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RELIABILITY ASSESSMENT OF PLASTIC SLABS

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Abstract—An integrated form of yield line theory incorporating finite elements and concave quadratic programming is developed. The branch and bound strategy used to solve this mathematical program is simple to operate because it involves only linear programming algorithms. It provides either an upper bound to the reliability index of the stochastic most relevant mechanism for continuous slabs or an exact solution when the yield lines of the collapse mode of the continuous model lie along element boundaries of the finite element model. Algorithms that enumerate other stochastic dominant modes are described and applied to assess the reliability of slabs. Illustrative examples are given.

1. INTRODUCTION

In the yield method [1] for the plastic limit analysis of slabs, potential collapse mechanisms are postulated and the corresponding load parameters evaluated using well-known methods. However, the task of generating suitable trial mechanisms for relatively complicated problems is far from simple and it is therefore desirable to develop an automatic procedure for deriving the collapse mechanism without first guessing such trial mechanisms. When the plastic limit analysis of slabs is tackled in mathematical programming form by means of finite elements, the programs are proven to be linear [2]. The solution of the linear program is the value of the load which causes plastic collapse of the finite element model. Since these models are based on deterministic behaviour, it is only possible to guarantee by using them that the structure resists up to a static loading assumed to represent the most unfavourable loading supposed to act during its life and the behaviour of the constituent materials. Most important in calculating the failure probability of structural systems—as the number of failure modes for large systems can be extraordinarily high—is the search for the stochastic most relevant failure mechanism. In order to avoid the difficult numerical integration of the probability density functions involved, the reliability index is obtained from the limit state equation using the first-order second moment approximation. Plastic limit analysis yields the deterministic relevant mechanism, which is not necessarily identical to the stochastic most important mechanism, that is, the mechanism with the smallest reliability index β and the highest probability of failure p_F. A reinforced concrete slab example is given to support this statement.

The important role of mathematical programming is being increasingly recognised in terms of both computational convenience and theoretical formulations. The assessment of the reliability of frames with plastic behaviour has been tackled in this way [3] and an extension is made in this paper to cover the finding of the stochastic most important mechanism of plates with linearised yield conditions through finite element models. It turns out that this nonconvex functional programming is equivalent to a quadratic concave programming that can be solved through a branch and bound strategy where the nodes of the combinatorial tree are linear programs. Examples are given that show the effectiveness of this technique. The yield line theory can be associated with a form of finite element representation in which the yield lines are restricted to element boundaries. Thus the natural kinematic variables for this model are the mechanism deformations. The global solution of this problem is the minimum distance from the limit-state surface to the origin of the reduced normal variables, applied load and bending moment resistances.

2. STRUCTURAL RELATIONS

2.1. *Nodal description of kinematics*

As the deformations are to be described through the displacements of the nodes of triangular finite elements, it follows that such deformations are necessarily compatible. According to the hypothesis of the yield line theory, when the collapse mechanism is activated, the finite elements behave as rigid, but angular discontinuities may be generated across the element sides whilst providing for continuity of vertical displacements. The rotations θ_i^e of the outward normals to the three edges of the single element of Fig. 1 may then be expressed in terms of the corner vertical displacements u_j in the following manner:

$$\begin{bmatrix} \theta_1^e \\ \theta_2^e \\ \theta_3^e \end{bmatrix} = \begin{bmatrix} -l/h_1 & b_1/(l_1 h_1) & a_1/(l_1 h_1) \\ a_2/(l_2 h_2) & -l/h_2 & b_2/(l_2 h_2) \\ b_3/(l_3 h_3) & a_3/(l_3 h_3) & -l/h_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \quad (1)$$

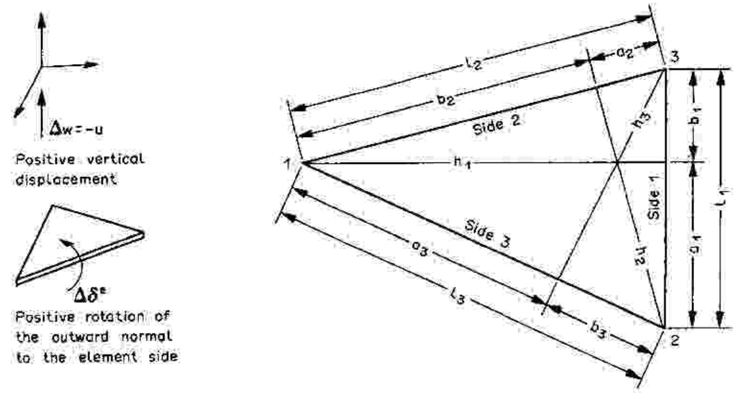

Fig. 1.

For an interelement side the total angular discontinuity θ_i is clearly the algebraic sum of the rotations θ_i^e of the two outward normals with respect to the two finite elements sharing such a side. If the boundary conditions are taken into account, then the number of nodal vertical displacements to be considered is merely the number (NC) of corner nodes free to undergo that form of displacement, and the vector θ will contain one component for each finite element side which can sustain the corresponding bending moment. If there are NS such angular discontinuities, the assemblage of relations (1) for all finite elements is thus readily performed and may be written in the following compact form:

$$\theta = Cu, \tag{2}$$

where C is an ($NS \times NC$) kinematic transformation matrix. Since these modal deformations are obtained as functions of the linearly independent displacements u it follows that such deformations are necessarily compatible and the rotation/displacement relations may be taken as the compatibility conditions.

2.2. *Constitutive relations*

A causal relationship between the statics and kinematics is encoded in the constitutive relations which model the material properties. In the present case, these constitutive relations consist of the yield conditions. The yield line theory considers a very simple yield criterion involving solely the normal bending moment and the normal angular discontinuity at every element side. The yield conditions impose limit values to the magnitudes of the total bending moments at the element side. If the unit length positive and negative bending moment capacities are, respectively, $\mathbf{m}*_i^+$ and $\mathbf{m}*_i^-$ for the ith element side, where the unit length bending moment is $\mathbf{m}_i$, then

$$-\mathbf{m}^-*_i \leqslant \mathbf{m}_i \leqslant \mathbf{m}^+*_i. \tag{3}$$

Now, if such an element side has a length l_i, the yield conditions in terms of total moments for the whole element side are:

$$-m*_i^- = -\mathbf{m}^-*_i l_i \leqslant m_i \leqslant \mathbf{m}^+*_i l_i = m^+*_i. \tag{4}$$

If α_i is the angle between the boundary side of length l_i and the y axis

$$m*_i^+ = l_i(\cos^2\alpha_i)m_x^+ + l_i(\sin^2\alpha_i)m_y^+ \tag{5a}$$

$$m*_i^- = l_i(\cos^2\alpha_i)m_x^- + l_i(\sin^2\alpha_i)m_y^-. \tag{5b}$$

The mechanism deformation can only take place at the element sides where the normal bending moment reaches one of its limiting values. That is to say, the angular discontinuity θ_i at the element side can only take a positive value, θ_i^+, when m_i is equal to $m*_i^+$ and it can only take a negative value, θ_i^-, when m_i is equal to $m*_i^-$.

The kinematic variables of this model are associated with mechanism modal deformations that are not necessarily total deformations before or after collapse. The modal deformations become total deformations when, in addition to identifying the collapse mechanism, the assumption is made that the element deformations are confined to plastic deformations at their boundaries. However, it should be emphasised that the latter assumption of element rigidity is unnecessary for the purposes of perfect plastic behaviour. Providing that at the incipient plastic collapse the displacements are small enough for the plastic analysis to be based on the undeformed geometry of the structure previous to the loading, elastoplastic deformations need not be considered. The plastic behaviour of slabs is not necessarily restricted to the finite element sides. Thus, the finite element modelling leads to a representation which is approximate for the slab.

3. RELIABILITY ASSESSMENT

3.1. *Collapse of ductile structural systems*

Since neither the loading nor the materials are deterministic, the reliability assessment of a structure has to take into account that during its design life the structure is generally subjected to a number of varying loads and their combinations and that its residence may deteriorate with time. In this context the problem arises of evaluating the conditional failure probability of the structural system given a certain load event. Variation of the parameters of the probability distributions, the respective types of distributions and the correlation among the variables involved has quite a significant effect on the results.

