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REFLECTIVE AND ORTHOGONAL HULLS

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Prefácio

Um dos mais importantes e frutuosos conceitos em Teoria das Categorias é o de subcategoria reflectiva. Por um lado, toda a subcategoria plena de uma categoria $\mathcal{X}$ que seja reflectiva partilha muitas das propriedades mais significativas de $\mathcal{X}$ (tais como existência e construção de limites, existência de colimites, etc.), por outro lado, é conhecido um considerável número de boas condições suficientes para que haja reflectividade. Para categorias plenas $\mathcal{A}$ de uma categoria $\mathcal{X}$ que não são reflectivas interessa determinar uma subcategoria plena de $\mathcal{X}$ que seja a menor de entre as que são reflectivas e contêm $\mathcal{A}$, chamada invólucro reflectivo de $\mathcal{A}$. Este é o tema central da presente dissertação. Duas questões se põem:

(1) Quando é que $\mathcal{A}$ tem um invólucro reflectivo?

(2) Como pode ser construído o invólucro reflectivo de $\mathcal{A}$, se ele existir?

Para (2), um caminho possível é formar o invólucro para limites de $\mathcal{A}$, i.e., a menor subcategoria plena de $\mathcal{X}$ fechada para limites e que contém $\mathcal{A}$. Se este invólucro é reflectivo, ele é um invólucro reflectivo de $\mathcal{A}$, mas a questão de determinar quando isto acontece tem-se revelado muito difícil (cf., por exemplo, [4], [22], [58], [73] e [77]). Portanto, nesta tese, optei por uma abordagem diferente baseada no conceito de ortogonalidade. Recordemos que um objecto A se diz ortogonal a um morfismo $f : X \to Y$ se a aplicação $hom(A, f) : hom(Y, A) \to hom(X, A)$ é uma bijecção. Para cada subcategoria plena $\mathcal{A}$ de uma categoria $\mathcal{X}$, denotamos por $\mathcal{A}^{\perp}$ a classe de todos os morfismos f em $\mathcal{X}$ ortogonais a todos os objectos de $\mathcal{A}$. É fácil concluir que se $\mathcal{A}$ é uma subcategoria reflectiva de $\mathcal{X}$, então $\mathcal{A}$ pode ser reconstruída a partir de $\mathcal{A}^{\perp}$ do seguinte modo: $\mathcal{A}$ é constituída por precisamente todos os objectos ortogonais a todos os morfismos em $\mathcal{A}^{\perp}$. Geralmente, para uma subcategoria plena $\mathcal{A}$, denotamos por $\mathcal{O}(\mathcal{A})$ o invólucro ortogonal de $\mathcal{A}$, i.e., a subcategoria plena de todos os objectos ortogonais a todos os $\mathcal{A}^{\perp}$-morfismos. Analogamente ao que acontece para o fecho para limites, quando a subcategoria $\mathcal{O}(\mathcal{A})$ é reflectiva, então ela é o invólucro reflectivo de $\mathcal{A}$. Por conseguinte, o invólucro ortogonal é também um bom candidato a ser o invólucro reflectivo. Na verdade, muitos dos invólucros reflectivos de subcategorias não reflectivas "do dia-a-dia" coincidem com o fecho para limites e, consequentemente, coincidem também com os respectivos invólucros ortogonais (visto que toda a subcategoria plena reflectiva é ortogonal). Contudo, o invólucro ortogonal pode ser simultaneamente reflectivo e diferente do fecho para limites. Em [56], J. Rosický apresenta um exemplo de uma categoria completa e cocompleta (na verdade, uma categoria monotopológica sobre $\mathcal{S}et$) que tem uma subcategoria plena fechada para limites que não é reflectiva e cujo invólucro ortogonal é reflectivo, logo o invólucro reflectivo. Por outro lado, é de salientar que para toda a categoria topológica com fibras pequenas sobre $\mathcal{S}et$, o invólucro reflectivo de uma subcategoria, caso exista, coincide com o invólucro ortogonal, mas não necessariamente com o fecho para limites (de acordo com 14.11 e 14.13). Portanto, o invólucro ortogonal pode constituir uma melhor abordagem do invólucro reflectivo do que o fecho para limites.

Assim, o conceito de ortogonalidade terá um lugar central nesta tese. A noção de ortogonalidade no sentido usado ao longo do presente estudo aparece já na literatura dos anos sessenta (cf. [52] e suas referências). Em 1972, D. Pumplün [52] observou que esta noção determina uma correspondência de Galois que induz um "operador de invólucro" que faz corresponder a cada subcategoria $\mathcal{A}$ de uma categoria $\mathcal{X}$ uma subcategoria - o invólucro ortogonal de $\mathcal{A}$ - que é uma boa aproximação do invólucro (mono)reflectivo de $\mathcal{A}$ em $\mathcal{X}$ e que tem grande parte das propriedades do invólucro (mono)reflectivo, mesmo se este não existir. Este conceito de ortogonalidade foi clarificado por P. J. Freyd e G. M. Kelly ([22]) que apresentaram uma definição de ortogonalidade entre um morfismo e um objecto de uma dada categoria (como em 1.1 a seguir). Desde então até agora o estudo desta noção, bem como o da sua relação com o conceito de reflectividade, tem-se desenvolvido. Nomeadamente, o chamado "Problema da Subcategoria Ortogonal", ou seja o problema de quando é que uma subcategoria ortogonal é reflectiva, tem merecido a atenção de vários matemáticos (cf. [22, 72, 79]). A nossa abordagem, em contraste com a de outros autores, parte de uma dada subcategoria plena ao invés de partir de uma dada classe de morfismos.

Para além do " Problema do Invólucro Reflectivo", investigamos também a relação deste com outros problemas tais como, por exemplo, a existência e caracterização do invólucro sólido de uma categoria concreta (Capítulo IV). Finalmente, a investigação feita sobre reflectividade e ortogonalidade conduzir-nos-á ao estudo de uma correspondente generalização sobre multi-reflectividade e multi-ortogonalidade (Capítulos V e VI).

Sumário

Apresentamos agora uma breve descrição do conteúdo desta dissertação.

O capítulo 0, "Preliminares", dá conta dos conceitos básicos existentes na literatura que são usados ao longo da dissertação. Outros conceitos conhecidos, mas menos estandardizados, são relembrados mais tarde à medida que forem sendo precisos.

No Capítulo I, "O invólucro ortogonal", começamos um estudo sistemático do invólucro reflectivo de uma subcategoria plena $\mathcal{A}$ de uma categoria $\mathcal{X}$ mediante o invólucro ortogonal $\mathcal{O}(\mathcal{A})$. Provamos, por exemplo, que numa categoria com colimites conexos as duas noções, invólucro ortogonal e invólucro reflectivo, coincidem se e só se a classe de morfismos $\mathcal{A}^{\perp}$ satisfaz a condição de conjunto solução (Teorema 2.10). Mostramos ainda que é possível estudar o invólucro ortogonal de $\mathcal{A}$ em qualquer subcategoria plena e reflectiva de $\mathcal{X}$ que contenha $\mathcal{A}$, em vez de na categoria $\mathcal{X}$ dada (Proposição 2.12). Este facto será usado várias vezes ao longo desta dissertação como um meio de obter descrições concretas do invólucro ortogonal. Por último, fazemos algumas considerações sobre classes firmes de morfismos, um conceito introduzido por G. Brümmer e E. Giuli ([12]). Uma classe $\mathcal{E}$ de morfismos diz-se firme sempre que existe alguma subcategoria plena e reflectiva tal que $\mathcal{E}$ é precisamente a classe de todos os morfismos que um reflector R transforma em isomorfismos, i.e.,
$\mathcal{E} = \{f \in Mor(\mathcal{X}) \mid Rf \text{ é um isomorfismo}\}$. O estudo do invólucro ortogonal desenvolvido nas anteriores secções deste capítulo é usado para caracterizar classes firmes de morfismos (Teorema 3.5).

O Segundo Capítulo é devotado ao estudo do operador de fecho ortogonal introduzido pela autora em [67] e será a principal ferramenta para a investigação dos invólucros ortogonais feita neste capítulo. O objectivo da definição do operador de fecho ortogonal é obter uma caracterização do invólucro ortogonal como sendo precisamente a subcategoria plena de todos os objectos fortemente fechados (Teorema 7.5) e uma caracterização de $\mathcal{A}^{\perp}$ como sendo uma classe de morfismos densos (Teorema 6.4). Aqui pressupomos que é dada uma classe "adequada" $\mathcal{M}$ de monomorfismos como um parâmetro adicional a $\mathcal{X}$ e $\mathcal{A}$. O operador de fecho ortogonal faz corresponder a cada $\mathcal{M}$-subobjecto $m : X \to Y$ a intersecção de todos os subobjectos m_g obtidos do seguinte modo: dado um morfismo arbitrário $g : X \to A$ com $A \in \mathcal{A}$, formamos a soma amalgamada $(g', \overline{m})$ do par (m, g) e denotamos por m_g a pré-imagem da $\mathcal{M}$-parte de $\overline{m}$ segundo g' (Definição 5.1). Para além do já mencionado papel deste operador de fecho, nomeadamente, na caracterização do invólucro ortogonal via fechamento (e na caracterização da classe $\mathcal{A}^{\perp}$ via densidade), ele é ainda usado para obter condições suficientes para que o invólucro ortogonal seja reflectivo (Theorem 8.1). Estes resultados são particularizados em algumas categorias básicas $\mathcal{X}$, e.g., a categoria dos espaços topológicos T_0 (Exemplos 8.8). Dada uma classe $\mathcal{M}$ de monomorfismos numa categoria $\mathcal{X}$, a maior subclasse de $\mathcal{M}$ estável para somas amalgamadas será representada por $PS(\mathcal{M})$. Para uma classe "adequada" $\mathcal{M}$, a subclasse $PS(\mathcal{M})$ desempenha um papel importante na caracterização do invólucro ortogonal e na determinação de condições suficientes para que ele seja reflectivo, por intermédio do operador de fecho ortogonal. Este facto motivou a Secção 9 que se dedica ao estudo da classe $PS(\mathcal{M})$ em categorias "do dia-a-dia". Em particular, caracterizamos $PS(\mathcal{M})$ para a classe $\mathcal{M}$ de todas as imersões em algumas subcategorias epireflectivas da categoria $\mathcal{T}op$ dos espaços topológicos e funções contínuas (Exemplos 9.5 e Proposição 9.9). Ao longo do capítulo, estabelecemos algumas relações entre o operador de fecho ortogonal e o já amplamente investigado operador de fecho regular (cf. [60], [18, 19, 20], [21] e suas referências), é o caso em 5.8, 7.8, 8.9, 9.7 e 9.8.

