Pareto solution of an inverse problem in elastoplasticity

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An elastic-perfectly plastic discretized structure subjected to given proportional loads, undergoes displacements, some of which are measured. On the basis of this experimental data the yield limits and the hardening coefficients are sought, whereas the elastic properties are known. A number of possible ways of tackling this inverse problem are outlined and discussed. The present paper contains results on the sensitivity analysis for elastoplastic problems in the case of discrete structures modeled by finite elements. This formulation covers situations where inaccuracies of practical significance with known statistical properties affect both the measurements and the modeling of the real system.

INTRODUCTION

Little attention has been paid to inverse problems in quasi-static elastoplasticity. However, there are practical reasons for the indirect identification (through in situ measurements on the overall response) of parameters which characterize the local resistance of complex systems, such as rock masses and other geotechnical formations. The calibration in this sense of an elastoplastic mathematical model embodying the Mohr-Coulomb yield criterion has been investigated.[1] A quite natural measure of discrepancy between the measured and the theoretical displacements is provided by the Euclidean norm of the difference vector. The minimization of this error with respect to the parameters appears to provide a way of identifying these parameters. Direct search techniques have been applied but involve, by its very nature, the solution of a sequence of analysis problems of the system for given parameters and, hence, turns out to be generally time consuming and costly. This special problem of identification under a suitable hypothesis of piecewise linear yield surfaces, and no local unstressing under increasing loads, is amenable to the minimization of a convex quadratic function under linear and complementary constraints. Exploiting this circumstance, the resistance identification was reduced to a particular problem in non-convex quadratic programming.[2] However, this algorithm is limited to structures where the hardening coefficients are assumed as constants.

The formulation proposed here treats the hardening moduli as parameters to be identified (together with the yield). In the inverse problem, it is convenient to have a suitable method for obtaining the sensitivity of the elastoplastic deformation, i.e. directional derivatives of response with respect to variation of design parameters. The sensitivity result can be used in its own right for the solution of the inverse problem and a sequential quadratic programming algorithm is suggested.

Alternatively, this mathematical program is set as a multicriteria optimization and a Pareto solution is sought. By using the maximum entropy formalism a solution may be found indirectly by the unconstrained minimization of a scalar function which is both continuous and differentiable and thus considerably easier to solve. The post-optimality analysis also shows the sensitivity of the parameters to identify with respect to perturbations of the measured displacements. The procedure developed is tested by means of a 50 element elastoplastic beam on an elastoplastic unilateral foundation.

THE ANALYSIS PROBLEM

The problem of elastoplastic analysis of structures modeled by finite elements can, under the usual assumptions of small displacements and deformations, be formulated as quadratic programming problems. For the sake of simplicity, reference will be made to truss-like structures. The matrix relations which govern the elastoplastic response of these structures are known to cover implicitly, just by re-interpretation of symbols, a
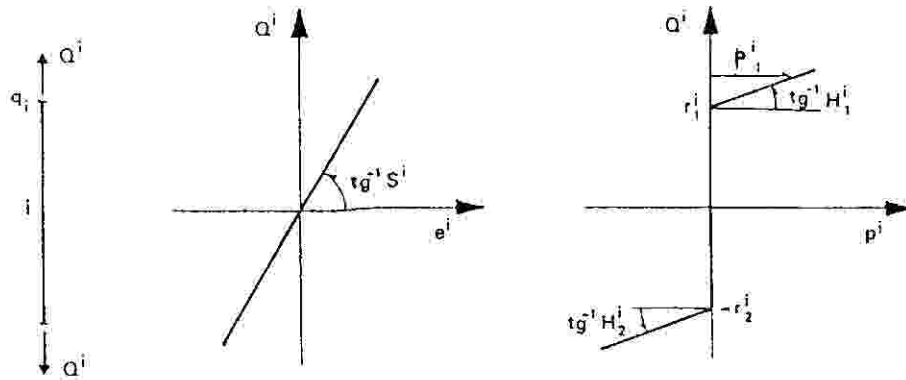
Fig. 1. Stress-strain relation.

broader category of discrete structural models of continua with piecewise yield loci. For the ith element, a reversible (path independent or holonomic) stress-strain relation of the type depicted in Fig. 1 is fully described analytically by the equations:

$$q^i = e^i + p^i = (S^i)^{-1}Q^i + p_1^i - p_2^i \tag{1a}$$

$$\phi_1^i = -Q^i + (r_1^i + H_1^i p_1^i) \geq 0 \tag{1b}$$

$$\phi_2^i = Q^i + (r_2^i + H_2^i p_2^i) \geq 0 \tag{1c}$$

$$p_1^i, p_2^i \geq 0; \quad \phi_1^i p_1^i = 0; \quad \phi_2^i p_2^i = 0 \tag{1d}$$

where q^i is the total generalized strain, S^i denotes the elastic stiffness, Q^i the generalized stress, p_j^i are non-negative measures of the inelastic strain termed plastic multipliers. ϕ_j^i are auxiliary variables, referred to as yield functions. r_j^i and H_j^i are non-negative constants, which define the yield limits and the hardening modulus for each i and $j = 1, 2$, respectively. Plastic flow implies energy dissipation and hence non-negative p_j^i. It can occur in a yield mode if, and only if, the corresponding yield limit is reached.