In the case of structures composed of ductile members such as components with elastic–perfectly plastic behaviour, the structural strength would be independent of the failure sequences of the component. For this class of structural systems, the collapse of a system would be through the formation of plastic mechanisms. In this regard, the performance function for the ith plastic mechanism for a frame may be defined as

$$Z_i = \sum_j a_{ij} m_j* - \sum_k b_{ik} \lambda_{kF}, \tag{6}$$

where j includes all the hinge moment capacities active in mechanism i, whereas k includes all loads active in mechanism i. The collapse of a system through mechanism i, therefore, is the event $F_i = (Z_i < 0)$. Then, if there are m possible failure mechanisms the probability of system collapse can be written as

$$p_F(C_s) = P(Z_1 < 0 \cup Z_2 < 0 \cup \cdots Z_m < 0) \tag{7a}$$

$$= P(F_1 \cup F_2 \cup \cdots \cup F_m). \tag{7b}$$

In general, Z_i are correlated random variables; consequently, calculation of the collapse probability, $p_f(C_s)$, will require the multiple integration of joint probability distributions of the correlated random variables Z_i, $i = 1, 2 \ldots m$. The mathematical program corresponding to the finding of the more important mechanism will be analysed next. The first-order second moment method [4] only requires the mean and standard deviation of the random variables probability distribution functions, namely loading and structural resistance. The reliability index, β_i, for the performance function Z_i is given by

$$\beta_i = \mu_{Zi} / \sigma_{Zi}, \tag{8}$$

where μ_{Zi} and σ_{Zi} are the mean and standard deviation of Z_i, respectively. As Z_i is a linear function of the number of random variables $m*$ and λ_F, the distribution of Z_i tends to normal (based on central limit theorem) irrespective of the individual distributions of the variables. Hence, using normal distribution for Z_i, the failure probability for mechanism i, p_{Fi}, is given by

$$p_{Fi} = \Phi(-\beta_i) = 1 - \Phi(\beta_i), \tag{9}$$

where Φ is the cumulative probability of a standardized normal variable.

3.2. *Computation of the reliability index*

By associating the rotations of the element sides with the rotations of the members represented by the same random variables $\theta*^+, \theta*^-$ through the incidence matrices J_θ^+, J_θ^- and the displacements of the point loads (or in the case of uniformly distributed load, deflections of the triangular elements centroids) linked by the same random variable $u*$ through J_u

$$\theta*^+ = J_\theta^+ \theta^+$$

$$\theta*^- = J_\theta^- \theta^-$$

$$u* = J_u u. \tag{10}$$

For statistically independent random normal variables, the identification of the stochastic most important mechanism consists of finding the position of the limit-state equation closer to the origin of the reduced normal variables. This amounts to minimizing the distance–reliability index, β [5]

$$\min \beta = \sqrt{((m*^{+\prime})^2 + (m*^{-})^{\prime 2} + (\lambda_F')^2)}. \tag{11}$$

The relationships linking the reduced normal variables to the normal variables are

$$m*^+ = \mu_{m+} + \sigma_{m+} m*^{+\prime}$$

$$m*^- = \mu_{m-} + \sigma_{m-} m*^{-\prime}$$

$$\lambda_F = \mu_F + \sigma_F \lambda_F', \tag{12}$$

where μ_{m+}, μ_{m-}, μ_F and σ_{m+}, σ_{m-}, σ_F are the mean and standard deviation of the random variables $m*^+$, $m*^-$ and λ_F, respectively. The limit-state function equates the external and internal work produced by each mechanism

$$Z(m*^+, m*^-, \lambda_F) = m*^{+\prime} \theta*^+ + m*^{-\prime} \theta*^- - \lambda_F' u*$$

$$= 0. \tag{13}$$

Substituting $m*$ and λ_F for eqn (9) in the limit-state equation yields

$$\sigma_{m+} m*^{+\prime} \theta*^+ + \sigma_{m-} m*^{-\prime} \theta*^- - \sigma_F \lambda_F' u*$$

$$= -\mu_{m+} \theta*^+ - \mu_{m-} \theta*^- + \mu_F u*. \tag{14}$$

In eqn (11) the plastic modal deformations of the mechanism are present as state variables. Since it is required to find the minimum distance from the origin of the reduced normal variates to the yield surface, the values of $m*^{+\prime}$, $m*^{-\prime}$ and λ'_F given by its minimum norm solution are

$$m*^{+\prime}=\frac{-\sigma_{m+}\theta*^{+}[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]}{(\sigma_{m+})^2(\theta*^{+})^2+(\sigma_{m-})^2(\theta*^{-})^2+(\sigma_F)^2(u*)^2} \tag{15a}$$

$$m*^{-\prime}=\frac{-\sigma_{m-}\theta*^{-}[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]}{(\sigma_{m+})^2(\theta*^{+})^2+(\sigma_{m-})^2(\theta*^{-})^2+(\sigma_F)^2(u*)^2} \tag{15b}$$

$$\lambda'_F=\frac{\sigma_F u*[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}\mu_F u*]}{(\sigma_{m+})^2(\theta*^{+})^2+(\sigma_{m-})^2(\theta*^{-})^2+(\sigma_F)^2(u*)^2}. \tag{15c}$$

For positive $[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]$, the identification of the stochastic most relevant mechanism consists of finding

$$\min\beta=\frac{[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]}{\sqrt{((\sigma_{m+})^2(\Delta\theta*^{+})^2+(\sigma_{m-})^2(\Delta\theta*^{-})^2+(\sigma_F)^2(u*)^2)}}. \tag{16}$$

If there are more random variables to be considered and the matrices $C_{\theta+}$, $C_{\theta-}$ and C_u represent the correlations between the yield lines and between the loads, respectively, the reliability index is given by

$$\beta=\frac{[\mu_{m+}t\theta*^{+}+\mu_{m+}t\theta*^{-}-\mu'_F u*]}{\sqrt{((\theta*^{+}\sigma_{m+}C_{\theta+}\sigma_{m+}\theta*^{+})+(\theta*^{-}\sigma_{m-}C_{\theta-}\sigma_{m-}\theta*^{-})+u*\sigma_F C_u\sigma_F u*)}}. \tag{17}$$

For random variables that are not correlated, the original variates may be transformed to a set of uncorrelated variables. The required transformation is necessarily dependent on the covariance matrix of the original variates, whose elements are the covariances between the pairs of variables. The required set of uncorrelated transformed variates can be obtained through an orthogonal transformation [6].

4. MATHEMATICAL PROGRAMMING FORMULATION

4.1. *Fractional programming*

The following optimisation problem is termed fractional programming [7]:

$$\min\{q(x)=n(x)/(d(x)|x\in S\}, \tag{18a}$$

where

$$\{x\in R^n|h_i(x)\leqslant b_i\quad i=1\ldots m\}, \tag{18b}$$

$n(x)$, $d(x)$ are non-negative, $d(x)$ is linear and $h_i(x)$ is convex.

Theorem. The fractional program (18) can be reduced to the program (19)

$$\min\{tn(y/t)|t>0,\,th_i(y/t)-tb_i\leqslant 0,\,td(y/t)\geqslant 1,\quad i=1\ldots m\} \tag{19a}$$

by applying the variable transformation

$$y=(1/d(x))x \tag{19b}$$

and $t=1/d(x)$ can be any positive number (say $t=1$).

