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SEARCH FOR THE GLOBAL OPTIMUM OF LEAST VOLUME TRUSSES

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This paper describes two types of strategies that find the global optimum of structures designed for minimum volume consumption. This bilinearly constrained problem may present multiple optima and some examples of this nonconvex behaviour are given. In the first method two branch and bound (B & B) based approaches are presented associated with suitable convex underestimating functions. The second is a cutting plane method and is derived as a generalization of Bender's algorithm; although the relaxed problems yielded are still nonconvex, they are amenable by standard codes for 0–1 mixed LP. Frequently structural engineers are confronted with only a limited set of discrete alternatives; both solution techniques presented here are combinatorial in nature and suitable to be cast into a discrete variable model, for which the algorithms converge to the global optimum in a much smaller number of steps.

KEYWORDS: Trusses, global optimum, branch and bound, Bender's decomposition

1 INTRODUCTION

In the 25th year since the paper by Schmidt at the Second Conference on Electronic Computation[1], it can be stated that mathematical optimization techniques have played a central role in the development of new approaches to structural design with the aid of computers, either for fixed (weight minimization problem) or for variable (shape optimization problem) geometry. In both cases the specific literature is so numerous that one may only tentatively refer to it emphasizing the reviews[2,3].

The methods that mostly have been proposed for the solution of the above problems can be divided into two different categories: the optimality criteria approach and the mathematical programming approach. It is generally accepted that both approaches at present show shortcomings that are mainly due to the nonlinear and nonconvex character of the optimization problem itself, being unable to guarantee the obtaining of a global as opposed to merely a local solution.

The minimum volume of trusses possessing a linear elastic behaviour is an example of such problems possessing a large field of practical applications and being themselves of intrinsic importance. In fact this one stress-resultant problem can be extended to the elasto-plastic behaviour of the members and the solution of both plate and frame problems (where the nonconvex behaviour becomes more notorious due to the higher nonlinearity of the problem).

Two strategies more appropriate to nonconvex optimization are presented here: The B & B approach can be applied to the solution of separable problems such as functions of one variable and their products. This technique can be accepted as competitive, since each resulting subproblem is solved by using a linear programming (LP) technique. Alternatively a generalization of Bender's algorithm using dual functions instead of dual variables allows the definition of a master problem equivalent to the minimum volume design. Even though the master problem is a 0–1 mixed LP, it has a special nature allowing it to be transformed into a LP in 0–1 variables.

2 EXAMPLES OF NONCONVEX BEHAVIOUR IN BILINEAR STRUCTURAL OPTIMIZATION PROBLEMS

The following simple examples illustrate some of the problems often encountered in optimal design:

2.1 *Grillage Design*

The grillage presented in Figure 1 is made up of orthogonal sandwich beams loaded normal to its plane. The geometry of the structure is assumed to be known including the number of beams, span lengths and support conditions. Constraints are imposed upon the bending stresses at node B, m_1, m_3 and in the critical section between nodes AB, m_2 and BC, m_4.

The optimal design problem is to find the cross sectional areas y_1 and y_2 such that:

$$\min l_1 y_1 + l_2 y_2 \tag{1}$$

$$\text{st } s_{1l} \leq s_1 = m_1/w_1 \leq s_{1u} \tag{2}$$

$$s_{2l} \leq s_2 = m_2/w_1 \leq s_{2u} \tag{3}$$

$$s_{3l} \leq s_3 = m_3/w_2 \leq s_{3u} \tag{4}$$

$$s_{4l} \leq s_4 = m_4/w_2 \leq s_{4u} \tag{5}$$

$$y_{1l} \geq y_{1l};\ y_2 \geq y_{2l} \tag{6}$$

By using the nodal-stiffness method the bilinear equation for the deflection at B, d_B is:

$$6E[I_1/(l_1/2)^3 + (I_2/(l_2/2)^3]\, d_B = \lambda \tag{7}$$

$$\lambda = 5(q_1 l_1 + q_2 l_2)/8 + P \tag{8}$$

$$s_{il} = -15.0 = -s_{iu};\ i = 1, \ldots, 4 \tag{9}$$

$$y_{jl} = 5.0 \qquad j = 1, 2 \tag{10}$$

in which E (the Young modulus) equals unity and λ is the nodal force.

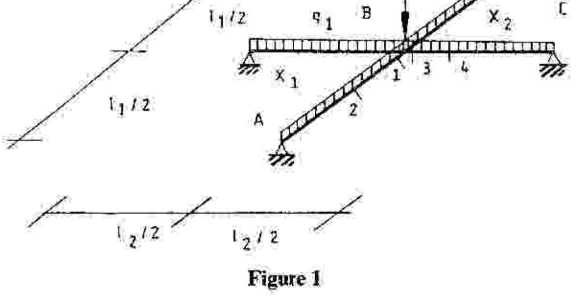

Figure 1

Case I Assume the following relationships:

$$q_1 = 0.8 = q_2; P = 0 \tag{11}$$

$$l_1 = 30.0; \quad l_2 = 31.0 \tag{12}$$

$$w_j = 0.5y_j; \quad I_j = 0.3y_j; j = 1, 2 \tag{13}$$

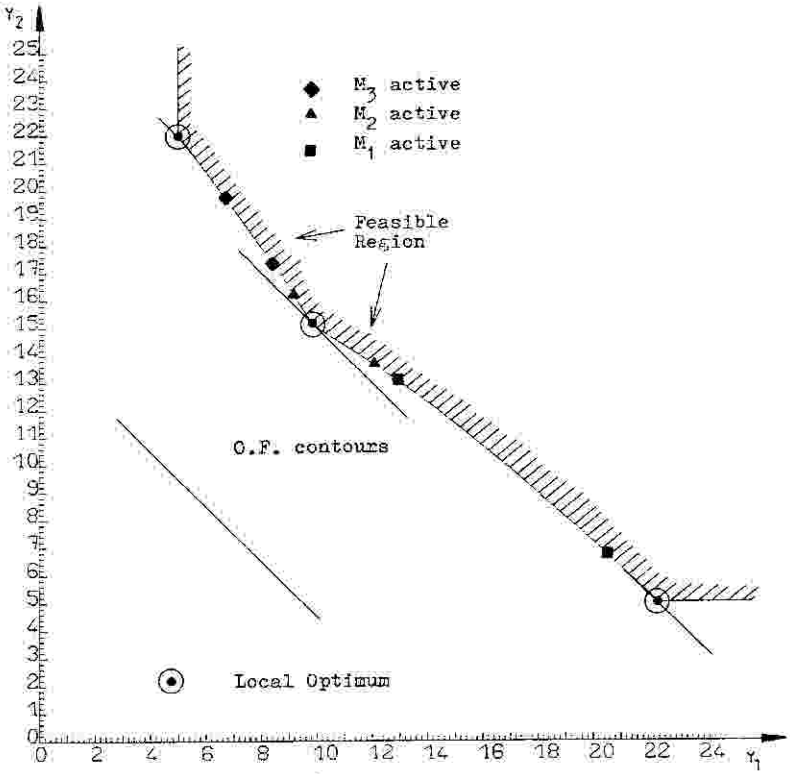

Figure 2

Three local optima are obtained; as shown in Figure 2.

