UNIVERSIDADE DE COIMBRA
INSTITUTO DE SISTEMAS E ROBÓTICA

TECHNICAL REPORT

Unique Solution for the Estimation of the Plücker Coordinates Using Radial Basis Functions

Author:
Pedro MIRALDO

Supervisor:
Prof. Dr. Helder ARAUJO
Prof. Dr. João QUEIRÓ

November, 2011

Unique Solution for the Estimation of the Plücker Coordinates Using Radial Basis Functions

Pedro Miraldo[†] and Helder Araujo
Institute for Systems and Robotics
Department of Electrical and Computer Engineering–Polo 2
University of Coimbra
3030-290 Coimbra, Portugal
{miraldo, helder}@isr.uc.pt

João Queiró
Department of Mathematics
University of Coimbra
3001-454 Coimbra, Portugal
jfqueiro@mat.uc.pt

[†] P. Miraldo wants to thank to *Portuguese Fondation for Science and Technology*, with grant SFRH/BD/49054/2008.

Contents

1 Notations and Background

1.1 Notation

Matrices are represented as bold capital letters (*eg.*$\mathbf{A} \in \mathbb{R}^{n\times m}$, n rows and m columns). Vectors are represented as bold small letters (*eg.*$\mathbf{a} \in \mathbb{R}^{n}$, n elements). By default, a vector is considered a column. Small letters (*eg.*a) represent one dimensional elements. By default, the jth column vector of $\mathbf{A}$ is specified as $\mathbf{a}_j$. The jth element of a vector $\mathbf{a}$ is written as a_j. The element of $\mathbf{A}$ in the line i and column j is represented as $a_{i,j}$. Regular capital letters (*eg.*A) indicate one dimensional constants.

1.2 Useful Algebra Tools

In this section we describe some algebra tools that will be useful in the remaining sections. For more information about their properties we suggest [7, 2, 3].

For a matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$

- $\mathcal{C}(\mathbf{A})$ – dimension of the column–space or *rank* of $\mathbf{A}$;
- $\mathcal{N}(\mathbf{A})$ – dimension of the null–space or nullity.
- $m = \mathcal{C}(\mathbf{A}) + \mathcal{N}(\mathbf{A})$
- An useful property of the *rank* is $\mathcal{C}(\mathbf{A}) = \mathcal{C}\left(\mathbf{A}^T\right)$.
- If $\mathbf{P}_1$ and $\mathbf{P}_2$ are two permutation matrices. Then $\mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{P}_1\mathbf{A}\mathbf{P}_2)$.
- If $\mathbf{B} \in \mathbb{R}^{k\times m}$ is *column full–rank* ($\mathcal{C}(\mathbf{B}) = m$) then $\mathcal{C}(\mathbf{BA}) = \mathcal{C}(\mathbf{A})$
- If a matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is partitioned as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_3 \end{pmatrix} \tag{1}$$

  then, its *eigenvalues* $\lambda(\mathbf{A}) = \lambda(\mathbf{A}_1) \cup \lambda(\mathbf{A}_3)$

1.3 *Kronecker* product

Let $\mathbf{U} \in \mathbb{R}^{m\times n}$ and $\mathbf{V} \in \mathbb{R}^{k\times l}$ and the equation

$$\mathbf{U}\mathbf{X}\mathbf{V}^T = \mathbf{C} \tag{2}$$

where $\mathbf{X} \in \mathbb{R}^{n\times l}$ is matrix of the system unknowns. It is possible to rewrite the previous equation as

$$(\mathbf{V} \otimes \mathbf{U})\,\mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{C}) \tag{3}$$

where $\otimes$ is the *Kronecker* product of $\mathbf{U}$ and $\mathbf{V}$ [2], with $[\mathbf{V} \otimes \mathbf{U}] \in \mathbb{R}^{mk \times nl}$, and $\text{vec}(\mathbf{X})$ is a nl–vector formed by stacking the columns of $\mathbf{X}$.

The *Kronecker* product is an useful tool to turn some systems linear. For $\mathbf{V} \in \mathbb{R}^{k \times l}$ and $\mathbf{U} \in \mathbb{R}^{m \times n}$ the *Kronecker* products

$$\mathbf{V} \otimes \mathbf{U} = \{v_{i,j}\mathbf{U}\} \in \mathbb{R}^{mk \times nl}. \tag{4}$$

2 Radial Basis Functions

Radial Basis Functions are are frequently used in approximating functions ($f : \mathbb{R}^2 \mapsto \mathbb{R}$) by means of least squares fitting. In these cases the interpolant equation can be written as

$$s(\mathbf{x}) = a_0 + \mathbf{a}_{\mathrm{x}}^T \mathbf{x} + \sum_{i=1}^{P} w_i \phi(||\mathbf{x} - \mathbf{c}_i||) = \underbrace{\begin{pmatrix} \boldsymbol{\phi}(\mathbf{x}) & \mathbf{p}(\mathbf{x}) \end{pmatrix}}_{\mathbf{r}(\mathbf{x})} \underbrace{\begin{pmatrix} \mathbf{w} \\ \mathbf{a} \end{pmatrix}}_{\mathbf{h_{wa}}} \tag{5}$$

where $\mathbf{x}$ and $\{\mathbf{c}_i\}$ belong to $\mathbb{R}^2$, $||.||$ is the *2–norm* of vectors, $\mathbf{p}(\mathbf{x}) = \begin{pmatrix} 1 & \mathbf{x}^T \end{pmatrix}$, $\boldsymbol{\phi}(\mathbf{x}) = (\phi_1(\mathbf{x}), \ldots, \phi_P(\mathbf{x}))$ where $\phi_i(\mathbf{x}) = \phi(||\mathbf{x} - \mathbf{c}_i||)$, $\mathbf{w} = \begin{pmatrix} w_1 & \ldots & w_P \end{pmatrix}^T$ and $\mathbf{a} = \begin{pmatrix} a_0 & \mathbf{a}_{\mathrm{x}}^T \end{pmatrix}^T$.

In this section we describe the typical problem of finding the unknown vector $\mathbf{h_{wa}}$ for a set of interpolant conditions

$$s(\mathbf{x}_i) = f(\mathbf{x}_i) \tag{6}$$

for $i = 1, \ldots, P$.

For a set $\{\mathbf{c}_i\}$, we define

$$\boldsymbol{\Phi} = \{\phi(||\mathbf{x}_i - \mathbf{c}_j||)\} \in \mathbb{R}^{P \times P}, \tag{7}$$

Wendland and *Buhamann* [8, 1] prove that, for $\{\mathbf{x}_i = \mathbf{c}_i\}$ where $i = 1, \ldots, P$, $\boldsymbol{\Phi}$ is *conditional positive definite*.

For *scattered* set $\{\mathbf{c}_i\}$, where $\{\mathbf{x}_i \neq \mathbf{c}_j\}$ for $i, j = 1, \ldots, P$, *Quak et al.*and *Sivakumar* and *Ward* [5, 6] prove that $\boldsymbol{\Phi}$ is *conditional positive definite*, where each *control point* has to be associated to a data point $\{\mathbf{x}_i\}$, that satisfies $d \leq q\epsilon$, where $0 < \epsilon \leq 1$, $d = \max\{||\mathbf{x}_i - \mathbf{c}_i||\}$ and $2q = \min_{j \neq i}\{||\mathbf{c}_i - \mathbf{c}_j||\}$. Quak *et al.*[5] also proved that $\phi_1(r) = (\beta_1^2 + r^2)^{1/2}$ and $\phi_2(r) = e^{-\beta_2 r^2}$ are good choices for *radial basis functions*, because, choosing an appropriate β_1 and β_2, they reduce the negative effects of small values of q and ϵ respectively.