4.2. *Identification of the stochastic most relevant mechanism*

For positive $[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]$ and the uncorrelated random variables $m*^{+}$, $m*^{-}$ and λ_F the identification of the stochastic most relevant mechanism consists of minimising the reliability index β given by eqn (16)

$$\min\beta=\frac{[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]}{\sqrt{((\sigma_{m+})^2(\theta*^{+})^2+(\sigma_{m-})(\theta*^{-})^2+(\sigma_F)^2(u*)^2)}} \tag{20a}$$

subject to the linear incidence equations (10)

$$\theta*^{+}=J_\theta^{+}\theta^{+}$$
$$\theta*^{-}=J_\theta^{-}\theta^{-}$$
$$u*=J_u u, \tag{20b}$$

the compatibility relations of the nodal description (2)

$$\theta=Cu \tag{20c}$$

and the sign constraints

$$\theta^{+}\geqslant 0,\theta^{-}\geqslant 0,u\geqslant 0,\theta*^{+}\geqslant 0,\theta*^{-}\geqslant 0,u*\geqslant 0. \tag{20d}$$

This mathematical program belongs to the class of fractional programming problems. Moreover, the minimisation of β shares its solutions with

$$\max\frac{1}{\beta}=\frac{\sqrt{((\sigma_{m+})^2(\theta*^{+})^2+(\sigma_{m-})^2(\theta*^{-})^2+(\sigma_F)^2(u*)^2)}}{[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]} \tag{21}$$

subject to the same constraints. Since $[\mu_{m+}\theta*^{+}+\mu_{m-}\theta*^{-}-\mu_F u*]$ is positive, and using the theorem stated in Sec. 4.1, eqn (20) shares its solutions with

$$\max 1/\beta^2=(\sigma_{m+})^2(\theta*^{+})^2+(\sigma_{m-})^2(\theta*^{-})^2+(\sigma_F)^2(u*)^2, \tag{22}$$

i.e.

$$\min -1/\beta^2=-(\sigma_{m+})^2(\theta*^{+})^2+(\sigma_{m-})^{-}(\theta*^{-})^2-(\sigma_F)^2(u*)^2 \tag{23a}$$

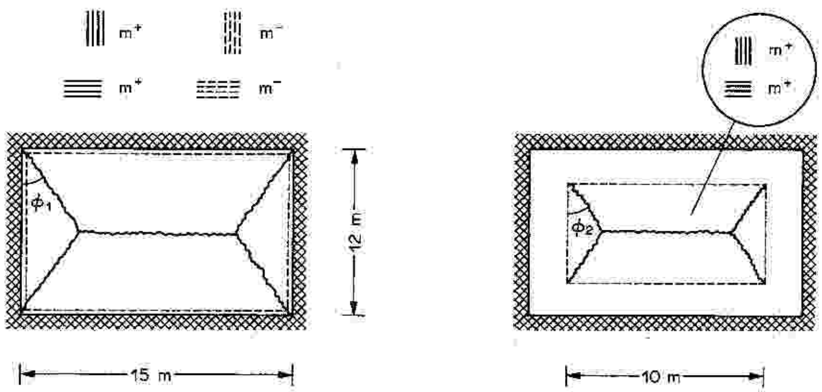

Fig. 2.

subject to

$$\mu_{m+}\theta*^{+} + \mu_{m-}\theta*^{-} - \mu_F u* = 1 \tag{23b}$$

$$\theta*^{+} = J_\theta^{+}\theta^{+}$$

$$\theta*^{-} = J_\theta^{-}\theta^{-}$$

$$u* = Ju \tag{23c}$$

$$\theta = Cu \tag{23d}$$

$$\theta^{+} \geqslant 0, \theta^{-} \geqslant 0, u \geqslant 0, \theta*^{+} \geqslant 0, \theta*^{-} \geqslant 0, u* \geqslant 0, \tag{23e}$$

which is a quadratic concave minimisation. This type of problem cannot be solved by convex programming techniques because of the possibility of nonglobal local minima. The global optimum of these programs gives the plastic deformations for the stochastic most important mechanism and the reduced random variables can be evaluated by using eqn (15) or

$$m*^{+} = -\sigma_{m+}\theta*^{+}\beta^2 \tag{24a}$$

$$m*^{-} = -\sigma_{m+}\theta*^{-}\beta^2 \tag{24b}$$

$$\lambda_F' = \sigma_F u*\beta^2. \tag{24c}$$

4.3. *Example of a slab where the deterministic most important mechanism is not necessarily identical to the stochastic relevant mode of collapse*

Limit load analysis at mean values yields the deterministic relevant mechanism, which may not be the same as the mechanism with the highest failure probability. As an example, consider a rectangular slab subject to a uniformly distributed load and with the top and bottom made of steel. If the top steel is curtailed, it is necessary to consider the alternative yield line mechanisms shown in Fig. 2. For $m* = m*^{+} = m*^{-}$, the deterministic load for the mechanism on the left is $\lambda_F = 0.2716m*$, whereas the mechanism on the right leads to $\lambda_F = 0.2889m*$. If the most important stochastic mechanism is intended, it is necessary to consider random variables such as the load λ_F and the bending moment resistances $m*^{+}$, $m*^{-}$. Assuming they are not correlated and Gaussian,

$$\mu_{m+} = 35\ \text{kN m/m} \qquad \Omega_{m+} = 0.15$$

$$\mu_{m-} = 35\ \text{kN m/m} \qquad \Omega_{m-} = 0.15$$

$$\mu_F = 5\ \text{kN/m}^2 \qquad \Omega_F = 0.30,$$

the mechanism on the left of Fig. 2 associated with $\beta = 2.493$ is less likely than the alternative collapse mode giving $\beta = 2.397$.

It is possible to guarantee that the deterministic relevant mode is equal to the stochastic most relevant mechanism only if Ω_{m+}, Ω_{m-} and Ω_F are small.

5. BRANCH AND BOUND SOLUTION PROCEDURE

5.1. *General methodology*

Two major classes of algorithms are appropriate to deal with nonconvexities. One belongs to the class of cutting plane methods [8] and, although being very elegant from a mathematical point of view, this technique requires a lot of expertise concerning the generation of the cutting planes. Alternatively, a branch and bound (B & B) solution procedure is described next in some detail. A similar B & B technique was employed by Phillips and Rosen [9] to solve concave quadratic minimisation problems with up to 50 nonlinear variables and 400 linear variables on a CRAY2. The general nonconvex domain is transformed in the B & B strategy into a sequence of intersecting convex domains by the use of underestimating convex functions. It is well known that a local solution to a problem possessing a convex objective function and being restricted by a convex domain is also its global solution.

The two main ingredients are a combinatorial tree with appropriately defined nodes and some upper and lower bounds to the final solution associated with each node of the tree. It is then possible to eliminate a large number of possible solutions without evaluating them. As the implicit enumeration program relies on an upper bound, its efficiency can be greatly improved by providing a good feasible initial solution.

A partial solution is said to be fathomed if the best feasible completion of the solution can be found or if it can be determined that, no matter how the design variables are chosen, it will be impossible to find a feasible completion of smaller distance than that previously found. If a partial solution is fathomed this means that all possible completions of this partial solution have been implicitly enumerated and therefore need not be explicitly enumerated. When the last node is fathomed the algorithm terminates with the optimum solution. Backtracking in the tree is performed so that no solution is repeated or omitted from consideration.