$$y_1 = 5.0; y_2 = 22.0; \text{OF} = 832.0$$
$$s_1 = -9.3; s_2 = 14.3; \quad s_3 = -15.0; s_4 = 15.0$$
$$y_1 = 9.9; y_2 = 15.1; \text{OF} = 765.1$$
$$s_1 = -14.8; s_2 = 15.0; \quad s_3 = -15.0; s_4 = 15.0$$
$$y_1 = 22.4; y_2 = 5.0; \text{OF} = 826.1$$
$$s_1 = -15.0; s_2 = 15.0; \quad s_3 = -6.3; s_4 = 13.0$$

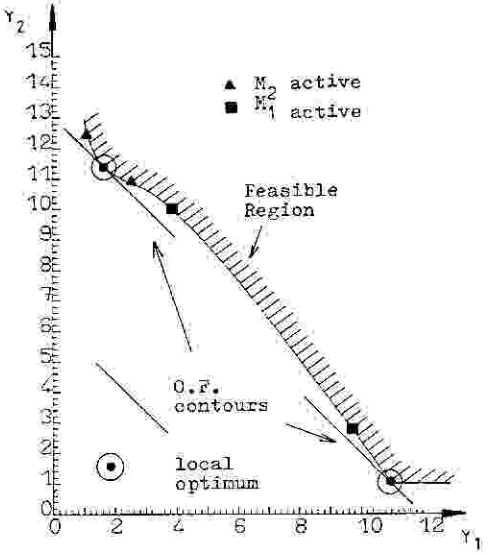

Figure 3

Case II Here:

$$q_1 = 1.2 = q_2; \quad p = 25.0 \tag{14}$$

$$l_1 = 21.0; \quad l_2 = 24.0 \tag{15}$$

$$w_j = 5.0y_j; \quad I_j = 25.0y_j \tag{16}$$

$$y_{jl} = 1.0; \quad j = 1.2 \tag{17}$$

$$s_{il} = -5.0 = -s_{iu}; i = 1, \ldots, 4 \tag{18}$$

In this case two distinct local minima would be obtained, as shown in Figure 3.

$$y_1 = 1.7; y_2 = 11.3; \text{OF} = 307.5$$
$$s_1 = -4.7; s_2 = 5.0; \quad s_3 = -4.7; s_4 = 4.7$$
$$y_1 = 11.0; y_2 = 1.0; \text{OF} = 255.0$$
$$s_1 = -5.0; s_2 = 5.0; \quad s_3 = 0.2; s_4 = 4.2$$

2.2 Weight Minimization of a Ten Bar Truss

The second example concerns the optimization of the statically indeterminate truss of Figure 4 loaded by two vertical forces and subject to stress and displacement constraints. Loading vector:

$$\lambda^T = [0 \quad 0 \quad 0 \quad -10 \quad 0 \quad 0 \quad 0 \quad -10]$$

The lower bound on all cross sections is 0.1 and the maxima allowable stresses are ± 2.5.

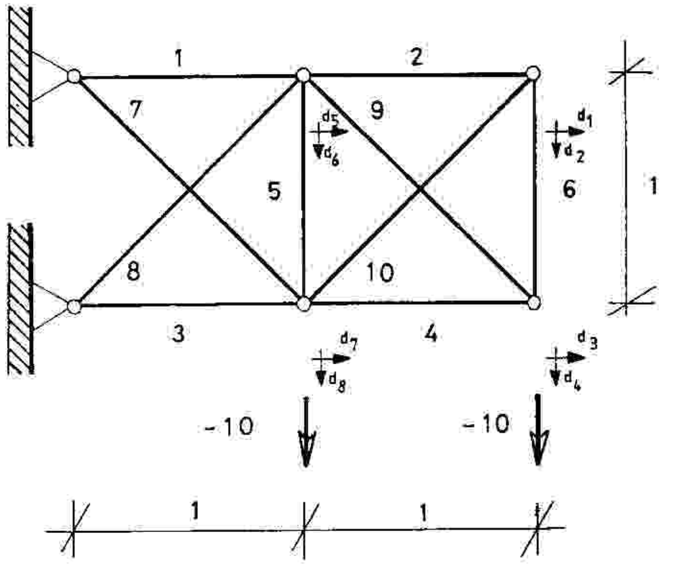

Figure 4

If the nodal displacements are limited to ± 3.5, two optimal solutions, both corresponding to statically indeterminate trusses can occur:

OF = 219.93

$$
\begin{aligned}
a_1 &= 48.7; a_2 = 0.1; a_3 = 35.6; a_4 = 24.1; \ a_5 = 0.1\\
a_6 &= 1.2; a_7 = 9.4; a_8 = 34.3; a_9 = 34.1; a_{10} = 0.1\\
s_1 &= 0.4; s_2 = -0.3; s_3 = -0.6; s_4 = -0.4; \ s_5 = 2.2\\
s_6 &= -0.0; s_7 = 2.9; s_8 = -0.8; s_9 = 0.8; s_{10} = 0.7\\
d_1 &= 0.2; d_2 = -3.5; d_3 = -1.0; d_4 = -3.5\\
d_5 &= 0.4; d_6 = -1.3; d_7 = -0.6; d_8 = -3.5
\end{aligned}
$$

OF = 223.34

$$
\begin{aligned}
a_1 &= 48.7; a_2 = 0.1; a_3 = 38.1; a_4 = 23.3; \ a_5 = 0.1\\
a_6 &= 0.1; a_7 = 13.7; a_8 = 33.1; a_9 = 33.0; a_{10} = 0.1\\
s_1 &= 0.4; s_2 = 0.0; s_3 = -0.5; s_4 = -0.4; \ s_5 = 1.3\\
s_6 &= 0.0; s_7 = 2.0; s_8 = -0.9; s_9 = 0.9; s_{10} = 0.0\\
d_1 &= 0.4; d_2 = -3.5; d_3 = -1.0; d_4 = -3.5\\
d_5 &= 0.4; d_6 = -1.3; d_7 = -0.5; d_8 = -2.6
\end{aligned}
$$

Both these points are solutions to the Kuhn–Tucker equations corresponding to different sets of Lagrange multipliers. Both locally minimize the volume of the structure. Any feasible direction linking these solutions would determine (at least) one point having a greater volume than the local optima.

