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APPROXIMATE DESIGN OF STRUCTURES USING PSEUDO-INVERSES

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Abstract—This paper describes a new method that can be used as a basis for the sizing of trusses and frames behaving elastically. Since in redundant structures the number of equations that needs to be solved is far smaller than the number of undeterminacies one takes for solution the structure giving the least norm. Its implementation is simple and the results obtained are reasonably close to the optimum design.

1. INTRODUCTION

A problem that has been much studied in the past decade is the weight minimization of a structure with fixed geometry and subjected to multiple constraints. As in the majority of the problems the solution domain is nonconvex and has plenty of subminima, the mathematical programming routines generally used to solve it can only guarantee convergence to a local minimum since they employ a convex approximation to the domain. Moreover, most of the algorithms generally available that linearize the domain do not allow an easy manipulation, the operator's help being needed at several stages of the solution procedure.

The grillage represented in Fig. 1 consists of two beams loaded as shown and subject to stress constraints. The interesting feature of the weight minimization problem is that it has several well-known optima [1].

The development of an alternative methodology that includes a few algebric concepts and yields, after a reduced member of reanalysis, a structure whose cost is close to the minimum volume solution, seems justified.

In this paper the author does not wish either to obtain the global optimum of the mathematical programming problem (that can only be achieved by adopting strategies more appropriate for nonconvex programming [2–3]) or study conditions under which an algorithm that linearizes the domain converges to the global optimum [4]. The procedure described here finds a structure such that under imposed behavioural constraints its volume is close to least volume design. It is based on the solution of an undetermined system of equations whose norm is smaller.

2. ELASTIC SYNTHESIS OF UNDETERMINED TRUSSES

The equations of static equilibrium are not by themselves sufficient for the evaluation of the member forces in a redundant structure. The problems of sizing undetermined trusses, besides possessing a large field of applications, are themselves of intrinsic importance. In fact this one stress resultant problem can be extended to include the members' elasto-plastic behaviour and the solution of both plate and frame problems.

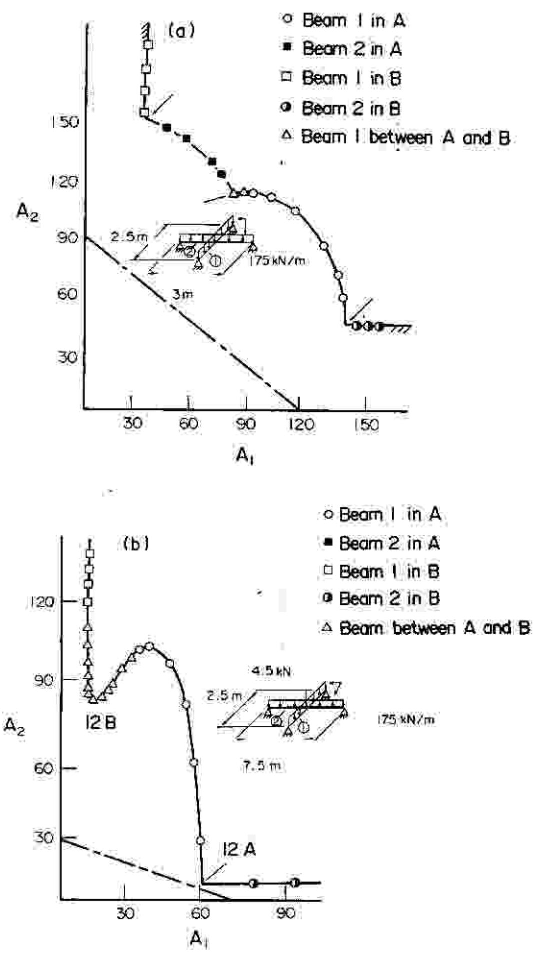

Fig. 1. (a) and (b) Grillage (allowable stresses 140 N/mm^2).

In this structural type one only considers member forces and it is intended to minimize the volume (weight/cost) of material to use, i.e.:

$$\min \{l\}^T \{a\}, \tag{1}$$

where $\{l\}$ is the vector of member lengths and $\{a\}$ is the m vector representing the design variables: member cross-sections.