5.2. *Convex underestimates of quadratic concave functions*

The convex envelope of a function over a closed convex set is the highest convex function which everywhere underestimates the function. For a quadratic concave function $-x^2$ given in the interval $[l, L]$ its convex underestimate is the affine function (linear plus a constant) passing through the endpoints of the given function graph (Fig. 3)

$$e(x) = -(l + L)x \times lL. \tag{25}$$

5.3. *Outside-in approach*

In the following the algorithm originally presented by Falk and Soland [10] for separable functions is outlined. Considering the quadratic concave program

$$\min z(x) = -x'x \tag{26a}$$

$$\text{st } Ax \geqslant b \tag{26b}$$

$$l \leqslant x \leqslant L, \tag{26c}$$

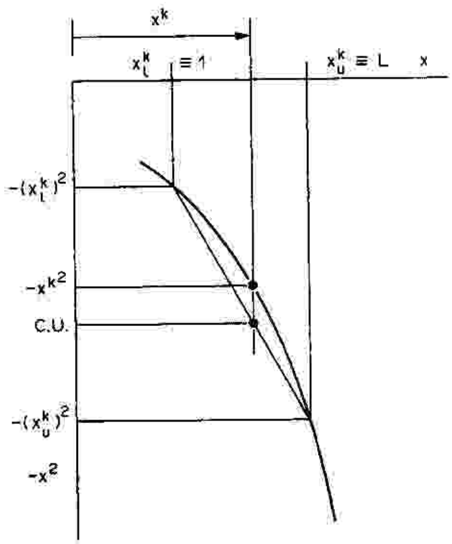

Fig. 3.

where $x = [\theta * u *]$, let x^p be the solution of the linear underestimating subproblem (P_p), with $k = 1$

$$\min e(x) = d_p' x + k_p \tag{27a}$$

$$\text{st } Ax \geqslant b \tag{27b}$$

$$l_p \leqslant x \leqslant L_p. \tag{27c}$$

If $e(x^p)$ does not coincide with $z(x^p)$, one may try to restrict the domain of the subproblem (P_p) in order to have a tighter objective function. Take as incumbent bound v the objective function of the original problem $z(x^p)$. (P_p) is replaced by a set of problems that bound the original problem in the sense that there exists one optimal solution $x*$ for at least one problem $j \in W^p$. Suppose an optimal solution to each such problem is obtained and let

$$x^s = \min_{j \in W^p} e(x^s) = d^t x^j + k. \tag{28}$$

If $e(x^s)$ does not coincide with $z(x^s)$, one of the problems of the bounding set is replaced by a set of new problems. If $z(x^s)$ is lower than the incumbent v, this upper bound will be updated. Let $p = p + 1$. The problem is repeated by a set W^p, such that $W^p = (W^{p-1} - \{s\}) \cup W^s$ contains an optimal solution of the original problem for at least one problem W^p.

For each problem $j \in W^p$, either although getting closer $e(x^j)$ does not coincide with $z(x^p)$, or $e(x^j) > v$. This is a condition ensuring that some progress towards the final solution is made.

The combinatorial tree has each node identified with a subproblem j. The problems that replace j in the bounding set W^p are pointed to by the branches directed outward from that node. At any intermediate point in the calculations, the set of the current bounding problem is identified with the set of nodes that are the leaves of the tree. Any leaf node of the tree whose bound is strictly less than v is active. Otherwise it is designated as terminated and need not be considered in any further computation. The B & B tree will be developed until every leaf can be terminated.

5.4. *Rule for splitting the intervals*

When the underestimate taken does not coincide with the function true value, it is also required to define (in an heuristic way) a refining rule for splitting the bounds on the selected variable: choose the index i of the variable that maximises the difference between the quadratic concave term and its affine underestimate. The corresponding interval is divided into new intervals $[l_i, x_i]$ and $[x_i, L_i]$. Therefore, as soon as a node is selected to be branched, the partition of its interval is only dependent on its solution value and is not related to other partitions

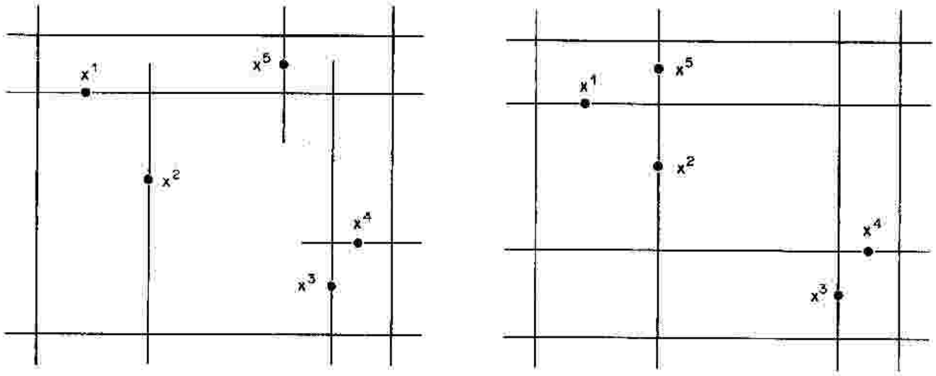

Fig. 4.

at the same level of the tree. This corresponds to a weaker form of Falk's convergence theorem not requiring the completion of the intervals partitioned (Fig. 4).

The intuitive motivation for these rules is that the convex envelopes computed for the new feasible regions will be brought closer to the value of the function at x^p by using this splitting rule than by using any other.

5.5. *Examples*

The following examples have been solved in [2] for plastic limit analysis and are used here to illustrate the reliability assessment procedure.

5.5.1. *Uniformly loaded isotropic square slab supported by columns at the four corners.* The first example is presented in some detail. The plate discretisation can be done in such a way that the correct yield lines may be activated, and consequently, the exact reliability index is obtained. The discretisation of Fig. 5 is then described by the following data:

No. of finite elements = 8

No. of corner nodes = 9

No. of element sides = 16.

The boundary conditions are easily taken into account by identifying the fixed corner nodes and the boundary element sides free to rotate. Thus

NC = no. of free corner nodes = 5

NS = no. of element sides not free to rotate = 8.

Assuming that the applied load and the moment resistances are random normal variables

$$\mu_{m+} = 100\ \text{kN m/m} \qquad \Omega_{m+} = 0.15$$

$$\mu_{m-} = 100\ \text{kN m/m} \qquad \Omega_{m-} = 0.15$$

$$\mu_F = 3.5\ \text{kN/m}^2 \qquad \Omega_F = 0.30.$$

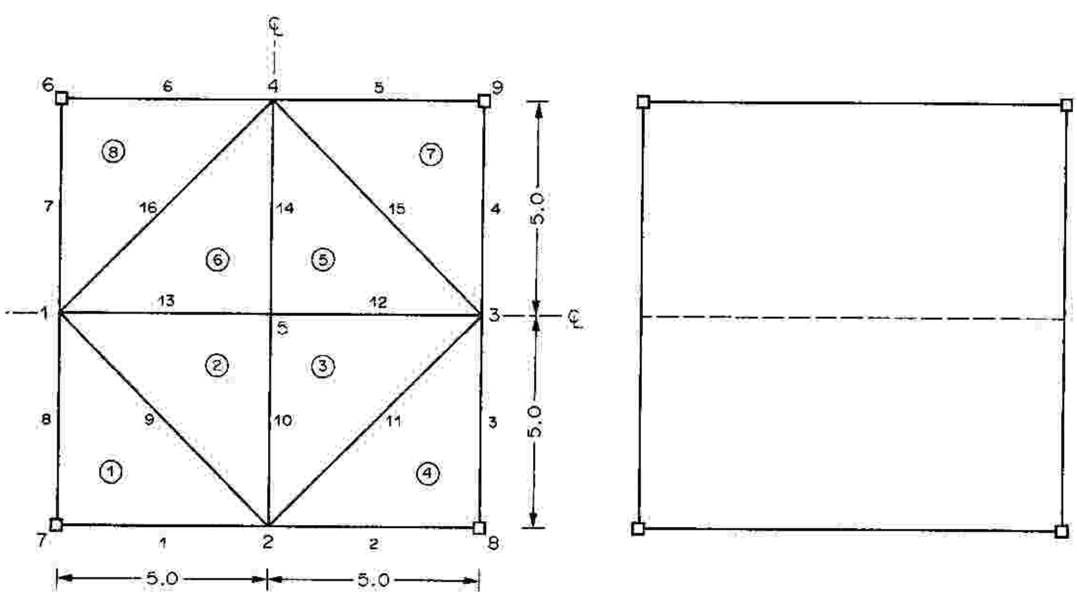

Fig. 5.

If they are uncorrelated, the quadratic concave programming is

$$\min z = \sigma_{m+}^{\ 2}\theta*^{-2} + \sigma_{m-}^{\ 2}\theta*^{+2} + \sigma_F^2 u*^2$$

$$\begin{array}{llllll} \text{st} & \mu_{m+}\theta*^{+} - \mu_{m-}\theta*^{-} - \mu_F u* & & & & = 1 \\ & \theta*^{+} & & -J_\theta\theta^{+} & & = 0 \\ & & \theta*^{-} & & -J_\theta\theta^{-} & = 0 \\ & & & u* & -J_u u & = 0 \end{array}$$

$$u \geqslant 0; \theta^{+} \geqslant 0; \theta^{-} \geqslant 0; u* \geqslant 0; \theta*^{+} \geqslant 0; \theta*^{-} \geqslant 0,$$

which has $M = 12$ constraints and $N = 24$ variables. If the coordinates of all corner nodes are provided, all data can be calculated.