Holonomic elastoplastic deformability laws which are represented, for one-component two yield mode elements, by the relation set (1) will be considered here in finite (not incremental) quantities. Such laws cover both truly reversible non-linear elastic cases and irreversible elastoplastic situations susceptible to the non-local-unstressing hypothesis under proportional loads. The relations for $i = 1, \ldots, m$ will be assembled for convenience in the following matrix relations,

$$q = (S)^{-1}Q + Np \tag{2a}$$

$$\phi = -N^t Q + Hp + r \geq 0 \tag{2b}$$

$$p \geq 0; \quad \phi^t p = 0 \tag{2c}$$

where S and H are diagonal matrices and N is a diagonal [1-I] matrix. Let u and F denote respectively vectors of the displacements of the free nodes (n degrees of freedom) and of the corresponding given independent nodal loads; Q and q will represent the m-vectors of generalized stresses and strains, respectively. The geometric compatibility and equilibrium equations read:

$$q = Cu \tag{3}$$

$$C^t Q = \alpha F \tag{4}$$

where C is an m by n matrix which depends only on the given layout of the structure and α is the load factor. For structures described by an elastoplastic stress-strain law with workhardening and for a given α, the resulting stress vector Q is given by the minimizer of

$$\text{Min } 1/2\,Q^t S^{-1} Q + 1/2 p^t Hp \tag{5a}$$

subject to,

$$C^t Q = \alpha F \tag{5b}$$

$$\phi = -N^t Q + Hp + r \geq 0 \tag{5c}$$

$$Q \text{ real}; \quad p \geq 0 \tag{5d}$$

This problem has a solution if the design makes the structure capable of carrying the given loads. The dual of eqns (5) is the convex quadratic program

$$\text{Min } 1/2 \begin{bmatrix} u^t & p^t \end{bmatrix} \begin{vmatrix} C^t SC & -C^t SN \\ -N^t SC & N^t SN + H \end{vmatrix} \begin{vmatrix} u \\ p \end{vmatrix}$$

$$+ \begin{bmatrix} \alpha F^t r^t \end{bmatrix} \begin{vmatrix} u \\ p \end{vmatrix} \tag{6a}$$

subject to,

$$u \text{ real}; \quad p \geq 0 \tag{6b}$$

By substitution of vectors q and Q, the relationships (2a), (3) and (4) lead to the following expression for the displacements:

$$u = \alpha u^e + Gp \tag{7a}$$

where,

$$u^e = K^{-1}F, \quad K = C^t SC \quad \text{and} \quad G = K^{-1} C^t SN \tag{7b}$$

The vector u^e represents the elastic displacement vector and marix G transforms vector p into the vector of plastic displacements. By setting

$$Q^e = SCK^{-1}F \tag{8a}$$

$$A = H + N^t SN - G^t KG \tag{8b}$$

one obtains the quadratic program,

$$\text{Min } 1/2\,\mathbf{p}^t\mathbf{A}\mathbf{p} - \mathbf{p}^t(\alpha\mathbf{N}^t\mathbf{Q}^e - r)\mathbf{p} \qquad (9a)$$

subject to,

$$\mathbf{p} \geq 0 \qquad (9b)$$

that is equivalent to eqn (6). At the optimum solution of the elastoplastic analysis problems, eqns (8) and (11), all the matrices and vectors involved are differentiable. Moreover the active constraints columns are linearly independent.

