Computers & Structures Vol. 41, No. 2, pp. 385–395, 1991
Printed in Great Britain.

0045-7949/91 $3.00 + 0.00
Pergamon Press plc

ENTROPY-BASED LEAST WEIGHT DESIGN OF ELASTOPLASTIC TRUSSES

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(Received 9 October 1990)

Abstract—Optimum design conceived as weight minimization is formulated with reference to discrete (truss-like) physically nonlinear structures under constraints concerning displacements, deformations and design parameter distributions. Basic assumptions are: (i) the cost function is linear in the design variables; (ii) no local unstressing occurs under the given proportional loading so that holonomic plastic laws can be adopted. This work presents a method that solves the corresponding complementarity programming problem by solving a minimax problem using the maximum entropy formalism. Some simple, but significant examples, illustrate the proposed method.

1. INTRODUCTION

In the literature dealing with optimization problems, minimum weight design was widely studied for both elastic and perfectly plastic structures; comprehensive surveys on both subjects can be found in [1, 2], respectively. Nevertheless, some cases deriving from nonlinear physical laws of materials were not treated. In this work, the elastoplastic linearly hardening model is considered and relevant constitutive relations are taken into account. A basic assumption, which is reasonably acceptable under proportional loading, is that the plastic constitutive law is holonomic, that is reversible and history-independent (no local unloading hypothesis). It is considered that the design variables will affect local strength and stiffness simultaneously. The assumed cost function linearly depends on strength parameters, the distribution of which results from the minimization under constraints grouping fundamental mechanics laws, bounds on displacements and possible technological and ductility constraints. For the sake of simplicity, theoretical developments are referred to trusses. A noteworthy aspect of the class of optimal design problem under consideration is that the mathematical formulation of the problem is characterized by a complementarity constraint requiring that between a certain pair of corresponding variables, at least one component must vanish. The research concerning the application of mathematical programming techniques in this area of structural engineering seems to be very limited. This class of problem has been solved in [3] where a branch and bound method associated with a quadratic programming approach was suggested. Since each node of the branch and bound combinatorial tree was associated with a member behaviour, its efficiency reduces exponentially as the number of potentially yielding members grow (*NP* hard problem). In [4] the elastoplastic design problem was tackled by an optimality criteria-based procedure and a small-scale example solved. Since nonlinearity and nonconvexity are a peculiar feature of this optimization problem, this latter approach is unable to guarantee the result of a global, as opposed to merely a local, solution.

This paper presents a method for elastoplastic truss sizing that has distinctive features. It is based on entropy and is a radically different alternative to mathematical programming techniques. Entropy is a natural measure of the amount of disorder (or information) in a system. High entropy values correspond to chaos and low entropy values to order. A similar definition of the entropy is known in information in information theory where it can be viewed as a quantitative measure of the information content of a system. In the case of optimization in general, the entropy can be interpreted as a quantitative measure for the degree of optimality. The objective function and the constraints define several criteria that need to be met: find values for the design variables which minimize the maximum violated criterium. The elastoplastic synthesis problem is set as a multicriteria optimization and a Pareto solution is sought. This minimax problem is discontinuous and nondifferentiable, both of attributes makes its numerical solution by direct means difficult. By using the maximum entropy formalism it is shown that its solution may be found indirectly by the unconstrained minimization of a scalar function which is both continuous and differentiable and thus considerably easier to solve.

2. PROBLEM FORMULATION

2.1. *Fundamental relations*

Let a truss-like structure be considered with the usual assumptions of small displacements and deformations. Let u and F denote respectively vectors of the displacements of the free nodes (n degrees of

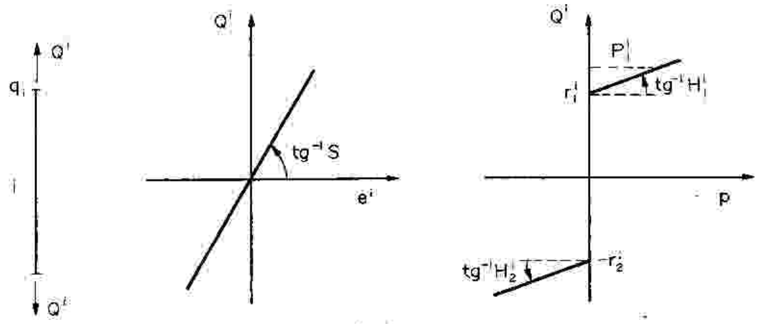

Fig. 1.

freedom) and of the corresponding given independent nodal loads; Q and q will represent the m-vectors of the bar forces and bar elongations, respectively. With these symbols, geometric compatibility and equilibrium can be expressed, respectively in the forms

$$q = Cu \tag{1}$$

$$C^tQ = F, \tag{2}$$

where C is a $m \times n$ matrix which depends only on the given layout of the structure. For each structural member i, the assumed piecewise linear holonomic constitutive law relating force to elongation is depicted in Fig. 1 and can be described as follows:

$$q^i = e^i + p^i = (S^i)^{-1}Q^i + p_1^i - p_2^i \tag{3a}$$

$$\phi_1^i = -Q^i + (r_1^i + H_1^i p_1^i) \geqslant 0 \tag{3b}$$

$$\phi_2^i = Q^i + (r_2^i + H_2^i p_2^i) \geqslant 0 \tag{3c}$$

$$p_1^i, p_2^i \geqslant 0; \quad \phi_1^i p_1^i = 0; \quad \phi_2^i p_2^i = 0, \tag{3d}$$

where p_j^i, ϕ_j^i represents the plastic multiplier and yield function, respectively. H_j^i is a nonnegative hardening modulus for each i and $j = 1, 2$ given by $a^iE^ik_j^i/l^i$ where k_j^i, a^i, E^i, l^i represent work hardening coefficients, cross-sectional area, Young's modulus and length of member i, respectively. The element S^i is the elastic stiffness a^iE^i/l^i, $r_j^i, j = 1, 2$ are the yield forces given by $\sigma_j^i a^i$ where σ_j^i are the compressive and tensile yield stresses for member i. It is convenient to assemble the above relationships for $i = 1, \ldots, m$ into the matrix form