O Capítulo III é dedicado à generalização do conceito de espaço sóbrio para espaços α-sóbrios, onde α é um ordinal, apresentada pela autora em [68]. Recordamos que os espaços sóbrios são importantes na topologia "livre de pontos", porque eles são precisamente os espaços topológicos que são caracterizados pelo reticulado local dos conjuntos abertos. Fazemos uso dos principais resultados do Capítulo II para provar que o "reticulado" das subcategorias epireflectivas da categoria $\mathcal{T}op_0$ dos espaços topológicos T_0 e aplicações contínuas contém uma classe própria bem ordenada, formada pelas categorias dos espaços α-sóbrios, onde α é um ordinal maior do que 1. Cada categoria desta classe é o invólucro reflectivo em $\mathcal{T}op_0$ do ordinal α equipado com a topologia de Alexandrov.

No Capítulo IV, "Invólucros Sólidos", que é essencialmente baseado em [66], estudamos condições sob as quais uma dada categoria concreta tem um invólucro sólido. Há uma ligação estreita entre invólucros sólidos e invólucros reflectivos porque uma categoria concreta é solida se e só se é reflectiva no seu completamento de MacNeille, ou equivalentemente, se e só se é uma subcategoria plena e reflectiva de alguma categoria topológica (cf. [37] and [71]). O estudo de invólucros sólidos aqui desenvolvido continua a investigação encetada por J. Rosický em [56, 57], sendo que a nossa abordagem é, contudo, completamente diferente. J. Rosický descobriu uma categoria concreta sobre $\mathcal{S}et$ que não tem invólucro sólido, apesar de ter uma extensão sólida finalmente densa. Trata-se de uma categoria muito interessante, cujas particularidades se revelam úteis em vários lugares da primeira parte desta tese. No Teorema 15.2, que foi inspirado por resultados de J. Adámek, J. Rosický e V. Trnková ([5], [7], [57]) sobre o Princípio de Vopěnka, estabelecemos que a existência de invólucros sólidos para todas as categorias concretas sobre $\mathcal{S}et$ com uma subcategoria pequena finalmente densa é equivalente ao Princípio Fraco de Vopěnka. Isto melhora o seguinte resultado devido a J. Rosický [57]: Assumindo o axioma (M) da não existência de uma classe própria de cardinais mensuráveis, existe uma categoria concreta sobre $\mathcal{S}et$ com uma subcategoria pequena finalmente densa que não tem invólucro sólido ((M) implica a negação do Princípio Fraco de Vopěnka ([7])).

Há subcategorias importantes em vários campos da Matemática cujo comportamento se assemelha ao das reflectivas, embora não o sendo; é o caso, por exemplo, da subcategoria plena dos corpos na categoria dos anéis comutativos com identidade. Tais exemplos levaram J.Kaput [44] a introduzir a noção de subcategoras localmente reflectivas. Este foi o ponto de partida para várias generalizações (e.g. [11], [17] and [74]) sob diferentes nomes. Aqui estudamos uma dessas noções, a de multi-reflectidade, que foi introduzida por R. Börger e W. Tholen em [11] e tem sido investigada por vários autores (e.g., [17, 74, 10, 61, 8]). Em [17] Y. Diers apresenta um estudo sistemático das subcategorias multi-reflectivas e fornece um grande número de exemplos. Uma multi-reflexão de um objecto X da categoria $\mathcal{X}$ na subcategoria plena $\mathcal{A}$ é uma fonte de morfismos com domínio X e codomínio em $\mathcal{A}$ universal no seguinte sentido: cada morfismo com domínio

X e codomínio em $\mathcal{A}$ se factoriza através de um único membro da fonte e, além disso, a factorização é única. Uma subcategoria $\mathcal{A}$ diz-se multi-reflectiva se todo o objecto de $\mathcal{X}$ tem uma multi-reflexão em $\mathcal{A}$. (Analogamente, se pode generalizar a noção de colimite para multicolimite, a de categoria sólida para categoria multi-sólida, etc.) Dedicamos os Capítulos V e VI ao estudo do invólucro multi-reflectivo de uma dada subcategoria plena $\mathcal{A}$, i.e., a menor subcategoria plena multi-reflectiva que contém $\mathcal{A}$. Estudamos também a conexão entre multi-reflectividade e propriedades tais como multicocompletude e multi-solidez. Na Secção 17 generalizamos os resultados de J. Adámek, H. Herrlich e J. Reiterman [3] que estabelecem que a cocompletude "quase" implica completude à questão de quando é que a multicocompletude implica a existência de limites conexos (Proposição 17.3 e Teorema 17.6) e vice-versa (Proposição 17.4). O Teorema 18.4 generaliza um resultado sobre solidez de W. Tholen [71] estabelecendo que uma categoria concreta e co-bempotenciada $(\mathcal{A}, U)$ sobre uma categoria-base multicocompleta é multi-sólida se e só se $\mathcal{A}$ é multicocompleta e U é um multi-adjunto direito. Este resultado melhora o Teorema 6.3 de [74] e é o principal resultado de [69].

O conceito de um objecto ortogonal a um morfismo também se generaliza naturalmente ao de um objecto A multi-ortogonal a uma fonte com domínio X: tal generalização significa que cada morfismo de X para A se factoriza de forma única através de um único membro da fonte. Uma subcategoria plena $\mathcal{A}$ diz-se multi-ortogonal se consistir precisamente em todos os objectos multi-ortogonais a uma dada colecção de fontes. No Capítulo VII, "Multi-reflectividade e Multi-ortogonalidade", estudamos uma generalização dos resultados sobre reflectividade e ortogonalidade dos Capítulos I e II no cenário das multi-reflectividade e multi-ortogonalidade. A noção de multi-ortogonalidade, introduzida, tanto quanto sei, por R. Börger [10], tem um papel central neste capítulo. Relacionamos multi-ortogonalidade com ortogonalidade via quasicategorias de completamento para produtos grandes e usamos esta relação para obtermos o Teorema 20.2 e a Proposição 20.4 que são uma generalização de, respectivamente, 2.10 e 2.12.2. Finalmente, generalizamos a definição de operador de fecho ortogonal de uma maneira que se revela mais apropriada para o estudo dos invólucros multi-reflectivos. Isto permite-nos caracterizar as fontes multi-ortogonais a uma dada subcategoria plena em termos de densidade (Proposição 22.8) e dar condições suficientes para que o invólucro multi-ortogonal seja multi-reflectivo e, além disso, caracterizá-lo em termos de fechamento (Teorema 23.4).

Observamos que existe uma diferença do ponto de vista da Teoria de Conjuntos entre as noções "multi" consideradas por Y. Diers [17] e outros autores (veja-se, por exemplo, [6], [8] e [61]) e as que nós consideramos: As multi-reflexões e os multicolimites de Y. Diers são indexados somente por conjuntos, enquanto que nós permitimos que eles sejam indexados por classes próprias. Os nossos resultados mais importantes permanecem válidos se obrigarmos a classe indexante de cada noção "multi" a ser precisamente um conjunto. A ideia de considerar classes em vez de conjuntos não é nova. Por exemplo, em [74], W. Tholen estudou as duas noções de multi-reflectividade, para conjuntos e classes em paralelo com outras generalizações de reflectividade. Mas o que parece ser clarificado nos últimos dois capítulos desta dissertação é que, ao contrário do que poderia parecer numa primeira impressão, em geral, não perdemos propriedades quando consideramos classes em vez de conjuntos, mesmo se por vezes a técnica usada nas demonstrações para o caso em que consideramos apenas conjuntos não funciona para o caso em que admitimos classes próprias (compare-se, por exemplo, a demonstração do Teorema 6.3 em [74] com o nosso Teorema 18.4). De facto, mais importante do que obter um resultado mais geral ao aceitar classes nas noções "multi", é o facto de que estas definições "grandes" e as técnicas usadas nas demonstrações sublinham o comportamento "local" destas noções e o facto de que apenas o "tamanho local" desempenha realmente um papel.

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Preface

One of the most important and fruitful concepts of Category Theory is that of reflective subcategory: On the one hand, a full subcategory of a category $\mathcal{X}$ which is reflective shares a lot of convenient properties of $\mathcal{X}$ (existence and construction of limits, existence of colimits, etc.), on the other hand, a number of good sufficient conditions for reflectivity are known. For full subcategories $\mathcal{A}$ of a category $\mathcal{X}$ which fail to be reflective it is interesting to study the smallest reflective subcategory of $\mathcal{X}$ containing $\mathcal{A}$, called a reflective hull of $\mathcal{A}$. This is the aim of the present dissertation. We turn to the question of

(1) When does $\mathcal{A}$ have a reflective hull?

and

(2) How can his reflective hull, if it exists, be constructed?