3 BRANCH AND BOUND METHODS

The existence of multiple solutions in the bilinearly constrained optimization problem leads us to consider methods of solution more appropriate for this nonconvex behaviour. The general nonconvex domain is transformed in the B & B strategy into a sequence of intersecting convex domains by the use of underestimating convex functions. It is well known that a local solution to a problem possessing a convex objective function and being restricted by convex inequalities is also its global solution.

The two main ingredients are a combinatorial tree with appropriately defined nodes and some upper and lower bounds to the final solution associated with each node of the tree. It is then possible to eliminate a large number of possible solutions without evaluating them. As the implicit enumeration program relies on an upper bound, its efficiency can be greatly improved by providing a good feasible initial solution.

A partial solution is said to be fathomed if the best feasible completion of the solution can be found or if it can be determined that, no matter how sections are assigned to the remaining free members it will be impossible to find a feasible completion of smaller cost than that previously found. If a partial solution is fathomed this means that all possible completions of this partial solution have been implicitly enumerated and therefore need not be explicitly enumerated. When the last node is fathomed the algorithm terminates with the optimum design. Backtracking in the tree is performed so that no solution is repeated or omited from consideration.

3.1 Transformation of Factorable Functions into Convex Underestimates

Bilinear expressions of the form

$$px + qy + kxy \tag{19}$$

are particular cases of the more general class of problems involving factorable functions. Since the convex underestimate of the linear part of the expression is trivial the envelope of the product term is needed. Let the function $f(x, y) = xy$ be defined in the rectangle of bounds

$$a \leq x \leq b; c \leq y \leq d \tag{20}$$

The function values at the corners must coincide with the underestimate taken. Since three points are enough to define a plane in 3-D, the convex underestimate will be taken as the z coordinate on the highest of the two planes defining a ridge through the two intermediate function values.

$$z_1 = cx + ay - ac \tag{21}$$

$$z_2 = dx + by - bd \tag{22}$$

$$z = \max\{z_1, z_2\} \leq f(x, y) = xy \tag{23}$$

The concave overestimating functions are accordingly defined as the z coordinate at the lowest of the two planes defining a ridge through the corners having the higher and lower function values

$$z_1^0 = dx + ay - ad \tag{24}$$

$$z_2^0 = cx + by - bc \tag{25}$$

$$z^0 = \min\{z_1^0, z_2^0\} \geq f(x, y) = xy \tag{26}$$

When these estimating functions are switched around to another quadrant, the approximations are suitably modified. The underestimating function z (and overestimating z^0) is not differentiable everywhere. There are several ways of handling this by altering the problem in order to create an equivalent LP. The simplest way involves the addition of some extra inequality constraints and variables. Considering the problem:

$$\min c^T x \tag{27}$$

$$\text{st } Ax \geq b \tag{28}$$

$$f^T x + \max\{g_1^T x, g_2^T x\} \leq h \tag{29}$$

$$p^T x + \min\{q_1^T x, q_2^T x\} \geq r \tag{30}$$

This is equivalent to the LP

$$\min c^T x \tag{31}$$

$$\text{st } Dx \geq e \tag{32}$$

$$f^T x + u \leq h \tag{33}$$

$$p^T x + v \geq r \tag{34}$$

$$u \geq g_1^T x \qquad u \geq g_2^T x \tag{35}$$

$$v \leq g_1^T x \qquad v \leq g_2^T x \tag{36}$$

Other formulations may be employed. The introduction of more variables is recommendable when the same nondifferentiable term will appear in a number of constraints.

When the bilinear constraints are equalities (equilibrium equations in the nodal-stiffness formulation), they should be replaced by a pair of inequalities. This amounts to duplication in the number of constraints and variables. Fortunately, it can be shown that in the structural optimization problem one of them is always nonactive and need not be considered. This reduces significantly the size of each LP.

3.2 Outside-In Approach

In the sequel the algorithm originally presented by Soland[4] for separable functions is outlined and it is shown that it can easily be extended to cover the convex underestimating functions described above. Let x^p be the solution of the linear underestimating subproblem (P_p), with $k = 1$

$$\min c^T x \tag{37}$$

$$\text{st } Ax \geq b \tag{38}$$

$$l_p \leq x \leq L_p \tag{39}$$

If x^p is not a feasible solution of the original problem, one may try to strengthen the constraints or to restrict the domain of the subproblem (P_p) in order to make the solution x^p feasible. (P_p) is replaced by a set of problems that bound the original problem in the sense that there exist one optimal solution x^* for at least one problem $j \in W^p$. Suppose an optimal solution to each such problem is obtained and let

$$x^s = \min_{j \in W^p} c^T x^j \tag{40}$$

If x^s is not a feasible solution of the original problem one of the problems of the bounding set is replaced by a set of new problems. Make $p = p + 1$. The problem s is replaced by a set W^p, such that $W^p = (W^{p-1} - \{s\}) \cup W^s$ contains an optimal solution of the original problem feasible for at least one problem W^p.

For each problem $j \in W^p$ either x^j is infeasible for j, or $c^T x^j > c^T x^*$. This is a condition ensuring that some progress towards the final solution is made.

The combinatorial tree has each node identified with a subproblem j. The problems that replace j in the bounding set W^p are pointed to by the branches directed outward from that node. At any intermediate point in the calculations, the set W^p of the current bounding problems is identified with the set of nodes that are the leaves of the tree.

Each node of the tree is associated with an incumbent bound v. Any leaf node of the tree whose bound is strictly less than v is active. Otherwise it is designated as terminated and need not be considered in any further computation. The B & B tree will be developed until every leaf can be terminated.

It is also required to define (in a heuristic way) a refining rule for splitting the bounds on the variables: Choose the index i of the variable that maximizes the difference between the factorable form and its convex underestimate out of the more violated constraint by x^s in the original problem. The corresponding interval is divided into two new intervals $[l_i, x_i]$ and $[x_i, L_i]$.

Therefore as soon as a node is selected to be branched, the partition of its interval is only dependent on its solution value and is not related to other partitions at the same level of the tree. This corresponds to a weaker form of the convergence theorem not requiring the completion of the intervals partitioned.