$$q = (S)^{-1}Q + p_1 - p_2 \tag{4a}$$

$$\phi_1 = -Q + (r_1 + H_1p_1) \geqslant 0 \tag{4b}$$

$$\phi_2 = Q + (r_2 + H_2p_2) \geqslant 0 \tag{4c}$$

$$p_1, p_2 \geqslant 0; \quad \phi_1p_1 + \phi_2p_2 = 0. \tag{4d}$$

Substituting vectors q and Q, the relationship (1), (2) and (4) can be written as

$$Ku - C^tSp_1 + C^tSp_2 = F \tag{5a}$$

$$-N^tSCu + (N^tSN + H)p + r \geqslant 0 \tag{5b}$$

$$\phi \geqslant 0; \quad p \geqslant 0; \quad \phi^tp = 0, \tag{5c}$$

where $K = C^tS\,C$ is the nonsingular stiffness matrix of the structure and $N = [I \ -I]$. From (5a) it follows that

$$u = u^e + GNp \tag{6}$$

where

$$u^e = K^{-1}F \quad \text{and} \quad G = K^{-1}C^tS. \tag{7}$$

The vector u^e represents elastic displacements and matrix G consists of the influence coefficients of the displacements due to imposed strains (elongations in the case of trusses).

The analysis of structures described by an elasto-plastic stress–strain law with work-hardening can be formulated as a convex quadratic programming problem. It is intended to minimize the resulting force vector Q and the plastic multiplier vector p in the structure

$$\text{Min}\ \tfrac{1}{2}Q^tS^{-1}Q + \tfrac{1}{2}p_1^tH_1p_1 + \tfrac{1}{2}p_2^tH_2p_2 \tag{8a}$$

subject to

$$C^tQ = F \tag{8b}$$

$$\phi_1 = -Q + H_1p_1 + r_1 \geqslant 0 \tag{8c}$$

$$\phi_2 = Q + H_2p_2 + r_2 \geqslant 0 \tag{8d}$$

$$Q\,\text{real}; \quad p_1 \geqslant 0; \quad p_2 \geqslant 0 \tag{8e}$$

or in the more compact form

$$\text{Min}\ \tfrac{1}{2}Q^tS^{-1}Q + \tfrac{1}{2}p^tHp \tag{9a}$$

subject to

$$C^t Q = F \tag{9b}$$

$$\phi = -N^t Q + Hp + r \geqslant 0 \tag{9c}$$

$$Q \text{ real}; \quad p \geqslant 0. \tag{9d}$$

We remark that for an elastic–perfectly plastic model with nonhardening behaviour $H = 0$ and (9) becomes

$$\text{Min} \tfrac{1}{2} Q^t S^{-1} Q \tag{10a}$$

subject to

$$C^t Q = F \tag{10b}$$

$$\phi = -N^t Q + r \geqslant 0 \tag{10c}$$

$$Q \text{ real}. \tag{10d}$$

This problem has a solution if the design makes the structure capable of carrying the given loads whatever p may be.

The dual of (8) is the convex quadratic programming problem

$$\text{Min} \tfrac{1}{2} [u^t p_1^t p_2^t] \begin{bmatrix} C^t SC & -C^t S & C^t S \\ -SC & S + H_1 & -S \\ SC & -S & S + H_2 \end{bmatrix} \times \begin{bmatrix} u & & u \\ p_1 & + [F^t r_1^t r_2^t] & p_1 \\ p_2 & & p_2 \end{bmatrix} \tag{11a}$$

subject to

$$u \text{ real}; \quad p_1 \geqslant 0; \quad p_2 \geqslant 0 \tag{11b}$$

the solution of which is the vector of nodal displacements u and the plastic multipliers p_1 and p_2. At the optimum solution of the elastoplastic analysis problems (8) and (11) all the matrices and vectors involved are differentiable. Moreover the columns of the simultaneously active constraints are linearly independent.

3. SYNTHESIS PROBLEMS

3.1. *Elastoplastic synthesis of trusses*

The synthesis problem arises when the cross-sectional area a are regarded as design variables. The cost function w is assumed to depend linearly on the design variables, through a given coefficient vector c. Thus, the optimum design problem is to minimize the function

$$\text{Min } w = c^t a. \tag{12a}$$

The design variables are possibly subject to variable linking (to reduce the number of different sections). These technological constraints are linear and can be expressed as

$$Ta = 0. \tag{12b}$$

The constraints,

$$a \geqslant a^L \tag{12c}$$

impose lower bounds a^L on the cross-sectional areas.

The bar stiffness is composed of elastic stiffness represented by S and, in the general, of plastic stiffness (hardening) denoted by H. It varies with a because all the parameters defining the member behaviour depend in general on the geometric characteristics of the cross-section

$$\phi_1 + Q - (r_1 + H_1 p_1) = 0 \tag{12d}$$

$$\phi_2 - Q - (r_2 + H_2 p_2) = 0 \tag{12e}$$

$$p_1, p_2 \geqslant 0; \quad \phi_1, \phi_2 \geqslant 0. \tag{12f}$$

A noteworthy aspect of the class of optimal design problem under consideration is that the mathematical formulation of the problem is characterized by a complementarity constraint requiring that between a certain pair of corresponding variables, ϕ_1, p_1 and ϕ_2, p_2 at least one component must vanish

$$\phi_1 p_1 + \phi_2 p_2 = 0 \tag{12g}$$

Serviceability conditions can be considered by placing upper bounds U on some or all of the displacement components

$$Bu = Bu^e + BG_1 p_1 + BG_2 p_2 \leqslant U \tag{12h}$$

$$Bu = Bu^e + BG_1 p_1 + BG_2 p_2 \geqslant -U, \tag{12i}$$

where B is a binary matrix. In the presence of limited ductility of the material, upper bounds Λ on the plastic multipliers (each of which measures the amount of yielding generated with respect to each yield mode) might also be imposed

$$p_1 \leqslant \Lambda_1; \qquad p_2 \leqslant \Lambda_2. \tag{12j}$$

Besides the nonlinearity steaming from the constraints (12d, e) and (12h), the complementary condition (12g) makes the optimum design problem (12) nonlinear and nonconvex.

This formulation covers situations where the design variables defining local strength parameters do and do not affect the elastic and hardening stiffness.