For (2), a possible way is to form a limit hull of $\mathcal{A}$, i.e., the smallest full subcategory of $\mathcal{X}$ closed under limits and containing $\mathcal{A}$. If the latter category is reflective, it is a reflective hull of $\mathcal{A}$, but the question of when this happens turns out to be very difficult (see, for instance, [4], [22], [58], [73] and [77]). In my thesis I therefore decided for a different approach based on the concept of orthogonality. Recall that an object A is said to be orthogonal to a morphism $f : X \to Y$ provided that $hom(A, -)$ turns f to an isomorphism. For every full subcategory $\mathcal{A}$ of a category $\mathcal{X}$ we denote by $\mathcal{A}^{\perp}$ the class of all morphisms f in $\mathcal{X}$ orthogonal to all $\mathcal{A}$-objects. It is not difficult to see that if $\mathcal{A}$ is a reflective subcategory of $\mathcal{X}$, then $\mathcal{A}$ can be reconstructed from $\mathcal{A}^{\perp}$ as follows: $\mathcal{A}$ consists of precisely all objects orthogonal to all morphisms in $\mathcal{A}^{\perp}$. For a general full subcategory $\mathcal{A}$, we denote by $\mathcal{O}(\mathcal{A})$ the orthogonal hull of $\mathcal{A}$, i.e., the full subcategory of all objects orthogonal to all $\mathcal{A}^{\perp}$-morphisms. Analogously to what happens to the limit-closure, whenever $\mathcal{O}(\mathcal{A})$ is reflective, it is the reflective hull of $\mathcal{A}$. So, the orthogonal hull is also a good approach to the reflective hull. Indeed, most of the known reflective hulls of everyday non-reflective subcategories coincide with the limit-closure and, consequently, they also coincide with the orthogonal hull (since every reflective subcategory is orthogonal). However, the orthogonal hull may be simultaneously reflective and different from the limit-closure. In [56], J. Rosický presents an example of a complete and cocomplete category (indeed, a monotopological category over $\mathcal{S}et$) which has a non-reflective limit-closed subcategory whose orthogonal hull is reflective, hence the reflective hull. On the other hand, for a fibre-small topological category over $\mathcal{S}et$, the reflective hull of a subcategory, if it exists, must coincide with the orthogonal hull, but it may not coincide with the limit-closure (see 14.11 below). So, the orthogonal hull may be a better approach to the reflective hull than the limit-closure.

Thus, orthogonality will be a central concept in this thesis. The notion of orthogonality in the sense we use has already been used in literature in the sixties (see [52] and references there). Subsequently, D. Pumplün [52] observed that this notion determines a Galois correspondence which induces a “hull operator” which assigns to each subcategory $\mathcal{A}$ of a category $\mathcal{X}$ a good approach to the (mono)reflective hull of $\mathcal{A}$ in $\mathcal{X}$ and which has most of the properties of the (mono)reflective hull, even if the latter does not exist. This concept of orthogonality was clarified by P. J. Freyd and G. M. Kelly ([22]) who presented a definition of orthogonality between a morphism and an object of a given category (as in 1.1 below). From then on, this notion and its rôle in the understanding of the concept of reflectivity were further developed. Namely, the so-called “Orthogonal Subcategory Problem”, that is, the problem of when an orthogonal subcategory is reflective, has drawn the attention of several mathematicians (cf. [22, 72, 79]). Our approach, in contrast to the other authors, is that we start with a given full subcategory, while they start with a given class of morphisms.

We also investigate the relationship between the “Reflective Hull Problem” to other problems such as, for instance, the existence and characterization of a solid hull of a concrete category (Chapter IV). Finally, the investigation done on reflectivity and orthogonality is generalized to a corresponding study of multireflectivity and multiorthogonality (Chapters V and VI).

Summary

We now present a short synopsis of the contents of the dissertation.

Chapter 0, "Preliminaries", summarizes the basic concepts found in literature which are used throughout the dissertation. Other less standard well-known concepts are recalled later as they are needed.

In Chapter I, "The Orthogonal Hull", we begin a systematic study of the reflective hull of a full subcategory $\mathcal{A}$ of a category $\mathcal{X}$ by means of its orthogonal hull $\mathcal{O}(\mathcal{A})$. We prove, for example, that in a category with connected colimits the two notions, orthogonal hull and reflective hull, coincide if and only if the collection $\mathcal{A}^{\perp}$ satisfies the solution set condition (Theorem 2.10). We also show that it is possible to study the orthogonal hull of $\mathcal{A}$ in, instead of the given category $\mathcal{X}$, any reflective full subcategory of $\mathcal{X}$ containing $\mathcal{A}$ (Proposition 2.12). This fact will be used many times throughout the dissertation in order to obtain concrete descriptions of the orthogonal hull. We finally turn to firm classes of morphisms, a concept introduced by G. Brümmer and E. Giuli ([12]). A class $\mathcal{E}$ of morphisms is called firm provided that, for some full reflective subcategory, $\mathcal{E}$ is precisely the class of all morphisms which a reflector R turns to isomorphisms, i.e., $\mathcal{E} = \{f \in Mor(\mathcal{X}) \mid Rf \text{ is an isomorphism}\}$. We use conditions defined in the present chapter in the study of orthogonal hulls in order to characterize firm classes of morphisms (Theorem 3.5).

The Second Chapter is devoted to the study of the orthogonal closure operator, which was introduced by the author in [67] and will be a main tool for the investigation of orthogonal hulls along this chapter. The aim of the definition of this closure operator is to obtain a characterization of the orthogonal hull as the full subcategory of all strongly closed objects (Theorem 7.5) and a characterization of $\mathcal{A}^{\perp}$ as a class of dense morphisms (Theorem 6.4). Here we assume that a "convenient" class $\mathcal{M}$ of monomorphisms is given as an additional parameter to $\mathcal{X}$ and $\mathcal{A}$. The orthogonal closure operator assigns to every $\mathcal{M}$-subobject $m : X \to Y$ the intersection of all subobjects m_g obtained as follows: $g : X \to A$ is an arbitrary morphism with $A \in \mathcal{A}$, then one forms a pushout $(g', \overline{m})$ of

(m, g) and denotes by m_g the pre-image of the $\mathcal{M}$-part of $\overline{m}$ under g' (Definition 5.1). Besides the already mentioned rôle of this closure operator, namely, a characterization of the orthogonal hull via closedness (and characterization of $\mathcal{A}^{\perp}$ via density), we also use it for giving sufficient conditions for the orthogonal hull to be reflective, thus, to be the reflective hull (Theorem 8.1). These results are specialized in some basic categories $\mathcal{X}$, e.g., the category of topological T_0-spaces (Examples 8.8). Given a class $\mathcal{M}$ of monomorphisms in a category $\mathcal{X}$, the greatest pushout-stable subclass of $\mathcal{M}$ will be denoted by $PS(\mathcal{M})$. For a "convenient" class $\mathcal{M}$, the subclass $PS(\mathcal{M})$ plays an important rôle in the characterization of the orthogonal hull and the determination of sufficient conditions for it to be reflective, via the orthogonal closure operator. This fact motivated section 9 which is concerned with the study of the class $PS(\mathcal{M})$ in "everyday" categories. In particular, we characterize $PS(\mathcal{M})$ for the class $\mathcal{M}$ of all embeddings in some epireflective subcategories of the category $\mathcal{T}op$ of topological spaces and continuous maps (Examples 9.5 and Proposition 9.9). Throughout the chapter, some links between the orthogonal closure operator and the widely investigated regular closure operator (cf. [60], [18, 19, 20], [21] and references there) are established, see 5.8, 7.8, 8.9, 9.7 and 9.8.

Chapter III is devoted to the author's generalization of sober spaces to α-sober spaces, where α is an ordinal, see [68]. We recall that sober spaces are important in point-free topology because they are precisely the spaces characterized by the frame of open sets. We make use of the main results of Chapter II to prove that the "lattice" of epireflective subcategories of the category $\mathcal{T}op_0$ of topological T_0-spaces and continuous maps contains a well ordered proper class, formed by categories $\mathcal{S}ob(\alpha)$ of α-sober spaces, where $\alpha \geq 2$ is an ordinal. Each category $\mathcal{S}ob(\alpha)$ of this class is the reflective hull in $\mathcal{T}op_0$ of the ordinal α eqquiped with the Alexandrov topology.

In Chapter IV, "Solid hulls", which is essentially based on [66], we study conditions under which a given concrete category has a solid hull. There is a close link between solid hulls and reflective hulls because a concrete category is solid if and only if it is reflective in its MacNeille completion, or equivalently, if and only if it is a reflective full subcategory of some topological category (cf. [37] and [71]) . The study of solid hulls continues the research initiated by J. Rosický [56, 57], however our approach is completely different. J. Rosický has presented a concrete category over $\mathcal{S}et$ which does not have a solid hull, although it has a finally dense solid extension. This is a very interesting category whose particularities turn out to be useful in several places in the first part of this thesis. In Theorem 15.2, which was inspired by results of J. Adámek, J. Rosický and V. Trnková ([5], [7], [57]) on the Vopěnka's Principle, we establish that the existence of solid hulls for all concrete categories over $\mathcal{S}et$ with a small finally dense subcategory is equivalent to the large-cardinal Weak Vopenka's Principle. This improves the following result due to J. Rosický [57]: Under the axiom (M) of the non existence of a proper class of measurable cardinals, there is a concrete category over $\mathcal{S}et$ with a small finally dense subcategory which does not have a solid hull ((M) implies the negation of Weak Vopenka's Principle ([7])).