Application The following problem is bilinearly constrained and similar to those representing the optimal design of trusses:

$$\min x_1 + x_2 + x_3 \tag{41}$$

$$\text{st } x_1 x_4 + x_3 x_6 = 0 \tag{42}$$

$$3x_1 x_4 + 1.2x_2 x_5 - x_3 x_6 = 10 \tag{43}$$

$$5x_4 + x_5 + x_6 \leq 2.5 \tag{44}$$

$$0.1 \leq x_1 \leq 5.0; 0.1 \leq x_2 \leq 5.0; 0.1 \leq x_3 \leq 5.0 \tag{45}$$

$$0 \leq x_4 \leq 2.5; 0 \leq x_5 \leq 2.5; -2.5 \leq x_6 \leq 0 \tag{46}$$

This mathematical problem presents three local optima, namely:

$$x_1 = 0.1; x_2 = 3.33; x_3 = 0.1; x_4 = 0; x_5 = 2.5; x_6 = 0$$
$$\text{OF} = 3.53$$
$$x_1 = 0.5; x_2 = 3.0; x_3 = 0.1; x_4 = 0.5; x_5 = 2.5; x_6 = -2.5$$
$$\text{OF} = 3.60$$
$$x_1 = 2.5; x_2 = 0.1; x_3 = 1.0; x_4 = 1.0; x_5 = 0; x_6 = -2.5$$
$$\text{OF} = 3.60$$

When it is intended to represent this nonconvex problem graphically, it is not possible to go beyond two equations in three unknowns, at least without being placed in a subspace of the solution space. Figure 5 represents a perspective view where all objective values below a given level are printed.

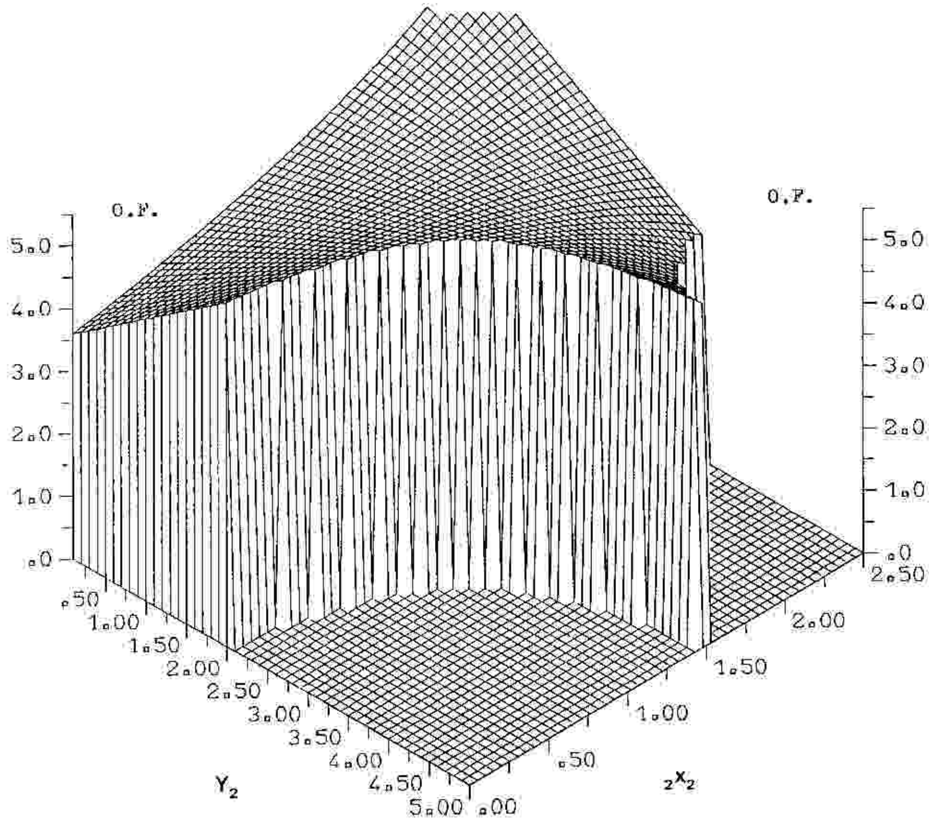

Figure 5 Plot for various y_2 and x_2 and keeping x_3 fixed.

Replacing the factorable functions by their convex underestimates (since the determination of the linear functions' convex envelopes is trivial) it is possible to obtain (in the more lengthy formulation) a LP in 10 variables and 15 inequality constraints. Any node of the B & B tree is defined for each set of bounds on the variables x_1 to x_6. The results of the branching strategy known as breadth first (choose the node with lower bound) are represented in the combinatorial tree of Figure 6.

Computational experience Several other convex underestimates were tried out[5] leading in some cases to problems with reduced dimensionality. Nevertheless the number of lower convex estimates (LCE) increased very significantly mainly due to the (poor) quality of the alternative underestimates employed.

Although tending to require less storage space, the number of LPs necessary to find the global optimum increased by 30–35% when an alternative search strategy known as depth first (as opposed to breadth first) is used. This strategy is: pick up the right hand sucessor of the current node, otherwise backtrack to the predecessor of the current node and reapply the rule.

The branching rule, requiring that the index of the variable to be partitioned is selected according to the terms belonging to the more violated constraints, has a more