3.2. *Minimax formulation*

In the context of the elastoplastic synthesis problem defined above, it is intended to minimize a whole set of goals such as the cost, nodal displacements, etc. by finding an optimal set of cross-sectional areas. The technological constraints (12b) are used in order to rewrite the problem in terms of the independent design variables. All the goals need to be cast in a normalized form. If $\mathbf{w}$ represents a reference cost, the relation (12a) becomes

$$c'a \leqslant \mathbf{w} \Rightarrow g_1(a) = \frac{c'a}{\mathbf{w}} - 1 \leqslant 0. \quad (13a)$$

The lower bounds on cross-sectional areas (12c) become

$$g_2(a) = -\frac{a}{a^L} + 1 \leqslant 0. \quad (13b)$$

Similarly, one has for the upper and lower bounds on the nodal displacements (12h, i)

$$g_3(a) = \frac{Bu}{U} - 1 \leqslant 0 \quad (13c)$$

$$g_4(a) = -\frac{Bu}{U} - 1 \leqslant 0. \quad (13d)$$

In the presence of limited ductility of the material

$$g_5(a) = -\frac{p_1}{\Lambda_1} - 1 \leqslant 0; \qquad g_6(a) = -\frac{p_2}{\Lambda_2} - 1 \leqslant 0. \quad (13e)$$

The sign constraints on ϕ_1, ϕ_2 and p_1, p_2 lead to

$$g_7(a) = -\frac{\Delta\phi_1}{\phi_1} - 1 \leqslant 0; \qquad g_8(a) = -\frac{\Delta\phi_2}{\phi_2} - 1 \leqslant 0 \quad (13f)$$

$$g_9(a) = -\frac{\Delta p_1}{p_1} - 1 \leqslant 0; \qquad g_{10}(a) = -\frac{\Delta p_2}{p_2} - 1 \leqslant 0. \quad (13g)$$

The complementarity constraint in normalized form becomes

$$g_{11}(a) = \frac{(\phi_1 + \Delta\phi_1)(p_1 + \Delta p_1) + (\phi_2 + \Delta_2)(p_2 + \Delta_2)}{\epsilon} - 1 \leqslant 0 \quad (13h)$$

and it is satisfied for sufficiently small ϵ (say 0.001).

The problem of finding values for the cross-sectional areas which minimize the maximum of the goals has the form

$$\min_a \max x_{k=1,c}(g_1, \ldots, g_k, \ldots, g_c) \quad (14)$$

and belongs to the class of minimax optimization. Several other optimum design problems, such as the minimization of plastic deformations with a constraint on weight can be transformed in similar minimax problem.

4. ENTROPY IN OPTIMIZATION PROCESSES

4.1. *Constrained nonlinear programming*

Entropy can be used to deduce desired results when only limited information is available. The general inequality constrained nonlinear programming problem

$$\operatorname{Min} f(x), \quad i = 1, \ldots, n \quad (15a)$$

$$\text{st } g_j(x) \leqslant 0 \quad \text{or} \quad g_j(x) + s_j = 0, \quad j = 1, \ldots, m \quad (15b)$$

was examined in [5]. An initial point was chosen and information is calculated about the objective and constraint functions, typically their numerical values and gradients at the design point. This numerical information was then used in a mathematical programming algorithm to infer where the next trial point should be placed so as to get closer to the constrained optimum of the problem. The new trial generates more information from which another point is inferred and eventually the solution is reached by this process of gathering better information and using it in an inference based algorithm. The essence of the method consisted in transforming problem (15) into an equivalent surrogate form

$$\operatorname{Min} f(x) \quad (16a)$$

$$\text{st } \Sigma_{j=1,M} \lambda_j g_j(x) = 0 \quad (16b)$$

$$\Sigma_{j=1,M} \lambda_j = 1 \quad (16c)$$

$$\lambda_j \geqslant 0 \quad (16d)$$

and using maximum entropy to obtain least biased estimates of the optimum values of the surrogate multipliers λ_j. In this two-phase method the absence of an explicit surrogate dual objective function is overcome by introducing the Shanon entropy [6] as a means of forcing iterations towards a saddle point. Each estimate lead to a new problem in the space of the x variables and generated information upon which to base an improved estimate of the optimum surrogate multipliers. The work was later extended by combining the two phases into a single phase consisting of solving a single unconstrained problem. The Lagrangian of problem (16) was augmented with an entropy term

$$\mathscr{L}' = f(x) + \alpha \Sigma_{j=1,m} \lambda_j g_j(x) + \mu(\Sigma_{j=1,m} \lambda_j - 1) - 1/\rho \Sigma_{j=1,m} \lambda_j \ln(\lambda_j), \quad (17)$$

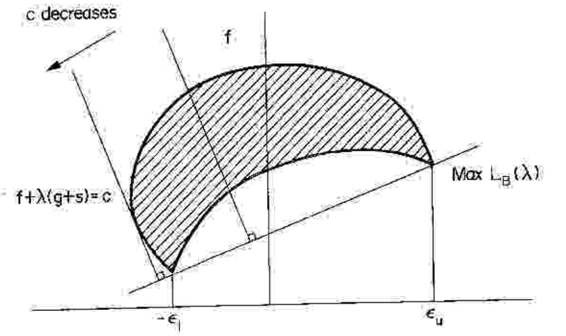

Fig. 2.

where the negative sign of the entropy term was necessary because of the nonnegativity of the summation. ρ is a positive constant. Using the stationarity conditions this reduces to

$$\mathrm{Min}_x f(x) + 1/\rho \ln \Sigma_{j=1,m} \exp[\rho\alpha g_j(x)], \quad (18)$$

ρ is an arbitrary positive constant with the entropy augmentation term and $\rho\alpha$ should be an increasing positive quantity. The $\ln \Sigma \exp(\cdot)$ form of the penalty term is particularly interesting, although it has not appeared before among many suggested penalty functions in the mathematical programming literature.

