 1.1. ■

We shall now request a context in which we shall be able to show that I-trivial extensions are I-normal, and that I-central and I-normal extensions coincide. Eventually, thanks to the comprehensive factorization, we shall associate with any regular epimorphism an I-central extension.

Definition 2.5. *Given a regular category $\mathbb{D}$, we call regular reflection any reg-epi reflection I which preserves the pullbacks of split epimorphisms along regular epimorphisms.*

We shall give a large class of examples in the next section. Such reflections I preserve in particular the kernel equivalence relations of split epimorphisms, and thus preserve the internal groupoids. This produces a functor still denoted by $I : Grd\mathbb{D} \rightarrow Grd\mathbb{C}$ for sake of simplicity.

Lemma 2.2. *Let $\mathbb{D}$ be a regular category and I a regular reflection. Any I-trivial map is I-normal.*

Proof: When $f : X \rightarrow Y$ is I-trivial, the pullback defining it produces the following whole rectangle of pullbacks of split epimorphisms along the regular epimorphism η_X:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\eta_{R[f]}"'] \arrow[rr, two heads, bend left, "R(\eta_X)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[r, "\chi"'] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] & R[If] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
X \arrow[r, two heads, "\eta_X"] \arrow[rr, two heads, bend right, "\eta_X"'] & IX \arrow[r, equal] & IX
\end{tikzcd}$$

It is preserved by I. This makes the right hand side squares pullbacks, and the factorization χ an isomorphism. Accordingly $p_0 : R[f] \to X$ is I-trivial and f I-normal. ■

Proposition 2.2. *Given a regular category $\mathbb{D}$ and a regular reflection I, any I-central map is I-normal. Accordingly the I-central extensions and I-normal extensions coincide.*

Proof: Suppose f is an I-central map and $h : Y' \twoheadrightarrow Y$ is the regular epimorphism along which f is pulled back onto an I-trivial map f'. Then consider the following diagram where all the left hand side squares are pullbacks:

$$\begin{tikzcd}
R[f'] \arrow[r, two heads, "R(g)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[f] \arrow[r, "\eta_{R[f]}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X' \arrow[r, two heads, "g"'] \arrow[d, "f'"'] & X \arrow[r, "\eta_X"'] \arrow[d, "f"] & IX \arrow[d, "If"] \\
Y' \arrow[r, two heads, "h"'] & Y \arrow[r, "\eta_Y"'] & IY
\end{tikzcd}$$

Since the upper ones are pullback of split epimorphisms along regular epimorphisms, they are preserved by I. Now the following diagram is globally the same as the previous one:

$$\begin{tikzcd}
R[f'] \arrow[r, two heads, "\eta_{R[f']}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f'] \arrow[r, two heads, "IR(g)"] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X' \arrow[r, two heads, "\eta_{X'}"'] \arrow[d, "f'"'] & IX' \arrow[r, two heads, "Ig"'] \arrow[d, "If'"] & IX \arrow[d, "If"] \\
Y' \arrow[r, two heads, "\eta_{Y'}"'] & IY' \arrow[r, two heads, "Ih"'] & IY
\end{tikzcd}$$

All the left hand squares are pullbacks since f' is I-trivial, and thus I-normal. We just noticed that the upper right hand side squares were pullbacks. Accordingly the upper rectangles are pullbacks. So this is also true for the upper rectangles in our first diagram, where moreover the left hand side ones are pullbacks. Accordingly, the upper right hand side squares are pullbacks as well, which make the projections $p_i : R[f] \to X$ I-trivial, and the map f I-normal. Accordingly any I-central extension is I-normal. ∎

Corollary 2.1. *Given a regular category $\mathbb{D}$ and a regular reflection I, any split epimorphism f which is I-central is I-trivial.*

Proof: Being split and I-central, f is an I-central extension, and thus an I-normal extension. On the other hand, being split, its kernel equivalence relation is preserved by I. According to Proposition 2.1, it is I-trivial. ∎

2.2. Associated universal I-central extension. We shall suppose now that $\mathbb{D}$ is an efficiently regular category. In this context we get two important precisions:

Proposition 2.3. *Let $\mathbb{D}$ be an efficiently regular category, and $I : \mathbb{D} \to \mathbb{C}$ a regular reflection. Then I is admissible, and the I-central extensions are stable under pullbacks.*

Proof: Consider the following diagram where the regular epimorphism ϕ is in $\mathbb{C}$ and the lower rectangle is a pullback:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\eta_{R([f]}"] \arrow[rr, two heads, bend left, "R(g)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[r, "IR(g)"] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] & R[\phi] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
X \arrow[r, two heads, "\eta_X"'] \arrow[rr, two heads, bend left, "g"] \arrow[d, two heads, "f"'] & IX \arrow[r, "Ig"'] \arrow[d, two heads, "If"] & A \arrow[d, two heads, "\phi"] \\
Y \arrow[r, two heads, "\eta_Y"'] & IY \arrow[r, equal] & IY
\end{tikzcd}$$

So are the corresponding two upper ones. The maps g and $R(g)$ being regular epimorphisms, these pullbacks are preserved by I. This makes the internal functor $IR(g) : IR[f] \to R[\phi]$ a discrete fibration. Since its codomain is an effective equivalence relation and $\mathbb{D}$ is efficiently regular, its domain $IR[f]$ is an effective equivalence relation whose quotient is If. Accordingly we have $IR[f] \simeq R[If]$. Since If is a regular epimorphism, the pullback properties of the upper right hand side squares can be shifted to the lower right hand side square by the Barr-Kock theorem, the lower right hand side square is a pullback as well, Ig is an isomorphism, and f is I-trivial. Now we can apply Lemma 2.1: so I-trivial extensions, and consequently I-normal maps and extensions, are stable under pullbacks. On the other hand we know that I-central extensions coincide with I-normal extensions. ■

Theorem 2.1. *Let $\mathbb{D}$ be an efficiently regular category, and $I : \mathbb{D} \to \mathbb{C}$ a regular reflection. Then with any extension $f : X \twoheadrightarrow Y$ is associated a universal I-central extension $\bar{f} : \bar{X} \twoheadrightarrow Y$. Moreover, we have $I\bar{X} \simeq IX$, $I\bar{f} \simeq If$ and $IR[\bar{f}] \simeq IR[f]$.*