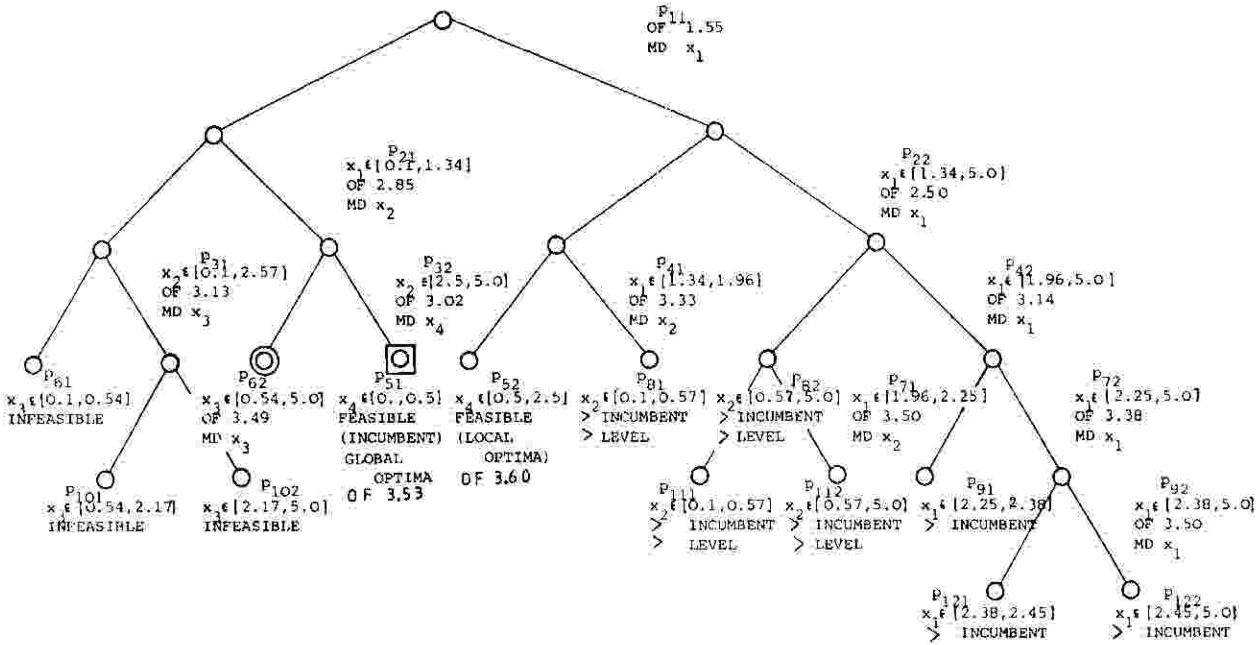
P11
OF 1.55
MD x1
P21
x1 ∈ [0.1,1.34]
OF 2.85
MD x2
P22
x1 ∈ [1.34,5.0]
OF 2.50
MD x1
P31
x2 ∈ [0.1,2.57]
OF 3.13
MD x3
P32
x2 ∈ [2.5,5.0]
OF 3.02
MD x4
P41
x1 ∈ [1.34,1.96]
OF 3.33
MD x2
P42
x1 ∈ [1.96,5.0]
OF 3.14
MD x1
P61
x3 ∈ [0.1,0.54]
INFEASIBLE
P62
x3 ∈ [0.54,5.0]
OF 3.49
MD x3
P51
x4 ∈ [0.,0.5]
FEASIBLE
(INCUMBENT)
GLOBAL
OPTIMA
OF 3.53
P52
x4 ∈ [0.5,2.5]
FEASIBLE
(LOCAL
OPTIMA)
OF 3.60
P81
x2 ∈ [0.1,0.57]
> INCUMBENT
> LEVEL
P82
x2 ∈ [0.57,5.0]
> INCUMBENT
> LEVEL
P71
x1 ∈ [1.96,2.25]
OF 3.50
MD x2
P72
x1 ∈ [2.25,5.0]
OF 3.38
MD x1
P101
x3 ∈ [0.54,2.17]
INFEASIBLE
P102
x3 ∈ [2.17,5.0]
INFEASIBLE
P111
x2 ∈ [0.1,0.57]
> INCUMBENT
> LEVEL
P112
x2 ∈ [0.57,5.0]
> INCUMBENT
> LEVEL
P91
x1 ∈ [2.25,2.38]
> INCUMBENT
P92
x1 ∈ [2.38,5.0]
OF 3.50
MD x1
P121
x1 ∈ [2.38,2.45]
> INCUMBENT
P122
x1 ∈ [2.45,5.0]
> INCUMBENT

Figure 6

marked influence in the increase of the number of LPs needing to be solved. By dividing the interval corresponding to the index of the variable chosen into half instead of partitioning it according to the value given in the LCE there is an increase in the number of LPs of 70–80%.

3.3 Inside-out Approach

An alternative B & B algorithm, originally presented by Reeves[6] and intended for all quadratic programming, can also be adapted to programmes in which the constraints and/or the objective function are reducible to factorable forms. It relies on performing each iteration over a subinterval of the original domain given by bounds on the variables. It consists of three basic steps:

The first step of each iteration is to determine a base point from which to branch and bound. Highly desirable points are local optima of the minimum volume design, although it is possible to start the algorithm with a point that may be not a local optimum (or even feasible). If this algorithm is used as a verification procedure then the local solution obtained by convex programming techniques is an ideal starting point.

Once a base point is obtained the second step consists of eliminating an interval surrounding it. For a feasible point x^V, where V represents the iteration number of the algorithm, an interval is eliminated for which x^V is the global solution of the optimization problem. It is composed of three basic substeps. First the interval under consideration is divided into subintervals around its base point. Next a region of each subinterval is defined over which the base point is global to the original problem. Finally the total elimination interval is formed from the union of the regions eliminated over the individual subintervals. For infeasible x^V an interval, for which the LP derived using convex underestimates (LCE) is not feasible, is found and eliminated.

In the last step the B & B section is entered. Uneliminated regions are partitioned in subintervals and a LCE problem is solved for each of them. These lower bounds are compared to the value of the best upper bound (incumbent solution). All previously uneliminated intervals with bounds which equal or exceed the incumbent are eliminated from further consideration. If any subset of the original domain remains, a new iteration is initiated over the uneliminated subinterval with the lower bound around a new base point. Figure 7 represents the different B & B strategies described.

The termination criteria for verifying the global minimum for a fixed set of bounds requires that the solution point y^V given by the LCE on the same interval should also

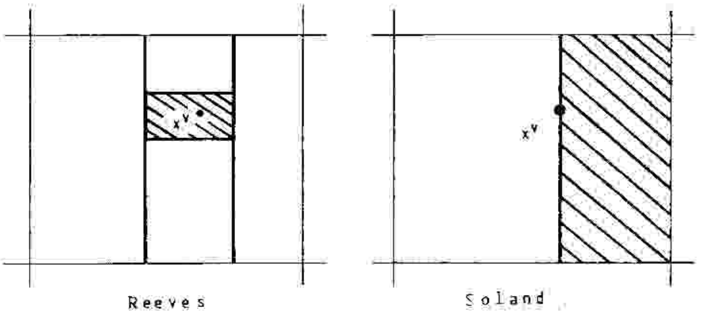

Figure 7

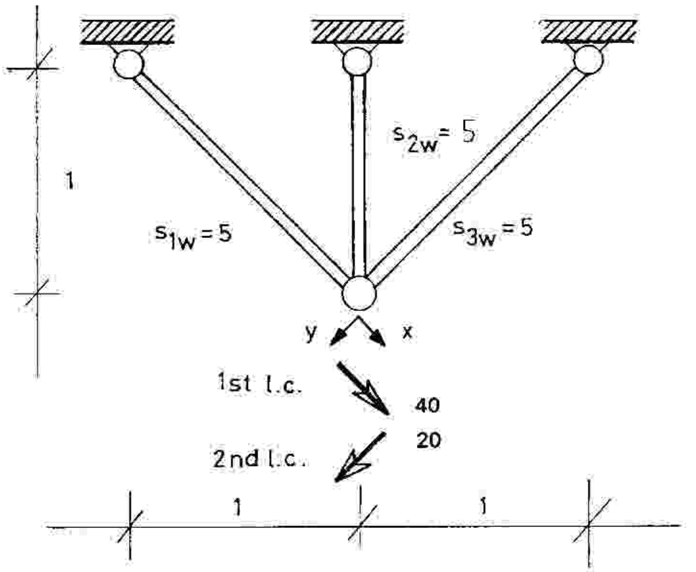

Figure 8

be feasible to the original problem. Considering the convex underestimates taken, this implies that all design variables are endpoints.