This objective is restricted by conditions associated with the structural topology that is kept constant by imposing lower limits on the member areas.

$$\{a\} \geqslant \{a_l\}. \tag{2}$$

In the nodal-stiffness format the nodal displacements caused by the external loading λ are given by

$$[A][k][A]^T\{d\} = [K]\{d\} = \{\lambda\}, \tag{3}$$

where $[A]$ is the direction cosine ($\beta \times m$), $[k]$ is a diagonal ($m \times m$) matrix whose elements are member stiffnesses (Ea_i/l_i), where E is Young's modulus—a constant—and $\{d\}$ is the β-vector of the state variables (nodal displacements).

Alternatively in the mesh-flexibility method the equilibrium equations can be derived by expressing the member forces $\{n\}$ separately in terms of the external loads and the unknown hyperstatic forces $\{p\}$.

$$\{n\} = [B_0]\{\lambda\} + [B]\{p\}. \tag{4}$$

A special feature of structures is that the product of the direction cosine matrix $[A]$ times $[B]$ is singular, i.e.:

$$[A][B] = 0. \tag{5}$$

The rows of $[A]$ span a subspace of dimension β whereas the columns of $[B]$ span a subspace of dimension α(α is the degree of static undeterminacy).

Under the assumption that the member behaves elastically, the member stresses satisfy their allowable limits:

$$\{\sigma_l\} \leqslant \{\sigma\} = [S][A]^T\{d\} \leqslant \{\sigma_u\}. \tag{6}$$

The vector $\{\sigma\}$ is a linear combination of nodal displacements, when the elements of the diagonal ($m \times m$) matrix $[S]$ are E/l_i.

The buckling constraints for the compressed members may be defined by the Euler–Johnson stability criterion. In long columns the lower bound constraint on σ_{jl} on member j can be substituted by:

$$-\sigma_{jl} - (\gamma\pi^2 E a_j^2)/l_j^2 \leqslant 0. \tag{7}$$

If the nodal displacements are limited,

$$-\{d_l\} \leqslant \{d\} \leqslant \{d_u\}, \tag{8}$$

therefore in the weight optimization of trusses one optimizes the linear objective function (1) subject to a set of bilinear equations (3), linear inequalities (2), (6), (8) and quadratic inequalities (7).

The minimum volume design of frames is essentially the same as for trusses. Here the matrices $[A]$ and $[k]$ are suitably adapted although keeping the same type of vectorial equations. Now the member stresses will be given by the quotient of the bending moment in the critical section divided by the section modulus. In some instances it is also necessary to impose restrictions related to the structural elements and often with the column footings.

In universal beams the cross sectional areas can be related to the section properties: section modulus and moment of inertia.

$$a_j = 0.77\, z_j^{2/3}; \quad a_j = 0.559\, I_j^{1/2}. \tag{9}$$

These expressions, although not changing one type of problem, just increase the nonlinearity.

3. MINIMUM NORM SOLUTION

In the system

$$[A]\{x\} = \{b\}, \tag{10}$$

where $[A]$ is a ($\beta \times m$) matrix ($m > \beta$) of rank β and $\{x\}$, (b) are m and β vectors respectively, there is an infinite number of solutions satisfying the equalities.

Any inverse $[A_*]$ such that

$$[A][A_*][A] = [A] \tag{11}$$

is called pseudo-inverse of A [8]. A pseudo-inverse will produce the minimum norm solution to the system (independently of $\{b\}$) if $[A_*]\{b\}$ has the minimum norm among all solutions to the system, i.e.:

$$\| A_* b \| = \min \| x \| \tag{12}$$

if the norm of $\| x \|$ is defined by the form

$$\| x \| = \sqrt{(\{x\}^T[C]\{x\})}, \tag{13}$$

where $[C]$ is a positive definite matrix of rank β. The minimum norm solution is unique [9].