Proof: Let $f : X \twoheadrightarrow Y$ be an extension. Consider the following diagram:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\eta_{R[f]}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X \arrow[r, two heads, "\eta_X"'] \arrow[d, "f"'] & IX \arrow[d, "If"] \\
Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

This way, we get an internal functor $\underline{\eta}_1 f : R[f] \to IR[f]$, with codomain the internal groupoid $IR[f]$. Take its associated comprehensive factorization which exists, since $R[f]$ is an effective equivalence relation and gives the following diagram according to Proposition 1.3:

$$\begin{tikzcd}
R[f] \arrow[r, "R(\hat{\eta})"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[\bar{f}] \arrow[r, two heads, "\bar{\eta}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X \arrow[r, "\hat{\eta}"'] \arrow[d, "f"'] & \bar{X} \arrow[r, two heads, "\bar{\eta}_0"'] \arrow[d, "\bar{f}"] & IX \arrow[d, "If"] \\
Y \arrow[r, equal] & Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

Accordingly the upper right hand side squares are pullbacks, so that the projections $p_i : R[\bar{f}] \to \bar{X}$ are I-trivial and the extension $\bar{f}$ is I-normal. As pullbacks of split epimorphism along regular epimorphisms, they are preserved by I, and consequently the image $I\underline{\bar{\eta}}_1$ of the discrete fibration $\underline{\bar{\eta}}_1$ is still a discrete fibration. This way we get the following diagram in $Grd\mathbb{D}$:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\underline{\eta}_1 f"] \arrow[d, "R(\hat{\eta})"'] & IR[f] \arrow[d, "IR(\hat{\eta})"] \\
R[\bar{f}] \arrow[ur, "\underline{\overline{\eta}}_1"'] \arrow[r, two heads, "\underline{\eta}_1 \bar{f}"'] & IR[\bar{f}] \arrow[u, bend right, "I\underline{\overline{\eta}}_1"']
\end{tikzcd}$$

The downward square commutes by naturality. We have $\underline{\overline{\eta}}_1.R(\hat{\eta}) = \underline{\eta}_1 f$ by construction, and $I\underline{\overline{\eta}}_1.\underline{\eta}_1\bar{f} = \underline{\overline{\eta}}_1$. From the first equality, we get $I\underline{\overline{\eta}}_1.IR(\hat{\eta}) = 1_{IR[f]}$. Thanks to the diagonally property associated with the comprehensive factorization, we shall get $IR(\hat{\eta}).\underline{\overline{\eta}}_1 = \underline{\eta}_1\bar{f}$ (and thus $IR(\hat{\eta}).I\underline{\overline{\eta}}_1 = 1_{IR[\bar{f}]}$), by checking it by composition with the final internal functor $R(\hat{\eta})$ and the discrete fibration $I\underline{\overline{\eta}}_1$, which is straightforward. Accordingly $IR(\hat{\eta})$ is an isomorphism of internal groupoids $IR[f] \simeq IR[\bar{f}]$, which determines the isomorphism of their quotient maps $I\bar{f} \simeq If$.

Now suppose we have a factorization $f = f'.h : X \to X' \to Y$ with f' an I-normal extension. We have then the following right hand side commutative square in $Grd\mathbb{D}$, with the internal functor $\underline{\eta}_1 f'$ a discrete fibration, since f' is an I-normal extension:

$$\begin{tikzcd}
 & R[f] \arrow[dl, "R(\hat{\eta})"'] \arrow[r, "R(h)"] \arrow[dd, "\underline{\eta}_1 f"'] & R[f'] \arrow[dd, "\underline{\eta}_1 f'"] \\
R[\bar{f}] \arrow[urr, dotted, "\underline{\tau}_1"'] \arrow[dr, "\underline{\overline{\eta}}_1"'] & & \\
 & IR[f] \arrow[r, "IR(h)"'] & IR[f']
\end{tikzcd}$$

The internal functor $R(\hat{\eta})$ being final and the internal functor $\underline{\eta}_1 f'$ being a discrete fibration, we have, thanks to the diagonality property associated with the comprehensive factorization system, a factorization $\underline{\tau}_1$ such that $\underline{\tau}_1.R(\hat{\eta}) = R(h)$ and $\underline{\eta}_1 f'.\underline{\tau}_1 = IR(h).\underline{\overline{\eta}}_1$. The first equality implies, at the level of objects, that $\tau_0 : \bar{X} \to X'$ is such that $\tau_0.\hat{\eta} = h$. Moreover the image by π_0 of this same equality gives $\pi_0(\underline{\tau}_1) = \pi_0(R(h)) = 1_Y$, which implies that $f'.\tau_0 = \bar{f}$. If we suppose for sake of simplicity that $IR(\hat{\eta})$ is an identity (and thus $\underline{\overline{\eta}}_1 = \underline{\eta}_1\bar{f}$), we have necessarily $I\underline{\tau}_1 = IR(h)$. Suppose we have another factorization $\tau : \bar{X} \to X'$ such that $\tau.\hat{\eta} = h$ and $f'.\tau = \bar{f}$. Then the internal functor $R(\tau) : R[\bar{f}] \to R[f']$ is such that $R(\tau).R(\hat{\eta}) = R(h) = \underline{\tau}_1.R(\hat{\eta})$ and consequently $IR(\tau) = IR(h) = I\underline{\tau}_1$. Thanks to the unicity of the diagonality condition, we get $R(\tau) = \underline{\tau}_1$ and $\tau = \tau_0$. This universal I-normal extension $\overline{f}$ is the universal I-central extension, since I-central extensions and I-normal extensions coincide. ■

3. The regular Mal'cev context

Recall that a finitely complete category $\mathbb{D}$ is said to be a Mal'cev category when any reflexive relation is an equivalence relation ([11], [12]). When $\mathbb{D}$ is a Mal'cev category, this is also the case for $Grd\mathbb{D}$. When furthermore $\mathbb{D}$ is regular [1], the regular epimorphisms in $Grd\mathbb{D}$ are the internal functors $\underline{f}_1$ which are levelwise regular epimorphism, and then $Grd\mathbb{D}$ is still a regular Mal'cev category [14]. This context will allow us to give many examples of regular reflections.