Application The optimum of the three bar truss of Figure 8 subject to stress constraints, allowing for a continuous variation of the area variables and undergoing two loading conditions obtained by a convex minimizer routine, is verified in this section.

Using the nodal stiffness description the original problem can be stated as follows:

$$\min \sqrt{2}a_1 + a_2 + \sqrt{2}a_3 \tag{47}$$

$$\text{st } \sqrt{2}/2a_1d_1^1 + 1/2a_2(d_1^1 + d_2^1) = 40 \tag{48}$$

$$1/2a_2(d_1^1 + d_2^1) + \sqrt{2}/2a_3d_2^1 = 0 \tag{49}$$

$$\sqrt{2}/2a_1d_1^2 + 1/2a_2(d_1^2 + d_2^2) = 0 \tag{50}$$

$$1/2a_2(d_1^2 + d_2^2) + \sqrt{2}/2a_3d_2^2 = 20 \tag{51}$$

$$0 \leq \sqrt{2}/2d_1^1 \leq 5; -5 \leq \sqrt{2}/2d_1^2 \leq 0 \tag{52}$$

$$0 \leq \sqrt{2}/2(d_1^1 + d_2^1) \leq 5; 0 \leq \sqrt{2}/2(d_1^2 + d_2^2) \leq 5 \tag{53}$$

$$-5 \leq \sqrt{2}/2d_2^1 \leq 0; 0 \leq \sqrt{2}/2d_2^2 \leq 5 \tag{54}$$

$$1 \leq a_1 \leq 11; 1 \leq a_2 \leq 4; 1 \leq a_3 \leq 5 \tag{55}$$

After approximating the nonconvex functions with the envelopes for factorable functions and setting the optimal design areas as one of the bounds in each resulting LP (LCE), for $V = 1$:

Step 1 The optimal value is

$$\text{OF} = 15.969; a_1 = 7.024; a_2 = 2.138; a_3 = 2.756$$

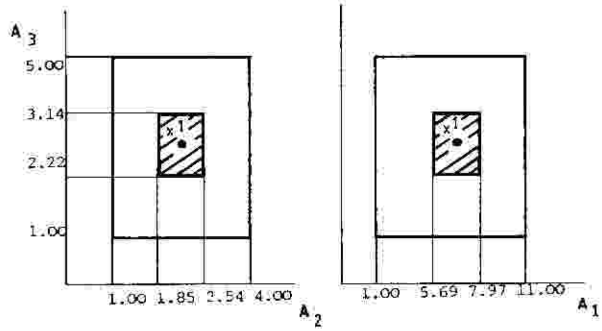

Figure 9

Step 2 First one tries to eliminate the entire range of a_j, i.e. $[a_{jl}, a_{ju}]$ for as many j as possible. The purpose of this is to determine whether or not we can reduce branching and bounding in Step 3, that is to minimize the total number of LCE.

By limiting the maximum infeasibility of the LCE solutions for which the scaled points would have a volume that exceeds the incumbent, the elimination interval was obtained after two runs of 8 LCE.

The total elimination interval is given by:

$$e_l^1 = [5.69 \quad 1.85 \quad 2.22]; e_u^1 = [7.97 \quad 2.54 \quad 3.14]$$

Step 3 One now takes the region remaining after the interval eliminated has been removed from the entire range and partitions it into $3 \times 2 = 6$ regions.

Subintervals 1, 2, 5 and 6 can be eliminated (1 and 5 because their LCE has no feasible solution, 2 and 6 because their lower bound exceeds the incumbent).

For the next iteration $V = 2$ of the algorithm one branches to interval 4 with the lowest bound. As the base point is infeasible, the procedure is therefore simplified. It is not necessary to make any subdivision and there is only one interval to be considered. The algorithm would proceed, terminating after 8 cycles comprising 78 LCE subproblems.

Discussion In many instances the first minimum found will be global. This approach tends to accelerate the remainder of the algorithm in which it is only verified whether the minimum is global or whether a better local minimum can be found by repeating

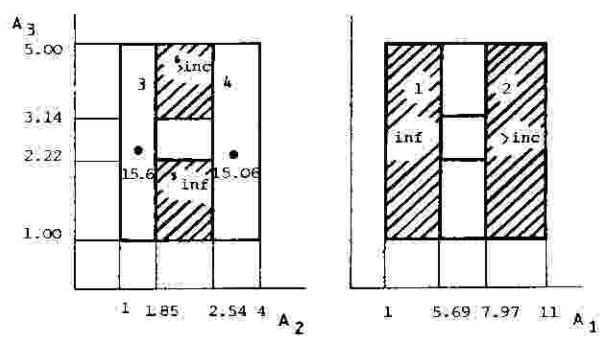

Figure 10

the process. Further obtaining a local minimum and eliminating the region around it increases the likelihood that the LCE will be either infeasible or with a solution value that will exceed the incumbent. Theoretically, in verifying an optimum, the number of elimination subintervals determined at Step 2 grows exponentially with the problem size. This appears to be a major drawback to apply this approach. A factor which tends to lessen the importance of the growth rate in the number of elimination subintervals is that as the problem size increases it is increasingly unlikely that all subintervals need to be investigated. Another criticism is related to the trial and error procedure for determining the endpoints for the elimination interval. An exponentially increasing number of LPs may need to be solved a number of times.

4 RESOURCE-DECOMPOSITION APPROACH

Bender's resource-decomposition algorithm, initially used[7] to solve convex and partially convex problems involving two types of variables, has an economic interpretation upon viewing the Lagrange multipliers as shadow prices in this decomposition scheme that is not truly decentralized. The central agency makes the final decisions by assigning optimal weights to subsystem proposals; moreover, old offers are never forgotten by the coordinating unit.