The work was pursued further to remove the difficulties associated with $p = \rho\alpha$ having to start small and increase positively, possibly causing overflow in $\exp[pg_j(x)]$ [7]. The constrained nonlinear programming problem is treated as

$$\mathrm{Min} f(x) \quad (19a)$$

$$\mathrm{st}\ A_p(x) = 0, \quad (19b)$$

where the single constraint $A_p(x)$ is called aggregate constraint

$$A_p(x) = 1/p \ln \Sigma_{j=1,M} \exp[pg_j(x)]$$
$$= g_k(x) + 1/p \ln \Sigma_{j=1,M} \exp\{p[g_j(x) - g_k(x)]\}, \quad (20)$$

where $g_k(x)$ is the maximum constraint. Problem (19) is then solved by an augmented Lagrangian algorithm. The exponents are now all nonpositive and p can be made very large immediately, without bothering with an increasing sequence for p.

For some time the validity of the above entropy-based methods upon whether it was valid to interpret the surrogate multipliers as probabilities. The proof of the validity of all the assumptions and of the methods turned out to be very simple requiring only the use of Cauchy's inequality (the arithmetic–geometric mean inequality). In consequence all the results can now be proved entirely deterministically and without recourse to probabilistic interpretation.

4.2. *Geometric interpretation*

The single phase entropy-based method (18) can be interpreted as a generalized Lagrangian method which fills duality gaps in nonconvex problems. If the mathematical programming (15) is considered and assuming for the sake of simplicity that only one constraint is active at the optimum solution. For any λ the minimization of $f(x) + \lambda[g(x) + s]$ corresponds to an hyperplane that supports the shaded domain represented in Fig. 2.

Also one cannot find a linear support of lower envelope of the set at a point having $[g(x) + s]$ between $-\epsilon_l$ and ϵ_u. On the other hand, if the Lagrangian

$$\mathscr{L} = f(x) + \Sigma_{j=1,m} \lambda_j(g_j + s_j) = f(x) + \lambda(g - s) \quad (21)$$

is considered, $\mathscr{L}$ corresponds to the intercept of supporting lines with slope $-\lambda$ on the f-axis and from Fig. 2 it can be seen that Max $\mathscr{L}$ is strictly less than $f(x^*)$. This explains why no duality gap exists in convex problems. By using a nonlinear support, such as augmenting the Lagrangian with the Shanon entropy term, the duality gap can be eliminated. The Lagrangian (17) is equivalent to

$$\mathscr{L}'' = f(x) + 1/\rho \ln \Sigma_{j=1,m} \exp\{\rho\alpha[g_j(x) + s_j]\} \quad (22)$$

and for values of λ_j between 0 and 1, $\mathscr{L}''$ (and $\mathscr{L}'$) becomes an exponential support which closes the duality gap as it can be shown in Fig. 3.

4.3. *Minimax optimization*

Following from the above work on scalar optimization [8] examined the role of maximum entropy in vector (multiobjective and multicriteria) and minimax optimization. Specifically it is shown that the minimax problem (14) can be solved indirectly by minimizing a continuous differentiable scalar optimization problem. In this section some of the theory behind this approach to minimax optimization is briefly described.

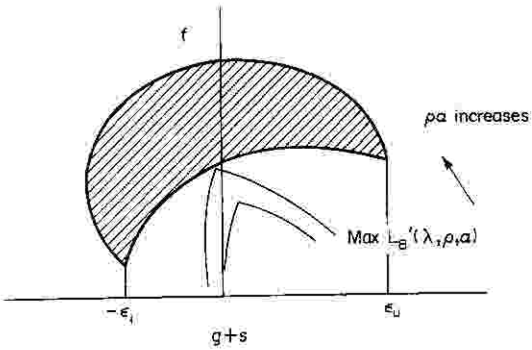

Fig. 3.

For any set of real, positive numbers U_j, $j = 1, \ldots, J$, and real $\rho \geq q \geq 1$, Jensen's inequality states that

$$(\Sigma_{j=1,m} U_j^\rho)^{1/\rho} \leq (\Sigma_{j=1,m} U_j^q)^{1/q}. \quad (23)$$

Inequality (23) means that the pth norm [9] of the set U decreases monotonically as its order, p, increases. Another important property of the pth norm is its limit as ρ tends towards infinity

$$\lim_{\rho\to\infty} (\Sigma_{j=1,m} U_j^\rho)^{1/\rho} = \mathrm{Max}_{j=1,m}\langle U_j\rangle. \quad (24)$$

Consider the minimax optimization problem

$$\mathrm{Min}_x\, \mathrm{Max}_{j=1,m}\langle g_j(x)\rangle \quad (25)$$

and Jensen's inequality. Let $U_j = \exp[g_j(x)]$, $j = 1, \ldots, m$ thus ensuring that $U_j > 0$, for all positive $g_j(x)$. Then

$$(\Sigma_{j=1,m} U_j^\rho)^{1/\rho} = \{\Sigma_{j=1,m} \exp[\rho g_j(x)]\}^{1/\rho}. \quad (26)$$

From (23)

$$\lim_{\rho\to\infty} \{\Sigma_{j=1,m} \exp[\rho g(x)]\}^{1/\rho} = \mathrm{Max}_{j=1,m}\langle g_j(x)\rangle. \quad (27)$$

Taking logarithms of both sides and noting that,

$$\log \lim(f) = \lim \log(f) \quad \text{and}$$
$$\log \mathrm{Max}(f) = \mathrm{Max} \log(f), \quad (28)$$

eqn (27) becomes

$$\lim_{\rho\to\infty} (1/\rho) \log\{\Sigma_{j=1,m} \exp[\rho g(x)]\} = \mathrm{Max}_{j=1,m}\langle g_j(x)\rangle. \quad (29)$$

Results (29) holds for any set of vectors $g(x)$, including that set which results from minimizing both sides of (25) over x. Thus (29) can be extended to:

$$\mathrm{Min}_x\, \mathrm{Max}_{j=1,m}\langle g_j(x)\rangle = \mathrm{Min}_x (1/\rho) \log\{\Sigma_{j=1,m} \exp[\rho g_j(x)]\} \quad (30)$$

with increasing ρ in the range $1 \leq \rho \leq \infty$. Result (30) shows that a Pareto solution of the minimax optimization problem can be obtained by the scalar minimization

$$\mathrm{Min}_x (1/\rho) \log\{\Sigma_{j=1,m} \exp[\rho g_j(x)]\} \quad (31)$$

with a sequence of values of increasingly large positive $\rho \geq 1$.