SENSITIVITY ANALYSIS

Elastoplastic analysis problems formulated as quadratic programming problems involve energy functionals, equilibrium and yield constraints that depend on the structural data and loading. The problem of determining the variation of structural response subject to variation of these parameters is considered in this section. A general result for discrete structures is presented and implications for the elastoplastic inverse problem are discussed. The general parametric quadratic program in the form

$$\text{Min } \psi(x, \epsilon) = 1/2\,x^t\mathbf{Q}(\epsilon)x - \mathbf{f}(\epsilon)^t x \qquad (10a)$$

subject to,

$$A_1(\epsilon)^t x - \mathbf{b}(\epsilon) = 0 \qquad (10b)$$

$$A_2(\epsilon)^t x - \mathbf{b}(\epsilon) \leq 0 \qquad (10c)$$

will be considered, where ϵ is a real positive parameter, x are real and the matrices $\mathbf{Q}(\epsilon)$ are symmetric and positive definite. Also, it is assumed that the matrices $\mathbf{Q}(\epsilon), A(\epsilon)^t = [A_1(\epsilon)^t A_2(\epsilon)^t]$ and the vectors $\mathbf{f}(\epsilon), \mathbf{b}(\epsilon)$ are differentiable at $\epsilon = 0$, with derivatives $\mathbf{Q}'$, A' and $\mathbf{f}'$, $\mathbf{b}'$. The Lagrange multipliers for problem (10) are given by the dual problem

$$\text{Min } \psi(x, \epsilon) = 1/2\,\mu^t\mathbf{P}(\epsilon)\mu - \mathbf{g}(\epsilon)^t\mu \qquad (11a)$$

with,

$$\mu_1 \text{ real}; \quad \mu_2 \geq 0 \qquad (11b)$$

where the matrix $\mathbf{P}(\epsilon)$ and the vector $\mathbf{g}(\epsilon)$ are differentiable with respect to ϵ and are given by,

$$\mathbf{P}(\epsilon) = A(\epsilon)^t\mathbf{Q}(\epsilon)^{-1}A(\epsilon);$$

$$\mathbf{g}(\epsilon) = A(\epsilon)^t\mathbf{Q}(\epsilon)^{-1}\mathbf{f}(\epsilon) - \mathbf{b}(\epsilon) \qquad (12)$$

In the dual problem $\mathbf{P}(\epsilon)$ and $\mathbf{g}(\epsilon)$ have derivatives $\mathbf{P}'$ and $\mathbf{g}'$, respectively. For $\epsilon \geq 0$ small enough, the right-derivative (sensitivity) μ' of the Lagrange multipliers is given as the unique solution of

$$\text{Min } 1/2\,v^t\mathbf{P}(o)v - [g' - P'\mu(o)]^t v \qquad (13a)$$

subject to,

$$v_i \text{ real} \quad \text{for} \quad \mu_1(o)_i \text{ real} \qquad (13b)$$

$$v_i > 0 \quad \text{for} \quad a(o)_i^t x - b(o)_i = 0 \quad \text{and} \quad \mu_2(o)_i > 0 \qquad (13c)$$

$$v_i \geq 0 \quad \text{for} \quad a(o)_i^t x - b(o)_i = 0 \quad \text{and} \quad \mu_2(o)_i = 0 \qquad (13d)$$

$$v_i = 0 \quad \text{for} \quad a(o)_i^t x - b(o)_i < 0 \qquad (13e)$$

The stationarity conditions can be used to find sensitivities of the primal variables x',

$$x' = Q(o)^{-1}[-Q'x(o) + f' - A'\mu(o) - A(o)\mu'] \qquad (14)$$

It should be emphasized that this procedure gives the right-derivatives. The left derivatives are the symmetric solutions if the set of active constraints with zero Lagrange multipliers is empty. In elastoplastic analysis of structures with positive strainhardening, the solution is unique and the active yield constraints are associated with positive plastic multipliers. The optimization procedure for the elastoplastic analysis problem can be used to provide the sensitivities as well.

NUMERICAL EXAMPLE

As an example, the plastic deformations and associated sensitivities are computed for the elastoplastic foundation represented in Fig. 2. The model has 26 degrees of freedom and consists of 50 deformable elements. Precisely 24 hinges where the flexural deformability of the beam is lumped, and 26 springs, account for the foundation deformability. The structural model is subjected to live loads only.

For the elastic hardening behaviour characteristic of the elements indicated in Fig. 3, the hardening stiffness equals 5% of the elastic stiffness specified for each element.

The deformation profile at $\alpha = 2\cdot13$ for the beam foundation is shown in Fig. 4, where the springs undergoing plastic strains are indicated as straight lines. It is worth noting that only the 20th hinge has been activated (in sagging bending), whereas the upper yield limit is just reached only in the 17th hinge, but no plastic rotation has developed here.