From Equation (5), for a set P of $\{\mathbf{x}_i\}$ we can write

$$\mathbf{s} = \underbrace{\begin{pmatrix} \boldsymbol{\Phi} & \mathbf{K}^T \end{pmatrix}}_{\mathbf{R}} \mathbf{h_{wa}} \tag{8}$$

where $\mathbf{K} \in \mathbb{R}^{3 \times P}$ is the stacking of $\mathbf{p}(\mathbf{x}_i)$ and $\mathbf{s} = \begin{pmatrix} s(\mathbf{x}_1) & \ldots & s(\mathbf{x}_P) \end{pmatrix}^T$.

From Equation (8), we have $P + 3$ unknowns and only P equations. To eliminate the extra degrees of freedom, additional constraints are needed. We use the additional constraints resulting from the conditional positive definiteness of the space of solutions of $\mathbf{w}$ [8]

$$\sum_{i=1}^{P} w_i \mathbf{p}(\mathbf{x}) = \mathbf{K}\mathbf{w} = \mathbf{0}. \tag{9}$$

Putting all together

$$\begin{pmatrix} \mathbf{s} \\ \mathbf{0} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{\Phi} & \mathbf{K}^T \\ \mathbf{K} & \mathbf{0} \end{pmatrix}}_{\mathbf{\Gamma}} \mathbf{h_{wa}} \tag{10}$$

which has only one solution when $\mathbf{\Gamma} \in \mathbb{R}^{P+3 \times P+3}$ is *full–rank*.

If $\mathcal{N}(\mathbf{\Gamma}) = 0$, $\mathcal{C}(\mathbf{\Gamma}) = P + 3$. Thus, computing the *null–space* of $\mathbf{\Gamma}$,

$$\begin{pmatrix} \mathbf{\Phi} & \mathbf{K}^T \\ \mathbf{K} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix} = \mathbf{0} \tag{11}$$

or

$$\mathbf{\Phi}\mathbf{v} + \mathbf{K}^T\mathbf{u} = \mathbf{0} \tag{12}$$

$$\mathbf{K}\mathbf{v} = \mathbf{0}. \tag{13}$$

The solution is only verified for $\mathbf{v} = \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$, which means that $\mathcal{N}(\mathbf{\Gamma}) = 0$ and $\mathcal{C}(\mathbf{\Gamma}) = P + 3$.

From [8, Section 8.5], if we pre–multiply the first Equation of Equation (12) by $\mathbf{v}^T$ we get

$$\mathbf{v}^T\mathbf{\Phi}\mathbf{v} + (\mathbf{K}\mathbf{v})^T \mathbf{u} = 0. \tag{14}$$

From Equation (13), $\mathbf{K}\mathbf{v} = \mathbf{0}$ which reduces Equation (14) to

$$\mathbf{v}^T\mathbf{\Phi}\mathbf{v} = 0. \tag{15}$$

We know from previous statements that $\mathbf{\Phi}$ is *conditional positive definite*, which means that $\mathbf{v}^T\mathbf{\Phi}\mathbf{v} > 0$ for any non–zero vector $\mathbf{v}$. As a consequence, Equation (15) is only verified for $\mathbf{v} = \mathbf{0}$.

Since we already proved that $\mathbf{v} = \mathbf{0}$, we can rewrite the Equation (12) as

$$\mathbf{K}^T\mathbf{u} = \mathbf{0}. \tag{16}$$

If the set $\{\mathbf{x}_i\}$, for $i = 1, \ldots, P$ with $P \geq 3$, forms a *full–column rank* matrix $\mathbf{K}^T$, $\mathcal{C}\left(\mathbf{K}^T\right) = 3$, Equation (16) is only verified for $\mathbf{u} = \mathbf{0}$, which implies $\mathcal{N}(\mathbf{\Gamma}) = 0$ and $\mathcal{C}(\mathbf{\Gamma}) = P + 3$.

3 Introduction

In this report we study and analyze the relationship between the number N of point correspondences $\{\mathbf{x}_i \mapsto \mathbf{p}_i\}$ required for the calibration and the rank of the calibration matrix described in [4].

The equation that represents the general imaging model (as described in [4]) can be written as

$$\mathbb{lR} = \mathbf{s}(\mathbf{x}) = \underbrace{\begin{pmatrix} \boldsymbol{\phi}(\mathbf{x}) & \mathbf{p}(\mathbf{x}) \end{pmatrix}}_{\mathbf{r}(\mathbf{x})} \underbrace{\begin{pmatrix} \mathbf{h}_{\mathbf{wa}}^{(1)} & \dots & \mathbf{h}_{\mathbf{wa}}^{(6)} \end{pmatrix}}_{\mathbf{H}_{\mathbf{wa}}} \tag{17}$$

where vectors $\mathbf{h}_{\mathbf{wa}}^{(i)}$, for $i = 1, \dots, 6$, are as in Equation (5).

The calibration parameters are computed by estimating a non–zero vector $vec(\mathbf{H}_{\mathbf{wa}})$ that satisfies

$$\underbrace{\begin{pmatrix} \mathbf{Q}(\mathbf{p}_1) \otimes \mathbf{r}(\mathbf{x}_1) \\ \mathbf{Q}(\mathbf{p}_2) \otimes \mathbf{r}(\mathbf{x}_2) \\ \vdots \\ \mathbf{Q}(\mathbf{p}_N) \otimes \mathbf{r}(\mathbf{x}_N) \\ \mathbf{D} \end{pmatrix}}_{\mathbf{M}} \operatorname{vec}(\mathbf{H}_{\mathbf{wa}}) = \mathbf{0} \tag{18}$$

where $\operatorname{vec}(\mathbf{H}_{\mathbf{wa}}) \in \mathbb{R}^{(6P+18)\times 1}$ is the stacking of $\mathbf{h}_{\mathbf{wa}}^{(i)}$ for $i = 1, \dots, 6$, and $\mathbf{Q}(\mathbf{p}_i)$ is the incident relation between a point in the world $\mathbf{p}_i \in \mathbb{R}^3$ and a line generated from an image point $\mathbf{x}_i$

$$\mathbf{Q}(\mathbf{p}_i) = \begin{pmatrix} [\mathbf{p}_i]_{\mathrm{x}} & -\mathbf{I} \\ \mathbf{0}^T & \mathbf{p}_i^T \end{pmatrix} \tag{19}$$

where $\mathbf{I}$ is the identity matrix, with dimensions 3×3, and $[\mathbf{a}]_{\mathrm{x}}$ is the matrix that linearizes the three dimensional exterior product as $[\mathbf{a}]_{\mathrm{x}}\, \mathbf{b} = \mathbf{a} \times \mathbf{b}$.

Since $\mathbb{lR} = \mathbf{s}(\mathbf{x})$, we see that the solution for $\mathbf{H}_{\mathbf{wa}}$ is up to a scale factor. Thus, to have a unique solution, we must have $\mathcal{N}(\mathbf{M}) = 1$ and the solution is any element of the right *null–space* exept the trivial solution $\operatorname{vec}(\mathbf{H}_{\mathbf{wa}}) = \mathbf{0}$.

4 *Rank* of matrix M

In this section, we study the relationship between the *rank* of matrix $\mathbf{M}$ (Equation (18)) and the number of point–correspondences (N), used in the calibration process.

Since permuting rows does not change the *rank* of a matrix, $\mathcal{C}(\mathbf{A}) = \mathbf{C}(\mathbf{ZM})$, for any permmutation matrix $\mathbf{Z}$, and we can study the *rank* of $\mathbf{A}$, instead of M.