There are important subcategories which, although they are not reflective, have a behaviour which resembles that of reflective subcategories: for example, the full subcategory of fields in the category of commutative unitary rings. Such examples have led J. Kaput [44] to introduce the notion of locally reflective subcategories. This was the starting point to various generalizations (e.g. [11], [17] and [74]) using different names. Here we study one of these notions, multireflectivity, which was introduced by R. Börger and W. Tholen in [11] and has been investigated by several authors (e.g., [17, 74, 10, 61, 8]). In [17], Y. Diers presents a systematic study of multireflective subcategories and provides a great number of examples. By a multireflection of an object X of a category $\mathcal{X}$ in a subcategory $\mathcal{A}$ is meant a source of morphisms with domain X and codomain in $\mathcal{A}$ universal in the following sense: every morphism with the domain X and a codomain in $\mathcal{A}$ factors through a member of the source and the factorization is unique. A subcategory $\mathcal{A}$ is called multireflective provided that every object of $\mathcal{X}$ has a multireflection in $\mathcal{A}$. (Analogously, one generalizes colimits to multicolimits, solidness to multisolidness, etc.) We dedicate Chapters V and VI to the study of a multireflective hull of a given full subcategory $\mathcal{A}$, i.e., the smallest full multireflective subcategory containing $\mathcal{A}$. We also study the connection of multireflectivity to properties such as multicocompleteness and multisolidness. In Section 17 we generalize the result of J. Adámek, H. Herrlich and J. Reiterman [3] that cocompleteness "almost" implies completeness to the question of when multicocompleteness implies the existence of connected limits (Proposition 17.3 and Theorem 17.6) and the other way round (Proposition 17.4). Theorem 18.4 generalizes a result on solidness of W. Tholen [71] by establishing that a cowellpowered concrete category $(\mathcal{A}, U)$ over a multicocomplete base-category is multisolid if and only if $\mathcal{A}$ is multicocomplete and U is a right multi-adjoint. This result improves Theorem 5.3 of [74] and is the main result of [69].

The concept of an object orthogonal to a morphism also naturally generalizes to that of an object A multiorthogonal to a source with domain X: it means that every morphism from X to A uniquely factors through a unique member of the source. A full subcategory $\mathcal{A}$ is called multiorthogonal provided that it consists of precisely all objects multiorthogonal to a given collection of sources. In Chapter VII, "Multireflectivity and multiorthogonality", we investigate a generalization of the results on reflectivity and orthogonality of Chapters I and II to the setting of multireflectivity and multiorthogonality. The notion of multiorthogonality, introduced, as far as I know, by R. Börger [10], plays a central rôle in this chapter. We relate multiorthogonality with orthogonality via free large-product completion quasicategories and we use these relationships to prove Theorem 20.2 and Proposition 20.4 which are a generalization of, respectively, 2.10 and 2.12.2. Finally, we give a generalization of the definition of orthogonal closure operator which shows to be more appropriate for the characterization of multireflective hulls. This enables us to characterize the sources multiorthogonal to a given full subcategory in terms of density (Proposition 22.8) and to give sufficient conditions for the multiorthogonal hull to be multireflective and, moreover, to be characterized in terms of closedness (Theorem 23.4).

We remark that there is a set-theoretic difference between the "multi" notions considered by Y. Diers [17] and other authors (see, for instance, [6], [8] and [61]) and the ones we consider: The multireflections and the multicolimits of Y. Diers are indexed only by sets, while we allow them to be indexed by classes. Our main results remain valid if we oblige the indexed class of each "multi" notion to be just a set. The idea of considering classes instead of sets is not new. For instance, in [74], W. Tholen studied the two notions of multireflectivity, for sets and for classes (which he called "strongly localizing reflectivity" and "strongly locally reflectivity", respectively) in parallel with other generalizations of reflectivity. But what seems to be clarified in the two last chapters of this dissertation is that, in contrast to what may appear at a first look, in general, we do not lose properties when we consider classes instead of sets, even if sometimes the technique used in the proof for the small case does not work in the large one (compare, for instance, the proof of Theorem 6.3 in [74] with our Theorem 18.4). In fact, more important than to have a more general result by accepting classes in the “multi” notions is the fact that these “large” definitions and the techniques used in the proofs stress the “local ” behaviour of these notions and the fact that only the “local smalness” plays really a rôle.

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Chapter 0

Preliminaries

Throughout this work we shall normally use script capitals $\mathcal{A}$, $\mathcal{B}$, $\ldots \mathcal{X}$, $\ldots$ to denote categories. A category is understood in the sense of [2]; in particular, if X and Y are objects of a category $\mathcal{X}$, then the family of all morphisms from X to Y, denoted by $\mathcal{X}(X,Y)$, is assumed to form a set. Otherwise we speak of quasicategories, even if the collection of objects is a set.

All subcategories are understood to be full and replete, unless anything is specified to the contrary.

A convenient reference for background in Category Theory is [2], whose terminology we use, in general.

Next, we recall some notions which will be used along the text and which cannot be found in [2], at least, with the details we need.

For a subcategory $\mathcal{A}$ of $\mathcal{X}$, an $\mathcal{X}$-morphism $f : X \to Y$ is $\mathcal{A}$-*cancellable* if, for each pair of morphisms g, $h : Y \to A$ with codomain in $\mathcal{A}$, the equality $g \cdot f = h \cdot f$ implies that $g = h$.

A class $\mathcal{E}$ of $\mathcal{X}$-morphisms is *right-cancellable* if, for any morphisms f and g, $f \in \mathcal{E}$ whenever $f \cdot g \in \mathcal{E}$ and $g \in \mathcal{E}$.

Dually, we have *left-cancellable* classes.

A category $\mathcal{X}$ is *connected* if it is non-empty and, for each pair X, Y of $\mathcal{X}$-objects, there is a finite family of $\mathcal{X}$-objects $X = X_0$, X_1, $\ldots$, $X_n = Y$ such that $\mathcal{X}(X_{i-1}, X_i) \cup \mathcal{X}(X_i, X_{i-1}) \neq \emptyset$, for $i = 1, 2, \ldots, n$. Thus, each category is the coproduct of connected categories. They are said to be its *connected components.*

A *connected colimit* is the colimit of a connected diagram, that is, of a diagram $D : I \to \mathcal{X}$, where f is a connected category. Dually, we get the notion of *connected limit.*

Let $(X_i)_I$ be a small family of objects of a category $\mathcal{X}$. The family $(X_i)_I$ is said to be a *terminal set* in $\mathcal{X}$ provided that for each $\mathcal{X}$-object Y there is a unique $i \in I$ such that $\mathcal{X}(Y, X_I) \neq \emptyset$ and, furthermore, there is a unique morphism from Y to X_i.

The family $(X_i)_I$ is said to be a *weakly terminal set* provided that for each $\mathcal{X}$-object Y there is some $i \in I$ such that $\mathcal{X}(Y, X_i) \neq \emptyset$.

If I is singular then the only object of the family is said to be a *terminal object* or a *weakly terminal object*, respectively.

The dual notions are *initial set*, *weakly initial set*, *initial object* and *weakly initial object.*

Chapter I

The orthogonal hull

We want to study reflective hulls of subcategories $\mathcal{A}$ of a given category $\mathcal{X}$. (Recall that all subcategories are assumed to be full and replete.) That is, we want to discuss the existence and characterize the objects of the smallest reflective subcategory $\overline{\mathcal{A}}$ of $\mathcal{X}$ containing $\mathcal{A}$. One approach is to start with the closure of $\mathcal{A}$ under limits (which is certainly contained in $\overline{\mathcal{A}}$ and can possibly be equal to $\mathcal{A}$). We find it more useful to work with the orthogonal hull of $\mathcal{A}$, i.e., the subcategory $(\mathcal{A}^{\perp})_{\perp}$ of all objects orthogonal to any morphism to which all $\mathcal{A}$-objects are orthogonal. This may be a better approach than the limit-closure: For instance, for a fibre-small topological category over $\mathcal{S}et$, the reflective hull of a subcategory, if it exists, must coincide with the orthogonal hull, but it may not coincide with the limit-closure (see 14.11 and 14.13 below). It is clear that whenever the orthogonal hull is reflective, it is the desired reflective hull.

In the first section of this chapter we collect basic definitions and properties on orthogonality, illustrated with several examples.

In Section 2, we characterize the subcategories of categories with connected colimits for which the orthogonal hull is reflective, then the reflective hull. We also show that if $\mathcal{A}$ is a subcategory of $\mathcal{B}$ and $\mathcal{X}$, and $\mathcal{B}$ is a reflective subcategory of $\mathcal{X}$, then the orthogonal hull of $\mathcal{A}$ in $\mathcal{B}$ coincides with the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$. This enables us to obtain some important results, such as, for instance, that if $\mathcal{A}$ is a subcategory of an $(\mathcal{E}, \mathbb{M})$-category $\mathcal{X}$ with connected colimits and such that the $\mathcal{E}$-reflective hull of $\mathcal{A}$ in $\mathcal{X}$ is cowellpowered, then the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ is its reflective hull.

The notion of firm classes of morphisms was introduced in [13] and [12] as an approach to a categorical concept of completion. In the last section of this chapter we relate this notion with the one of orthogonality and characterize firm classes in categories with connected colimits.

Troughout the chapter, $\mathcal{X}$ denotes a given category.