If one set of variables y in the bilinearly constrained minimization is held fixed the resulting problem in the x set of variables becomes a much easier optimization task. Although the former problem is not convex in the x and y variables jointly, by fixing y renders it so in x (a LP). The key idea that enables the problem (P):

$$\min c^T y \tag{56}$$

$$\text{st } g_i^T x + x^T H_i y + f^T y \geq b_i;\ i = 1, \ldots, m \tag{57}$$

$$0 \leq y \leq y_{\max} \qquad \text{i.e. } y \in Y \tag{58}$$

$$x_l \leq x \leq x_u \qquad \text{i.e.: } x \in X \tag{59}$$

to be viewed as a problem in the x-space is the concept of projection:

$$\min v(x) \tag{60}$$

$$\text{st } x \in X \cap V \tag{61}$$

where

$$v(x) = \text{infimum } c^T y \tag{62}$$

$$\text{st } g_i^T x + x^T H_i y + f^T y - b_i \geq 0 \tag{63}$$

$$0 \leq y \leq y_{\max} \tag{64}$$

and

$$V = \{x: g_i^T x + x^T H_i y + f^T y - b_i \geq 0 \text{ for some } y \in Y\} \tag{65}$$

Note that $v(x)$ is the optimal value of (P) for fixed x, and evaluating $v(x)$ is much easier than solving the bilinearly constrained problem itself.

Denoting by $(P(\underline{x}))$ the optimization problem:

$$\min c^T y \tag{66}$$

$$\text{st } g_i^T x + x^T H_i y + f^T y - b_i \geq 0;\ i = 1, \ldots, m \tag{67}$$

$$0 \leq y \leq y_{\max} \tag{68}$$

The set V consists of those values x for which $(P(\underline{x}))$ is feasible and $X \cap V$ can be thought of as the projection of the feasible region of (P) onto X-space. The difficulty with the use of the latter as a route for solving (P) is that the function v and the set V are only known implicitly via their definitions.

In order to overcome this difficulty, a cutting plane method is devised that builds up approximations to v and V. The central idea is to use linear duality theory applied to v and V after projecting the original problem. The master problem (MP) will be solved via a process of relaxation that generates dominating approximations to v and V. This is accomplished by obtaining the optimal multiplier vectors for $(P(\underline{x}))$ corresponding to various trial values of x and adding new cuts to the relaxed master problem as needed.

4.1 *Formulation of the Master Problem*

The master problem which is equivalent to (P) is originated by a sequence of three manipulations.

(A) Project (P) onto x resulting in $(P(\underline{x}))$.

(B) Invoke the natural dual representation of V in terms of the intersection of a collection of regions that contain it.

(C) Invoke the natural dual representation of v in terms of a pointwise infimum of a collection of functions that dominate it.

By using these it is possible to define the following MP:

$$\min \eta \tag{69}$$

$$\text{st } \eta - \sum_{i=1,m} u_i^s(b_i - g_i^T x) + \sum_{j=1,n} [-c_j + \sum_{i=1,m} u_i^s(x^T H_i)_j]^+ y_{j\max} \geq 0 \tag{70}$$

$$- \sum_{i=1,m} u_i^c(b_i - g_i^T x) + \sum_{j=1,n} \left[\sum_{i=1,m} u_i^c(x^T H_i)_j \right]^+ y_{j\max} \geq 0 \tag{71}$$

$$\sum_{i=1,m} u_i^c = 1; u_i^s \geq 0; u_i^c \geq 0 \tag{72}$$

where u_i^c and u_i^s are optimal multiplier vectors for cut and support functions, respectively.

However the master problem is of theoretical interest only since it has an enormous number of constraints. But it can be solved via a series of subproblems: at each iteration a relaxed version of the MP containing only few of the constraints of type (71) and (72) is solved. The solution $(\underline{\eta}, \underline{x})$ will be tested for feasibility in the initially unrelaxed master problem by solving the subproblem $(P(\underline{x}))$ or its dual and either new cuts or support functions will be added until a termination criteria shows that a solution of acceptable accuracy has been obtained.

Both the support and cut functions define a piecewise concave region and each relaxed master problem (RMP) will consist of a minimization over a piecewise concave region (i.e.: nonconvex programming) represented in Figure 11. The disjunctive terms in both support and cut functions can be reformulated by introducing binary variables δ so that the RMP becomes a standard mixed 0–1 LP.

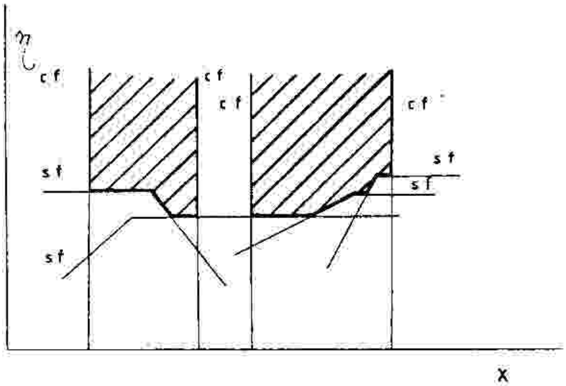

Figure 11

Let L and U be the lower and upper bounds on the affine expression in each term:

$$L \leq f^T x + e \leq U \tag{73}$$

$$0 \leq \delta \leq 1 \tag{74}$$

By introducing a new variable r the disjunction $[f^T x + e]^+$ can be linearized:

$$r \leq \delta U; r \leq (\delta - 1)L + (f^T x + e) \tag{75}$$

The algorithm previously presented requires that each y variable be situated in the non-negative half-space so that x is the vector of member stresses. It is evident that a bar is subjected to either compression or tension, but not to both at the same time. Each member stress may be restricted to vary in the half-space that will be non-positive if the member is compressed or non-negative if the member is tensioned. Now if y and x represent state and design variables, respectively:

$$[u^T H x_j - c]_j + y_{j\max}; 0 \leq y_j \leq y_{j\max} \tag{76}$$

can be transformed into:

$$[(u^T H x_j - c)_j (y_{j\min} + y_{j\max})]^+; y_{j\min} \leq y_j \leq y_{j\max} \tag{77}$$

where either $y_{j\min}$ or $y_{j\max}$ will be zero.

4.2 *Acceleration Algorithm for Structural Optimization*

One of the most basic properties of a truss type of structure is the scaling invariance of the stress/displacement resultant vector: The internal forces in a statically indeterminate structure are a function of the cross section. However, if all areas are multiplied by a positive scaling factor ρ the member forces remain unchanged and all member stresses are multiplied by $1/\rho$. The nodal displacements would also be affected by a factor $1/\rho$ since they can be represented by a linear combination of the member stresses. These scaling properties can be used to devise a simplified version of the R-D algorithm.

By fixing a set x of areas, a unique set of stresses can be determined by inverting the equilibrium equations. Assuming that a feasible set of state variables was determined by using a scaling factor of $0 < \rho < 1$, the structural volume can be reduced until it eventually touches the boundary of the stress/displacement space. Alternatively, if the stress/displacements are outside their rectangle of bounds, the design variables can be multiplied by $\rho > 1$ and the stress resultant vector will be linearly reduced until it fits its bounds. Therefore the vector of areas ρx has at least one member fully stressed or a displacement at its boundary.