The incidence vectors are simply

$$J_\theta = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$$

$$J_u = [16.67\ 16.67\ 16.67\ 16.67\ 16.67].$$

The kinematic transformation matrix for every finite element may be first determined, and then assembled according to the element connectivity. Considering, for example, finite element 2 (Fig. 5), and using eqn (1), the kinematic transformation reads

$$\begin{bmatrix} \theta_{10} \\ \theta_{13} \\ \theta_9 \end{bmatrix} = \begin{bmatrix} -1/5 & 0 & 1/5 \\ 0 & -1/5 & 1/5 \\ 1/5\sqrt{2} & 1/5\sqrt{2} & -\sqrt{2}/5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

The kinematic transformation matrix C is then assembled for the system of finite elements.

$$\begin{bmatrix} \theta_9 \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ \theta_{14} \\ \theta_{15} \\ \theta_{16} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/5 & \sqrt{2}/5 & 0 & 0 & -\sqrt{2}/5 \\ -1/5 & 0 & -1/5 & 0 & 2/5 \\ 0 & \sqrt{2}/5 & \sqrt{2}/5 & 0 & -\sqrt{2}/5 \\ 0 & -1/5 & 0 & -1/5 & 2/5 \\ 0 & -1/5 & 0 & -1/5 & 2/5 \\ -1/5 & 0 & -1/5 & 0 & 2/5 \\ 0 & 0 & \sqrt{2}/5 & \sqrt{2}/5 & -\sqrt{2}/5 \\ \sqrt{2}/5 & 0 & 0 & \sqrt{2}/5 & -\sqrt{2}/5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}.$$

The results of the branching strategy known as breadth first (choosing the node with lower bound) are represented in the combinatorial tree of Fig. 6.

The non-zero vertical displacements at corner nodes are

$$u_1 = 0.00444; \quad u_3 = 0.00444; \quad u_5 = 0.00444.$$

The non-zero element side angular discontinuities are

$$\theta_{12} = 0.00178; \quad \theta_{13} = 0.00178.$$

The reduced normal variables can be obtained using eqn (24)

$$\theta*^{+} = 0.01778- \quad \theta*^{-} = 0; \quad u* = 0.222222$$

$$m*^{+'} = -\sigma_{m+}\theta*^{+}\beta^2 = -2.12$$

$$m*^{-'} = -\sigma_{m-}\theta*^{-}\beta^2 = 0.0$$

$$\lambda_{F'} = \sigma_F u*\beta^2 = 1.86$$

and from eqn (12) the normal variables are

$$m*^{+} = \mu_{m+} + \sigma_{m+}m*^{+'} = 68.14$$

$$m*^{-} = \mu_{m-} + \sigma_{m-}m*^{-'} = 100$$

$$\lambda_F = \mu_F + \sigma_F\lambda_{F'} = 5.45.$$

A yield line is formed along sides 12 and 13, as represented in Fig. 5, associated with the reliability index $\beta = 1.932$. Another optimal solution with the same safety factor exists with a yield line formed along sides 10 and 14. No use was made of the symmetric features of the problem, but it is readily seen that it would have been sufficient to consider 1/8 of the slab.

5.5.2. *Centrally loaded isotropic clamped square slab*. Due to the symmetry of the problem, only 1/8 of the slab need to be considered. Take as statistically independent and Gaussian the random variables

$$\mu_{m+} = 100 \text{ kN m/m} \qquad \Omega_{m+} = 0.15$$

$$v_{m-} = 100 \text{ kN m/m} \qquad \Omega_{m-} = 0.15$$

$$\mu_F = 650 \text{ kN} \qquad \Omega_F = 0.30.$$

Since a circular yield line pattern is expected, the very simple discretisation of Fig. 7 is sufficient for an estimate, $\beta = 2.927$, of the reliability index, along with the yield line pattern represented in Fig. 8.

The discretisation in Fig. 9 gives a lower reliability index, $\beta = 2.6366$, which is closer to the value $\beta = 2.568$ that can be obtained from an analysis using a circular yield line.

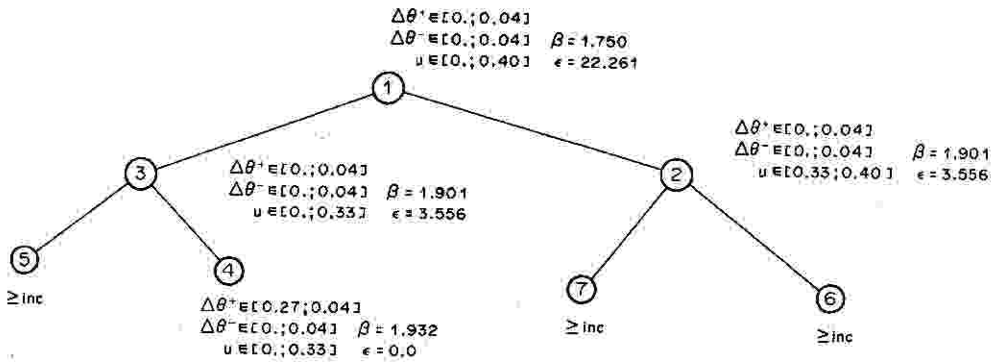

Fig. 6.

5.5.3. *Bäcklund slab.* The previous examples were relatively simple but illustrated the main features of the procedure. The uniformly loaded isotropic slab shown in Fig. 10 contains much of the complexity of real design problems. Its plastic limit analysis with determinate behaviour has been previously studied by Bäcklund [11], who used a step-by-step procedure, and in [2], where linear programming was employed to find the collapse load. As an illustration of the quadratic concave programming solution procedure described in the present paper when applied to this relatively complex example, the discretisation of Fig. 11 was adopted, using 60 finite elements. Consider the uncorrelated and Gaussian random variables

$$\mu_{m+} = 20\ \text{kN m/m} \qquad \Omega_{m+} = 0.15$$

$$\mu_{m-} = 20\ \text{kN m/m} \qquad \Omega_{m-} = 0.15$$

$$\mu_F = 4\ \text{kN/m}^2 \qquad \Omega_F = 0.30.$$

Since the supporting walls can be taken as a clamped edge, this slab can be divided into three distinct parts. The finite element modelling of these and the corresponding yield line patterns are represented in Figs 12–14.

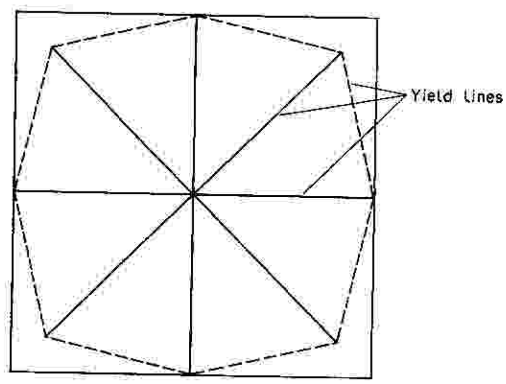

Fig. 8.

If the discretisation of Fig. 11 is used one is penalised in the CPU time by the larger size of each *LP* that has to be solved in each node: $NS = 77$, $NC = 31$; $N = 188$; $M = 81$; no. iterations = 16; CPU time = 8 min; $\beta = 2.674$.

The minimum intervals of variation of the non-linear variables can be found by solving a multiple-cost-row linear program. When these smaller intervals are used, the procedure becomes much more efficient.