From Equation (18) and Equation (4), we can find a matrix $\mathbf{A} = \mathbf{ZM}$ as Equation (20). where $\mathbf{Z}$ is a permutation matrix, $\mathbf{p}_i = \left(p_i^{(1)}, p_i^{(2)}, p_i^{(3)}\right)$ and $\mathbf{r}_i = \mathbf{r}(\mathbf{x}_i)$, where $\mathbf{r}(\mathbf{x}_i)$ is as described in Section 2.

$$
\mathbf{A} = \begin{pmatrix}
\mathbf{0} & -p_1^{(3)}\mathbf{r}_1 & p_1^{(2)}\mathbf{r}_1 & \mathbf{r}_1 & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & -p_N^{(3)}\mathbf{r}_N & p_N^{(2)}\mathbf{r}_N & \mathbf{r}_N & \mathbf{0} & \mathbf{0} \\
p_1^{(3)}\mathbf{r}_1 & \mathbf{0} & -p_1^{(1)}\mathbf{r}_1 & \mathbf{0} & \mathbf{r}_1 & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_N^{(3)}\mathbf{r}_N & \mathbf{0} & -p_N^{(1)}\mathbf{r}_N & \mathbf{0} & \mathbf{r}_N & \mathbf{0} \\
-p_1^{(2)}\mathbf{r}_1 & p_1^{(1)}\mathbf{r}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-p_N^{(2)}\mathbf{r}_N & p_N^{(1)}\mathbf{r}_N & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_N \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & p_1^{(1)}\mathbf{r}_1 & p_1^{(2)}\mathbf{r}_1 & p_1^{(3)}\mathbf{r}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & p_N^{(1)}\mathbf{r}_N & p_N^{(2)}\mathbf{r}_N & p_N^{(3)}\mathbf{r}_N \\
 & & & \mathbf{D} & &
\end{pmatrix} \tag{20}
$$

We define $\mathbf{E}$ and $\mathbf{F}$ as

$$
\mathbf{E} = \begin{pmatrix}
\mathbf{0} & -p_1^{(3)}\mathbf{r}_1 & p_1^{(2)}\mathbf{r}_1 & \mathbf{r}_1 & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & -p_N^{(3)}\mathbf{r}_N & p_N^{(2)}\mathbf{r}_N & \mathbf{r}_N & \mathbf{0} & \mathbf{0} \\
p_1^{(3)}\mathbf{r}_1 & \mathbf{0} & -p_1^{(1)}\mathbf{r}_1 & \mathbf{0} & \mathbf{r}_1 & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_N^{(3)}\mathbf{r}_N & \mathbf{0} & -p_N^{(1)}\mathbf{r}_N & \mathbf{0} & \mathbf{r}_N & \mathbf{0} \\
-p_1^{(2)}\mathbf{r}_1 & p_1^{(1)}\mathbf{r}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-p_N^{(2)}\mathbf{r}_N & p_N^{(1)}\mathbf{r}_N & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_N
\end{pmatrix} \quad \text{and} \quad \mathbf{F} = \begin{pmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & p_1^{(1)}\mathbf{r}_1 & p_1^{(2)}\mathbf{r}_1 & p_1^{(3)}\mathbf{r}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & p_N^{(1)}\mathbf{r}_N & p_N^{(2)}\mathbf{r}_N & p_N^{(3)}\mathbf{r}_N
\end{pmatrix} \tag{21}
$$

where $\mathbf{E} \in \mathbb{R}^{3N \times 6P+18}$, $\mathbf{F} \in \mathbb{R}^{N \times 6P+18}$ and we can rewrite $\mathbf{A}$ as

$$
\mathbf{A} = \begin{pmatrix} \mathbf{E} \\ \mathbf{F} \\ \mathbf{D} \end{pmatrix}. \tag{22}
$$

We can see that the rows of $\mathbf{F}$ are linear dependent on the rows of $\mathbf{E}$.

4.1 Proof that matrix M (in Equation (18)) can have *rank* $6P + 17$

Since the rows of the $\mathbf{F}$ are linearly dependent on the rows of $\mathbf{E}$, we ignore the rows of $\mathbf{F}$ for the rest of the section. Thus, we consider the matrix $\mathbf{A}^{(1)} \in \mathbb{R}^{3N+18 \times 6P+18}$

$$
\mathbf{A}^{(1)} = \mathbf{Z}^{(1)} \begin{pmatrix} \mathbf{E}^{(1)} \\ \mathbf{D} \end{pmatrix} \tag{23}
$$

and if we define $\mathbf{D} \in \mathbb{R}^{18 \times 6P+18}$ as

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & -\mathbf{\Xi}_1\mathbf{P}_1 & \mathbf{\Xi}_2\mathbf{P}_1 & \mathbf{P}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{\Xi}_1\mathbf{P}_1 & \mathbf{0} & -\mathbf{\Xi}_3\mathbf{P}_1 & \mathbf{0} & \mathbf{P}_1 & \mathbf{0} \\ -\mathbf{\Xi}_2\mathbf{P}_1 & \mathbf{\Xi}_3\mathbf{P}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_1 \\ \mathbf{0} & -\mathbf{\Xi}_4\mathbf{P}_2 & \mathbf{\Xi}_5\mathbf{P}_2 & \mathbf{P}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{\Xi}_4\mathbf{P}_2 & \mathbf{0} & -\mathbf{\Xi}_6\mathbf{P}_2 & \mathbf{0} & \mathbf{P}_2 & \mathbf{0} \\ -\mathbf{\Xi}_5\mathbf{P}_2 & \mathbf{\Xi}_6\mathbf{P}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_2 \end{pmatrix} \tag{24}$$

where

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{K}_1 & \mathbf{0} \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} \mathbf{K}_2 & \mathbf{0} \end{pmatrix} \tag{25}$$

$\mathbf{P}_i \in \mathbb{R}^{3 \times P+3}$ and $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{R}^{3 \times P}$ are the stacking of the set $\{\mathbf{p}(\mathbf{x}_i)\}$ for $i = 1, \ldots, P$, and $\{\mathbf{p}(\mathbf{x}_i)\}$ for $i = P+1, \ldots, 2P$ respectively. Matrices $\mathbf{\Xi}_i \in \mathbb{R}^{3 \times 3}$ are random diagonal matrices, where $\xi_j^{(i)}$, for $j = 1, 2, 3$ are their diagonal elements.

We see that $\mathbf{E}^{(1)} \in \mathbb{R}^{3N \times 6P+18}$ and $\mathbf{D} \in \mathbb{R}^{18 \times 6P+18}$. Thus, to have $\mathcal{C}\left(\mathbf{A}^{(1)}\right) = 6P + 17$, we need at least $N = 2P$.

For a permutation matrix $\mathbf{Z}^{(1)}$, $\mathbf{E}$ with $N = 2P$ and $\mathbf{D}$ as in Equation (24), we define $\mathbf{A}^{(1)}$ as in Equation (26).

We can express $\mathbf{A}^{(1)}$ as a block of $P + 3 \times P + 3$ matrices

$$\left(\mathbf{A}^{(1)}\right)^T = \begin{pmatrix} \mathbf{0} & \mathbf{\Gamma}_1^T\mathbf{T}_1 & -\mathbf{\Gamma}_1^T\mathbf{D}_1 & \mathbf{0} & \mathbf{\Gamma}_2^T\mathbf{T}_2 & -\mathbf{\Gamma}_2^T\mathbf{D}_2 \\ -\mathbf{\Gamma}_1^T\mathbf{T}_1 & \mathbf{0} & \mathbf{\Gamma}_1^T\mathbf{S}_1 & -\mathbf{\Gamma}_2^T\mathbf{T}_2 & \mathbf{0} & \mathbf{\Gamma}_2^T\mathbf{S}_2 \\ \mathbf{\Gamma}_1^T\mathbf{D}_1 & -\mathbf{\Gamma}_1^T\mathbf{S}_1 & \mathbf{0} & \mathbf{\Gamma}_2^T\mathbf{D}_2 & -\mathbf{\Gamma}_2^T\mathbf{S}_2 & \mathbf{0} \\ \mathbf{\Gamma}_1^T & \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_2^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_1^T & \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_2^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_1^T & \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_2^T \end{pmatrix} \tag{27}$$

where $\mathbf{D}_i, \mathbf{T}_i, \mathbf{S}_i \in \mathbb{R}^{P+3 \times P+3}$ are diagonal matrices, whose diagonal elements are equal to respectively $p_n^{(m)}$ and to corresponding elements of diagonal matrices $\mathbf{\Xi}_i$ ($\xi_j^{(i)}$, with $j = 1, \ldots, 3$). For instance, diagonal matrix $\mathbf{T}_1$ is

$$\mathbf{T}_1 = \begin{pmatrix} p_1^{(3)} & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & p_P^{(3)} & 0 & 0 & 0 \\ 0 & \ldots & 0 & \xi_1^{(1)} & 0 & 0 \\ 0 & \ldots & 0 & 0 & \xi_2^{(1)} & 0 \\ 0 & \ldots & 0 & 0 & 0 & \xi_3^{(1)} \end{pmatrix}. \tag{28}$$