4.4. *Scalar function optimization*

Problem (31) is unconstrained and differentiable which, in theory, gives a wide choice of possible numerical solution methods. However, since the goal functions $g_j(x)$ do not have explicit algebraic form in most cases, the strategy adopted was to solve (31) by means of an iterative sequence of explicit approximation models. An explicit approximation can be formulated by taking Taylor series expansions of all the goal functions $g_j(x)$ truncated after the linear term. This gives

$$\min 1/\rho \log \Sigma_{j=1,c} \exp\{\rho[g(a_0) + \Sigma_{i=1,d}(\partial g/\partial a)_{a_0}(a - a_0)]\}, \quad (32)$$

where d and c are the number of design variables a (member cross-sections) and goal functions $g_j(a)$, respectively.

5. SENSITIVITY ANALYSIS

To formulate and solve the explicit approximation problem (32), numerical values are required for all the goal functions and their derivatives with respect to the design variables. The truss volume is known explicitly and need not be considered further. However, member forces, plastic multipliers and nodal displacements are implicit functions of a. Given some design variables the analysis of the truss will yield numerical values for δ, p and u. One way of evaluating the derivatives is to a postoptimality analysis, as follows.

The matrices S, H_1, H_2 and vectors r_1, r_2 are differentiable with respect to each of the design variables with derivatives S^{a^i}, $H_1^{a^i}$, $H_2^{a^i}$. By considering the dual problem of elastoplastic analysis and its solution $[u^t p_1^t p_2^t]$, the right-derivative $[u^{a^{it}} p_1^{a^{it}} p_2^{a^{it}}]$ for a positive Δa^i is given as the unique solution of the quadratic programming problem

$$\mathrm{Min}\, \tfrac{1}{2}[u^{a^{it}} p_1^{a^{it}} p_2^{a^{it}}] \begin{bmatrix} C^tSC & -C^tS & C^tS \\ -SC & S+H_1 & -S \\ SC & -S & S+H_2 \end{bmatrix} \times \begin{bmatrix} u^{a^i} \\ p_1^{a^i} \\ p_2^{a^i} \end{bmatrix} + [F^t (r_1^{a^i} - r_1)^t (r_2^{a^i} - r_2)^t] \begin{bmatrix} u^{a^i} \\ p_1^{a^i} \\ p_2^{a^i} \end{bmatrix} \quad (33a)$$

subject to

u^{a^i} is real

$$p_1^{a^i} = 0 \quad \text{for} \quad \phi_1^i = -Q^i + (r_1^i + H_1^i p_1^i) > 0 \quad (33b)$$

$$p_2^{a^i} = 0 \quad \text{for} \quad \phi_2^i = Q^i + (r_2^i + H_2^i p_2^i) > 0 \quad (33c)$$

$$p_1^{ai} < 0 \quad \text{for} \quad \phi_1^i = -Q^i + (r_1^i + H_1^i p_1^i) = 0 \quad \text{(33d)}$$

$$p_2^{ai} < 0 \quad \text{for} \quad \phi_2^i = Q^i + (r_w^i + H_2^i p_2^i) = 0. \quad \text{(33e)}$$

The sensitivity Q^{ai} can be directly evaluated from the compatibility equations

$$Q^{ai} = SCu^{ai} + Sp_1^{ai} - Sp_2^{ai} \quad (34)$$

and the yield functions are given by

$$\phi^{ai} = -N^t Q^{ai} + H^{ai} p^{ai} + r^{ai}. \quad (35)$$

It should be emphasized that this procedure gives the right-derivatives. The left-derivatives are the symmetric solutions but with the same constraint set. Therefore, the sensitivity analysis programming problem is dealing with differentiable plastic multipliers when the set defined by the corresponding yield functions $N^tQ + Hp + r > 0$ is empty and with differentiable yield functions when the set of corresponding plastic multipliers is empty. Since this is not generally the case, this procedure is only able to generate one-sided derivatives and (32) should be solved by an algorithm for nondifferentiable optimization. But since both p and ϕ are semismooth [10], methods for nondifferentiable optimization can be employed.

6. ALGORITHM FOR THE OPTIMIZATION OF ELASTOPLASTIC TRUSSES

6.1. *Control parameter* ρ

The previous sections have examined the major elements of the design method—the minimax optimum design formulation, the entropy-based Pareto solution and the sensitivity analysis of the structure. The minimax optimization algorithm requires a sequence of positive values of ρ increasing towards infinity. Many different schemes are possible depending on the convergence rate and stability of the algorithm upon the particular sequence chosen. In particular, an analogy with the physical process of annealing can be made. Annealing consists of heating up a solid until it melts followed by cooling it down until it cristallizes in a state with a perfect lattice. During the process the free energy of the solid is minimized. Practice shows that the cooling must be done carefully in order not to trap locally optimal lattice structures with crystal imperfections. Now, by establishing a correspondence between the cost function and the free energy and between the solution and the physical states one can introduce a solution method based on a simulation of this process. In the case of optimization in general, the entropy can be interpreted as a quantitative measure for the degree of optimality. During the execution of the algorithm the expected cost and entropy decreases monotonically providing optimality is reached at each value of the control parameter. The value of ρ is estimated by an iterative procedure which makes the objective function of problem (31) stationary with respect to ρ; i.e. to iteratively solve

$$\rho = \frac{\{\log \Sigma_{i=1,c} \exp[\rho g(a_0)]\}\{\Sigma_{i=1,c} \exp[\rho g(a_0)]\}}{\Sigma_{i=1,c} g(a_0) \exp[\rho g(a_0)]}. \quad (36)$$

From this expression it can be seen that ρ increases as the unfeasibility of the current design decreases, i.e. increasing ρ tends to enforce feasibility. Also expression (36) requires the starting point to be unfeasible in order to make $\rho \geqslant 0$. This is a very restrictive condition, because the algorithm works even when it starts from a point that meets all the criteria. It has been observed that a satisfactory behaviour for the procedure is to set $1 \leqslant \rho \leqslant 50$ in the first iteration, and this value was increased in subsequent iterations. The choice of ρ should reflect the number of violated criteria and its algebraic values. A large ρ would produce a solution with smaller cost but more unfeasible and a smaller ρ needs to be chosen to deal with large criteria violations.