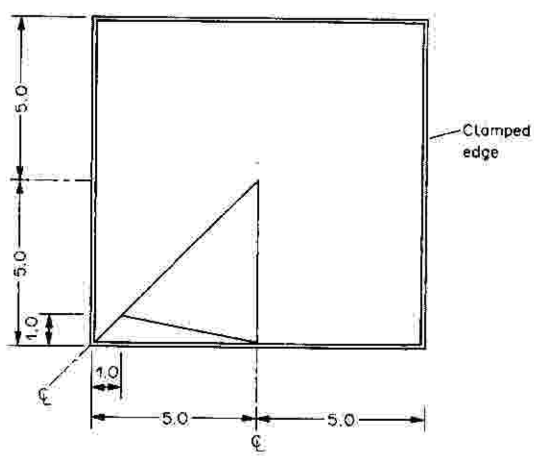

Fig. 7. $NS = 5$; $NC = 2$; $N = 15$; $M = 9$; no. iterations = 15; CPU time = 2 sec.

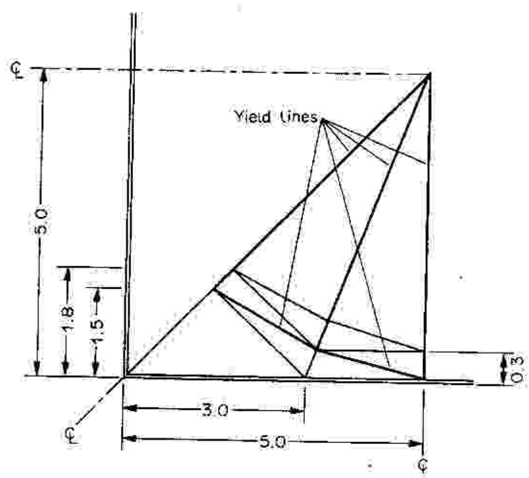

Fig. 9. $NS = 17$; $NC = 6$; $N = 43$; $M = 21$; no. iterations = 13; CPU time = 12 sec.

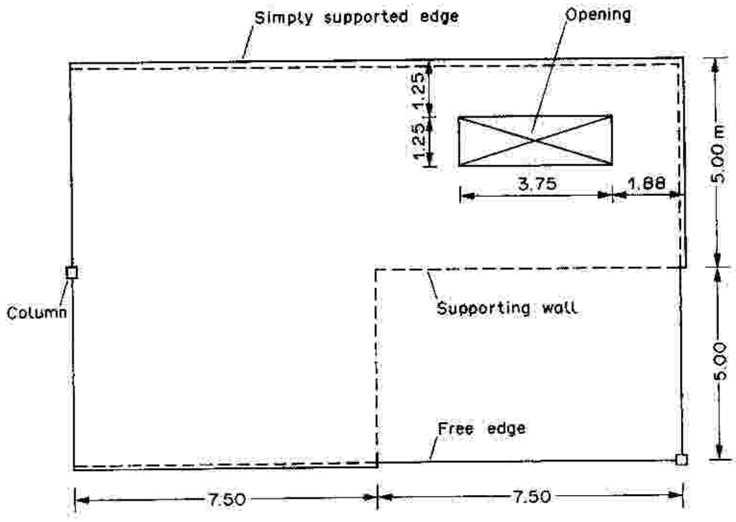

Fig. 10.

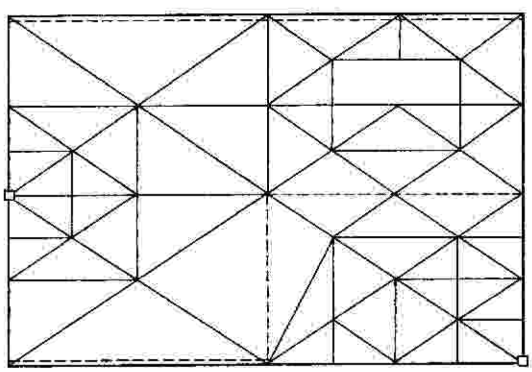

Fig. 11.

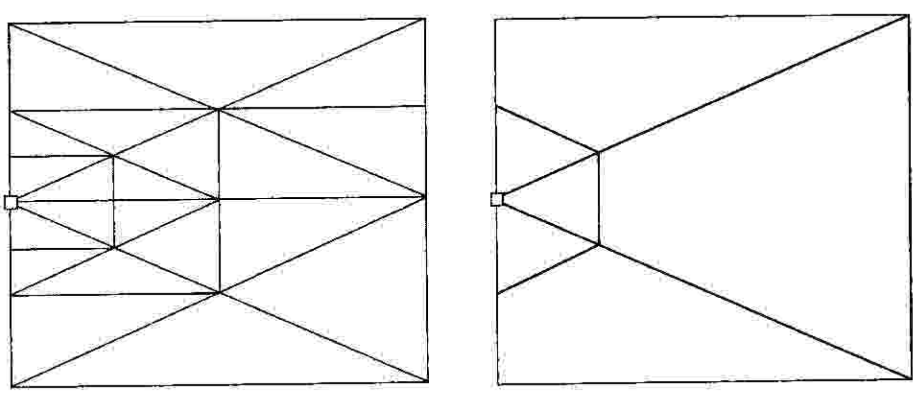

Fig. 12. $NS = 29$; $NC = 11$; $N = 72$; $M = 33$; no. iterations = 13; CPU time = 33 sec; $\beta = 2.674$.

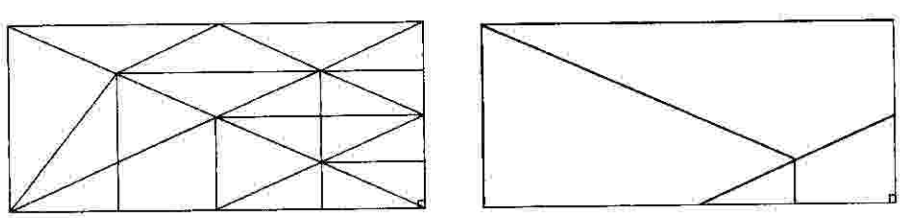

Fig. 13. $NS = 29$; $NC = 12$; $N = 73$; $M = 33$; no. iterations = 13; CPU time = 38 sec; $\beta = 3.843$.

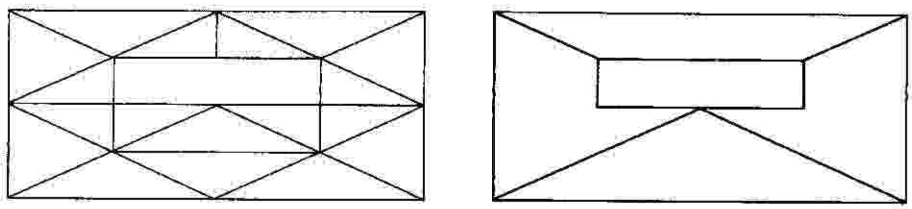

Fig. 14. $NS = 24$; $NC = 8$; $N = 59$; $M = 28$; no. iterations = 17; CPU time = 26 sec; $\beta = 6.667$.

6. OVERALL PROBABILITY OF FAILURE WITH RESPECT TO PLASTIC COLLAPSE

6.1. *First- and second-order bounds*

In reliability analysis with respect to plastic collapse, the reinforced concrete structure can be modelled as a system with the individual collapse modes in series (the structure will fail if any collapse mode occurs). An individual collapse mode can be modelled as a parallel system of yield lines (a failure mode will occur if all the yield lines related to this mode occur).

Collapse mode events are usually correlated through loading and resistances, so an exact evaluation of the probability (7) is impractical, or even impossible to perform numerically. For this reason, several investigators considered this problem by finding either bounds for p_F or approximate solutions. A first estimate of p_F can be found through the well-known first-order bounds proposed by Cornell [12]

$$\max_{\text{all } i}\,[P(F_i)] \leqslant p_F \leqslant 1 - \prod_{i=1}^{m}[1 - P(F_i)]. \tag{29}$$

The lower bound, which represents the probability of occurrence of the most critical mode (dominant mode) is obtained by assuming the mode failure events F_i to be perfectly dependent, and the upper bound is derived by assuming independence between mode failure events.