Matrices $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are

$$\mathbf{\Gamma}_1 = \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{P}_1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Gamma}_2 = \begin{pmatrix} \mathbf{R}_2 \\ \mathbf{P}_2 \end{pmatrix} \tag{29}$$

$$
\mathbf{A}^{(1)} = \begin{pmatrix}
\mathbf{0} & -p_1^{(3)}\mathbf{r}_1 & p_1^{(2)}\mathbf{r}_1 & \mathbf{r}_1 & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & -p_P^{(3)}\mathbf{r}_P & p_P^{(2)}\mathbf{r}_P & \mathbf{r}_P & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\boldsymbol{\Xi}_1\mathbf{P}_1 & \boldsymbol{\Xi}_2\mathbf{P}_1 & \mathbf{P}_1 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -p_{P+1}^{(3)}\mathbf{r}_{P+1} & p_{P+1}^{(2)}\mathbf{r}_{P+1} & \mathbf{r}_{P+1} & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & -p_{2P}^{(3)}\mathbf{r}_{2P} & p_{2P}^{(2)}\mathbf{r}_{2P} & \mathbf{r}_{2P} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\boldsymbol{\Xi}_4\mathbf{P}_2 & \boldsymbol{\Xi}_5\mathbf{P}_2 & \mathbf{P}_2 & \mathbf{0} & \mathbf{0} \\
p_1^{(3)}\mathbf{r}_1 & \mathbf{0} & -p_1^{(1)}\mathbf{r}_1 & \mathbf{0} & \mathbf{r}_1 & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_P^{(3)}\mathbf{r}_P & \mathbf{0} & -p_P^{(1)}\mathbf{r}_P & \mathbf{0} & \mathbf{r}_P & \mathbf{0} \\
\boldsymbol{\Xi}_1\mathbf{P}_1 & \mathbf{0} & -\boldsymbol{\Xi}_3\mathbf{P}_1 & \mathbf{0} & \mathbf{P}_1 & \mathbf{0} \\
p_{P+1}^{(3)}\mathbf{r}_{P+1} & \mathbf{0} & -p_{P+1}^{(1)}\mathbf{r}_{P+1} & \mathbf{0} & \mathbf{r}_{P+1} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_{2P}^{(3)}\mathbf{r}_{2P} & \mathbf{0} & -p_{2P}^{(1)}\mathbf{r}_{2P} & \mathbf{0} & \mathbf{r}_{2P} & \mathbf{0} \\
\boldsymbol{\Xi}_4\mathbf{P}_2 & \mathbf{0} & -\boldsymbol{\Xi}_6\mathbf{P}_2 & \mathbf{0} & \mathbf{P}_2 & \mathbf{0} \\
-p_1^{(2)}\mathbf{r}_1 & p_1^{(1)}\mathbf{r}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-p_P^{(2)}\mathbf{r}_P & p_P^{(1)}\mathbf{r}_P & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_P \\
-\boldsymbol{\Xi}_2\mathbf{P}_1 & \boldsymbol{\Xi}_3\mathbf{P}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_1 \\
-p_{P+1}^{(2)}\mathbf{r}_{P+1} & p_{P+1}^{(1)}\mathbf{r}_{P+1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_{P+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-p_{2P}^{(2)}\mathbf{r}_{2P} & p_{2P}^{(1)}\mathbf{r}_{2P} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{r}_{2P} \\
-\boldsymbol{\Xi}_5\mathbf{P}_2 & \boldsymbol{\Xi}_6\mathbf{P}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_2
\end{pmatrix} \in \mathbb{R}^{6P+18\times 6P+18} \tag{26}
$$

where $\mathbf{\Gamma}_i \in \mathbb{R}^{P+3\times P+3}$, and

$$\mathbf{R}_1 = \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_P \end{pmatrix} \quad \text{and} \quad \mathbf{R}_2 = \begin{pmatrix} \mathbf{r}_{P+1} \\ \vdots \\ \mathbf{r}_{2P} \end{pmatrix} \tag{30}$$

where $\mathbf{R}_i \in \mathbb{R}^{P\times P+3}$ and $\mathbf{P}_1$ and $\mathbf{P}_2$ are as in Equation (25).

We assume that the conditions described in Section 2 are met for $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, which means that these matrices are *full–rank*.

Let us define a matrix

$$\mathbf{N} = \mathbf{G}_1 \left(\mathbf{A}^{(1)}\right)^T \mathbf{G}_2 \tag{31}$$

where

$$\mathbf{G}_1 = \begin{pmatrix} \left(\mathbf{\Gamma}_1^T\right)^{-1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{\Gamma}_1^T\right)^{-1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \left(\mathbf{\Gamma}_1^T\right)^{-1} \end{pmatrix} \quad \text{and} \quad \mathbf{G}_2 = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}^{-1} \end{pmatrix} \tag{32}$$

$\mathbf{G}_1, \mathbf{G}_2, \in \mathbb{R}^{6P+18\times 6P+18}$ are *full–rank* matrices, and with $\mathbf{L} = \left(\mathbf{\Gamma}_1^T\right)^{-1} \mathbf{\Gamma}_2^T$. The pre or post–multiplication by any *full–rank* matrix does not change the *rank* of a matrix. Thus, $\mathcal{C}(\mathbf{N}) = \mathcal{C}\left(\left(\mathbf{A}^{(1)}\right)^T\right)$ and $\mathcal{C}\left(\mathbf{A}^{(1)}\right) = \mathcal{C}\left(\left(\mathbf{A}^{(1)}\right)^T\right)$.

From Section 1.2, we can see that $\mathcal{C}(\mathbf{N}) + \mathcal{N}(\mathbf{N}) = 6P+18$. Thus, if we want $\mathcal{C}\left(\mathbf{A}^{(1)}\right) = \mathcal{C}(\mathbf{N}) = 6P+17$, we must have $\mathcal{N}(\mathbf{N}) = 1$. As a result, we need to prove that the *nullity* of $\mathbf{N}$ is one, where $\mathbf{N}$ is

$$\mathbf{N} = \begin{pmatrix} \mathbf{0} & \mathbf{T}_1 & -\mathbf{D}_1 & \mathbf{0} & \mathbf{L}\mathbf{T}_2\mathbf{L}^{-1} & -\mathbf{L}\mathbf{D}_2\mathbf{L}^{-1} \\ -\mathbf{T}_1 & \mathbf{0} & \mathbf{S}_1 & -\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1} & \mathbf{0} & \mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} \\ \mathbf{D}_1 & -\mathbf{S}_1 & \mathbf{0} & \mathbf{L}\mathbf{D}_2\mathbf{L}^{-1} & -\mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{33}$$

which means that $\mathbf{N}\mathbf{v} = \mathbf{0}$ has a one dimensional subspace of solutions.