Proposition 3.1. *Let $\mathbb{D}$ be a regular Mal'cev category. The functor $\underline{\Sigma}_1 : Grd\mathbb{D} \to Req\mathbb{D}$ preserves pullbacks of split epimorphisms along any map. Accordingly the functor $\underline{\Sigma}_1$ is a regular reflection and the pullback of split epimorphisms along any map exists in $Gref\mathbb{D}$.*

Proof: This a straightforward consequence of Lemma 2.5.7 in [2]. ■

Recall Lemma 2.5.6 in [2]:

Lemma 3.1. *Let $\mathbb{D}$ be a regular Mal'cev category. Given a commutative diagram of split epimorphisms:*

$$\begin{tikzcd}
X' \arrow[r, two heads, "x"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\
Y' \arrow[r, two heads, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, shift right, "s"']
\end{tikzcd}$$

where x (and thus y) is a regular epimorphism, then the factorization $(f', x) : X' \to Y' \times_Y X$ is a regular epimorphism.

From that we get then a very powerful observation:

Proposition 3.2. *Let $\mathbb{D}$ be a regular Mal'cev category. Suppose the following whole rectangle is a pullback and the left hand side square is a commutative* *square of split epimorphisms:*

$$\begin{tikzcd}
X' \arrow[r, two heads, "x"] \arrow[d, shift right, "f'"'] & X \arrow[r, "u"] \arrow[d, shift right, "f"'] & U \arrow[d, "\phi"] \\
Y' \arrow[r, two heads, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[r, "v"'] \arrow[u, shift right, "s"'] & V
\end{tikzcd}$$

If x (and thus y) is a regular epimorphism, then the two squares are pullbacks.

Proof: By Lemma 3.1, the factorization $(f', x) : X' \twoheadrightarrow Y' \times_Y X$ is a regular epimorphism. But it is also a monomorphism, since $(f', u \cdot x)$ is a monomorphism, thus it is an isomorphism. This proves that the left hand side square is a pullback. Since y is a regular epimorphism, then Corollary 1.1 allows us to conclude that the right hand side square is also a pullback. ■

This result was stated in [15] (see Lemma 1.1) in the stricter context of exact categories. Thanks to this last property, we are going to show that, in the Mal'cev context, the reg-epi reflections I satisfy some significant left exact properties which allows us to recover some aspects of regular reflections. This will eventually lead, by increasing gradually the assumptions, to the preservation of pullbacks of split epimorphisms along regular epimorphisms, namely to the property of being a regular reflection. Let us begin by the following:

Proposition 3.3. *Let $\mathbb{D}$ be a regular Mal'cev category, and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Consider any pullback of split epimorphisms:*

$$\begin{tikzcd}
X' \arrow[r, "x"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\
Y' \arrow[r, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, shift right, "s"']
\end{tikzcd}$$

then the factorization γ towards the following pullback P is a regular epimorphism:

$$\begin{tikzcd}
IX' \arrow[rr, "Ix"] \arrow[dd, shift right, "If'"'] \arrow[dr, two heads, "\gamma"] & & IX \arrow[dd, shift right, "If"'] \\
& P \arrow[ur] \arrow[dl] & \\
IY' \arrow[rr, "Iy"'] \arrow[uu, shift right] & & IY \arrow[uu, shift right, "Is"']
\end{tikzcd}$$

If moreover Ix is a monomorphism, then I preserves this pullback.

Proof: This is a consequence of the fact that the factorization $\eta : X' \to P$ between the two pullbacks, induced by the three regular epimorphisms η_X, η_Y and $\eta_{Y'}$, is a regular epimorphism in a Mal'cev category by Lemma 2.5.7 in [2]. When moreover Ix is a monomorphism, then γ is also a monomorphism. Consequently it is an isomorphism, and I preserves the pullback in question. ■

Then we get to the next important point:

Proposition 3.4. *Let $\mathbb{D}$ be a regular Mal'cev category, and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. The functor I preserves pullbacks of pairs of split epimorphism. Accordingly it preserves kernel equivalence relations of split epimorphisms, and the image $I(\underline{X}_1)$ of any internal groupoid $\underline{X}_1$ is an internal groupoid.*

Proof: Using the same notation as that of the last proposition, suppose the map y is split by ξ. It suffices to prove that the factorization γ is a monomorphism, which we shall obtain by proving that the two kernel equivalence relations $R[\eta_{X'}]$ and $R[\eta]$ are the same. Consider now the following diagram:

$$\begin{tikzcd}
R[\eta_{X'}] \arrow[rr, shift left, "R(x)"] \arrow[dd, shift right, "R(f')"'] \arrow[dr, "\bar{\gamma}"'] & & R[\eta_X] \arrow[ll, shift left] \arrow[dd, shift right, "R(f)"'] \arrow[dl, shift left] \\
& R[\eta] \arrow[ur, shift left] \arrow[dl, shift left] & \\
R[\eta_{Y'}] \arrow[uu, shift right] \arrow[rr, shift right, "R(y)"'] \arrow[ur, shift left] & & R[\eta_Y] \arrow[ll, shift right] \arrow[uu, shift right, "R(s)"']
\end{tikzcd}$$

The lower quadrangle is a pullback of split epimorphisms since it is constructed from the pullbacks defining P and X'. The factorization $\bar{\gamma}$ comes from the factorization γ. It is a monomorphism since it compares the two kernel equivalence relations $R[\eta_{X'}]$ and $R[\eta]$. But also it makes the two squares of split epimorphism commute, and since $\mathbb{D}$ is a Mal'cev category, it is a regular epimorphism. Accordingly this map γ is an isomorphism, and, as expected, we have $R[\eta_{X'}] \simeq R[\eta]$. ■

Proposition 3.5. *Let $\mathbb{D}$ be a regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. If a morphism f is I-trivial, then we have $IR[f] \simeq R[If]$. In other words, the functor I preserves the kernel equivalence relation of any I-trivial morphism. Accordingly, any I-trivial map (and a fortiori any I-trivial extension) is I-normal.*

Proof: Consider the following diagram with f I-trivial:

$$\begin{tikzcd}
R[\eta_{R[f]}] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "\bar{\chi}"'] & R[f] \arrow[r, two heads, "\eta_{R[f]}"] \arrow[d, equal] & IR[f] \arrow[d, "\chi"] \\
R[R(\eta_X)] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[f] \arrow[r, two heads, "R(\eta_X)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[If] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
R[\eta_X] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "R(f)"'] & X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"] & IX \arrow[d, "If"] \\
R[\eta_Y] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

Since the lower right hand side square is a pullback, any square of the two lower levels are pullbacks. This implies that the map $R(\eta_X)$ is a regular epimorphism. There is a factorization χ, since $R[If]$ lies in $\mathbb{C}$, which consequently is also a regular epimorphism.