1 Orthogonality

Definitions 1.1 An $\mathcal{X}$-morphism $f : X \to Y$ and an $\mathcal{X}$-object Z are said to be *orthogonal* to each other, written $f \perp A$, provided that for each morphism $g : X \to Z$, there is a unique $g' : Y \to Z$ such that the triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, "g'"] \\ Z & \end{tikzcd}$$

is commutative.

For every subcategory $\mathcal{A}$ of $\mathcal{X}$, we denote by $\mathcal{A}^{\perp}$ the class of all $\mathcal{X}$-morphisms which are orthogonal to $\mathcal{A}$, that is, all morphisms f such that $f \perp A$ for all $A \in Obj(\mathcal{A})$.

Given a class $\mathcal{E}$ of $\mathcal{X}$-morphisms, we denote by $\mathcal{E}_{\perp}$ the subcategory of all $\mathcal{X}$-objects which are orthogonal to $\mathcal{E}$, i.e., all $\mathcal{X}$-objects such that $f \perp X$ for all $f \in \mathcal{E}$.

A subcategory $\mathcal{B}$ of $\mathcal{X}$ is said to be *orthogonal* if $\mathcal{B} = \mathcal{E}_{\perp}$ for some class $\mathcal{E}$ of $\mathcal{X}$-morphisms.

We shall write $\mathcal{A}^{\perp_{\mathcal{X}}}$ and $\mathcal{E}_{\perp_{\mathcal{X}}}$ every time that the reference to the category $\mathcal{X}$ is convenient.

Let $\mathcal{A}$ be a subcategory of a category $\mathcal{X}$. A reflection from an $\mathcal{X}$-object X to $\mathcal{A}$ is a morphism $X \xrightarrow{r} A$ with codomain in $\mathcal{A}$ and such that

(o) each morphism with domain X and codomain in $\mathcal{A}$ is uniquely factorizable by r.

The condition (o) means that the morphism r is orthogonal to $\mathcal{A}$. So, an $\mathcal{A}$-reflection is an $\mathcal{A}^{\perp}$-morphism with codomain in $\mathcal{A}$.

The following Propositions 1.2 and 1.4 collect some properties on orthogonality which can be found in [22], [72] and [58].

Proposition 1.2

1. *The pair of maps $((-)_{\perp}, (-)^{\perp})$ establishes a (contravariant) Galois connection between the conglomerate of all classes of $\mathcal{X}$-morphisms and the conglomerate of all subcategories of $\mathcal{X}$, both ordered by inclusion, that is, if $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{X}$ and $\mathcal{E}$ and $\mathcal{F}$ are classes of $\mathcal{X}$-morphisms, then:*
   - $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow \mathcal{A}^{\perp} \supseteq \mathcal{B}^{\perp}$
   - $\mathcal{E} \subseteq \mathcal{F} \Longrightarrow \mathcal{E}_{\perp} \supseteq \mathcal{F}_{\perp}$
   - $\mathcal{A} \subseteq \mathcal{E}_{\perp} \Longleftrightarrow \mathcal{E} \subseteq \mathcal{A}^{\perp}$
2. *For every subcategory $\mathcal{A}$ and for the following assertions*
   (a) *$\mathcal{A}$ is reflective,*
   (b) *$\mathcal{A}$ is orthogonal,*
   (c) *$\mathcal{A}$ is closed under limits,*

   we have that $(a) \Rightarrow (b) \Rightarrow (c)$.
3. *For every family $(\mathcal{E}_i)_I$ of classes of morphisms,*
$$\bigcap_{i\in I}(\mathcal{E}_i)_{\perp} = (\bigcup_{i\in I} \mathcal{E}_i)_{\perp}. \qquad \square$$

From 1.2.1, it follows that, given a subcategory $\mathcal{A}$ of $\mathcal{X}$, the subcategory $(\mathcal{A}^{\perp})_{\perp}$ is the smallest orthogonal subcategory of $\mathcal{X}$ containing $\mathcal{A}$.

Definition 1.3 For every subcategory $\mathcal{A}$ of $\mathcal{X}$, the subcategory $(\mathcal{A}^{\perp})_{\perp}$ is called the *orthogonal hull* of $\mathcal{A}$ in $\mathcal{X}$ and it will be denoted by $\mathcal{O}(\mathcal{A})$.

From the definition of $\mathcal{A}^{\perp}$ it is clear that all morphisms in $\mathcal{A}^{\perp}$ are $\mathcal{A}$-cancellable. The following proposition lists some other useful properties of $\mathcal{A}^{\perp}$.

Proposition 1.4 *For every subcategory $\mathcal{A}$ of $\mathcal{X}$ we have that:*

1. *$\mathcal{A}^{\perp}$ contains all isomorphisms and is closed under composition.*

2. *If $f \cdot g \in \mathcal{A}^{\perp}$ and g is $\mathcal{A}$-cancellable, then $f \in \mathcal{A}^{\perp}$. Thus, $\mathcal{A}^{\perp}$ is right-cancellable.*

3. *$\mathcal{A}^{\perp}$ is left-cancellable.*

4. *$\mathcal{A}^{\perp}$ is closed under pushouts, i.e., if the diagram*

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g"] & \bullet \arrow[d, "\overline{g}"] \\ \bullet \arrow[r, "\overline{f}"] & \bullet \end{tikzcd}$$

   is a pushout and $f \in \mathcal{A}^{\perp}$, then $\overline{f} \in \mathcal{A}^{\perp}$.

5. *$\mathcal{A}^{\perp}$ is closed under multiple pushouts, i.e., if the diagrams*

$$\begin{tikzcd} X \arrow[r, "e_i"] \arrow[d, "e"] & X_i \arrow[dl, "d_i"] \\ E & \end{tikzcd} \quad , i \in I,$$

   represent a multiple pushout and $e_i \in \mathcal{A}^{\perp}$ for all $i \in I$, then $e \in \mathcal{A}^{\perp}$. □

Examples 1.5 In the following examples, for each subcategory $\mathcal{A}$ of a category $\mathcal{X}$, the corresponding class $\mathcal{A}^{\perp}$ and subcategory $\mathcal{O}(\mathcal{A})$ are described.

1. (cf. [13] and [34]) Let $\mathcal{T}op_0$ be the category of topological T_0-spaces and continuous maps.

   - A morphism $f : X \to Y$ in $\mathcal{T}op_0$ is called *b-dense* if each $y \in Y$ satisfies the condition

     (b) for each open set H in Y, if $y \in H$, then $\overline{\{y\}} \cap H \cap f(X) \neq \emptyset$,

     or, equivalentely, the condition

     (b') for all open sets H and H' in Y such that $H \cap f(X) = H' \cap f(X)X$, we have that $y \in H$ iff $y \in H'$.

- A topological space X is called *sober* if every non-empty irreducible closed set of X (i.e., a closed set which cannot be written as a union of two proper closed subsets) is the closure of a unique point.

Let $\mathcal{A}$ be the subcategory of $\mathcal{T}op_0$ whose objects are the Sierpiński spaces. Then the orthogonal hull of $\mathcal{A}$ is the subcategory $\mathcal{S}ob$ of all sober spaces, since $\mathcal{S}ob$ is simultaneously the limit-closure and the reflective hull of $\mathcal{A}$ in $\mathcal{T}op_0$ (see [63] and [50]). On the other hand, $\mathcal{A}^{\perp}$ is the class of all b-dense embeddings. Indeed, $\mathcal{T}op_0$ is the epireflective hull of the Sierpiński space $\mathbf{S} = (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$ in $\mathcal{T}op$ and $\mathcal{T}op$ is an $(Epi, InitialMonoSource)$-category; hence, it follows from 2.17.3 below that every $\mathcal{T}op_0$-morphism orthogonal to $\mathbf{S}$ must be an embedding and an epimorphism in $\mathcal{T}op_0$. Then, since the epimorphisms in $\mathcal{T}op_0$ are just b-dense morphisms, as proved by S. Baron [9], every such morphism is a b-dense embedding. Conversely, let $X \xrightarrow{m} Y$ be a b-dense embedding in $\mathcal{T}op_0$ and let $X \xrightarrow{f} \mathbf{S}$ be a continuous map. Let H be an open set of Y such that $X \cap H = f^{-1}(\{1\})$. Then the continuous map $\overline{f} : Y \to \mathbf{S}$, defined by

$$\overline{f}(y) = \begin{cases} 1 & \text{if} \quad y \in H \\ 0 & \text{if} \quad y \notin H \end{cases}$$

is such that $\overline{f} \cdot m = f$. Furthermore, since each $y \in Y$ satisfies condition (b'), it immediately follows that $\overline{f}$ is unique.

The following examples 2. and 3. follow from 3.8 of [12].

2. Let $\mathcal{T}ych$ be the category of Tychonoff spaces and continuous maps and let $\mathcal{A}$ be the subcategory of $\mathcal{T}ych$ whose objects are all spaces homeomorphic to the closed unit interval $\mathbf{I} = [0, 1]$ with the euclidean topology. An embedding $X \xrightarrow{m} Y$ is said to be a C^*-embedding provided that every continuous function $X \xrightarrow{f} \mathbf{I}$ can be extended to a continuous function $Y \xrightarrow{\overline{f}} \mathbf{I}$. We recall that two subsets Z and W of a topological space X are said to be *completely separated* provided that there is some continuous map $g : X \to \mathbf{I}$ such that $g(Z) = 0$ and $g(W) = 1$. We recall further that an embedding $X \xrightarrow{m} Y$ is a C^*-embedding iff every pair of completely separated sets in X is completely separated in Y (see [78]).

In this case, $\mathcal{A}^{\perp}$ is the class of all dense C^*-embeddings and $\mathcal{O}(\mathcal{A})$ is the subcategory $\mathcal{HC}omp$ of compact Hausdorff spaces.