We consider that $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_6) \in \mathbb{R}^{6P+18}$ where $\mathbf{v}_i \in \mathbb{R}^{P+3}$. From the three last rows of Equation (33), we see that the *null–space* of $\mathbf{N}$ must verify

$$\mathbf{v}_1 = -\mathbf{v}_4 \tag{34}$$

$$\mathbf{v}_2 = -\mathbf{v}_5 \tag{35}$$

$$\mathbf{v}_3 = -\mathbf{v}_6. \tag{36}$$

Getting the second, fifth and sixth row of equations of matrix $\mathbf{N}$ and the third, fifth and sixth row of equations of matrix $\mathbf{N}$ respectively, we can define the following constraints

$$\begin{pmatrix} -\mathbf{T}_1 & \mathbf{0} & \mathbf{S}_1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = - \begin{pmatrix} -\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1} & \mathbf{0} & \mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}_4 \\ \mathbf{v}_5 \\ \mathbf{v}_6 \end{pmatrix} \tag{37}$$

and

$$\begin{pmatrix} \mathbf{D}_1 & -\mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = - \begin{pmatrix} \mathbf{L}\mathbf{D}_2\mathbf{L}^{-1} & -\mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}_4 \\ \mathbf{v}_5 \\ \mathbf{v}_6 \end{pmatrix}. \tag{38}$$

If the diagonal elements of $\mathbf{D}_1$ and $\mathbf{T}_1$ are different from zero, we can define matrices $\mathbf{B}$ and $\mathbf{C}$ as

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \mathbf{B} \begin{pmatrix} \mathbf{v}_4 \\ \mathbf{v}_5 \\ \mathbf{v}_6 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \mathbf{C} \begin{pmatrix} \mathbf{v}_4 \\ \mathbf{v}_5 \\ \mathbf{v}_6 \end{pmatrix}. \tag{39}$$

Using Appendix A.2, we obtain

$$-\begin{pmatrix} -\mathbf{T}_1 & \mathbf{0} & \mathbf{S}_1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} -\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1} & \mathbf{0} & \mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} = -\begin{pmatrix} -\mathbf{T}_1^{-1} & \mathbf{0} & \mathbf{T}_1^{-1}\mathbf{S}_1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} -\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1} & \mathbf{0} & \mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} = \underbrace{-\begin{pmatrix} \mathbf{T}_1^{-1}\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1} & \mathbf{0} & -\mathbf{T}_1^{-1}\mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} + \mathbf{T}_1^{-1}\mathbf{S}_1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}}_{\mathbf{B}} \tag{40}$$

and

$$-\begin{pmatrix} \mathbf{D}_1 & -\mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{L}\mathbf{D}_2\mathbf{L}^{-1} & -\mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} = -\begin{pmatrix} \mathbf{D}_1^{-1} & \mathbf{D}_1^{-1}\mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{L}\mathbf{D}_2\mathbf{L}^{-1} & -\mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} = \underbrace{-\begin{pmatrix} \mathbf{D}_1^{-1}\mathbf{L}\mathbf{D}_2\mathbf{L}^{-1} & -\mathbf{D}_1^{-1}\mathbf{L}\mathbf{S}_2\mathbf{L}^{-1} + \mathbf{D}_1^{-1}\mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}}_{\mathbf{C}}. \tag{41}$$

From Section 1.2, the sets of *eigenvalues* of $\mathbf{B}$ and $\mathbf{C}$ are respectively

$$\lambda(\mathbf{B}) = \lambda\left(-\mathbf{T}_1^{-1}\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1}\right) \cup \lambda\left(\begin{pmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}\right) \quad \text{and} \quad \lambda(\mathbf{C}) = \lambda\left(-\mathbf{D}_1^{-1}\mathbf{L}\mathbf{D}_2\mathbf{L}^{-1}\right) \cup \lambda\left(\begin{pmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}\right) \tag{42}$$

and we define $\mathbf{\Sigma_B}$, $\mathbf{\Sigma_C}$ as diagonal matrices, whose diagonal elements are the *eigenvalues* of $\mathbf{B}$ and $\mathbf{C}$ respectively

$$\mathbf{\Sigma_B} = \begin{pmatrix} \mathbf{\Sigma}_{-\mathbf{T}_1^{-1}\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma_C} = \begin{pmatrix} \mathbf{\Sigma}_{-\mathbf{D}_1^{-1}\mathbf{L}\mathbf{D}_2\mathbf{L}^{-1}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{pmatrix}. \tag{43}$$

We can see that the solutions for Equations (37) that verify Equations (34), (34) and (36), are defined by the *eigenvectors*, that correspond to the *eigenvalues* $\lambda(\mathbf{B})$ that are equal to -1. On the other hand, solutions for Equations (38) that verify Equations (34), (34) and (36), are defined by the *eigenvectors*, that correspond to the *eigenvalues* $\lambda(\mathbf{C})$ that are equal to -1.

If we consider that $\mathbf{T}_i$ and $\mathbf{D}_i$ are random matrices, we can conclude that the probability of $\lambda\left(\mathbf{T}_1^{-1}\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1}\right) \cap \lambda(-\mathbf{I}) = \varnothing$ and $\lambda\left(\mathbf{D}_1^{-1}\mathbf{L}\mathbf{D}_2\mathbf{L}^{-1}\right) \cap \lambda(-\mathbf{I}) = \varnothing$ is equal to one.

From Appendix A.3, we conclude that the matrices that correspond to the stacking of *eigenvectors* (*eigenvectors matrices*), $\mathbf{V}$ and $\mathbf{U}$ ($\mathbf{B} = \mathbf{V}\mathbf{\Sigma_B}\mathbf{V}^{-1}$ and $\mathbf{C} = \mathbf{U}\mathbf{\Sigma_C}\mathbf{U}^{-1}$) have the form

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^{(1)} & \mathbf{0} & \mathbf{V}^{(2)} \\ \mathbf{0} & \mathbf{V}^{(3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}^{(4)} \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} \mathbf{U}^{(1)} & \mathbf{U}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^{(3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{U}^{(4)} \end{pmatrix} \tag{44}$$

where $\mathbf{V}, \mathbf{U} \in \mathbb{R}^{3P+9\times 3P+9}$.

Since we are only interested in *eigenvectors* associated to *eigenvalues* equal to -1, we only consider the subspaces generated from matrices

$$\hat{\mathbf{V}} = \begin{pmatrix} \mathbf{0} & \mathbf{V}^{(2)} \\ \mathbf{V}^{(3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^{(4)} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{U}} = \begin{pmatrix} \mathbf{U}^{(2)} & \mathbf{0} \\ \mathbf{U}^{(3)} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^{(4)} \end{pmatrix} \tag{45}$$

where $\hat{\mathbf{V}}, \hat{\mathbf{U}} \in \mathbb{R}^{3P+9\times 2P+6}$

However, we want solutions that verify $\mathbf{N}\mathbf{v} = \mathbf{0}$, which means that they must belong to both $\hat{\mathbf{V}}$ and $\hat{\mathbf{U}}$ subspaces. As a result, solutions must belong to the intersection of subspaces defined by $\hat{\mathbf{V}}$ and $\hat{\mathbf{U}}$.

From Appendix A.3 and Appendix A.4, we conclude that the intersection subspace is defined by the column space of

$$\mathbf{W} = \begin{pmatrix} * \\ \mathbf{I} \\ \mathbf{K} \end{pmatrix} \tag{46}$$

where $\mathbf{W} \in \mathbb{R}^{3P+9\times P+3}$. This means that, any linear combination of $\mathbf{W}$ columns ($\mathbf{W}\mathbf{a}$ for any $\mathbf{a} \neq \mathbf{0}$) is a solution for Equations (39) that verifies Equation (34), (35) and (36) where

$$\mathbf{v} = (*, \mathbf{a}, \mathbf{K}\mathbf{a}, *, -\mathbf{a}, -\mathbf{K}\mathbf{a}) \tag{47}$$

for any vector $\mathbf{a} \in \mathbb{R}^{P+3}$ different from zero.