On the other hand, the factorization $\bar{\chi}$, induced by χ, is a monomorphism since it compares two equivalence relations. It is also a regular epimorphism in the Mal'cev category $\mathbb{D}$ since it is, inside a square of split epimorphism (the splittings being given by the maps s_0), a factorization towards a pullback (given by the middle left hand square), see Theorem 2.2.9 in [2]. Accordingly $\bar{\chi}$ is an isomorphism, we have $R[\eta_{R[f]}] \simeq R[R(\eta_X)]$ and, $\eta_{R[f]}$ being a regular epimorphism, the factorization χ is a monomorphism. Being also a regular epimorphism, this χ is an isomorphism, and we get $IR[f] \simeq R[If]$. According to Proposition 2.1, the I-trivial map is I-normal. ■

From Propositions 3.4 and 2.1, it is straightforward that, in this context, *any I-normal split epimorphism is I-trivial.* In the last step of this section, we shall extend a bit this kind of observation:

Lemma 3.2. *Suppose $\mathbb{D}$ is a regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Given a commutative diagram, where t is a split epimorphism,* *and f' is I-normal:*

$$\begin{tikzcd}
& R[f] \arrow[r, shift right, "R(t)"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[f'] \arrow[l, shift right, "R(s)"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
R[t] \arrow[ur, "\tau"] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, shift right, "t"'] \arrow[d, "f"'] & X' \arrow[l, shift right, "s"'] \arrow[d, "f'"] \\
& Y \arrow[r, equal, "1_Y"'] & Y
\end{tikzcd}$$

then the factorization τ is I-trivial.

Proof: The following square is a pullback of split epimorphisms:

$$\begin{tikzcd}
R[f] \arrow[r, shift right, "R(t)"'] & R[f'] \arrow[l, shift right, "R(s)"'] \\
R[t] \arrow[u, "\tau"] \arrow[r, shift right, "t.p_1"'] & X' \arrow[l, shift right, "s_0.s"'] \arrow[u, "s_0"']
\end{tikzcd}$$

Since t is a split epimorphism, we have $IR[t] = R[It]$ and the map $I\tau$ is necessarily a monomorphism. According to Proposition 3.3, this pullback is preserved by I. On the other hand, the map f' being I-normal, the map $p_0 : R[f'] \to X'$ and, consequently, the map $s_0 : X' \to R[f']$ are I-trivial. Accordingly τ is I-trivial, being the pullback of an I-trivial map under a pullback which is preserved by I. ■

Proposition 3.6. *In the commutative diagram of the previous lemma, when f and f' are I-normal, then the split epimorphism t is I-trivial.*

Proof: When f is I-normal, the map $p_0 : R[t] \to X$ is the result of the composition of two I-trivial maps: $R[t] \xrightarrow{\tau} R[f] \xrightarrow{p_0} X$. Then it is I-trivial, and t is I-normal. Being also split, this map t is I-trivial, as mentioned earlier. ■

3.1. The Birkhoff reflections.

We shall now request a slightly stronger assumption for the reg-epi reflection I.

Definition 3.1. *Let $j : \mathbb{C} \rightarrowtail \mathbb{D}$ be a full replete inclusion and $\mathbb{D}$ a regular category. We shall say that a reflection $I : \mathbb{D} \to \mathbb{C}$ is a Birkhoff reflection, when it is a reg-epi reflection such that for any regular epimorphism $f : X \twoheadrightarrow$*

Y the factorization $R(f)$ is a regular epimorphism:

$$\begin{tikzcd}
R[\eta_X] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] \arrow[d, "R(f)"'] & X \arrow[r, two heads, "\eta_X"] \arrow[d, "f"'] & IX \arrow[d, "If"] \\
R[\eta_Y] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & Y \arrow[r, two heads, "\eta_Y"'] & IY
\end{tikzcd}$$

When I is a Birkhoff reflection, the right hand square above is a pushout. Accordingly $\mathbb{C}$ is stable under regular epimorphism and is certainly a regular category. Since $\mathbb{C}$ is also stable under monomorphism, we conclude that $\mathbb{C}$ is a Birkhoff subcategory of $\mathbb{D}$ in the sense of [19]. The reg-epi reflection $\underline{\Sigma}_1 : Grd\mathbb{E} \to Req\mathbb{E}$ is not a Birkhoff reflection, since $Req\mathbb{E}$ is not stable under regular epimorphism inside $Grd\mathbb{E}$ (see the projections $\underline{\epsilon}_1\underline{X}_1$).

When $\mathbb{D}$ is a Mal'cev category, the previous condition is equivalent to *"for any regular epimorphism $f : X \twoheadrightarrow Y$ the factorization $R(\eta_X)$ is a regular epimorphism"*. When $\mathbb{D}$ *is an exact Mal'cev category*, we have a converse of our previous observation: *if $\mathbb{C}$ is stable under regular epimorphism, then any reg-epi reflection is a Birkhoff reflection.* So any Birkhoff subcategory $\mathbb{C}$ of an exact Mal'cev category $\mathbb{D}$ determines a Birkhoff reflection. When $\mathbb{D}$ is an exact Mal'cev category which is finitely cocomplete, this is precisely the case of the subcategory $M\mathbb{D}$ of *commutative objects*, as we shall recall below.

Lemma 3.3. *Let $\mathbb{D}$ be a regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. It is a Birkhoff reflection if and only if, given any regular epimorphism $f : X \twoheadrightarrow Y$, the internal groupoid $IR[f]$ has an effective support. Accordingly it is a Birkhoff reflection if and only if I preserves the internal groupoids with effective support.*

Proof: Suppose I is a Birkhoff reflection. Consider the following diagram:

$$\begin{tikzcd}
R[f] \arrow[rr, shift left, "p_0"] \arrow[rr, shift right, "p_1"'] \arrow[dr, "R(\eta_X)"'] \arrow[d, "\eta_{R[f]}"'] & & X \arrow[r, two heads, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
IR[f] \arrow[r, dotted, "\chi"'] & R[If] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & IX \arrow[r, two heads, "If"'] & IY
\end{tikzcd}$$

The map $R(\eta_X)$ being a regular epimorphism, so is the dotted factorization which consequently makes $R[If]$ the effective support of the internal groupoid $IR[f]$. As a consequence I preserves any internal groupoid with effective support.