The pseudo-inverse giving the minimum norm solution is:

$$[A_*] = [C]^{-1}[A]^T([A][C]^{-1}[A]^T)^{-1}. \tag{14}$$

The solution of $[A]\{x\} = \{b\}$ under the norm is equivalent to the mathematical programming problem:

$$\min 1/2\{x\}^T[C]\{x\} \tag{15}$$

$$st\ [A]\{x\} = \{b\}$$

$$\{x\} \text{ unrestricted.}$$

Therefore the pseudo-inverse giving the least norm can also be obtained from the Lagrangian.

4. PHYSICAL INTERPRETATION

4.1 *Cauchy–Schwarz inequality*

If x and y are any two vectors in an inner product space,

$$(\{x\},\{y\})^2 \leqslant (\{x\},\{x\})(\{y\},\{y\}). \quad (16)$$

4.2 *Decomposition of a vector space*

Theorem 1. A closed convex subset of a Hilbert space H contains a unique vector of smallest norm.

Theorem 2. Let M be a closed linear subspace of a Hilbert space, let $\{x\}$ be a vector not in M and let d be the distance for $\{x\}$ to M. Then there exists a unique vector $\{x_0\}$ in M such that:

$$\|x - x_0\| = d. \quad (17)$$

Theorem 3. If $[P]$ and $[T]$ are projections on closed linear subspaces M and N of H, then M is orthogonal to N if $[P][T]=[T][P]=0$. Noting that the equilibrium equations can be expressed in both the nodal-stiffness format

$$[A]\{n\} = \{\lambda\} \quad (18)$$

and employing the mesh-flexibility relations,

$$\{n\} = [B_0]\{\lambda\} + [B]\{p\}, \quad (19)$$

any set $\{n\}$ can be decomposed into two orthogonal components $[P]\{n\}$ and $[T]\{n\}$ so that the ranks of $[P]$ and $[T]$ are α and β, respectively.

Hence,

$$([P]\{n\},[T]\{n\}) = 0 \quad (20)$$

and

$$[P]\{n\} + [T]\{n\} = \{n\}. \quad (21)$$

Premultiplying (19) by $[P]$ and $[T]$, where T is given by

$$[T] = [A]^T([A][A]^T)^{-1}[A] = [A_*], \quad (22)$$

yields

$$[P]\{n\} = [P]\{B_0\}\{\lambda\} + [B]\{p\} \quad (23)$$

$$[T]\{n\} = [T][B_0]\{\lambda\}, \quad (24)$$

since $[A][P]=0$, $[A][T]=[A]$ and $[A][B]=0$. Therefore the vector $[P]\{n\}$ is self equilibrated and in particular the vector $[P][B_0]\{\lambda\}$ is self equilibrated at the load level $\{\lambda\}$. Also the vector $[T]\{n\}=\{n_f\}$ is in equilibrium with $\{\lambda\}$ as it is shown in Fig. 2.

$$\begin{aligned}\{n_f\} &= [T]\{n\} = [T][B_0]\{\lambda\} \\ &= [A]^T([A][A]^T)^{-1}[A]\{n\} \\ &= [A]^T([A][A]^T)^{-1}\{\lambda\}. \end{aligned} \quad (25)$$

Furthermore, $\{n_f\}$ is uniquely determined given either $\{\lambda\}$ or any $\{n\}$. Clearly the vector $[T]\{n\} = \{n_f\}$ in the subspace M is the vector of smallest length stated in Theorem 1. In fact, using the Cauchy–Schwartz inequality with $\{x\}=\{n_f\}=[T]\{n\}$ and $\{y\}$ equal to an arbitrary $\{n\}$, yields

$$([T]\{n\},\{n\})^2 \leqslant ([T]\{n\},[T]\{n\})(\{n\},\{n\}) \quad (26)$$

or

$$\{n\}^T[T]\{n\} = \{n_f\}^T\{n_f\} \leqslant \{n\}^T\{n\}, \quad (27)$$

where

$$\{n_f\}^T\{n_f\} = \sum_{j=1,n} (n_{fj})^2 = \|n_f\|^2. \quad (28)$$

According to relation (27) in a statically determinate structure $\|n_f\|$ attains its maximum value, i.e. $n_f^0 = n^0 = n$, which implies that $\|n_f^0\| = \|n^0\|$. It may be infered that in a statically redundant structure the area defined by the $\{n_f\}$ distribution is a minimum among all alternative distributions $\{n\}$ in equilibrium with $\{\lambda\}$.