3. For $\mathcal{X}$=$\mathcal{T}ych$ and $\mathcal{A}$ the subcategory of all spaces homeomorphic to the real line $\mathbf{R}$, we have that $\mathcal{A}^{\perp}$ consists of all dense C-embeddings (i.e., dense embeddings which can extend all continuous functions with codomain in $\mathbf{R}$) and $\mathcal{O}(\mathcal{A})$ is the subcategory $\mathcal{RC}omp$ of all real compact spaces.

4. Let $\mathcal{HU}nif$ be the category of Hausdorff uniform spaces and uniformly continuous maps and let $\mathcal{A}$ be the subcategory of Cauchy-complete Hausdorff uniform spaces. Hence, $\mathcal{A}$ is orthogonal, since it is reflective. On the other hand, since $\mathcal{HU}nif$ is the epireflective hull of $\mathcal{A}$, it follows from 2.17.3 below that all morphisms in $\mathcal{A}^{\perp}$ must simultaneously be epimorphisms and embeddings. Consequently, since in $\mathcal{HU}nif$ the epimorphisms are just the dense uniformly continuous functions (see [51] or [34]) and every embedding extends each uniformly continuous function with codomain in a Cauchy-complete Hausdorff uniform space (see, for instance, [78]), it follows that the class $\mathcal{A}^{\perp}$ is just the class of all dense embeddings.

5. Let $\mathcal{X}$ be the category $\mathcal{M}et$ of all metric spaces and non-expansive maps and let $\mathcal{A}$ be the subcategory whose objects are the complete metric spaces. Then, by using an argument analogous to the one used in example 4., we conclude that $\mathcal{A}^{\perp}$ consists of all dense embeddings and $\mathcal{O}(\mathcal{A})$ is just $\mathcal{A}$ (see also [34]).

6. Similarly, if we consider the category $\mathcal{N}orm$ of normed spaces and non-expansive maps and its subcategory $\mathcal{B}an$ of Banach spaces, then we have that $\mathcal{B}an^{\perp}$ is the class of all dense embeddings and $\mathcal{O}(\mathcal{B}an)$=$\mathcal{B}an$.

7. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}op$ and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ which contains the indiscrete two-point space $\{0,1\}$. Furthermore, let $\mathcal{A}$ be initially dense in $\mathcal{X}$, that is, for each space X in $\mathcal{X}$ there is an $\mathcal{X}$-source $(X \xrightarrow{f_i} A_i)_I$ with codomain in $\mathcal{A}$ which is initial, i.e., for every space Y in $\mathcal{X}$ a map $Y \xrightarrow{g} X$ is continuous iff all $f_i \cdot g$ are continuous. Then $\mathcal{A}^{\perp}$ consists of isomorphisms only and, consequently, $\mathcal{O}(\mathcal{A}) = \mathcal{X}$. In fact, let the morphism $X \xrightarrow{f} Y$ belong to $\mathcal{A}^{\perp}$. Then f is a bijection: an injection because every map from X to $\{0,1\}$ must be factorized through f, and a surjection because every map from X to $\{0,1\}$ can be factorized through f in an

unique way. On the other hand, the fact that $\mathcal{A}$ is initially dense in $\mathcal{X}$ implies that the morphism $X \xrightarrow{f} Y$ is initial. Indeed, let Z be an $\mathcal{X}$-object and let $Z \xrightarrow{g} X$ be a map between the underlying sets of Z and X such that $f \cdot g$ is a continuous map. Let $(X \xrightarrow{f_i} A_i)_I$ be an initial source with codomain in $\mathcal{A}$ and, for each $i \in I$, let $Y \xrightarrow{\overline{f}_i} A_i$ be the morphism such that $\overline{f}_i \cdot f = f_i$. Then $f_i \cdot g = \overline{f}_i \cdot f \cdot g$ is continuous for all $i \in I$ and, thus, g is continuous. Therefore, since every initial bijection is an isomorphism, it follows that every morphism in $\mathcal{A}^{\perp}$ is an isomorphism.

Two examples of categories $\mathcal{X}$ and $\mathcal{A}$ satisfying these conditions are the following:

(a) $\mathcal{X}$ is the category $FinGen$ of finitely generated topological spaces and continuous maps and $\mathcal{A}$ is the subcategory of finite topological spaces.

(b) $\mathcal{X}$ is the category $\mathcal{C}omp\mathcal{G}en$ of compactly generated topological spaces and continuous maps and $\mathcal{A}$ is the subcategory of all compact spaces.

8. Let $\mathcal{T}f\mathcal{A}b$ be the category of torsion-free abelian groups and group homomorphisms.

A morphism $f : A \to B$ in $\mathcal{T}f\mathcal{A}b$ is said to be *T-dense* if the factor group $B/f(A)$ is a torsion group.

Let $\mathcal{A}$ be the subcategory of $\mathcal{T}f\mathcal{A}b$ whose objects are all divisible torsion-free abelian groups. Then $\mathcal{A}^{\perp}$ consists of all T-dense monomorphisms and $\mathcal{O}(\mathcal{A})$=$\mathcal{A}$ (see [12]).

2 Orthogonal hulls and reflective hulls

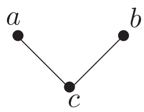
Definition 2.1 Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. A reflective subcategory $\mathcal{B}$ of $\mathcal{X}$ is said to be a *reflective hull* of $\mathcal{A}$ in $\mathcal{X}$ provided that it contains $\mathcal{A}$ and is contained in any other reflective subcategory which contains $\mathcal{A}$.

If $\mathcal{E}$ is a class of $\mathcal{X}$-morphisms, $\mathcal{B}$ is said to be a *$\mathcal{E}$-reflective hull* of $\mathcal{A}$ if it is $\mathcal{E}$-reflective and is contained in every $\mathcal{E}$-reflective subcategory containing $\mathcal{A}$.

A convenient reference on reflective subcategories, reflective hulls and the related problem on the reflectivity of the intersection of reflective subcategories, is the survey paper [73] and references there.

By 1.2, the orthogonal hull is a good candidate to be the reflective hull. Indeed, in all examples given in 1.5 we have that $\mathcal{O}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$. However, this does not always occur. Next, we show a simple example of a subcategory which has a reflective hull different from its orthogonal hull.

Example 2.2 Let $\mathcal{X}$ be the poset



considered as a category. If $\mathcal{A}$ is the subcategory of $\mathcal{X}$ having only the object a, then $\mathcal{A}$ is orthogonal and $\mathcal{X}$ is the reflective hull of $\mathcal{A}$, but the orthogonal hull of $\mathcal{A}$ is $\mathcal{A}$ itself.

It can be argued that the category $\mathcal{X}$ in Example 2.2 is not a “nice” category; for instance, it is neither cocomplete nor complete. However, even very “reasonable” categories may have subcategories such that neither the limit-closure nor the orthogonal hull are reflective and, moreover, which do not have a reflective hull. Actually, V. Trnková, J. Adámek and J. Rosický proved in [77] that the category $\mathcal{T}op$ of topological spaces and continuous maps has a subcategory which does not have a reflective hull, although it is an orthogonal subcategory. Another important example is the following:

Example 2.3 ([4]) Let $Bi\mathcal{T}op$ be the category of bitopological spaces and bicontinuous maps: objects are triples (X, τ, υ) where X is a set and τ and υ are topologies on X; a morphism $f : (X, \tau, \upsilon) \longrightarrow (X', \tau', \upsilon')$ is a map from X to X' which is continuous with respect to the first topologies and with respect to the second topologies. The subcategory $Bi\mathcal{C}om$ of all bitopological spaces with both topologies compact Hausdorff is the intersection of two reflective subcategories: the subcategory of all bitopological spaces whose first topology is compact Hausdorff, and the subcategory of all bitopological spaces whose second topology is compact Hausdorff. However, $Bi\mathcal{C}om$ is not reflective, and, consequently, it does not have a reflective hull.

Remark 2.4 For several categories, the existence of the reflective hull of $\mathcal{A}$ forces that hull to be precisely the orthogonal hull of $\mathcal{A}$:

Let $\mathcal{X}$ be a category such that, for each morphism f, the subcategory $\{f\}_\perp$ is reflective. This holds, for example, in each locally presentable category and also in the category $\mathcal{T}op$ of topological spaces (see [22] and [77]). Then, by 1.2, it follows that the reflective hull of every subcategory $\mathcal{A}$ of $\mathcal{X}$, if it exists, coincides with the orthogonal hull of $\mathcal{A}$. In fact, concerning $\mathcal{T}op$, this is a particular case of a more general result (see 14.11 below).

Several results on conditions under which an orthogonal subcategory of the form $\{f\}_\perp$ is reflective are given in [22] and [72] (see also [58]).

From 1.2.2, it is clear that, whenever the limit-closure is reflective, then it coincides with the orthogonal hull. However, $\mathcal{O}(\mathcal{A})$ may be reflective without coinciding with the limit-closure of $\mathcal{A}$. It was J. Rosický who found out an example of a category $\mathcal{X}$ with very good properties and, yet, having a limit-closed subcategory $\mathcal{A}$ such that $\mathcal{O}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ but $\mathcal{A} \neq \mathcal{O}(\mathcal{A})$. It is described next.