Conversely suppose the internal groupoid $IR[f]$ has an effective support R. Since the map If is the coequalizer of $IR[f]$, it is the effective quotient of R, and we get $R = R[If]$. So $R[If]$ is the support of the internal groupoid $IR[f]$. Accordingly the factorization χ is a regular epimorphism, which implies that $R(\eta_X)$ is a regular epimorphism. ■

Corollary 3.1. *Let $\mathbb{D}$ be a regular Mal'cev category. Any exact reg-epi reflection $I : \mathbb{D} \to \mathbb{C}$ is a Birkhoff reflection (a functor being called exact when it preserves the kernel equivalence relations of any regular epimorphism).*

Proposition 3.7. *Let $\mathbb{D}$ be a regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection. Then I is admissible. Accordingly the I-trivial extensions, I-normal maps and extensions, I-central extensions are stable under pullbacks.*

Proof: Consider the following diagram with the regular epimorphism ϕ in $\mathbb{C}$ and lower rectangle being a pullback:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "\eta_{R([f]}"] \arrow[rr, bend left, two heads, "R(g)"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[r, "IR(g)"] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] & R[\phi] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
X \arrow[r, two heads, "\eta_X"'] \arrow[rr, bend left, two heads, "g"] \arrow[d, "f"'] & IX \arrow[r, two heads, "Ig"'] \arrow[d, "If"] & A \arrow[d, "\phi"] \\
Y \arrow[r, two heads, "\eta_Y"'] & IY \arrow[r, equal] & IY
\end{tikzcd}$$

So are the corresponding two upper ones. The maps η_X and $\eta_{R([f]}$ being regular epimorphisms, then, since $\mathbb{D}$ is a Mal'cev category, any of the upper squares are pullbacks by Proposition 3.2. This makes the internal functor $IR(g) : IR[f] \to R[\phi]$ a discrete fibration. Since its codomain is an equivalence relation, its domain $IR[f]$ is an equivalence relation, which is effective since I is a Birkhoff reflection. Accordingly we have $IR[f] \simeq R[If]$. The pullback properties of the upper squares can be shifted to the lower squares by the Barr-Kock theorem, so Ig is an isomorphism and f is I-trivial. The last point is a consequence of Lemma 2.1. ■

This result was stated in [23] (See Theorem 3.5 of) with slightly different assumptions. Another important consequence is the following:

Corollary 3.2. *In the conditions of the previous proposition, a regular epimorphism f is I-cartesian if and only if it is I-trivial.*

Proof: We already noticed that an I-trivial map is certainly I-cartesian. Conversely, consider a regular epimorphism $f : X \twoheadrightarrow Y$. According to the previous proposition, the pullback $\bar{f}$ of If along η_Y is I-trivial and thus I-cartesian above $I\bar{f} = If$. If, moreover, f is I-cartesian above If, then, since the maps f and $\bar{f}$ are I-cartesian above the same map $If = I\bar{f}$, they are the same up to isomorphism and f is I-trivial. ■

Eventually, we get a large class of examples of regular reflection:

Proposition 3.8. *Let $\mathbb{D}$ be a regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection. Then the functor I preserves the pullback of split epimorphism along regular epimorphisms; in other words, I is a regular reflection.*

Proof: We can use the proof of Proposition 3.4, where x and y are now just regular epimorphisms. Knowing that $R(x)$ and $R(y)$ are regular epimorphisms since x and y are such:

$$\begin{tikzcd}
R[\eta_{X'}] \arrow[rr, two heads, "R(x)"] \arrow[dr, two heads, "\bar{\gamma}"'] \arrow[dd, shift right, "R(f')"'] & & R[\eta_X] \arrow[dd, shift right, "R(f)"'] \\
& R[\eta] \arrow[ur, two heads] \arrow[dl, shift left] & \\
R[\eta_{Y'}] \arrow[uu, shift right] \arrow[ur, shift left] \arrow[rr, two heads, "R(y)"'] & & R[\eta_Y) \arrow[uu, shift right, "R(s)"']
\end{tikzcd}$$

the monomorphic factorization $\bar{\gamma}$ is still a regular epimorphism by Lemma 3.1. ■

This result was already noticed [16] in the much more restricted context of semi-abelian categories, i.e. pointed finitely cocomplete exact protomodular categories [20].

3.2. Associated universal I-central and I-normal extension. According to Section 2.2, when $\mathbb{D}$ is an efficiently regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection (and thus a regular reflection), with any extension we can associate an I-central extension. The Mal'cev context brings an important precision:

Lemma 3.4. *Let $\mathbb{D}$ be an efficiently regular Mal'cev category, $I : \mathbb{D} \to \mathbb{C}$ a Birkhoff reflection and $\underline{f}_1 : \underline{X}_1 \twoheadrightarrow \underline{Y}_1$ a regular epimorphic internal functor. Then the comprehensive factorization of $\underline{f}_1$, when it exists, is such that the* *internal functor* $\underline{\hat{f}}_1$ *is a regular epimorphism:*

$$\begin{tikzcd}
X_1 \arrow[r, two heads, "\hat{f}_1"] \arrow[rr, bend left, "f_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T_1 \arrow[r, two heads, "\overline{f}_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X_0 \arrow[u] \arrow[r, two heads, "\hat{f}_0"'] \arrow[rr, bend right, "f_0"'] & T_0 \arrow[u] \arrow[r, two heads, "\overline{f}_0"'] & Y_0 \arrow[u]
\end{tikzcd}$$

Proof: Let us go back to the structural diagram above Proposition 1.2. When $\mathbb{D}$ is a regular Mal'cev category, then the internal functor $\underline{\psi}_1 : Dec\underline{X}_1 \to \underline{U}_1$ is a regular epimorphism, since it is a levelwise regular epimorphism. For instance, the map $\psi_0 : X_1 \to U_0$ is given by the following factorization towards the pullback U_0:

$$\begin{tikzcd}
X_1 \arrow[rr, two heads, "f_1"] \arrow[dr, "\psi_0"] \arrow[dd, shift right=2, "d_1"'] & & Y_1 \arrow[dd, shift right=2, "d_1"'] \\
& U_0 \arrow[ur] \arrow[dl, "\delta_1"] & \\
X_0 \arrow[uu, dotted, shift right=2] \arrow[rr, two heads, "f_0"'] & & Y_0 \arrow[uu, dotted, shift right=2]
\end{tikzcd}$$

and it is a regular epimorphism according to Lemma 3.1. The same argument holds for ψ_1. Accordingly $\hat{f}_0 = \pi_0(\underline{\psi}_1)$ is a regular epimorphism (similarly for $\hat{f}_1$), thus $\underline{\hat{f}}_1$ is a regular epimorphism in $Grd\mathbb{D}$, i.e. a levelwise epimorphic internal functor. ∎