Since the relaxed master problem gives a nondecreasing lower bound on the final solution at each iteration, the resulting state variables will be located outside their bounds unless the design is optimal. The infeasible set of areas define the lower bound as $l^T x$ and a scaling factor $\rho > 1$ that makes the solution feasible is determined. The dual ray associated with the infeasible set of areas x can be obtained by multiplying by ρ the dual variables corresponding to the bilinear equilibrium equations, while the multipliers related to the remaining linear active constraints will remain unchanged. The relaxed master program resulting from these considerations (RAC):

$$\min \eta = l^T x \tag{78}$$

$$\text{st} - \sum_{i=1,m} u_i^e b_i + \sum_{j=1,n} \left\{ \left[\sum_{i=1,m} u_i^e (x^T H_i + f_i)_j (y_{jl} + y_{ju}) \right] \right\}^+ \geq 0 \tag{79}$$

$$x \in X; e = 1, \ldots, e_f \tag{80}$$

can also be converted either into a standard 0–1 mixed LP or a complementarity programming problem.

4.3 Application to the Design of Trusses with Discrete Variables

The acceleration algorithm described above (RAC) is particularly suited to be used for structural synthesis with discrete design variables (RAD), becoming a 0–1 mixed LP. The discrete nature of the problem makes the scaled problem generally infeasible in the design variables' space, making the upper bounding stopping criterion meaningless. We remark that the discrete nature of the cross sections makes RAD more efficient than RAC due to the sluggishness of the latter algorithm near a feasible solution.

The three bar truss solved previously will serve the purpose of showing how this cutting plane method works. Here the member sizes differ by a constant step size and the cost is proportional to the area of the cross section, i.e.: a_1, a_2, a_3 are integers. In the relaxed master program, let:

$$x_1 = a_1; x_2 = a_2; x_3 = a_3$$

represent the design variables(areas) and

$$y_1 = d_1^1; y_2 = d_2^1; y_3 = d_1^2; y_4 = d_2^2$$

the resulting displacements.

Initialization Let UB $= +\infty$, $\varepsilon = 0.000$ (since one ends up with the exact solution) and LB $= 3.828$ corresponding to $a_1 = a_2 = a_3 = 1$. From $d^k = K^{-1}\lambda^k (k = 1, 2)$:

$$d^1 = [40.0 - 16.6]^T; d^2 = [-8.3 - 20.0]^T$$

In order to find a feasible set of nodal displacements the scaling factor $\rho = 5.657$ would increase the design variables until the set of stresses would fit within their bounds; this amounts to fully stressing member 1 under the first loading condition. The constraint $e_f = 1$ that cuts off the initial point is:

$$-32.929 + (5x_1 + 2.071x_2 - 7.071)^+ + (-2.071x_2 + 2.071x_3)^+ \geq 0$$

Linearizing the constraint, either by introducing 0–1 variables or by introducing real variables and imposing complementarity, the optimal solution of the relaxed master program, $\eta = \text{LB}$ is 13.728:

$$x_1 = 8;\ x_2 = 1;\ x_3 = 1$$

Solving the equilibrium equations,

$$d^1 = [6.724 \quad -2.784]^T;\ d^2 = [-1.392 \quad 17.148]^T$$

so that $\rho = 2.42$, corresponding to a displacement d_2^2 at its upper bound. The new constraint is given by:

$$-10.069 + [0.35x_1 - 2.784x_2]^+ + [4.235x_3 - 2.784x_2 - 7.071]^+ \geq 0$$

The resulting RAD has for solution LB = 16.142 and

$$x_1 = 7;\ x_2 = 2;\ x_3 = 3$$

$\rho = 1.005$, corresponding to an upper bound on d_1^1. A new constraint should be added to the RAD.

The algorithm terminates after at the fourth iteration (with four cutting planes considered in the RAD), for which the $\eta = \text{LB} = 16.728$ and $x_1 = 7;\ x_2 = 4;\ x_3 = 2$ so that $\rho = 0.996 < 1$, making the UB = 16.728 = LB.

Discussion In general the idea of strengthening the LP relaxation of a 0–1 mixed LP is thought to be very important for a B & B based method. However a straightforward implementation need not work well. Cuts could be used in the OF in a Lagrangian fashion but the need for preserving the structure of the relaxed LP should be considered. Bender's conventional decomposition procedure could be used to solve each relaxed master problem with the advantage of splitting the size of the problem. Moreover it offers the possibility of making sequences of related runs in much less computer time compared with doing each run independently. Moreover, cuts devised to solve one problem can often be revised with little or no work so as to be valid in a modified version of the same problem enhancing the reoptimization capability.

5 CONCLUSIONS

In many cases the solution of a structural synthesis problem with continuous variables found by using an algorithm which obtains local minima is the global solution. The present strategies should be viewed as verification procedures rather than as global optimum-seeking algorithms. Then their use, which is longer in computer time than an algorithm which directly tries to obtain local solutions, can be controlled by those who formulate the problem. The introduction of discrete design variables renders the methods described more tractable, mainly because the convergence to the global optimum is assured in a finite number of steps.

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REFERENCES

1. L. A. Schmit and R. A. Mallett, "Structural Synthesis and Design Parameter Hierarchy", *J. Struct. Div. ASCE*, **89**, 269–299, (1963).
2. V. B. Venkayya, "Structural Optimization: a Review and some Recommendations", *Int. J. Num. Meth. Eng.*, **13**, 203–228, (1978).
3. B. H. V. Topping, "Shape Optimization of Skeletal Structures: a review", *J. Struct. Eng.*, **109**, 1933–1949, (1983).
4. R. M. Soland, "Algorithm for Separable Nonconvex Programming Problem: Nonconvex Constraints", *Man. Sci.*, **17**, 759–773, (1971).
5. L. M. C. Simões, *Exhaustive Search for the Global Optimum of Least Volume Structures*, Ph. D. Thesis, University of London, (1982).
6. G. R. Reeves, "Global Optimization in Nonconvex All Quadratic Programming", *Man. Sci.*, **17**, (1973).
7. A. M. Geoffrion and G. W. Graves, "Multicomodity Distribution Systems Design by Bender's Decomposition", *Man. Sci.*, **20**, 822–844, (1974).
8. G. P. McCormick, "Future directions in Mathematical Programming", in NATO-ASI on *Engineering Plasticity*, University of Waterloo, 1977, M. Z. Cohn and G. Maier (Ed.), Pergamon Press, (1979).