The bounds (29) can be improved by taking into account the probabilities of joint failure events such as $P(F_i \cap F_j)$, which means the probability that both events F_i and F_j will simultaneously occur. The resulting closed-form solutions for the lower and upper bounds are as follows:

$$p_F \geqslant P(F_1) + \sum_{i=2}^{m} \max\left\{\left[P(F_i) - \sum_{j=1}^{i-1} P(F_i \cap F_j)\right]; 0\right\} \tag{30a}$$

$$p_F \leqslant \sum_{i=1}^{m} P(F_i) - \sum_{i=2}^{m} \max_{j<i} P(F_i \cup F_j). \tag{30b}$$

The above bounds can be further approximated using Ditlevsen's method of conditional bounding [13]. This is accomplished by using a Gaussian distribution space in which it is always possible to determine three numbers β_i, β_j and ρ_{ij} for each pair of collapse modes F_i and F_j such that if $\rho_{ij} > 0$ (i.e. if F_i and F_j are positively dependent)

$$P(F_i \cap F_j) \geqslant \max\left\{\Phi(-\beta_j)\Phi\left(-\frac{\beta_i - \beta_j\rho_{ij}}{\sqrt{(1-\rho_{ij}^2)}}\right);\ \Phi(-\beta_i)\cdot\Phi\left(-\frac{\beta_j - \beta_i\rho_{ij}}{\sqrt{(1-\rho_{ij}^2)}}\right)\right\} \tag{31a}$$

$$P(F_i \cap F_j) \leqslant \Phi(-\beta_j)\cdot\Phi\left(-\frac{\beta_i - \beta_j\rho_{ij}}{\sqrt{(1-\rho_{ij}^2)}}\right) + (-\beta_i)\cdot\Phi\left(-\frac{\beta_j - \beta_i\rho_{ij}}{\sqrt{(1-\rho_{ij}^2)}}\right), \tag{31b}$$

in which β_i and β_j are the safety indices of the ith and the jth failure modes, p_{ij} is the correlation coefficient between the ith and the jth failure modes, and $\Phi(\,)$ is the standardised normal probability distribution function.

The probabilities of the joint events $P(F_i \cap F_j)$ in eqns (30a) and (30b) are then approximated with the appropriate sides of eqns (31a) and (31b). For example, if F_i and F_j are positively dependent, for the lower bound (30a) it is necessary to use the approximation given by the upper bound (31b) and for the upper bound (30b) it is necessary to use the approximation given by the lower bound (31a).

Moses and Kinser [14] have shown that the overall probability of collapse of a system, can be expressed in the following way:

$$p_F = P(F_1) + \sum_{i=2}^{m} a_i P(F_i), \tag{32a}$$

where

$$a_i = P(S_1 \cap S_2 \cap \cdots \cap S_{i-1} \mid F_i) \tag{32b}$$

is the conditional probability that the first $i-1$ modes survive given that mode i occurs. Note that the failure modes are arranged so that $P(F_1) \geqslant P(F_2) \geqslant \cdots \geqslant P(F_i) \geqslant \cdots \geqslant P(F_m)$ because the value of the conditional probability (32a) depends on the ordering of failure modes.

The method introduced by Vanmarcke [15] reduces the number of survival events in relation (32a) to one, such that

$$a_i \leqslant \min_{j=1}^{i-1} P(S_j \mid F_i) = a_i^*. \tag{33}$$

Therefore, $a_1^* = a_1 = 1$, $a_2^* = a_2$ and

$$P(F_1) + \sum_{i=2}^{m} a_i P(F_i) \tag{34}$$

is an upper bound to the overall probability of collapse. Using a first-order approach, Vanmarcke introduced a useful approximation of the conditional probability $P(S_j|F_i)$ in terms of the safety indices β_i and β_j and of the coefficient of correlation ρ_{ij} between the failure modes F_i and F_j, as follows:

$$P(S_j|F_i) \cong 1 - \frac{\Phi[\max(\beta_j/|\rho_{ij}|;\, \beta_i)]}{\Phi[\beta_i]}, \tag{35}$$

in which it is assumed that the probability of occurrence of the ith mode $P(F_i) = \Phi[\beta_i]$ depends on β_i only.

The identification of all the significant collapse methods of a ductile structural system is necessary in the analysis and evaluation of the system reliability, including the evaluation of the corresponding bounds. The dominant modes are easily selected if the failure probabilities of all possible modes of ductile structural systems can be evaluated. The modes of a small system can be found by simple investigation, but this may be very tedious or even impossible for large systems. Methods which automate the search for other stochastic dominant mechanisms will be described next.

6.2. *Vertex enumeration and ranking*

Murty's method [16] can be used for ordering the extreme points of a linear domain. It is based on a theorem which states that if $x^1 \ldots x^r$ are r best points, x^{r+1} will be an adjacent point of one of the first

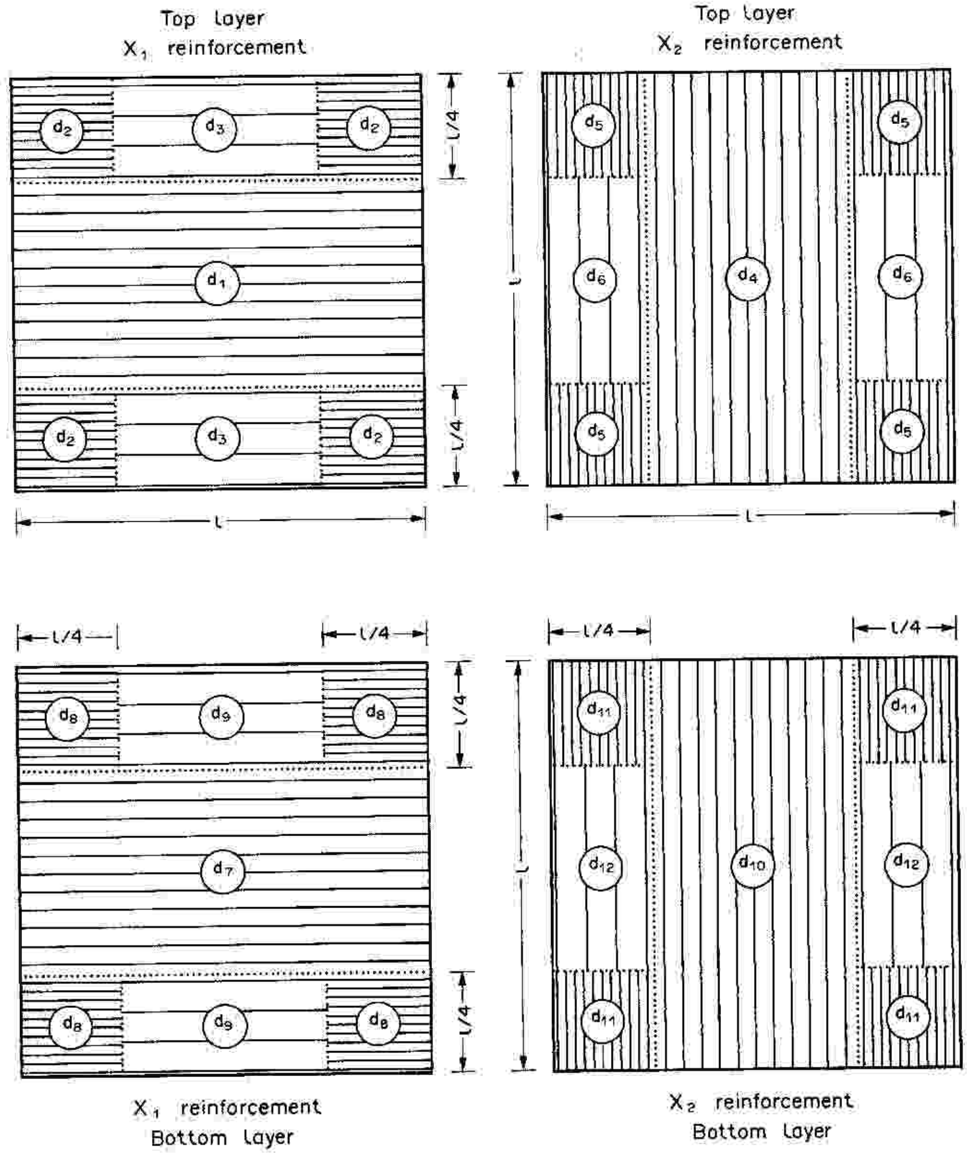

Fig. 15. Design regions for the steel reinforcement.