6.2. *The algorithm*

0. Take as an initial design a uniform force distribution—fully-stressed design. This method also works with other starting points and its performance improves when some of the criteria are not met.

1. Do the sensitivity analysis. Consider only the affine terms of the Taylor series of the c criteria with respect to the starting point a_0. The quasi-Newton routine NAG E04JAF proved to be more efficient to solve the explicit problem (32) than the other algorithms tried. It uses differences on the gradients of the function and its first derivatives to computer the approximations required. The evaluation of the derivatives by analytic means is not competitive because they are given by sums of combinations of exponential terms, being computationally expensive.

2. Analyse the truss subject to the loading conditions.

   (a) If the starting point is feasible and the solution of the optimization leads to a unfeasible design, make a linear interpolation in order to find a solution closer to the boundary of the domain and use it in the next iteration. The nearly feasible design required can also be achieved either by limiting the design variable changes or by reducing the parameter ρ.

   (b_1) If the solution of the optimization leads to a truss with least cost and that meets all criteria keep it as the new optimum solution. Use this solution as a starting point for the next iteration.

   (b_2) If the starting point is feasible, the solution found is feasible and the difference in costs is small, increase ρ. If the difference in costs is still negligible, stop the algorithm and use the incumbent design as the optimum.

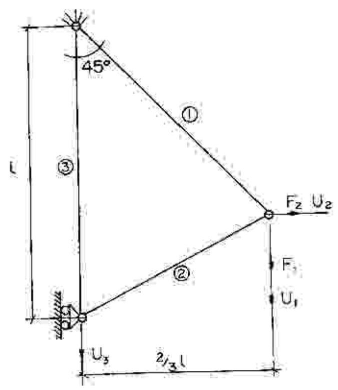

Fig. 4.

(c) If the starting point is feasible and the result of the optimization leads to a structure of higher cost, increase the parameter ρ and repeat the iteration of problem (32) with the same data.

(d_1) If the starting point is not feasible and the result of the optimization violates more the criteria stop the algorithm and use the incumbent design as the optimum.

(d_2) If the starting point is not feasible and the result of the optimization violates less the criteria reduce the parameter ρ and repeat the iteration of problem (32) with the same data.

7. NUMERICAL EXAMPLES

Some numerical examples were solved in order to illustrate the procedure described above.

7.1. *Example 1*

The isostatic truss depicted in Fig. 4 and described in [4] is considered. The elastoplastic synthesis problem consists of

$$\text{Min } 2\sqrt{2}/3a_1 + \sqrt{5}a_2 + a_3$$

subject to

$$(2F_1 + F_2)\sqrt{2}/3 - \sigma_1^1 a_1 - Ha_1/(2l\sqrt{2}/3)p_1^1 + \phi_1^1 = 0$$

$$-(-F_1 + F_2)\sqrt{5}/3 - \sigma_2^2 a_2 - Ha_2/(l\sqrt{5}/3)p_2^2 + \phi_2^2 = 0$$

$$(F_1 - F_2)/3 - \sigma_1^3 a_3 - Ha_3/l\, p_1^3 + \phi_1^3 = 0$$

$$u_1 = 0.9428[(2F_1 + F_2)4l/(9Ea_1) + p_1^1] - 0.7454[(-F_1 + F_2)5l/(9.Ea_2) - p_2^2] + 0.3333[(F_1 - F_2)l/(9Ea_3) + p_1^3] \leqslant U$$

$$u_2 = 0.4714[(2F_1 + F_2)4l/(9Ea_1) + p_1^1] + 0.7454[(-F_1 + F_2)5l/9.Ea_2) - p_2^2] - 0.3333[(F_1 - F_2)]l/(9Ea_3) + p_1^3] \leqslant U$$

$$u_3 = (F_1 - F_2)l/(9Ea_3) + p_1^3 \leqslant U$$

$$p_1^1\phi_1^1 + p_2^2\phi_2^2 + p_1^3\phi_1^3 = 0$$

$$a_1/l_2, \quad a_2/l^2, \quad a_3/l^2 \geqslant 0.01$$

and the sign constraints

$$p_1^1, \phi_1^1, p_2^2, \phi_2^2, p_1^3, \phi_1^3 \geqslant 0.$$

The following parameters are assumed

$$F_1 = 0.04 \times 10^{-6}; \quad F_2/F_1 = 0.9$$

$$U/l = 1/400; \quad H/E = 1/6; \quad \sigma_1/E = \sigma_2/E = 0.0015$$

For each bar the hardening branch possibly activated may be selected. Take for starting point the elastic truss such that all bars are at their yield limits. In this problem, the sensitivities can be directly derived. Take as starting point

$$a_1/l^2 = 36.4; \quad a_2/l^2 = 2.0; \quad a_3/l^2 = 0.9; \quad w/l^3 = 36.71$$

is infeasible because $u = 1.07U$. A design satisfying all the constraints could be found by increasing proportionally all the member sizes to

$$a_1/l^2 = 38.69; \quad a_2/l^2 = 2.12; \quad a_3/l^2 = 0.96; \quad w/l^3 = 39.205.$$

By making $\rho = 10$ in the first iteration of the algorithm, the solution

$$a_1/l^2 = 34.93; \quad a_2/l^2 = 3.0; \quad a_3/l^2 = 1.35; \quad w/l^3 = 37.42$$

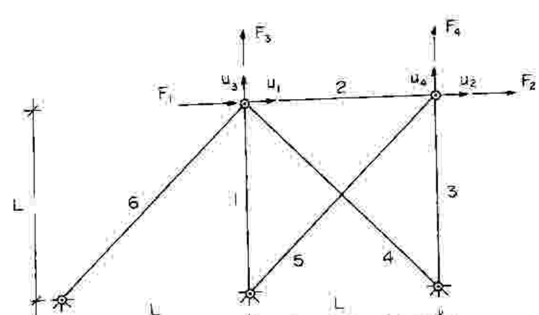

Fig. 5.