From that, we get immediately:

Corollary 3.3. *Let* $\mathbb{D}$ *be an efficiently regular Mal'cev category and* $I : \mathbb{D} \to \mathbb{C}$ *a Birkhoff reflection. Then, given any extension* f*, the projection to the universal associated* I*-central extension is a regular epimorphism:*

$$\begin{tikzcd}
X \arrow[rr, two heads, "\hat{\eta}"] \arrow[dr, two heads, "f"'] & & \bar{X} \arrow[dl, two heads, "\bar{f}"] \\
& Y &
\end{tikzcd}$$

Actually, in the context of efficiently regular categories, when I is only a reg-epi reflection, we get the two following relevant pieces of information:

Proposition 3.9. *Let $\mathbb{D}$ be an efficiently regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Then I is admissible.*

Proof: We can follow exactly the proof of Proposition 3.7. The only difference is the reason why the equivalence relation $IR[f]$ is effective. This time, it comes from the fact that the discrete fibration $IR(g) : IR[f] \to R[\phi]$ lies in an efficiently regular category. ■

Proposition 3.10. *Let $\mathbb{D}$ be an efficiently regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Then with any extension $f : X \twoheadrightarrow Y$ is associated a universal I-normal extension $\bar{f} : \bar{X} \twoheadrightarrow Y$. Moreover, we have $I\bar{X} \simeq IX$, $I\bar{f} \simeq If$ and $IR[\bar{f}] \simeq IR[f]$, and the map $\hat{\eta} : X \twoheadrightarrow \bar{X}$ is a regular epimorphism.*

Proof: In the proof of Theorem 2.1, the assumption *"I is a regular reflection"* was mainly used since it implied the coincidence of I-central and I-normal extensions, but the construction itself delt with the associated I-normal extension. We can mimic here, step by step, the proof of this theorem. The main point was that the regular reflection I preserved the following pullbacks of split epimorphisms along regular epimorphism:

$$\begin{tikzcd}
R[\bar{f}] \arrow[r, two heads, "\bar{\eta}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & IR[f] \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
\bar{X} \arrow[r, two heads, "\bar{\eta}_0"'] & IX
\end{tikzcd}$$

But, in the Mal'cev context, thanks to Proposition 3.2, this is true for this particular pullback even when I is only a reg-epi reflection. ■

4. Preservation of products

Given a full replete inclusion $j : \mathbb{C} \rightarrowtail \mathbb{D}$, we investigated the preservation by a reflection $I : \mathbb{D} \to \mathbb{C}$ of certain kinds of pullbacks. We shall have a look now to the preservation of products. Of course when $\mathbb{D}$ is pointed (i.e. when it has a zero object), any terminal map is split and any product is a special case of a pullback of a split epimorphism along a split epimorphism. So we shall be interested in the non pointed case. An object X has a *global support* when the terminal map $X \twoheadrightarrow 1$ is a regular epimorphism.

4.1. The reg-epi reflections.

Proposition 4.1. *Suppose $\mathbb{D}$ is a regular category and $I : \mathbb{D} \to \mathbb{C}$ an admissible reflection. If the projection $p_X : X \times C \twoheadrightarrow X$ is a regular epimorphism, with $C \in \mathbb{C}$, it is I-trivial, and we have $I(X \times C) \cong IX \times C$. This is the case when C has a global support, or when there is a map $x : X \to C$. In particular, the projection $p_X : X \times IX \twoheadrightarrow X$ is I-trivial, and we have $I(X \times IX) \cong IX \times IX$.*

Proof: Since the terminal map $C \to 1$ is in $\mathbb{C}$, the regular epimorphism p_X, as a pullback of this map, is I-trivial according to Lemma 2.1. So both the following left hand side square and the following rectangle are pullbacks:

$$\begin{tikzcd}
X \times C \arrow[r, two heads, "\eta_{X\times C}"] \arrow[rr, two heads, bend left, "\eta_X \times 1_C"] \arrow[d, two heads, "p_X"'] & I(X \times C) \arrow[r, two heads, "(Ip_X, Ip_C)"] \arrow[d, two heads, "Ip_X"] & IX \times C \arrow[d, two heads, "p_{IX}"] \\
X \arrow[r, two heads, "\eta_X"'] & IX \arrow[r, equal] & IX
\end{tikzcd}$$

Accordingly the right hand square is a pullback (by Lemma 3.1). Consequently, $I(X \times C) \cong IX \times C$. ■

We shall be now interested in the objects X such that $I(X \times X) \simeq IX \times IX$.

Proposition 4.2. *Let $\mathbb{D}$ be an efficiently regular category and $I : \mathbb{D} \to \mathbb{C}$ a regular reflection. If X is an object such that $I(X \times X) \simeq IX \times IX$ and if X and Y have global supports, then the regular reflection I preserves their product.*

Proof: Consider the following diagram where Y has a global support. Its left hand part is made of pullbacks of split epimorphism along a regular epimorphism:

$$\begin{tikzcd}
X \times X \times Y \arrow[r, shift left, "p_1 \times 1_Y"] \arrow[r, shift right, "p_0 \times 1_Y"'] \arrow[d, two heads, "p_{(X\times X)}"'] & X \times Y \arrow[r, two heads, "p_Y"] \arrow[d, two heads, "p_X"] & Y \arrow[d, two heads] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] & X \arrow[r, two heads] & 1
\end{tikzcd}$$

Accordingly this left hand part is preserved by I, as pullbacks, and produces a vertical discrete fibration:

$$\begin{tikzcd}
I(X \times X \times Y) \arrow[r, shift left, "I(p_1 \times 1_Y)"] \arrow[r, shift right, "I(p_0 \times 1_Y)"'] \arrow[d, "Ip_{(X \times X)}"'] & I(X \times Y) \arrow[r, two heads, "Ip_Y"] \arrow[d, "Ip_X"] & IY \arrow[d] \\
I(X \times X) \arrow[r, shift left, "Ip_1"] \arrow[r, shift right, "Ip_0"'] & IX \arrow[r, two heads] & 1
\end{tikzcd}$$

Since $I\nabla X = \nabla IX$ is an effective equivalence relation and $\mathbb{D}$ efficiently regular, the upper horizontal internal groupoid is an effective equivalence relation. If, moreover X has a global support, then $p_Y : X \times Y \twoheadrightarrow Y$ is a regular epimorphism and is the quotient of the upper equivalence relation in our first diagram. Accordingly Ip_Y is the quotient of our upper effective equivalence relation in the second one. Then the Barr-Kock theorem makes the right hand side square a pullback which shows that I preserves the product in question. ■

We are now looking for those objects X such that $I(X \times X) \simeq IX \times IX$. For that let us call *$I$-normal* an object X which has an I-normal terminal map $X \to 1$.