However, from the first row of equations of $\mathbf{N}$, Equation (47) must verify

$$\mathbf{T}_1\mathbf{v}_2 + \mathbf{L}\mathbf{T}_2\mathbf{L}^{-1}\mathbf{v}_5 = \mathbf{D}_1\mathbf{v}_3 + \mathbf{L}\mathbf{D}_2\mathbf{L}^{-1}\mathbf{v}_6, \tag{48}$$

which from Equation (47) is equal to

$$\underbrace{\left(\mathbf{T}_1 - \mathbf{L}\mathbf{T}_2\mathbf{L}^{-1}\right)}_{\mathbf{F}(\mathbf{T}_{1,2})}\mathbf{a} = \underbrace{\left(\mathbf{D}_1 - \mathbf{L}\mathbf{D}_2\mathbf{L}^{-1}\right)}_{\mathbf{F}(\mathbf{D}_{1,2})}\mathbf{K}\mathbf{a}. \tag{49}$$

From Section A.1, the previous assumptions that $\lambda\left(\mathbf{T}_1^{-1}\mathbf{L}\mathbf{T}_2\mathbf{L}^{-1}\right) \cap \lambda\left(-\mathbf{I}\right) = \varnothing$ and $\lambda\left(\mathbf{D}_1^{-1}\mathbf{L}\mathbf{D}_2\mathbf{L}^{-1}\right) \cap \lambda\left(-\mathbf{I}\right) = \varnothing$ and assuming that $\mathbf{S}_1, \mathbf{S}_2$ are random matrices which implies that the probability of $\lambda\left(\mathbf{S}_1^{-1}\mathbf{L}\mathbf{S}_2\mathbf{L}^{-1}\right) \cap \lambda\left(-\mathbf{I}\right) = \varnothing$ is one, we see that $\mathcal{C}\left(\mathbf{F}\left(\mathbf{T}_{1,2}\right)\right) = P + 3$, $\mathcal{C}\left(\mathbf{F}\left(\mathbf{D}_{1,2}\right)\right) = P + 3$ and $\mathcal{C}\left(\mathbf{K}\right) = P + 3$. Thus, the constraint corresponding to Equation (48) can be rewritten as

$$\mathbf{K}^{-1}\mathbf{F}\left(\mathbf{D}_{1,2}\right)^{-1}\mathbf{F}\left(\mathbf{T}_{1,2}\right)\mathbf{a} = \mathbf{a}. \tag{50}$$

As a result, we can see that the dimension of the *null–space* of $\mathbf{N}$ is equal to the number of *eigenvalues* $\lambda\left(\mathbf{K}^{-1}\mathbf{F}\left(\mathbf{D}_{1,2}\right)^{-1}\mathbf{F}\left(\mathbf{T}_{1,2}\right)\right)$ that are equal to 1.

4.2 The set $\lambda\left(\mathbf{K}^{-1}\mathbf{F}\left(\mathbf{D}_{1,2}\right)^{-1}\mathbf{F}\left(\mathbf{T}_{1,2}\right)\right)$

In the previous section, we saw that $\mathcal{C}\left(\mathbf{A}^{(1)}\right) = \mathcal{C}\left(\mathbf{N}\right)$. On the other hand, we see that the $\mathcal{N}\left(\mathbf{N}\right)$ is equal to the number of *eigenvalues* $\lambda\left(\mathbf{K}^{-1}\mathbf{F}\left(\mathbf{D}_{1,2}\right)^{-1}\mathbf{F}\left(\mathbf{T}_{1,2}\right)\right) \cap \lambda\left(\mathbf{I}\right)$ and, since $\mathbf{N}$ is a square matrix, we know that $6P + 18 = \mathcal{C}\left(\mathbf{N}\right) + \mathcal{N}\left(\mathbf{N}\right)$. As a result, $\mathcal{C}\left(\mathbf{N}\right) = 6P + 17$, implies $\mathcal{N}\left(\mathbf{N}\right) = 1$, which means that $\lambda\left(\mathbf{K}^{-1}\mathbf{F}\left(\mathbf{D}_{1,2}\right)^{-1}\mathbf{F}\left(\mathbf{T}_{1,2}\right)\right)$ must have one *eigenvalue* equal to 1.

$\mathbf{\Gamma}_i$ are matrices that depend on a vector $\mathbf{d}$. As a result, if we consider random elements of $\mathbf{d}$, it is expected that the number of *eigenvalues* $\lambda\left(\mathbf{K}^{-1}\mathbf{F}\left(\mathbf{D}_{1,2}\right)^{-1}\mathbf{F}\left(\mathbf{T}_{1,2}\right)\right) \cap \lambda\left(\mathbf{I}\right) = \varnothing$.

However, we, intentionally chose matrix $\mathbf{D}$ as in Equation (24). Therefore matrix $\mathbf{D}$ has the following rows

$$\mathbf{Y} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\xi}_1^{(1)} & \boldsymbol{\xi}_1^{(2)} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\xi}_1^{(1)} & \mathbf{0} & -\boldsymbol{\xi}_1^{(3)} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ -\boldsymbol{\xi}_1^{(2)} & \boldsymbol{\xi}_1^{(3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & -\boldsymbol{\xi}_1^{(4)} & \boldsymbol{\xi}_1^{(5)} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\xi}_1^{(4)} & \mathbf{0} & -\boldsymbol{\xi}_1^{(6)} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ -\boldsymbol{\xi}_1^{(5)} & \boldsymbol{\xi}_1^{(6)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{51}$$

where $\mathbf{Y} \in \mathbb{R}^{(6\times 6P+18)}$,

$$\boldsymbol{\xi}_1^{(i)} = \begin{pmatrix} \xi_1^{(i)} & \ldots & \xi_1^{(i)} & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{1} = \begin{pmatrix} 1 & \ldots & 1 & 0 & 0 & 0 \end{pmatrix} \tag{52}$$

with $\boldsymbol{\xi}_1^{(i)}, \mathbf{1} \in \mathbb{R}^{(1\times P+3)}$.

One concludes that $\mathcal{C}(\mathbf{Y}) = \mathcal{C}(\bar{\mathbf{Y}})$ where

$$\bar{\mathbf{Y}} = \begin{pmatrix} \mathbf{0} & -\xi_1^{(1)} & \xi_1^{(2)} & 1 & \mathbf{0} & \mathbf{0} \\ \xi_1^{(1)} & \mathbf{0} & -\xi_1^{(3)} & \mathbf{0} & 1 & \mathbf{0} \\ -\xi_1^{(2)} & \xi_1^{(3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \\ \mathbf{0} & -\xi_1^{(4)} & \xi_1^{(5)} & 1 & \mathbf{0} & \mathbf{0} \\ \xi_1^{(4)} & \mathbf{0} & -\xi_1^{(6)} & \mathbf{0} & 1 & \mathbf{0} \\ -\xi_1^{(5)} & \xi_1^{(6)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{pmatrix} \tag{53}$$

and $\mathcal{C}(\mathbf{Y}) = \mathcal{C}(\bar{\mathbf{Y}}) = 5$.

Since the rows of $\mathbf{Y}$ will be the columns of $\left(\mathbf{A}^{(1)}\right)^T$, we see that $\mathcal{C}\left(\left(\mathbf{A}^{(1)}\right)^T\right) \leq 6P + 17$ which means that we have one *eigenvector* of $\lambda\left(\mathbf{K}^{-1}\mathbf{F}\left(\mathbf{D_{1,2}}\right)^{-1}\mathbf{F}\left(\mathbf{T_{1,2}}\right)\right)$ equal to 1.

Thus, for random elements of the diagonal matrices $\mathbf{D}_i, \mathbf{T}_i, \mathbf{S}_i$ and random vector $\mathbf{d}$, we have $\mathcal{N}(\mathbf{N}) = 1$ with probability one, which implies $\mathcal{C}\left(\mathbf{A}^{(1)}\right) = \mathcal{C}\left(\left(\mathbf{A}^{(1)}\right)^T\right) = \mathcal{C}(\mathbf{N}) = 6P + 17$.

5 Conclusions

To obtain the *rank* of the matrix $\mathbf{M}$ we write

$$\mathbf{M} = \mathbf{Z}^{(2)}\mathbf{A}^{(2)} \tag{54}$$

where the matrix $\mathbf{A}^{(2)}$ is as

$$\mathbf{A}^{(2)} = \begin{pmatrix} \mathbf{A}^{(1)} \\ \mathbf{F} \end{pmatrix} \tag{55}$$

and $\mathbf{A}^{(1)}$ is as in Equation (26) and $\mathbf{Z}^{(2)}$ is a permutation matrix.