5. REANALYSIS

Since the procedure presented here is iterative it is convenient to use a form allowing one to obtain easily an approximation to the inverse of the positive definite structural matrix $[K]_{i+1}$. Due to the geometric layout of the members and because for the statics case the loading is assumed not to change, the values of $[A]$, $\{\lambda\}$ remain the same.

Note that:

$$[\Delta K] = [K]_{i+i} - [K]_i. \quad (29)$$

Formulating the expansion

$$[K]_{i+1}^{-1} = [K]_i^{-1} - [K]_i^{-1}[\Delta K]_i[K]_i^{-1} + [K]_i^{-1}[\Delta K]_i[K]_i^{-1}[\Delta K]_i[K]_i^{-1} - \cdots \quad (30)$$

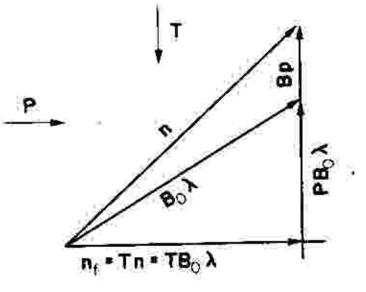

Fig. 2

will avoid the costly inverse operations, since $[K]_i^{-1}$ is obtained in the basic analysis [6]. The rate of convergence of (30) will depend on the individual element stiffness changes Δk_j. In order to assure convergence for values of Δk_j above some cutoff, say 0.4, a scheme can be derived that compounds the change at a particular rate for a finite number of times until the total Δk_j is achieved. If c_j' is the compounding rate then:

$$c_j' = 1 - (1 - \Delta k_j)^{1/n}. \tag{31}$$

The use of this allows reduction of the CPU needed to invert the structural matrix several times [7].

One of the most basic properties of a trussed type of structure is the 'scaling invariance' of the stress-resultants vector $\{n\}$. The member loads of a statically determinate truss are obtained from the equilibrium equations and are independent of the member cross-sections $\{a\}$. If the structure is redundant the internal forces are a function of the cross-sections $\{a\}$. However, the forces remain unchanged if all the areas are multiplied by the same scaling factor ρ ($\rho > 0$):

$$\{n\}(\rho\{a\}) = \{n\}(\{a\}). \tag{32}$$

The stress-resultants are thus homogeneous functions of degree $n = 0$ in the design variables. The stress in any member j of the structure $\sigma_j = n_j/a_j$ is therefore a homogeneous function of degree $n - 1$ in $\{a\}$ [5]

$$\sigma_j(\rho\{a\}) = 1/\rho\,\sigma_j(\{a\}). \tag{33}$$

The nodal displacements would also be affected by a factor of $1/\rho$ since they can be represented by a linear combination of the member stresses.

Therefore, by fixing a set $\{a\}$ of areas, a unique set of stresses can be determined by inverting the equilibrium equations. Assuming that a feasible set of state variables was determined by using a scaling factor of $0 < \rho < 1$, the structure's volume could be reduced until it eventually touches the boundary of the stress/displacement space. Alternatively, if the stress/displacements are outside their rectangle of bounds, the design variables could be multiplied by $\rho > 1$ and the stress resultant vector would be linearly reduced until it fits its bounds. Therefore the vector of areas $\rho\{a\}$ has at least one member fully stressed or a displacement at its boundary.

6. ALGORITHM

The algorithm presented here gives approximations close to the optimal design given by mathematical programming routines.

Step 1

(1.1) Iteration 1

By making $y_j = l_j n_j$ a starting point the minimum norm $\{y\}$ obtained from the undeterminate system of equilibrium equations is

$$[A]\{n\} = \{\lambda\}, \tag{34}$$

which is equivalent to

$$[B]_1\{y\} = \{\lambda\}. \tag{35}$$

The solution of the latter is in equilibrium with $\{\lambda\}$ and has its least norm given by the product

$$\{y\} = [B]_1^T([B]_1[B]_1^T)^{-1}\{\lambda\}, \tag{36}$$

where the elements of $[B]_1$ are equal to the quotient of the corresponding elements belonging to the direction cosine matrix $[A]$ divided by the member lengths l_j associated to the column j of $[A]$. The initial set of design variables is given by

$$a_j = \max\{y_j/(l_j\sigma_{jl}), y_j/(l_j\sigma_{ju})\}. \tag{37}$$

With these cross-sections it is possible to analyse the truss and proportionate the size in order to have a number fully stressed (or a nodal displacement at a boundary).