Figure 5 represents the nodal displacements obtained at different locations by using the sensitivity information
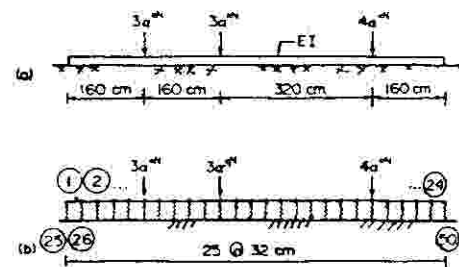

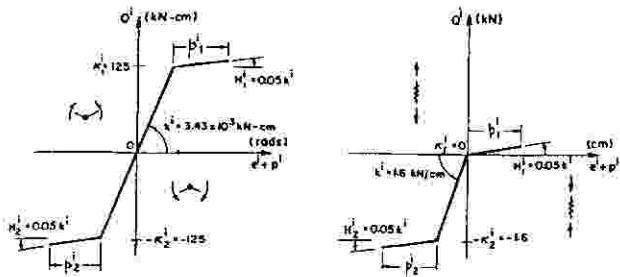**Fig. 2.** Elastoplastic beam on elastoplastic foundation.

Fig. 3

for variations of the beam sagging moment, foundation yield limit and hardening moduli.

The approximations agree with actual results, given that the derivatives of the beam deformations do not change significantly. Since the quadratic coefficients for the sensitivity problem are the same as for the analysis problem, it is necessary to assemble only one stiffness matrix.

THE INVERSE ELASTOPLASTIC PROBLEM

The inverse elastoplastic problem can be described as follows. Whereas the elastic stiffnesses are all known, the strainhardening coefficients defining the diagonal matrix H and the element resistances r will be the parameters to identify. The yield limits depend linearly on some unknown parameters gathered in vector R,

$$r = BR \qquad (15)$$

For instance, the m structural members might be *'a priori'* subdivided into g groups of equal members, in each of which the $2g$ resistances are unknown; then R becomes an identification or colocation matrix of order $2m \times 2g$ with binary entries. The diagonal elements of matrix A in eqn (8b) are directly related to the hardening matrix H. Some (say d) displacements u_{hk}^M are measured in t tests performed on the structure along a loading process, say at levels $\alpha_1, \ldots \alpha_t$. This experimental information $(u_{hk}^m; h = 1, \ldots, d; k = 1, \ldots, t)$ on the overall response of the system to be exploited to determine the unknown parameters R governing the local elemental strength. Let u_{hk}^c indicate the calculated displacements, i.e. the values which would be supplied
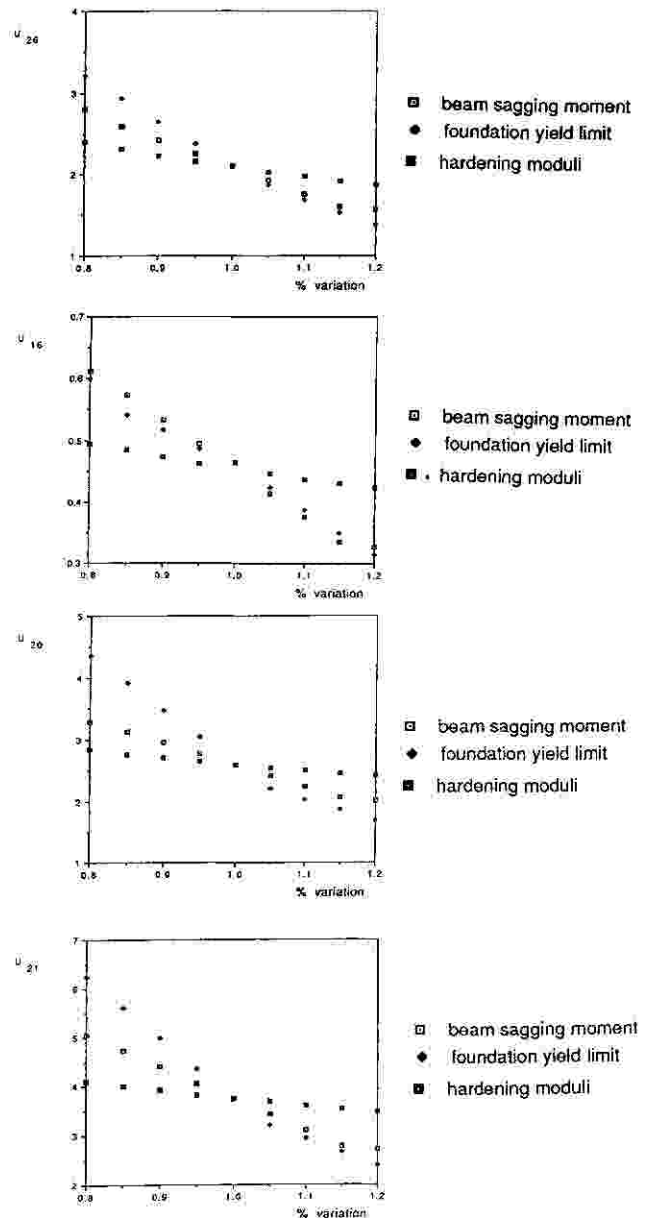

Fig. 5. Nodal displacements.

by the governing relations set (mathematical model) under the same loads for the same displacement components subjected to measurements, generally depends on the values of parameters fed into those relations.