In Section 4, we saw that each of the rows of $\mathbf{F}$ is linearly dependent on the rows of $\mathbf{E}$, which are included in matrix $\mathbf{A}^{(1)}$. Thus, we can write $\mathcal{C}\left(\mathbf{A}^{(2)}\right) = \mathcal{C}\left(\mathbf{A}^{(1)}\right) = 6P + 17$.

Since the permutation of rows does not change the *rank* of a matrix, we can write $\mathcal{C}(\mathbf{M}) = \mathcal{C}\left(\mathbf{A}^{(2)}\right) = 6P + 17$.

Appendices

A Some Matrix Results

A.1 *Rank* of $\mathbf{D}_1 - \mathbf{L}\mathbf{D}_2\mathbf{L}^{-1}$

Considering diagonal *full–rank* matrices $\mathbf{D}_1, \mathbf{D}_2 \in \mathbb{R}^{P\times P}$ and a generic *full–rank* $\mathbf{L} \in \mathbb{R}^{P\times P}$.

If we write a matrix $\mathbf{M} \in \mathbb{R}^{2P\times 2P}$ as

$$\mathbf{M} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{L}\mathbf{D}_1\mathbf{L}^{-1} & -\mathbf{L}\mathbf{D}_1\mathbf{L}^{-1} + \mathbf{D}_2 \end{pmatrix}, \tag{56}$$

we see that $\mathcal{C}(\mathbf{M}) = \mathcal{C}(\mathbf{I}) + \mathcal{C}(-\mathbf{LD}_1\mathbf{L}^{-1} + \mathbf{D}_2)$. If we post–multiply $\mathbf{M}$ by any *non–singular* matrix, the *rank* of the resulting matrix will be the same as the *rank* of $\mathbf{M}$. As a result, we define

$$\mathbf{N} = \mathbf{M}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \tag{57}$$

where $\mathcal{C}(\mathbf{N}) = \mathcal{C}(\mathbf{M})$ and

$$\mathbf{N} = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{LD}_1\mathbf{L}^{-1} & \mathbf{D}_2 \end{pmatrix}. \tag{58}$$

We can see that the *null–space* of $\mathbf{N}$ must satisfy

$$\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{LD}_1\mathbf{L}^{-1} & \mathbf{D}_2 \end{pmatrix}\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \mathbf{0}, \tag{59}$$

which can be rewritten as

$$\begin{cases} \mathbf{v}_1 = -\mathbf{v}_2 \\ -\mathbf{D}_2^{-1}\mathbf{LD}_1\mathbf{L}^{-1}\mathbf{v}_1 = \mathbf{v}_2 \end{cases} \tag{60}$$

and $\mathcal{N}(\mathbf{N}) = n$, where n is the number of *eigenvalues* of $-\mathbf{D}_2^{-1}\mathbf{LD}_1\mathbf{L}^{-1}\mathbf{v}_1$ equal to one.

If do not exist *eigenvalues* equal to one, then $\mathcal{N}(\mathbf{N}) = 0$, which implies $\mathcal{N}(\mathbf{M}) = \mathcal{N}(\mathbf{N}) = 2P$ and $\mathcal{C}(-\mathbf{LD}_1\mathbf{L}^{-1} + \mathbf{D}_2) = P$.

A.2 Inverse of Matrices

In this section we describe how to get the inverses of the matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_4 \end{pmatrix} \tag{61}$$

where $\mathbf{A}$ is *full–rank*.

The inverse must satisfy $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, thus

$$\begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \\ \mathbf{X}_4 & \mathbf{X}_5 & \mathbf{X}_6 \\ \mathbf{X}_7 & \mathbf{X}_8 & \mathbf{X}_9 \end{pmatrix}\begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{62}$$

We can define the tree next systems

$$\begin{cases} \mathbf{X}_1\mathbf{A}_1 = \mathbf{I} \\ \mathbf{X}_4\mathbf{A}_1 = \mathbf{0} \\ \mathbf{X}_7\mathbf{A}_1 = \mathbf{0} \end{cases}, \quad \begin{cases} \mathbf{X}_2\mathbf{A}_3 = \mathbf{0} \\ \mathbf{X}_5\mathbf{A}_3 = \mathbf{I} \\ \mathbf{X}_8\mathbf{A}_3 = \mathbf{0} \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{X}_1\mathbf{A}_2 + \mathbf{X}_3\mathbf{A}_4 = \mathbf{0} \\ \mathbf{X}_4\mathbf{A}_2 + \mathbf{X}_6\mathbf{A}_4 = \mathbf{0} \\ \mathbf{X}_7\mathbf{A}_2 + \mathbf{X}_9\mathbf{A}_4 = \mathbf{I} \end{cases}. \tag{63}$$

From the first system, we get $\mathbf{X}_7 = \mathbf{X}_4 = \mathbf{0}$ and $\mathbf{X}_1 = \mathbf{A}_1^{-1}$. From the second system, we get $\mathbf{X}_2 = \mathbf{X}_8 = \mathbf{0}$ and

$\mathbf{X}_5 = \mathbf{A}_3^{-1}$. Since $\mathbf{X}_4 = \mathbf{X}_7 = \mathbf{0}$, we can rewrite the third system as

$$\begin{cases} \mathbf{X}_1\mathbf{A}_2 + \mathbf{X}_3\mathbf{A}_4 = \mathbf{0} \implies \mathbf{X}_3 = -\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ \mathbf{X}_6\mathbf{A}_4 = \mathbf{0} \\ \mathbf{X}_9\mathbf{A}_4 = \mathbf{I} \end{cases} \tag{64}$$

and we can write $\mathbf{X}_6 = \mathbf{0}$ and $\mathbf{X}_9 = \mathbf{A}_4^{-1}$.

Finally, we can write

$$\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{A}_1^{-1} & \mathbf{0} & -\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ \mathbf{0} & \mathbf{A}_3^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_4^{-1} \end{pmatrix}. \tag{65}$$

Using the same method, we can prove that

$$\mathbf{B}^{-1} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_4 \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{B}_1^{-1} & -\mathbf{B}_1^{-1}\mathbf{B}_2\mathbf{B}_3^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_3^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_4^{-1} \end{pmatrix}. \tag{66}$$

A.3 *Eigenvector Matrices*

Suppose we want to know the structure of the *eigenvector matrix* ($\mathbf{V}_\mathbf{A}$) of a matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{pmatrix} \tag{67}$$

where $\mathbf{A}$ is *full–rank*.

We know that $\mathbf{V}_\mathbf{A}$ must satisfy $\mathbf{A}\mathbf{V}_\mathbf{A} = \mathbf{V}_\mathbf{A}\boldsymbol{\Sigma}_\mathbf{A}$, where $\boldsymbol{\Sigma}_\mathbf{A}$ is a diagonal matrix whose diagonal elements are $\lambda(\mathbf{A})$. Thus

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{A}_2 \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \\ \mathbf{X}_4 & \mathbf{X}_5 & \mathbf{X}_6 \\ \mathbf{X}_7 & \mathbf{X}_8 & \mathbf{X}_9 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \\ \mathbf{X}_4 & \mathbf{X}_5 & \mathbf{X}_6 \\ \mathbf{X}_7 & \mathbf{X}_8 & \mathbf{X}_9 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{-\mathbf{I}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma}_{-\mathbf{I}} \end{pmatrix}. \tag{68}$$

Using this representation we can define the system

$$\begin{cases} \mathbf{A}_1\mathbf{X}_1 + \mathbf{A}_2\mathbf{X}_7 = \mathbf{X}_1\boldsymbol{\Sigma}_{\mathbf{A}_1} \\ -\mathbf{X}_4 = \mathbf{X}_4\boldsymbol{\Sigma}_{\mathbf{A}_1} \\ -\mathbf{X}_7 = \mathbf{X}_7\boldsymbol{\Sigma}_{\mathbf{A}_1} \end{cases}. \tag{69}$$

If we consider that matrix $\mathbf{A}_1$ is a random matrix, the probability of $\lambda(\mathbf{A}_1) \cap \lambda(-\mathbf{I}) = \varnothing$ is equal to one, which from Equation (69) implies that $\mathbf{X}_4 = \mathbf{X}_7 = \mathbf{0}$ and $\mathbf{X}_1 = \mathbf{V}_{\mathbf{A}_1}$ where $\mathbf{V}_{\mathbf{A}_1}$ is the *eigenvector matrix* of $\mathbf{A}_1$.