Step 2

(2.1) Iteration $i + 1$

Using the vectors $\{a\}_i$ and $\{\sigma\}_i$ obtained at iteration i, find the minimum norm solution $\{x\}_{i+1}$ of the system:

$$[B]_i\{x\} = \{\lambda\}, \tag{38}$$

where $[B]_i = [A][D]_i$ and $[D]$ is a diagonal matrix whose elements are the products of the member stresses at iteration i, σ_{ji} times the square root of the variables x_{ji} and divided by the member length l_j. Using $\{x\}_{i+1}$ the new set of areas $\{a\}_{i+1}$ can be easily accessed, i.e.

$$a_{ji+1} = x_j/l_j. \tag{39}$$

(2.2) Reanalyse the structure using $\{a\}_{i+1}$ and proportionate it in order to find a new solution within the domain.

(2.3) Convergence:

(a) If

$$\sum_{j=1,m} (a_{ji+1} - a_{ji})^2 < \epsilon$$

or the solution deteriorates, terminate.

(b) Else, go to (2.1).

7. APPLICATIONS

The following examples are referred to in the literature as a basis for comparison of the efficiencies of algorithms used.

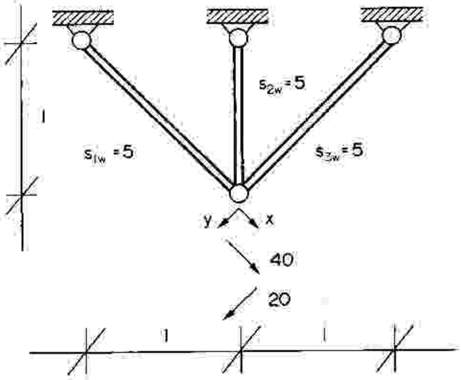

Fig. 3

7.1 Three bar truss subjected to a single loading condition

Consider the structure represented in Fig. 3.

STEP 1

$$A = \begin{vmatrix} 1 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 1 \end{vmatrix}.$$

Let

$$B_1 = \begin{vmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{vmatrix},$$

hence

$$y^T = [28.3 \quad 28.3 \quad 0.0] \Rightarrow a^T$$

$$= [4.0 \quad 5.7 \quad 0.0]$$

$$\sigma^T = [5.0 \quad 5.0 \quad 0.0].$$

By imposing lower bounds on the members' cross-sections in order to keep the topology fixed and proportionating the areas such as to have a feasible design:

$$a^T = [4.0 \quad 5.7 \quad 0.0]$$

$$\sigma^T = [5.0 \quad 5.0 \quad 0.0].$$

STEP 2

$$B_2 = \begin{vmatrix} 8.4 & 8.4 & 0.0 \\ 0.0 & 8.4 & 0.0 \end{vmatrix}$$

leads to

$$a^T = [4.0 \quad 5.7 \quad 0.0]$$

$$\sigma^T = [5.0 \quad 5.0 \quad 0.0],$$

that is coincident with the value obtained previously, itself leading to the true optimum.

7.2 Three bar truss undergoing two alternative loading conditions

In this problem in the first iteration take as an estimate for the member forces

$$n_j = \max\{y^1_{jl}/l_j, y^2_{jl}/l_j\}.$$

The $[B]_{*i+1}$ matrix is composed of two matrices $[B]^1_{i+1}$ and $[B_{i+1}]^2$ that are defined using the stresses arising in the members due to each of the loading conditions:

$$\begin{aligned}\{x\} &= [B]_{*i+1}\{\lambda\} \\ &= [B]^{1T}_{i+1}([B]^1_{i+1}[B]^{1T}_{i+1} + [B]^2_{i+1}[B]^{2T}_{i+1})^{-1}\{\lambda\}^1 \\ &\quad + [B]^{2T}_{i+1}([B]^1_{i+1}[B]^{1T}_{i+1} \\ &\quad + [B]^2_{i+1}[B]^{2T}_{i+1})^{-1}\{\lambda\}^2. \end{aligned} \quad (40)$$

The solution obtained after three iterations is 1.5% greater than the optimal solution:

	a_1	a_2	a_3
	6.97	2.28	2.81
Optimum	7.02	2.14	2.76

7.3 Ten bar truss

Consider the truss represented in Fig. 4.