Example 2.5 ([56]) Let $\mathcal{X}$ be the category defined as follows

- objects: pairs (X, x) where X is a set and x is either the empty map

$$\emptyset \xrightarrow{x} X$$

  or a function from the class of all ordinals to X

$$Ord \xrightarrow{x} X$$

  such that, if $x(i) = x(k)$ for some pair (i, k) with $i < k$, then, $j \geq i \Rightarrow x(j) = x(i)$;

- morphisms: $f : (X, x) \to (Y, y)$ where $f : X \to Y$ is a map for which, whenever x is a map from Ord to X, so is y and

$$f(x(i)) = y(i),\ i \in Ord.$$

The category $\mathcal{X}$ is

- fibre-small, i.e. for every set X, the pairs (X, x) which are objects of $\mathcal{X}$ form a set, not a proper class;

and

- monotopological, i.e. if $(X \xrightarrow{f_i} X_i)_I$ is a monosource in $\mathcal{S}et$ and (X_i, x_i) is an $\mathcal{X}$-object, $i \in I$, then there is a unique x such that $((X,x) \xrightarrow{f_i} (X_i, x_i))_I$ is an initial source.

The fact that $\mathcal{X}$ is fibre-small and monotopological over $\mathcal{S}et$ implies that it is complete, cocomplete, well-powered and an $(Epi, InitialMonoSource)$-category (see, for instance, [2]).

Let $\mathcal{A}$ be the subcategory of $\mathcal{X}$ consisting of all $\mathcal{X}$-objects (X,x) such that x is not the empty map.

Let us define, for each $\mathcal{X}$-object (X,x),

$$\|x\| = \begin{cases} 0 & \text{if } x \text{ is the empty map} \\ min\{k \in Ord \,|\, j \geq k \Rightarrow x(j) = x(k)\} & \text{otherwise.} \end{cases}$$

It is obvious that, for each $\mathcal{X}$-morphism $f : (X,x) \to (Y,y)$ with $\|x\| \neq 0$, the inequality $\|x\| \geq \|y\|$ holds.

Now, let $f : (X,x) \to (Y,y)$ be a morphism in $\mathcal{A}^{\perp}$. Then, it is easily seen that, on the one hand, from the $\mathcal{A}$-cancellability of f, the restriction and corestriction of $f : X \to Y$ to $X \backslash Im(x)$ and $Y \backslash Im(y)$ must be a bijection and, on the other hand, since every morphism with domain (X,x) and codomain in $\mathcal{A}$ is factorizable trough f, one must have $\|x\| \leq \|y\|$. Consequently, the class $\mathcal{A}^{\perp}$ is just the class of all $\mathcal{X}$-isomorphisms.

Therefore $\mathcal{O}(\mathcal{A})$ is the whole category $\mathcal{X}$, and this is the reflective hull of $\mathcal{A}$. But it is different from the limit-closure of $\mathcal{A}$, which is $\mathcal{A}$ itself, since this subcategory is closed under limits.

Given an $\mathcal{X}$-object X and a subcategory $\mathcal{A}$ of $\mathcal{X}$, let

$$X/\mathcal{A}^{\perp}$$

denote the category whose objects are all $\mathcal{X}$-morphisms orthogonal to $\mathcal{A}$ and with domain X and whose morphisms are all

$$s : (X \xrightarrow{f} Y) \longrightarrow (X \xrightarrow{f'} Y')$$

such that $s : Y \to Y'$ is an $\mathcal{X}$-morphism and $s \cdot f = f'$. Since $[\mathcal{O}(\mathcal{A})]^\perp = \mathcal{A}^\perp$ (see 1.2.1), it is immediate that if $\mathcal{O}(\mathcal{A})$ is reflective, then for each $\mathcal{X}$-object X the reflection from X to $\mathcal{O}(\mathcal{A})$ is a terminal object of the category $X/\mathcal{A}^\perp$.

So, it is natural to inquire into the converse. As a matter of fact, the existence of a terminal object in the category $X/\mathcal{A}^\perp$ for each $X \in Obj(\mathcal{X})$ does not guarantee the reflectivity of $\mathcal{O}(\mathcal{A})$. By way of illustration, consider $\mathcal{X}$ and $\mathcal{A}$ as in Example 2.2.

However, we are going to show that, under suitable conditions the desired converse happens. For that, instead of considering the orthogonal hull of a given category, we first deal with an orthogonal subcategory $\mathcal{E}_\perp$ for a given class $\mathcal{E}$ of morphisms.

For any class $\mathcal{E}$ of $\mathcal{X}$-morphisms, the notation $X/\mathcal{E}$ has the same meaning as above, that is, it denotes the subcategory of the comma category $X \downarrow \mathcal{X}$ whose objects are the morphisms belonging to $\mathcal{E}$. We find conditions for a class $\mathcal{E}$ of $\mathcal{X}$-morphisms such that the existence of a weakly terminal object of $X/\mathcal{E}$ for each $\mathcal{X}$-object X implies the reflectivity of $\mathcal{E}_\perp$. Thus, we get an answer to the so-called Orthogonal Subcategory Problem ([22, 72, 79]).

Definitions 2.6 For a class $\mathcal{E}$ of morphisms in a category $\mathcal{X}$, we consider the following conditions:

Coequalizer condition. If $c : C \to D$ is a coequalizer of a family of morphisms $(f_i : B \to C)_I$ such that, there exist $e \in \mathcal{E}$ and h with $f_i \cdot e = h$ for all $i \in I$, then $c \in \mathcal{E}$.

Fill-in condition. Given morphisms $f \in \mathcal{E}$ and g with the same domain, there are morphims f' and g' such that $f' \in \mathcal{E}$ and $g' \cdot f = f' \cdot g$.

Pseudoreflectivity condition. For each $X \in \mathcal{X}$, the category $X/\mathcal{E}$ has a weakly terminal object. (This weakly terminal object is called an *$\mathcal{E}$-pseudoreflection* of X.)

Lemma 2.7 *For any subcategory $\mathcal{A}$ of a category $\mathcal{X}$, we have that:*

1. *$\mathcal{A}^\perp$ satisfies the coequalizer condition;*

2. *$\mathcal{A}^\perp$ satisfies the fill-in condition whenever $\mathcal{X}$ has pushouts;*

3. *if $\mathcal{A}$ is reflective in $\mathcal{X}$, then $\mathcal{A}^\perp$ satisfies the fill-in and the pseudoreflectivity conditions.*

Proof.

1. If $e \in \mathcal{A}^{\perp}$, $(f_i : B \to C)_I$ is a family of morphisms such that $f_i \cdot e = h$ for every $i \in I$, and $c : C \to D$ is the coequalizer of $(f_i : B \to C)_I$, let $g : C \to A$ be a morphism with codomain in $\mathcal{A}$. Then $g \cdot f_i \cdot e = g \cdot f_j \cdot e$ and, since $e \in \mathcal{A}^{\perp}$, this implies that $g \cdot f_i = g \cdot f_j$ for every $i, j \in I$. Thus, there is a unique morphism g' such that $g' \cdot c = g$.

2. If $\mathcal{X}$ has pushouts, it follows from 1.4.4 that $\mathcal{A}^{\perp}$ fulfils the fill-in condition.

3. If $\mathcal{A}$ is reflective in $\mathcal{X}$, given morphisms $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$, with $f \in \mathcal{A}^{\perp}$, let $Z \xrightarrow{r_Z} RZ$ be the $\mathcal{A}$-reflection of Z; then, since $RZ \in \mathcal{A}$, there is a unique morphism $\overline{g} : Y \to RZ$ such that $\overline{g} \cdot f = r_Z \cdot g$; furthermore, $r_Z \in \mathcal{A}^{\perp}$. It is immediate that $\mathcal{A}^{\perp}$ fulfils the pseudoreflectivity condition. $\square$

Remark 2.8 The above three conditions are independent, in the sense that none of them is implied by the others. Indeed:

Coequalizer and fill-in conditions $\not\Rightarrow$ pseudoreflectivity condition: If $\mathcal{X}$ is a cocomplete category and $\mathcal{A}$ is a subcategory of $\mathcal{X}$ then $\mathcal{E} = \mathcal{A}^{\perp}$ obviously satisfies the coequalizer and the fill-in conditions. But, if the orthogonal hull of $\mathcal{A}$ is not reflective, then $\mathcal{A}^{\perp}$ does not fulfil the pseudoreflectivity condition, as we can conclude from 2.10 below. And this is the case if $\mathcal{X}$ and $\mathcal{A}$ are, for instance, the categories of example 2.3.

Coequalizer and pseudoreflectivity conditions $\not\Rightarrow$ fill-in condition: Let $\mathcal{X}$ and $\mathcal{A}$ be as in example 2.2. Then, $\mathcal{A}^{\perp}$ trivially satisfies the coequalizer codition since all (multiple) coequalizers in $\mathcal{X}$ are isomorphisms. On the other hand, the categories $a/\mathcal{A}^{\perp}$ and $b/\mathcal{A}^{\perp}$ consists just of 1_a and 1_b, respectively, so they have a terminal object. The category $c/\mathcal{A}^{\perp}$ consists of the identity 1_c and the morphism $c \to a$; the latter morphism is clearly a terminal object of $c/\mathcal{A}^{\perp}$. Consequently, $\mathcal{A}^{\perp}$ also satisfies the pseudoreflectivity condition. But it does not fulfil the fill-in condition, since the diagram

$$\begin{tikzcd} c \arrow[r] \arrow[d] & a \\ b & \end{tikzcd}$$

cannot be “completed”.

Fill-in and pseudoreflectivity conditions $\not\Rightarrow$ coequalizer condition: Let $\mathcal{X} = \mathcal{S}et$ and let $\mathcal{E}$ be the class of all injective maps. Clearly $\mathcal{E}$ does not satisfy the coequalizer condition but it satisfies the fill-in condition and, for each set X, the identity map is an $\mathcal{E}$-pseudoreflection of X.