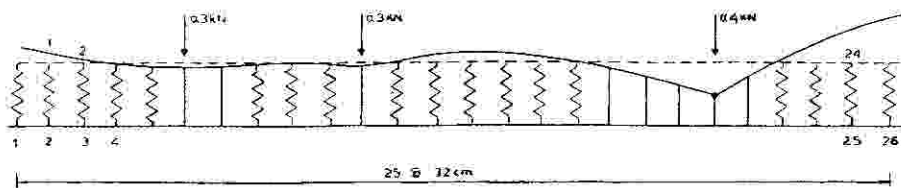A quite natural measure of discrepancy between the

Fig. 4. Deformation profile at $\alpha = 2 \cdot 13$.

measured and the theoretical displacements is provided by the Euclidean norm of the difference vector or its square (should measures exhibit different levels of confidence, then different weighting coefficients would be appropriate).

$$F = \Sigma_{h=1,d}\Sigma_{k=1,t}(u_{hk}^m - u_{hk}^c)^2 \tag{16}$$

The minimization of this error with respect to the parameters appears to provide a way of identifying them.

Entropy-based weighting coefficients

The maximum entropy formalism is a fundamental concept in information theory.[3] In general terms it is concerned with establishing what logical, unbiased inferences can be drawn from available information. On the basis of the information available, there is no logical justification for a criterion which unduly favours one specific coefficient rather than another. In view of Shannon's interpretation of the entropy function it is entirely logical to maximize the entropy of the weighting coefficients w_{hk}. These are obtained by solving the maximum entropy mathematical problem:

$$\text{Max}\, S/K = -\Sigma_{h=1,d}\Sigma_{k=1,t}w_{hk}\ln w_{hk} \tag{17a}$$

subject to,

$$\Sigma_{h=1,d}\Sigma_{k=1,t}w_{hk} = 1 \tag{17b}$$

$$\Sigma_{h=1,d}\Sigma_{k=1,t}w_{hk}g_{hk} = \epsilon \tag{17c}$$

$$w_{hk} \geq 0 \tag{17d}$$

S is the Shannon entropy, K is a positive constant and the g_{hk} represent the levels of confidence (or the square of the discrepancy between measured and theoretical displacements).

Equation (17c) has an unexpected value of zero. The entropy maximization problem has an algebraic solution for w_{hk}:

$$w_{hk} = \frac{\exp[\beta g_{hk}]}{\Sigma_{h=1,d}\Sigma_{k=1,t}\exp[\beta g_{hk}]} \tag{18}$$

in which β, the Lagrange multiplier for eqn (17c) is closely related to ϵ and can be found by substituting result (18) into eqn (17c). It may be deduced that for ϵ to approach zero from above, β must be chosen as positive.

VARIOUS APPROACHES TO THE INVERSE PROBLEM

In this section a number of possible ways of tackling the identification problem formulated earlier is envisaged and briefly outlined. Let **D** denote a binary $d \times n$ matrix which selects, among the n displacement components, those subjected to measurements; thus, through eqn (7a) the **d** vector of calculated values for the kth test can be expressed in the form,

$$u_k^c = D(\alpha_k u^e + Gp_k) \tag{19}$$

Making use of eqn (19) in (16), the function to minimize becomes a quadratic form in the plastic multipliers only.

$$\text{Min}\, F = \Sigma_{k=1,t}(p_k^t M p_k + b_k^t p_k + c_k) \tag{20}$$

where,

$$M = G^t D^t D G; \quad b_k = 2G^t D^t(\alpha_k D u^e - u_k^m);$$

$$c_k = (u_k^{mt} - \alpha_k u^{et} D^t)(\alpha_k D u^e - u_k^m)$$

It is worth noting that **F** is convex (as matrix **M** is positive semidefinite) and does not depend on the parameters R. These intervene in the minimization constraints, which are directly supplied by eqn (2) on the analysis problem and read:

$$\phi_k = -\alpha_k N^t Q^e + A(H)p_k + BR \geq 0 \tag{21a}$$

$$p_k \geq 0; \quad \phi_k^t p_k = 0 \tag{21b}$$

The constrained optimization problem to which the identification of the local resistance parameters has been cast, is characterized by a complementary constraint requiring that between a certain pair of corresponding variables at least one component must vanish. Its peculiarity rests on the fact that the constraints are all linear except the complementary condition. Besides the non-linearity stemming from the product $A(H)p_k$ in eqn (21a), the complementary condition makes the parameter identification problem non-linear and non-convex. For simplicity sake, the experimental data is assumed to be derived from a single loading condition in the following sections.