Lemma 4.1. *Suppose $\mathbb{D}$ is a regular category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. An I-normal object X is such that $I(X \times X) \simeq IX \times IX$ if and only if X is in $\mathbb{C}$.*

Proof: It is clear that if X is in $\mathbb{C}$ we have $I(X \times X) \simeq IX \times IX$. Conversely let X be an I-normal object such that $I(X \times X) \simeq IX \times IX$. Consider the following diagram:

$$\begin{tikzcd}
X \times X \arrow[r, two heads, "\eta_{(X \times X)}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & I(X \times X) \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X \arrow[r, two heads, "\eta_X"'] \arrow[d, "q"'] & IX \arrow[d, "Iq"] \\
Q \arrow[r, two heads, "\eta_Q"'] \arrow[d, tail] & IQ \arrow[d, tail] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

The support Q of X, as a subobject of 1, is such that η_Q is an isomorphism. Moreover $I\nabla X \simeq \nabla IX$ is an effective equivalence relation, and the upper squares are pullbacks since X is I-normal. Accordingly the middle square is a pullback, η_X is an isomorphism, and X is in $\mathbb{C}$. ■

Proposition 4.3. *Let $\mathbb{D}$ be an efficiently regular Mal'cev category and $I : \mathbb{D} \to \mathbb{C}$ a reg-epi reflection. Then $I(X \times X) \simeq IX \times IX$ for all objects X if and only if all I-normal objects are in $\mathbb{C}$.*

Proof: Suppose all I-normal objects are in $\mathbb{C}$. Given any object X, take the comprehensive factorization of the internal functor $\nabla X \to I\nabla X$:

$$\begin{tikzcd}
X \times X \arrow[r, two heads] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & \bar{X} \times \bar{X} \arrow[r, two heads] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & I(X \times X) \arrow[d, shift right, "Ip_0"'] \arrow[d, shift left, "Ip_1"] \\
X \arrow[r, two heads] \arrow[d, "q"'] & \bar{X} \arrow[r, two heads] \arrow[d, "\bar{q}"] & IX \arrow[d, "Iq"] \\
Q \arrow[r, equal] & Q \arrow[r, two heads, "\eta_Q"'] & IQ
\end{tikzcd}$$

It is the same construction as the one used for the associated I-normal extension of Proposition 3.10. Then $\bar{X}$ is I-normal, and thus in $\mathbb{C}$. We have moreover $IX \simeq I\bar{X} = \bar{X}$, and thus $I(X \times X) \simeq I(\bar{X} \times \bar{X}) = \bar{X} \times \bar{X} = IX \times IX$. Conversely suppose we have $I(X \times X) \simeq IX \times IX$ for any X. This holds for any I-normal object X which must be in $\mathbb{C}$ according to the previous lemma. ■

4.2. The reflection to the commutative objects. It appears that the context of Mal'cev categories $\mathbb{D}$ particularly fits with the notion of commutator of equivalence relations [9], see also [22]. It is then possible to define an object X in $\mathbb{D}$ as being *commutative* when we have $[\nabla X, \nabla X] = 0$, i.e. when the commutator $[\nabla X, \nabla X]$ is trivial, or, equivalently, when the object X is equipped with a (unique possible) Mal'cev operation $p : X \times X \times X \to X$. We shall denote by $M\mathbb{D}$ the subcategory of the commutative objects X in $\mathbb{D}$ and by $j : M\mathbb{D} \rightarrowtail \mathbb{D}$ the inclusion functor. The subcategory $M\mathbb{D}$ is stable under finite limits and under subobjects. It is a naturally Mal'cev category in the sense of [21]: any object is endowed with a natural Mal'cev operation, or, equivalently, any reflexive graph is an internal groupoid, or, again equivalently, any pair, R and S, of equivalence relations on the same object X is such that $[R, S] = 0$, i.e. admits a centralizing double equivalence relation.

When moreover $\mathbb{D}$ is regular, $M\mathbb{D}$ is stable under regular epimorphisms, and is consequently a Birkhoff subcategory of $\mathbb{D}$. If $\mathbb{D}$ is efficiently regular, then so is $M\mathbb{D}$. When moreover $\mathbb{D}$ is finitely cocomplete, the inclusion $j : M\mathbb{D} \rightarrowtail \mathbb{D}$ admits a reg-epi reflection (see [7]), denoted by $M : \mathbb{D} \to M\mathbb{D}$. The subcategory $M\mathbb{D}$ being a Birkhoff subcategory, *when* $\mathbb{D}$ *is exact this reflection is a Birkhoff reflection.* In this section we shall suppose that $\mathbb{D}$ is a finitely cocomplete efficiently regular Mal'cev category. Let us begin by recalling the following results of [10].

Lemma 4.2. *Let* $\mathbb{A}$ *be an efficiently regular naturally Mal'cev category, and* $f : X \to Y$ *a morphism in* $\mathbb{A}$*. Then there is an object* $N[f]$ *in* $\mathbb{A}$ *such that the following right hand side square is a pullback:*

$$\begin{tikzcd}
\Theta \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[d, shift right=3, "p_0"'] \arrow[d, dotted, shift left=3, "p_1"] & R[f] \arrow[r, two heads, "\nu(f)"] \arrow[d, shift right=3, "p_0"'] \arrow[d, dotted, shift left=3, "p_1"] & N[f] \arrow[d] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u] & X \arrow[r] \arrow[d, "f"'] \arrow[u] & 1 \\
& Y &
\end{tikzcd}$$

It (actually the pair $(N[f], \nu(f))$*) is called the* metakernel *of the map* f*.*

Proof: The centralizing double equivalence relation on the left hand side comes from the fact that in the naturally Mal'cev category $\mathbb{A}$ we always have $[\nabla X, R[f]] = 0$. Any commutative square, on this side, is a pullback. So the maps p_0 produce a discrete fibration between the two horizontal equivalence relation. Now, the lower one, ∇X, is effective and, since $\mathbb{A}$ is efficiently regular, it is the same for the upper one which, consequently, admits a quotient $\nu(f) : R[f] \twoheadrightarrow N[f]$. The Barr-Kock theorem implies that the right hand side square is a pullback. ■

The terminology comes from the fact that, when $\mathbb{A}$ is pointed and, thus, additive, this metakernel $N[f]$ coincides with the kernel $K[f]$ of the map f. When f is a terminal map $X \to 1$, we call its metakernel the *direction* of the object X.