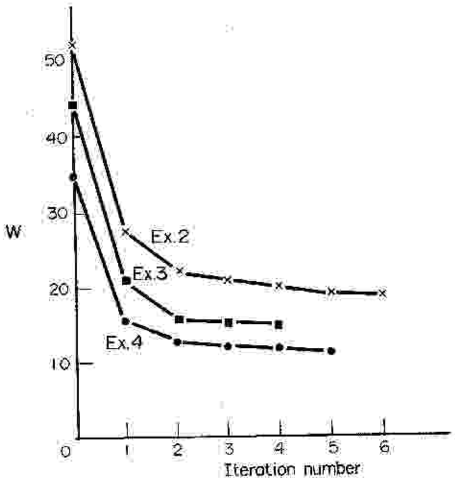

Fig. 6. Total weight obtained in the elastoplastic optimum design.

is feasible. ρ is increased to 100 in the second iteration because the new base point is feasible giving

$$a_1/l^2 = 35.9; \quad a_2/l^2 = 2.68; \quad a_3/l^2 = 1.19; \quad w/l^3 = 37.12.$$

This solution satisfies the constraints and is not far from the exact optimum

$$a_1/l^2 = 35.4; \quad a_2/l^2 = 3.0; \quad a_3/l^2 = 1.33; \quad w/l^3 = 36.91.$$

The purely elastic solution that complies with the suitable constraints on the stresses and displacements is

$$a_1/l^2 = 36.5; \quad a_2/l^2 = 2.3; \quad a_3/l^2 = 1.0; \quad w/l^3 = 37.1.$$

If the algorithm is started with a feasible point where the member areas are proportional to the member forces

$$a_1/l^2 = 50; \quad a_2/l^2 = 3; \quad a_3/l^2 = 1.25; \quad w/l^3 = 36.71$$

the algorithm (with constant $\rho = 100$) would converge in five iterations to

$$a_1/l^2 = 35.89; \quad a_2/l^2 = 2.81; \quad a_3/l^2 = 1.20; \quad w/l^3 = 37.13.$$

7.2. *Example 2*

Examples 2–4 refer to the six-bar truss depicted in Fig. 5 and used in [3] as a testbed. An elastic–perfectly plastic member behaviour was adopted. Each bar has a single yielding mode, with H_1^i in compression that is representative of the Euler buckling load in the plane of the axes and has unlimited tensile strength. Assuming that for sandwich cross-sections the force r_1 and the elastic stiffness S_i are proportional to a_i

$$r^1 = r^2 = r^3 = 3a_1; \qquad r^4 = r^5 = r^6 = 2a_2$$

$$S^1 = S^2 = S^3 = 3a_1; \qquad S^4 = S^5 = S^6 = 2a_2$$

and it is assumed that all bars possess unlimited tensile strength. Lower bounds on the cross-sectional areas $a^L = 0.01$ were imposed on all bars and all displacements cannot exceed in modulus 4, $U = [4\,4\,4\,4]$. In this example, it is required that all bars of the same length have the same cross-sectional area. Therefore, the structural optimization problem has only two independent design variables.

The cost function will be defined by the vector,

$$[3\,3\,3\,2\sqrt{2}\,2\sqrt{2}\,2\sqrt{2}].$$

The truss carries the load $[9\,0\,0\,0]$

$$a_1 = a_2 = a_3 = a'$$

$$a_4 = a_5 = a_6 = a''.$$

The feasible design $[a'\,a''] = [2\,4]$ with the corresponding cost 51.941 is assumed as the starting point. Since the initial design is inside the feasible domain it is desirable to use a large ρ. By keeping a constant $\rho = 50$ throughout the algorithm, the solution [0.904 1.405] with cost 20.082 and where bars 2 and 4 develop plastic behaviour is obtained at the fourth iteration. This result could be improved further (less than 1%) in the subsequent iterations by increasing ρ to 100 and 200, respectively.

The elastoplastic solution has a cost 35.7% lower than the purely elastic solution complying with the

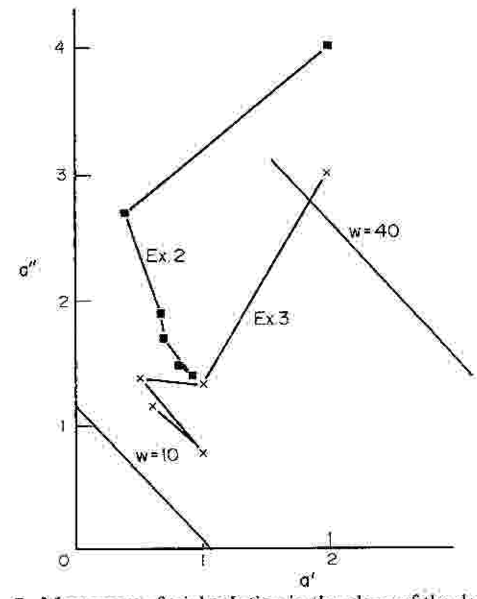

Fig. 7. Movement of trial solution in the plane of the design variables.