Nonconvex parametric quadratic program

If the hardening coefficients in matrix **H** are assumed as given constants and the identification problem is conceived as a constrained minimization with respect to vectors **p** and **R**, the only real source of mathematical and numerical difficulties is represented by the complementary requirement. At first, a quite natural way of trying to circumvent this difficulty is to augment the objective function with the complementary condition, removing it from the constraints:

$$\text{Min}\,[p^t R^t]\begin{bmatrix}(M+\rho A) & \rho/2B \\ \rho/2B^t & 0\end{bmatrix}\begin{bmatrix}p \\ R\end{bmatrix}$$

$$+(b-\rho N^t Q^e)^t p + c \tag{22a}$$

subject to,

$$Ap + BR \geq N^t Q^e \tag{22b}$$

$$p \geq 0 \tag{22c}$$

where **p** is a real positive constant. It can be stated that problem (20) subject to (21) is equivalent to the non-convex **QP** problem (22) in the sense that both

minimum points coincide. Reference 2 provides a two-phase method for obtaining the global optimum. Now, if the hardening coefficients are regarded as parameters to be identified, then inequality constraints (22b) become non-linear and the non-linearity of the objective function increases in view of the parametric nature of $A(H)$. The sensitivity analysis procedure described previously for the general convex parametric QP can be adapted here to identify a (at least) local solution.

Sequential quadratic programming

The minimization of the quadratic error function **F** in eqn (16) with respect to both **R** and **H** can be done numerically. The objective function $\mathbf{F}(R, H)$ is given implicitly via the elastoplastic analysis programs. $\mathbf{u}(R^o, H^o)^c$ is the displacement vector of the structure characterized by the yield limits **R** and hardening moduli **H** in the dual program (6) (or in eqn (9)). Therefore, **F** is a continuously differentiable function of u^c. The derivatives of **F** with respect to the parameters can be computed by using the results of the parametric quadratic program (13).

$$\text{Min } \mathbf{F} = \Sigma_{h=1,d}\left[u_h^m - u_h(R^o, H^o)^c \Sigma_{j=1,g_1}\left(\frac{\partial u_h^c}{\partial R_j}\right)_o \Delta R_j \right.$$
$$\left. + \Sigma_{j=1,g_2}\left(\frac{\partial u_h^c}{\partial H_j}\right)_o \Delta H_j\right]^2 \tag{23}$$

For this problem where there are only simple range constraints imposed on $(\Delta R_j, \Delta H_j)$, computational results can be obtained by the use of standard sequential quadratic programming. By letting $x^l = [R^l H^l]$, eqn (23) can be written,

$$\text{Min } \mathbf{F} = \Sigma_{i=1,n}\Sigma_{j=1,n}\Sigma_{h=1,d}\left(\frac{\partial u_h^c}{\partial x_i}\right)_o\left(\frac{\partial u_h^c}{\partial x_j}\right)_o \Delta x_i \Delta x_j$$
$$+ \Sigma_{i=1,n}\Sigma_{h=1,d}\left(\frac{\partial u_h^c}{\partial x_i}\right)_o[u_h^m - u_h(x^o)^c]\Delta x_i$$
$$+ \Sigma_{h=1,d}[u_h^m - u_h(x^o)^c]^2 \tag{24}$$

Solving eqn (24) for particular numerical values of $u_h(x^o)^c$ and $(\partial u_h^c/\partial x_i)_{x^o}$ forms only an iteration. The solution vector x^l of such an iteration represents a new set of parameters which must be analysed and gives new values for $u_h(x^l)^c$ and $(\partial u_h^c/\partial x_i)_{x^l}$ to replace those corresponding to x^o in (24). Iterations continue until changes in the design variables become small.

Minimax formulation

The information provided by the plastic multipliers and yield functions reduces the dependence on the stipulated bounds for the parameter changes in each iteration. The

mathematical programming algorithm described in this section consists of solving a minimax problem that is found by rewriting the objective function (16) and the constraints (21) as goals in a normalized form. If **F** represents a reference error, and **p**, ϕ the corresponding plastic multipliers and yield functions, relation (16) becomes

$$\Sigma_{h=1,d}(u_h^m - u_h^c)^2 \le F \Rightarrow g_1(x)$$
$$= \frac{(u_h^m - u_h(x)^c)^2}{F} - 1 \le 0 \tag{25a}$$

The sign constraints which impose limits on the varitions of non-zero multipliers **p** and yield functions ϕ lead to,

$$g_2(x) = -\frac{\Delta p(x)}{p} - 1 \le 0 \tag{25b}$$

$$g_3(x) = -\frac{\Delta\phi(x)}{\phi} - 1 \le 0 \tag{25c}$$

where ϕ is given in eqn (2b). The sensitivity result is obtained by considering the primal as well as its dual problem and sensitivities are given for both the primal and dual variables. The components of $\partial p/\partial x$ and $\partial\phi/\partial x$ are obtained from the piecewise solution of the quadratic program and hence are discontinuous piece-wise functions of x. The complementary constraint is checked after each iteration.