Let $a_{jl} = 0.1$ and $|\sigma_{jl}| = \sigma_{ju} = 2.5$.

After four iterations one reaches a solution 2% greater than the optimum:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
8.2	0.1	8.4	3.9	0.1	0.1	6.0	5.6	5.6	0.1
Optimum									
7.9	0.1	8.1	3.9	0.1	0.1	5.7	5.6	5.6	0.1

7.4 Ten bar truss where the displacements are limited

This is a well-known nonconvex problem, where $|\{d_l\}| = \{d_u\} = 3.5$. After three iterations one reaches

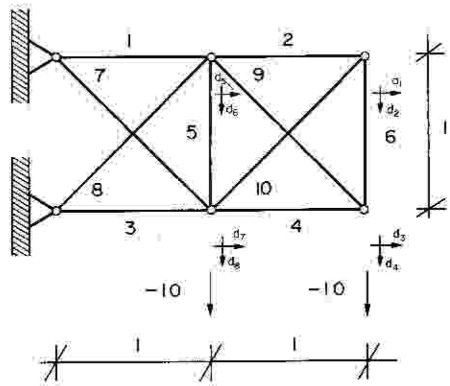

Fig. 4

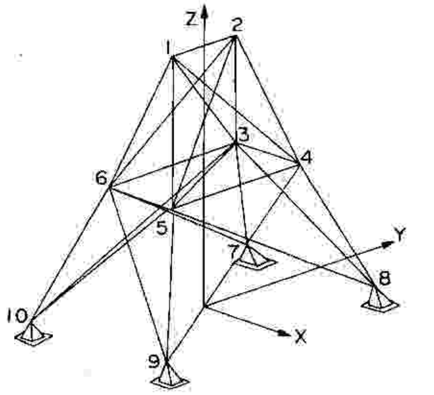

Fig. 5

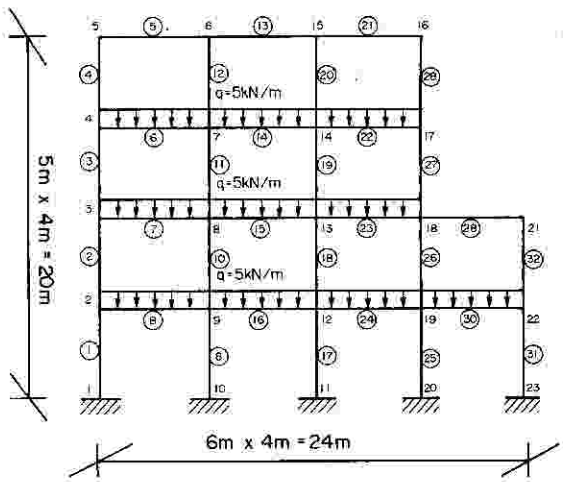

Fig. 6

a solution 15% greater than the optimum:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
46	0.1	47	23	0.1	0.1	33	32	32	0.1
Optimum									
49	0.1	36	24	0.1	1.0	9	34	34	0.1

7.5 *Space truss*

Consider the truss represented in Fig. 5.

The solution obtained after three iterations is 3% greater than the optimal solution.

7.6 *Frame* (see Fig. 6)

After three iterations one reaches a solution 3% greater than the optimum.

8. CONCLUSIONS

In this paper a new methodology was presented for the sizing of structures. It reveals itself to be of great simplicity, leading to very good approximations to the optimum given by mathematical programming routines [10]. If accuracy is desired it provides them with a good starting point. Slabs can also be designed in this way by adapting the Mindlin theory for its description.

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