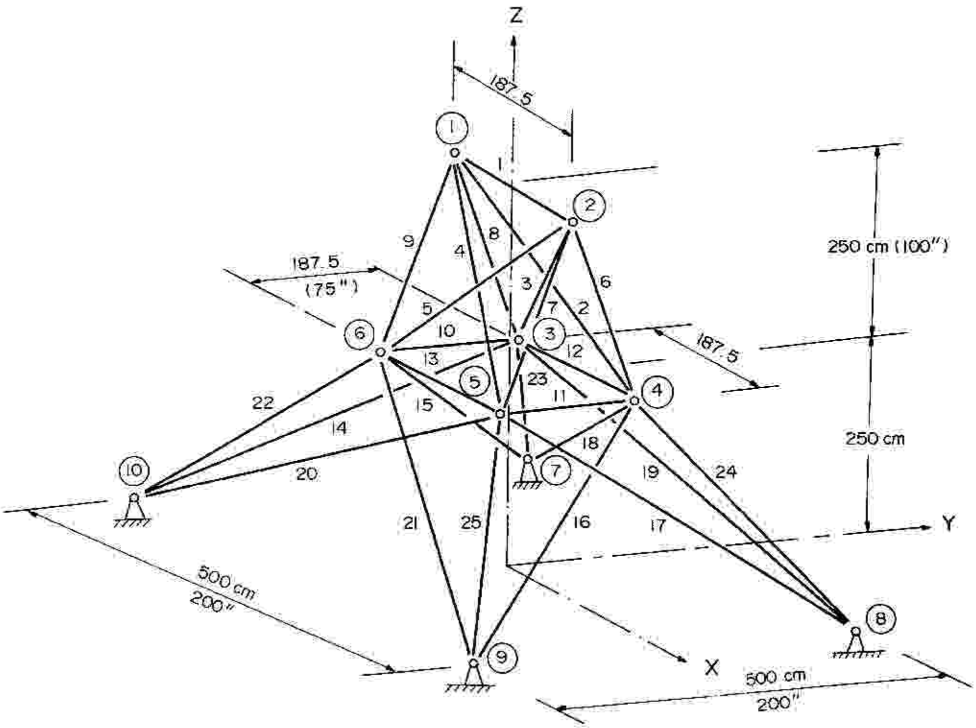

Fig. 8. Twenty-five-bar space truss.

suitable constraints on the stresses and displacements is [0.01 3.173] with the cost $w = 27.01$. The plastic limit synthesis *LP* leads to [1.495 0.01] with the cost $w = 13.54$.

7.3. *Example 3*

This problem is similar to example 2, but now the loading vector is [7 −3 0 0] and the initial feasible design $[a' \; a''] = [2 \; 3]$ with the corresponding cost 43.456 was adopted.

The solution [0.617 1.167] with cost 15.450 and where bars 2 and 4 yield is obtained in the fourth iteration. The parameter ρ was 50 except in the third iteration where it was reduced to 5 in order to produce a feasible design. Alternatively, it is possible to use $\rho = 50$ throughout the procedure. The nearly feasible design required in the third iteration can be achieved either by limiting the design variable changes or by interpolating the results of the third and fourth iterations. The purely elastic solution is [1.128 1.276] with the cost $w = 20.991$ and the plastic limit synthesis gives [1.162 0.01] with the cost $w = 10.54$ (see Fig. 6).

7.4. *Example 4*

In this example, the truss was considered under the same load vector as in example 2, but the cross-sectional areas of all six bars were assumed as independent design variables. The procedure started with an initial feasible design $a = [2\;2\;2\;4\;1\;1]$ with the corresponding cost 34.970 was assumed as the starting point. The design [0.01 0.01 0.01 3.202 0.01 0.675] with cost 11.085 that is also the solution in the elastic range was obtained after five iterations by using $\rho = 50$. The plastic limit design is [0.01 0.01 0.01 3.161 0.01 0.01] with the cost $w = 9.09$ (see Fig. 7).

7.5. *Example 5*

Figure 8 shows the 25-bar truss and its design specification, Table 1, gives member grouping and compressive stress limits. This truss carries the load,

$$[1\;10\;-5\;0\;10\;-5\;0.5\;0\;0\;0.5\;0\;0\;0\;0\;0\;0\;0\;0]_.$$

The cost function is defined by the vector of bar lengths and an elastic–perfectly plastic member behaviour was assumed. The truss was solved for the same stress constraints and minimum cross-sectional areas and for different nodal displacement limits.

Table 1. Details regarding 25-bar truss

Member number	Stress limits: Upper stress limit (psi)	Lower stress limit (psi)
1	40.000	−35.092
2–5	40.000	−11.590
6–9	40.000	−17.305
10–13	40.000	−35.092
14–17	40.000	−6.759
18–21	40.000	−6.959
22–25	40.000	−11.082

Imposing displacement limits of 0.50″ on all free nodes, the solution (in^2)

0.071 0.673 1.974 0.01 0.646 0.553 0.398 2.720

with weight 3649 was found after five iterations by using a constant value for $\rho = 100$. We remark that although an elastoplastic behaviour was possible, this design is entirely elastic. The algorithm is able to start from a feasible point inside the domain. The elastic design,

0.071 0.673 1.974 0.01 0.646 0.553 0.398 2.72

with weight 3439 was taken as the starting point and it was intended to find the truss with least weight such that the maximum nodal displacements are limited to 0.75″. The solution

0.06 0.337 1.54 0.183 0.80 0.495 0.274 1.694

with weight 2449 develops a plastic behaviour, the bars 14, 16 and 19 yielding under compression. Three iterations were required to find this design where ρ was 100, 20 and 100, respectively. The reduction of ρ in the second iteration results from obtaining a design with $\rho = 100$ which is unable to carry the applied loading.

The algorithm also works when the starting point is an infeasible solution. Consider the nodal displacements limited to 0.65″ and take for starting point the solution for $U = 0.75$. The solution

0.056 0.437 1.620 0.135 0.739 0.544 0.236 2.085

with weight 2733 develops a plastic behaviour in bar 19. It was found after four iterations where ρ was made 20 in the first three iterations and was reduced to 10 in the last in order to find a feasible solution.

8. CONCLUSIONS

This paper explored the possibility informational entropy maximization processes may have a place in the development of new algorithms for solving mathematical programming problems. The present work represents one new direction which might possibly result in improved techniques in the future.

The least weight design of materially nonlinear trusses was cast as a minimax problem that allows the simultaneous optimization and control of many different goals. The entropy-based solution method solves this problem by an unconstrained scalar optimization involving only one control parameter. The multipliers are automatically evaluated by maximizing the entropy, that is: by reducing the uncertainty in their evaluation based on the previous behaviour of the mathematical model. The ρ parameter is chosen according to the position of the current design with respect to the problem domain. It is not possible to guarantee that this algorithm converges to more than a local optimum of this nonconvex problems. The remaining local solutions can be enumerated either by selecting a different sequence for the control parameter or by trying out different starting points. However, in all the examples tried out, these procedures lead to designs with approximately the same cost. In conclusion, the performance of the maximum entropy based algorithms for the optimal design of trusses with elastoplastic behaviour is very encouraging providing a smooth convergence.

Rigid hardening models have a practical interest when the real local behaviour exhibits a strong hardening and when plastic strains prevail over elastic strains. They can be also dealt with by the method described in this work.

Acknowledgement—The author wishes to thank the financial support given by JNICT (Junta Nacional de Investigação Científica e Tecnológica, Proj. 87 230).

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