The problem of finding values for the cross sectional areas which minimize the maximum of the goals has the form,

$$\min_x \max_{k=1,c}(g_1, \ldots, g_k \cdots g_c) \tag{26}$$

and belongs to the class of minimax optimization. The procedure used to solve this problem is a recently developed entropy-based approach. The minimax problem (26) is discontinuous and non-differentiable, of which both attributes make its numerical solution by direct means difficult. In Ref. 4 it is shown that the minimax solution may be found indirectly by the unconstrained minimization of a scalar function which is both continuous and differentiable, and is thus considerably easier to solve:

$$\text{Min}_x \text{Max}_{k \in K}\langle g_k(x)\rangle = \text{Min}(1/\rho)$$
$$\log\{\Sigma_{k=1,c} \exp[\rho g_k(x)]\} \tag{27}$$

over variable x with a sequence of values of increasingly large positive $\rho \ge 1$. The scalar function minimization allows the use of algorithms for convex optimization. The strategy adopted was to solve the implicit optimization problem by means of an iterative sequence of explicit approximation models. An explicit approximation can be formulated by taking Taylor series expansions of all the goal functions in problem (27), truncated after the linear term for $g_2(x)$ and $g_3(x)$ and

the quadratic term in the case of $g_1(x)$:

$$\text{Min}(1/\rho) \log\left\{\Sigma_{k=2,c} \exp \rho\left[g_k(x^o) + \Sigma_{i=1,N}\left(\frac{\partial g_k}{\partial x_i}\right)_o \Delta x_i\right]\right.$$
$$+ \exp \rho\left[g_1(x^o) + \Sigma_{i=1,N}\left(\frac{\partial g_1}{\partial x_i}\right)_o \Delta x_i\right]$$
$$\left. + \frac{1}{2}\Sigma_{i=1,n}\Sigma_{j=1,n}\left(\frac{\partial g_1}{\partial x_i}\right)_o\left(\frac{\partial g_1}{\partial x_j}\right)_o \Delta x_i \Delta x_j\right\} \qquad (28)$$

that is solved iteratively until changes in the design parameters become small.

Discussion

In order to identify $\mathbf{R} + \mathbf{H}$ parameters, one obviously needs at least $(\mathbf{R} + \mathbf{H})t$ independent measured values. This condition is not sufficient: if in the experiments the plastic properties are not activated, the yield limits are nowhere reached and lower bounds on the parameters are provided by the elastic stresses related to the solution. Practically, the solution is highly dependent upon the number and positions of the measurements. Measurements should be made where the discrepancies are potentially higher: in the neighborhood of the most critical points both in compression and tension in an unsymmetrical way.

Normally the number $\mathbf{R} + \mathbf{H}$ of independent parameters to identify is much less than the number $(2m)t$ of the variables $\mathbf{p}$ which characterize the dimensions of the non-convex quadratic parametric program. Clearly the size of this identification problem increases almost proportionally with the number of t different tests and should be ruled out. This is in contrast with both the sequential quadratic programming technique and the minimax formulation which are rather insensitive to the test number. In the sequential quadratic programming approach, the nodal displacements $\mathbf{u}$ are approximated by first order Taylor series at the current parameters (R^o, H^o) yielding a quadratic program in the parameter changes $(\Delta R, \Delta H)$. Since the solution largely depends on the assumed bounds, move limits should be small to avoid an erratic behaviour. The minimax problem is less dependent on the bounds stipulated for the parameter changes, providing a smoother convergence. Hence the number of cycles of analysis/optimization in SQP is potentially greater than in the case of the minimax approach. On the other hand, the quasi-Newton algorithm used to solve eqn (28) is less efficient than the routine used for quadratic programming. The algorithm used to minimize quadratic functions is subject only to upper and lower bounds employed for the elastoplastic analysis, sensitivity analysis and SQP uses partial LDL^t factorization. The computational times are comparable to those required for the factorization of the quadratic coefficients matrix.