extreme points. The new point is distinct from the first r and minimizes the objective function giving $-1/\beta^2$ among all the remaining extreme points. All the adjacent extreme points are found from the canonical tableau corresponding to the linear domain $Ax = b$—i.e. eqns (23b)–(23e)—by bringing one by one all the nonbasic variables—only those corresponding to rotations in the critical sections—into the

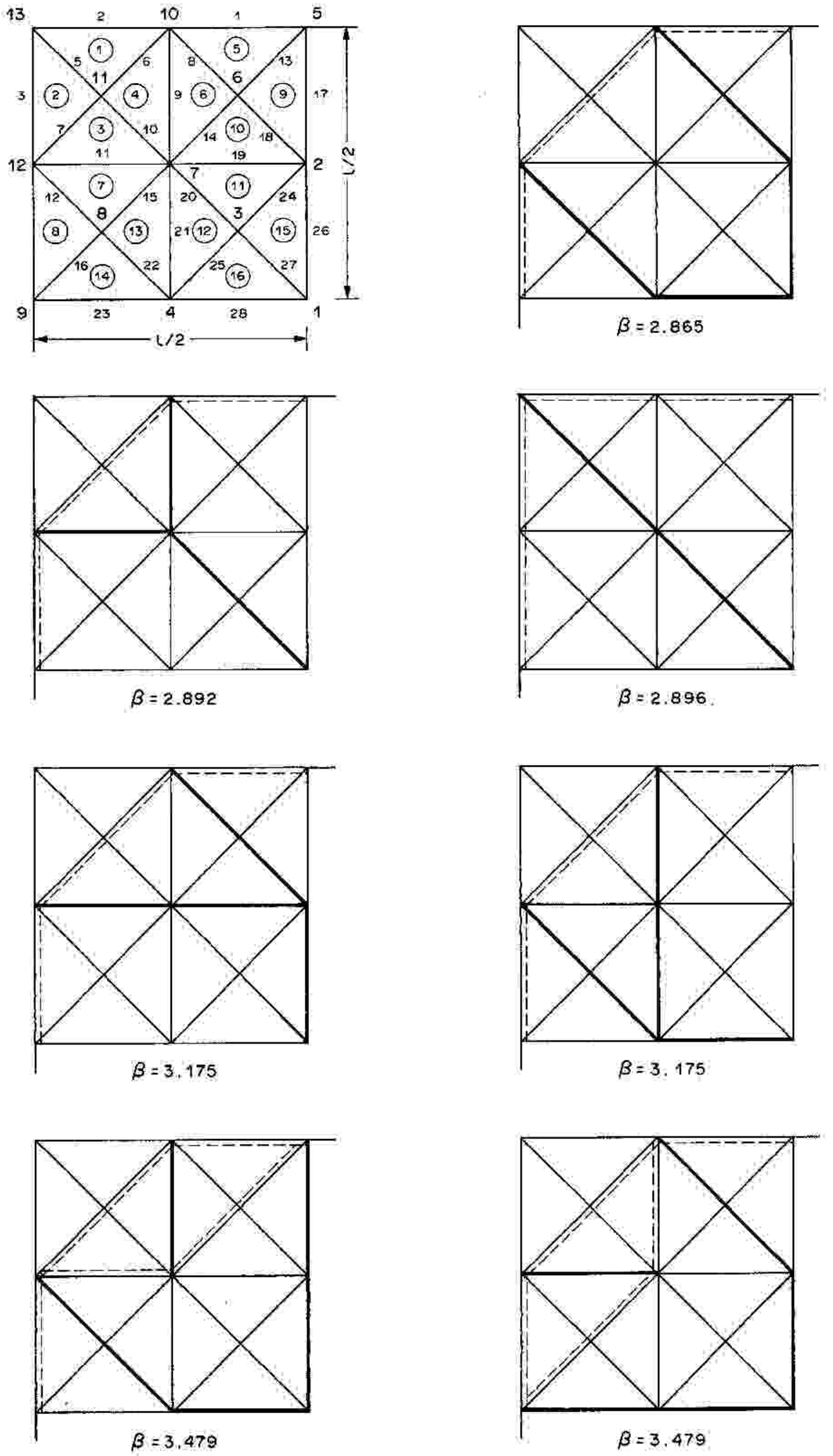

Fig. 16. Finite element discretisation most relevant mechanisms.

basis. The procedure is repeated until either a prespecified number of extreme points is found or all the extreme points in the ranked sequence whose objective value gives a reliability index which is less than a prespecified value of β_{max} are obtained. This method amounts to an implicit enumeration of the basis corresponding to different collapse mechanisms and its efficiency reduces as the number of kinematically admissible mechanisms increases.

6.3. *Branch and bound tree*

This strategy is an adaptation of Soland's algorithm for separable piecewise convex programming problems [17], where a strong branching rule is employed: the number of nodes created at each stage from an intermediate node is equal to the number of critical sections participating in the mechanism associated with the intermediate node. The result obtained at any node is a lower bound on those obtained by branching from it and, if the reliability index associated with that node is larger than a prespecified value, that the leaf of the combinatorial tree can be terminated. Nothing prevents the same mode being obtained by branching at different nodes. Nevertheless, the data available concerning mechanisms already obtained can be used to predict the solutions of a large number of nonconvex problems created by branching at intermediate nodes and this makes the procedure reasonably efficient.

6.4. *Example: uniformly loaded clamped square slab*

The isotropic symmetry of the material properties and the symmetric features of the slab geometry, reinforcement and loading allow for the consideration of one only quarter of the slab ($l = 10$ m). Three regions are considered in the layout of the steel reinforcement, as represented in Fig. 15, in the total of 12 random bending moments of resistance (kN m/m): $d_1 = d_4 = 13.33$, $d_2 = d_5 = d_8 = d_{11} = 20$; $d_3 = d_6 = 15$; $d_7 = d_{10} = 26.67$; $d_9 = d_{12} = 0$.

It is assumed that random loading ($\mu_F = 5$ kN/m^2; $\Omega_F = 0.3$) and the design variables ($\Omega_{m+} = \Omega_{m-} = 0.15$) are Gaussian and statistically independent. The discretisation of the selected quarter of the slab is achieved using 16 finite elements in such a way that the interelement sides are located along lines of reinforcement transition, i.e. along the lines in between distinct regions. Out of the 17 stochastic most important mechanisms considered, only those for which $\beta \leqslant 3.5$ are represented in Fig. 16.

The probabilities of failure of the reinforced concrete slab given by the Cornell, Ditlevsen and Vanmarcke methods are, respectively, $0.00208 \leqslant p_F \leqslant 0.00894$, $0.00225 \leqslant p_F \leqslant 0.00403$, and $p_F \leqslant 0.00375$. These results indicate that the last of these gives a narrow upper bound on the reliability of the slab.

7. CONCLUSIONS

According to the yield line theory the slab collapses due to the formation of lines (yield lines) along which the slab folds when a mechanism can be activated. When the finding of the stochastic most relevant mechanism of slabs, discretised by a triangular finite element representation in which the yield lines are restricted to element sides, is tackled in mathematical programming form, linear programming algorithms can be employed in each iteration of the branch and bound strategy.

The global solution of this problem is the minimum distance from the limit-state surface to the origin of the reduced normal variables that are applied load and bending moment resistances. This is the exact value of the reliability index of the continuous model stochastic most relevant mechanism, when the yield lines of the collapse mode of the latter lie along element boundaries of the former model. If this is not the case, the finite element model reliability index is an upper bound to the β value corresponding to the continuous model. All the remaining local solutions of the algorithm correspond to collapse mechanisms with a lower probability of failure and algorithms that enumerate other stochastic dominant modes were described. Due to the high correlation between modes of failure, Vanmarcke's method is ideally suited to assessment of the reliability of slabs, giving a narrow upper bound.

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