The remaining equations from Equation (68) must verify

$$\left\{\begin{array}{l}\mathbf{A}_1\mathbf{X}_2+\mathbf{A}_2\mathbf{X}_8=-\mathbf{X}_2\\ -\mathbf{X}_5=-\mathbf{X}_5\\ -\mathbf{X}_8=-\mathbf{X}_8\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\mathbf{A}_1\mathbf{X}_3+\mathbf{A}_2\mathbf{X}_9=-\mathbf{X}_3\\ -\mathbf{X}_6=-\mathbf{X}_6\\ -\mathbf{X}_9=-\mathbf{X}_9\end{array}\right. . \tag{70}$$

We are interested in the subspace of *eigenvectors*. Thus, we can define a set of *eigenvector* basis where $\mathbf{X}_8=\mathbf{X}_6=\mathbf{0}$ and $\mathbf{X}_5=\mathbf{X}_9=\mathbf{I}$

$$\mathbf{V}_{\mathbf{A}}=\begin{pmatrix}\mathbf{V}_{\mathbf{A}_1} & \mathbf{0} & \left(-\mathbf{I}-\mathbf{A}_1\right)^{-1}\mathbf{A}_2\\ \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{pmatrix}. \tag{71}$$

If we apply the same method to the matrix

$$\mathbf{B}=\begin{pmatrix}\mathbf{B}_1 & \mathbf{B}_2 & \mathbf{0}\\ \mathbf{0} & -\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{I}\end{pmatrix} \tag{72}$$

and considering that $\mathbf{B}_1$ is a random matrix ($\lambda\left(\mathbf{B}_1\right)\cap\lambda\left(-\mathbf{I}\right)=\varnothing$), we get

$$\mathbf{V}_{\mathbf{B}}=\begin{pmatrix}\mathbf{V}_{\mathbf{B}_1} & \left(-\mathbf{I}-\mathbf{B}_1\right)^{-1}\mathbf{B}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{pmatrix}. \tag{73}$$

A.4 Intersection Subspace

In this section, we study the intersection subspace between *eigenvector matrices* $\mathbf{V}_{\mathbf{A}}$ of Equation (71) and $\mathbf{V}_{\mathbf{B}}$ of Equation (73), that correspond to *eigenvalues* equal to minos one.

Since we are only interested in the *eigenvectors* that correspond to *eigenvalues* equal to minos one, from Appendix A.3, we can define

$$\hat{\mathbf{V}}_{\mathbf{A}}=\begin{pmatrix}\mathbf{0} & \left(-\mathbf{I}-\mathbf{A}_1\right)^{-1}\mathbf{A}_2\\ \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix} \quad \text{and} \quad \hat{\mathbf{V}}_{\mathbf{B}}=\begin{pmatrix}\left(-\mathbf{I}-\mathbf{B}_1\right)^{-1}\mathbf{B}_2 & \mathbf{0}\\ \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}, \tag{74}$$

and the basis for the intersection subspace can be obtained from the solution of the following Equation

$$\underbrace{\begin{pmatrix}\mathbf{0} & \mathbf{A}_3^{-1}\mathbf{A}_2 & \mathbf{B}_3^{-1}\mathbf{B}_2\\ \mathbf{I} & \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{I} & \mathbf{0}\end{pmatrix}}_{\mathbf{M}}\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \mathbf{X}_3\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{X}_4\end{pmatrix} \tag{75}$$

where $\mathbf{M}$ is *full-rank*, $\mathbf{A}_3=-\mathbf{I}-\mathbf{A}_1$ and $\mathbf{B}_3=\mathbf{I}-\mathbf{B}_1$. Note that $\mathbf{A}_1$ and $\mathbf{B}_1$ are random matrices which means that the probability of $\lambda(\mathbf{A}_1) \cap \lambda(\mathbf{I}) = \varnothing$ and $\lambda(\mathbf{B}_1) \cap \lambda(\mathbf{I}) = \varnothing$ is one and, from Appendix A.1, we know that $\mathbf{A}_3$ and $\mathbf{B}_3$ have inverses.

The subspace of solution for Equation (75) can be defined as

$$\begin{cases} \mathbf{X}_1 = -\mathbf{K} \\ \mathbf{X}_3 = \mathbf{K} \\ \mathbf{X}_2 = -\mathbf{A}_2^{-1}\mathbf{A}_3\mathbf{B}_3^{-1}\mathbf{B}_2\mathbf{K} \\ \mathbf{X}_4 = -\mathbf{A}_2^{-1}\mathbf{A}_3\mathbf{B}_3^{-1}\mathbf{B}_2\mathbf{K} \end{cases}, \tag{76}$$

for any *non–singular* matrix $\mathbf{K}$.

We are interested in defining the basis for the intersection subspace. Thus, we can write $\mathbf{K} = \mathbf{I}$ and

$$\begin{cases} \mathbf{X}_1 = -\mathbf{I} \\ \mathbf{X}_3 = \mathbf{I} \\ \mathbf{X}_2 = -\mathbf{A}_2^{-1}\mathbf{A}_3\mathbf{B}_3^{-1}\mathbf{B}_2 \\ \mathbf{X}_4 = -\mathbf{A}_2^{-1}\mathbf{A}_3\mathbf{B}_3^{-1}\mathbf{B}_2 \end{cases}. \tag{77}$$

Using $\mathbf{X}_1$ and $\mathbf{X}_2$ we can determine the intersection subspace from

$$\begin{pmatrix} \mathbf{0} & \mathbf{A}_3^{-1}\mathbf{A}_2 \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} = \begin{pmatrix} -\mathbf{B}_3^{-1}\mathbf{B}_2 \\ -\mathbf{I} \\ -\mathbf{A}_2^{-1}\mathbf{A}_3\mathbf{B}_3^{-1}\mathbf{B}_2 \end{pmatrix}. \tag{78}$$

References

[1] Martin Buhmann. *Radial Basis Functions: Theory and Implementations*. Cambridge University Press, The Edinburgh Building, Cambridge CB2 8RU, UK, 2003. 3

[2] Gene H. Golub and Charles F. Van Loan. *Matrix computations (3rd ed.)*. Johns Hopkins University Press, Baltimore, MD, USA, 1996. 2, 3

[3] Roger A. Horn and Charles R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, The Edinburgh Building, Cambridge CB2 8RU, UK, 1991. 2

[4] Pedro Miraldo, Helder Araujo, and J. Queiró. Point–based calibration using a parametric representation of the general imaging model. *In ICCV*, 2011. 4, 5

[5] E. Quak, N. Sivakumar, and D. J. Ward. Least squares approximation by radial functions. *SIAM Journal on Mathematical Analysis*, 24(4):1043–1066, 1993. 3

[6] N. Sivakumar and J. D. Ward. On the least squares fit by radial functions to multidimensional scattered data. *Numerische Mathematik*, 65:219–243, 1993. 10.1007/BF01385749. 3

[7] Gilbert Strang. *Linear Algebra and Its Applications (2nd ed.)*. Academic Press, Oval Road, London NW1 7DX, 1980. 2

[8] Holger Wendland. *Scattered Data Approximation*. Cambridge University Press, The Edinburgh Building, Cambridge CB2 8RU, UK, 2005. 3, 4