INFLUENCE OF INACCURACIES IN MEASUREMENTS

As long as it is '*a priori*' known that there are no inaccuracies in measurements and modeling of the identification problem is treated in purely deterministic terms, the minimum of the error function is zero. In real-life situations inaccuracies of practical significance affect both the measurements and the modeling of the real system. One of the procedures for filtering such 'noises' is the post-optimality analysis of the quadratic program giving the error function (24) with respect to each parameter in turn. If the coefficients a_h represent such sensitivities, the parameter changes can be obtained by the linear approximation:

$$\Delta x = \Sigma_{h=1,d} a_h \Delta u_h \qquad (29)$$

If the inaccuracies have known statistical properties, the mean and variance of the parameter changes are given by

$$\mu_{\Delta x} = \Sigma_{h=1,d} a_h \mu_{\Delta x_h} \qquad (30a)$$
$$\sigma_{\Delta x^2} = \Sigma_{h=1,d} a_h^2 \sigma_{\Delta x_h^2}$$
$$+ \Sigma_{h=1,d}\Sigma_{i=1,d; i \neq h} a_h a_i \rho_{hi} \sigma_{\Delta x_h} \sigma_{\Delta x_i} \qquad (30b)$$

where ρ_{ij} is the correlation coefficients between Δu_h and Δu_i. Inaccuracies due to other instrumental errors and modeling can be treated in a similar way.

NUMERICAL EXAMPLE

The vertical displacements of the hinge points of the beam on elastoplastic foundation represented in Fig. 2 are assumed to be measurable and the two yield limits of the hinges, the compressive strength of the springs and the hardening coefficients are to be identified on the basis of those measurements. In principle it should be possible to identify the yield limits, if the corresponding yield modes are activated and if the number of measured displacements is not less than the number of unknowns. Different starting points were used and both the sequential quadratic programming and the minimax formulation gave results in two to three iterations. As stated before, the quality of these solutions depends on the location and number of the measurements and this can be seen in Table 1. It should be noted that the hogging yield limit of the beam cannot be identified correctly on the basis of the available information, since there are no positive plastic rotations in the simulated experiment.

Since the parameter identification process depends on the precision of the measures, it is appropriate to test the sensitivity of the method with respect to possible measurement errors. An investigation has been carried out by giving a 5% increment to every measurement.

Table 1

Measured displacements	Foundation yield limit	Beam sagging moment	Hardening moduli
All	0·0%	0·1%	0·1%
17,19,21,23,24,26	0·0	0·1	0·1
14,15,16,17	0·4	1·0	5·4
17,18,19,21	0·1	0·3	1·6
19,20,21,22	0·0	0·1	0·1
19,20,21	0·0	0·1	0·1
17,18,21	0·1	0·3	1·8

The resulting errors in the parameters are 0·6, 1·3 and 3·4%, respectively. It can be seen that the effect on the parameter identification is much less than the order of magnitude of the measurement errors in the case of the foundation yield limit and the sagging limit moment and smaller in the case of the hardening coefficient. For some other combinations of measurement discrepancies, the error involving this parameter might be higher, because the displacements are rather insensitive with respect to hardening moduli changes (as can be seen in Fig. 5). In order to partially simulate measurement errors the generated 'measured' displacements were rounded off at the first decimal point. Also in this case the disturbances on the identified parameters proved to be negligible (0·7, 0·1 and 0·1%, respectively). The procedure has also worked in a subsequent numerical test, where the number of parameters to be identified is increased by also assuming the zero tensile yield limit of the spring to be unknown.

CONCLUSIONS

The present paper contains results on the sensitivity analysis for elastoplastic problems in the case of discrete structures and structures modeled by finite elements. The result shows that determination of the sensitivities can be based on the solution of an associated quadratic programming with an unchanged quadratic term but with changed linear terms and constraints which are given by the derivatives of the matrices involved as well as by the solution of the primal analysis problem. The inverse problem of identifying yield limits on the basis of information on displacement response to given loads has been tackled here in the context of discrete structural models with holonomic elasto-plastic piecewise-linear laws governing the local deformability. As the dual variables in the elastoplastic analysis problems have a physical interpretation, (eg. displacements and plastic strains) the sensitivities for the dual variables are also of interest in the parameter identification context. Various solution procedures resting on mathematical programming methods, all capable of exploiting the peculiar mathematical features of the proposed formulation have been devised and discussed. Inaccuracies primarily due to the approximations embodied in the model and instrumental 'noises' affecting the measures have been considered by the procedure employed for the sensitivity analysis. The examples reported show that the yield limits and hardening moduli are relatively insensitive with respect to perturbation of the measured displacements, provided these are chosen in location and number such that they are affected by the local yielding processes.

ACKNOWLEDGEMENTS

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