Proposition 4.4. *Let $\mathbb{A}$ be an efficiently regular naturally Mal'cev category, and $\underline{X}_1$ an internal groupoid in $\mathbb{A}$. Then the following square is a pullback:*

$$\begin{tikzcd}
X_1 \arrow[r, "\nu(d_0).s_1"] \arrow[d, "d_0"'] & N[d_0] \arrow[d] \\
X_0 \arrow[r] & 1
\end{tikzcd}$$

Proof: Consider the following two pullbacks:

$$\begin{tikzcd}
X_1 \arrow[r, "s_1"] \arrow[d, "d_0"'] & R[d_0] \arrow[r, "\nu(d_0)"] \arrow[d, "p_0"'] & N[d_0] \arrow[d] \\
X_0 \arrow[r, "s_0"'] & X_1 \arrow[r] & 1
\end{tikzcd}$$

We shall set $\lambda_{\underline{X}_1} = \nu(d_0).s_1$. ■

Proposition 4.5. *Let $\mathbb{D}$ be a finitely cocomplete efficiently regular Mal'cev category. Then the reg-epi reflection $M : \mathbb{D} \to M\mathbb{D}$ is such that for any object X we have $M(X \times X) \simeq MX \times MX$.*

Proof: According to Proposition 4.3, we have to show that any M-normal object is commutative. For that consider the following diagram, with X an M-normal object:

$$\begin{tikzcd}
\Theta \arrow[r, dotted, shift left, "p_1"] \arrow[r, dotted, shift right, "p_0"'] \arrow[d, dotted, shift right, "p_0"'] \arrow[d, dotted, shift left, "p_1"] & X \times X \arrow[r, "\eta_{(X\times X)}"] \arrow[d, shift right, "p_0"'] \arrow[d, dotted, shift left, "p_1"] & M(X \times X) \arrow[r, "\lambda_{M\nabla X}"] \arrow[d, shift right, "M(p_0)"'] \arrow[d, dotted, shift left, "M(p_1)"] & N[M(p_0)] \arrow[d] \\
X \times X \arrow[r, dotted, shift left, "p_1"] \arrow[r, dotted, shift right, "p_0"'] & X \arrow[r, "\eta_X"'] & MX \arrow[r] & 1
\end{tikzcd}$$

The middle square is a pullback since X is M-normal, and the right hand side one too according to the previous proposition. So that the right hand side rectangle is a pullback. Its completion by the horizontal kernel equivalence relations produces the centralizing double relation which shows that we have $[\nabla X, \nabla X] = 0$, and that X is commutative. ■

Proposition 4.6. *Let $\mathbb{D}$ be a finitely cocomplete exact Mal'cev category. Suppose X and Y are two objects with global support. Then the Birkhoff reflection M preserves their product.*

Proof: The category $\mathbb{D}$ being an exact Mal'cev category, the reg-epi reflection M is a Birkhoff reflection, and thus a regular reflection. So we can apply Proposition 4.2. ■

4.3. A remark on central extensions. In the Mal'cev context, a morphism $f : X \to Y$ is classically said to be *central* when we have $[R[f], \nabla X] = 0$. The following proposition gathers part of Theorem 4.6 of [23] and Theorem 6.1 of [15]:

Proposition 4.7. *Let $\mathbb{D}$ be a finitely cocomplete efficiently regular Mal'cev category. A map f is central if and only if it is M-normal. When $\mathbb{D}$ is exact, f is a central extension if and only if f is an M-central extension.*

Proof: Suppose f is M-normal. Consider the following rectangle:

$$\begin{tikzcd}
R[f] \arrow[r, "\eta_{R[f]}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dotted, "p_1"] & M(R[f]) \arrow[r, "\lambda_{M(R[f])}"] \arrow[d, shift right, "M(p_0)"'] \arrow[d, shift left, dotted, "M(p_1)"] & N[M(p_0)] \arrow[d] \\
X \arrow[r, "\eta_X"'] & MX \arrow[r] & 1
\end{tikzcd}$$

It is a pullback as made of two pullbacks. The completion by the horizontal kernel equivalences produces the double centralizing relation which gives us $[\nabla X, R[f]] = 0$. Conversely suppose f central. Then the following left hand side double centralizing relation

$$\begin{tikzcd}
\Theta \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & R[f] \arrow[r, two heads, "q"] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & C \arrow[d] \\
X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[u] & X \arrow[r] \arrow[u] \arrow[d, "f"'] & 1 \\
 & Y &
\end{tikzcd}$$

makes a vertical discrete fibration which, via the quotient q of the upper horizontal effective equivalence relation, produces the right hand side pullback. Let us show that C in $M\mathbb{D}$: the right hand side square being a pullback, the image of $R[p_0]$ by the regular epimorphism q is $q(R[p_0]) = \nabla C$; from $[\nabla X, R[f]] = 0$, we get $[R[f], R[f]] = 0$ and then $[R[p_0], R[p_0]] = 0$. Whence, according to [9], $[q(R[p_0]), q(R[p_0])] = 0$, and thus: $[\nabla C, \nabla C] = 0$ which means $C \in M\mathbb{D}$. Finally, since the category $\mathbb{D}$ is efficiently regular, and, thus, the reg-epi reflection M is admissible, the map f is thus M-normal

(Lemma 2.1). When D is exact, M is a Birkhoff reflection, and the M-normal and M-central extensions coincide. ■

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Dominique Bourn
Laboratoire de Mathématiques Pures et Appliquées, Université du Littoral, 50 rue F. Buisson B.P. 699, 62228 Calais, France
E-mail address: `Dominique.Bourn@lmpa.univ-littoral.fr`

Diana Rodelo
Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, and Centre for Mathematics of the University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: `drodelo@ualg.pt`