Proposition 2.9 *Let $\mathcal{X}$ be a category with multiple coequalizers. If a class $\mathcal{E}$ of $\mathcal{X}$-morphisms is closed under composition and satisfies the coequalizer, the fill-in and the pseudoreflectivity conditions, then $\mathcal{E}_\perp$ is $\mathcal{E}$-reflective in $\mathcal{X}$.*

Proof. Let $X \in \mathcal{X}$ and $d : X \to Y$ be an $\mathcal{E}$-pseudoreflection of X. If $c : Y \to C$ is a coequalizer of the family $(h_i)_I$ of all morphisms $h_i : Y \to Y$ which satisfy the equality $h_i \cdot d = d$, then, from the coequalizer condition and the fact that $\mathcal{E}$ is closed under composition, the morphism $e = c \cdot d$ belongs to $\mathcal{E}$. We show now that $e : X \to C$ is a terminal object of $X/\mathcal{E}$. If f is a $\mathcal{E}$-morphism with domain X, then there is some morphism t such that $t \cdot f = e$. If a morphism t' also fulfils $t' \cdot f = e$, let $g = coeq(t, t')$. From the coequalizer condition and the fact that $\mathcal{E}$ is closed under composition, $g \cdot e \in \mathcal{E}$. Then there exists a morphism n such that $n \cdot g \cdot e = d$, i.e.,

$$n \cdot g \cdot c \cdot d = d.$$

Hence, $n \cdot g \cdot c = h_i$ for some i and so, the equality $c \cdot n \cdot g \cdot c = c$ holds, which implies that

$$c \cdot n \cdot g = 1.$$

Thus, g is an isomorphism and so $t = t'$.

Now, in order to conclude that $e : X \to C$ is a universal morphism from X to $\mathcal{E}_\perp$, it suffices to show that C belongs to $\mathcal{E}_\perp$. Given morphisms f and g with the same domain such that $f \in \mathcal{E}$ and the codomain of g is C, the fill-in condition assures the existence of morphisms $f' \in \mathcal{E}$ and g' such that $g' \cdot f = f' \cdot g$. Since $\mathcal{E}$ is closed under composition, $f' \cdot e$ belongs to $\mathcal{E}$. As $e : X \to C$ is terminal, there is a morphism t such that $t \cdot f' \cdot e = e$ and, again because e is terminal in $X/\mathcal{E}$, the morphism $t \cdot f'$ is the identity 1_C. Then, for $r = t \cdot g'$, we have that

$$r \cdot f = t \cdot g' \cdot f = t \cdot f' \cdot g = g.$$

To show the uniqueness, let $r' \cdot f = r \cdot f$ and let q be a coequalizer of (r, r'). Then, from the coequalizer condition, $q \in \mathcal{E}$, and, since $\mathcal{E}$ is closed under composition, $q \cdot e$ belongs to $\mathcal{E}$ and so, there is some morphism p such that $p \cdot q \cdot e = e$. Thus $p \cdot q = 1$ and, consequently, $r = r'$. $\square$

Now, we can give an answer to the question, raised in the introduction, concerning the reflectivity of the orthogonal hull of a subcategory.

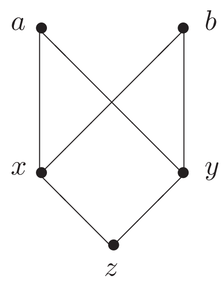
A class $\mathcal{E}$ of $\mathcal{X}$-morphisms is said *to satisfy the solution set condition* whenever, for each $X \in Obj(\mathcal{X})$, the category $X/\mathcal{E}$ has a weakly terminal set.

Theorem 2.10 *If $\mathcal{X}$ has connected colimits then the orthogonal hull of a subcategory $\mathcal{A}$ is the reflective hull of $\mathcal{A}$ in $\mathcal{X}$ if and only if $\mathcal{A}^{\perp}$ satisfies the solution set condition.*

Proof. If $\mathcal{O}(\mathcal{A})$ is reflective, then it is clear that $\mathcal{A}^{\perp}$ satisfies the above condition. Conversely, for $X \in \mathcal{X}$, assume that $(f_i : X \to A_i)_I$ is a weakly terminal set of the category $X/\mathcal{A}^{\perp}$. Then, the multiple pushout $f : X \to C$ of $(f_i : X \to A_i)_I$ is an $\mathcal{A}^{\perp}$-pseudoreflection of X. It is clear that $\mathcal{A}^{\perp}$ is closed under composition and, by 2.7, it fulfils the coequalizer and the fill-in conditions. Therefore, by 2.9, $\mathcal{O}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{X}$. $\square$

We may have subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$ such that $\mathcal{A}$ is contained in $\mathcal{B}$ but the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ is different from the orthogonal hull of $\mathcal{A}$ in $\mathcal{B}$, even when $\mathcal{B}$ is orthogonal in $\mathcal{X}$, as the following example shows.

Example 2.11 Let $\mathcal{X}$ be the poset



and let $\mathcal{A}$ and $\mathcal{B}$ be the full subcategories whose set of objects is $\{a\}$ and $\{a, b\}$, respectively. It is easy to check that $\mathcal{B}$ is orthogonal in $\mathcal{X}$. However, the orthogonal hull of $\mathcal{A}$

in $\mathcal{X}$ is different from the orthogonal hull of $\mathcal{A}$ in $\mathcal{B}$, the former being $\mathcal{A}$, and the latter being $\mathcal{B}$.

In contrast, we prove now that the equality holds whenever $\mathcal{B}$ is reflective in $\mathcal{X}$.

Proposition 2.12 *If $\mathcal{B}$ is a reflective subcategory of $\mathcal{X}$ and $\mathcal{A}$ is a subcategory of $\mathcal{B}$, the following hold:*

1. *For a reflector $R : \mathcal{X} \to \mathcal{B}$,*

$$\mathcal{A}^{\perp_{\mathcal{X}}} = \{f \in Mor(\mathcal{X}) \mid Rf \in \mathcal{A}^{\perp_{\mathcal{B}}}\}.$$

2. *The orthogonal hull of $\mathcal{A}$ in $\mathcal{B}$ coincides with the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$.*

Proof.

1. Given an $\mathcal{X}$-morphism $f : X \to Y$, let $r_X : X \to RX$ and $r_Y : Y \to RY$ be reflections of X and Y in $\mathcal{B}$, respectively; then

$$Rf \cdot r_X = r_Y \cdot f.$$

On the one hand, if $f \in \mathcal{A}^{\perp_{\mathcal{X}}}$, then from the fact that $r_X, r_Y \in \mathcal{A}^{\perp_{\mathcal{X}}}$ and that $\mathcal{A}^{\perp_{\mathcal{X}}}$ is closed under composition and is right-cancellable (see 1.4) it follows that Rf belongs to $\mathcal{A}^{\perp_{\mathcal{X}}}$ and, since $\mathcal{A}^{\perp_{\mathcal{B}}} = \mathcal{A}^{\perp_{\mathcal{X}}} \bigcap Mor(\mathcal{B})$, Rf belongs to $\mathcal{A}^{\perp_{\mathcal{B}}}$.

On the other hand, if Rf belongs to $\mathcal{A}^{\perp_{\mathcal{B}}}$, then it belongs to $\mathcal{A}^{\perp_{\mathcal{X}}}$, and since $\mathcal{A}^{\perp_{\mathcal{X}}}$ is closed under composition and left-cancellable (by 1.4) it follows that $f \in \mathcal{A}^{\perp_{\mathcal{X}}}$.

2. Since $\mathcal{B}$ is reflective in $\mathcal{X}$, by 1.2 we obtain that

$$(\mathcal{A}^{\perp_{\mathcal{X}}})_{\perp_{\mathcal{X}}} \subseteq (\mathcal{A}^{\perp_{\mathcal{X}}})_{\perp_{\mathcal{X}}} = \mathcal{B}; \tag{1}$$

from the fact that $\mathcal{A}^{\perp_{\mathcal{B}}} \subseteq \mathcal{A}^{\perp_{\mathcal{X}}}$, and using 1.2.1, we conclude that

$$(\mathcal{A}^{\perp_{\mathcal{X}}})_{\perp_{\mathcal{X}}} \subseteq (\mathcal{A}^{\perp_{\mathcal{B}}})_{\perp_{\mathcal{X}}}; \tag{2}$$

and so, the inclusions 1 and 2 imply that

$$(\mathcal{A}^{\perp_{\mathcal{X}}})_{\perp_{\mathcal{X}}} \subseteq (\mathcal{A}^{\perp_{\mathcal{B}}})_{\perp_{\mathcal{B}}}.$$

To prove the reverse inclusion, let $B \in (\mathcal{A}^{\perp_{\mathcal{B}}})_{\perp_{\mathcal{B}}}$. We want to show that B belongs to $(\mathcal{A}^{\perp_{\mathcal{X}}})_{\perp_{\mathcal{X}}}$, i.e., that B is orthogonal to each $\mathcal{A}^{\perp_{\mathcal{X}}}$-morphism. Let $X \xrightarrow{f} Y$ belong to $\mathcal{A}^{\perp_{\mathcal{X}}}$, let h be a morphism from X to B and let $h' : RX \to B$ be the morphism which fulfils $h' \cdot r_X = h$. By 1., $Rf \in \mathcal{A}^{\perp_{\mathcal{B}}}$ and thus there is a unique morphism $h^* : RY \to B$ such that $h^* \cdot Rf = h'$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "r_X"] \arrow[dd, "h"'] & Y \arrow[dr, "r_Y"] & \\
& RX \arrow[r, "Rf"] \arrow[dl, "h'"'] & RY \arrow[dll, "h^*"] \\
B & &
\end{tikzcd}$$

Hence, for $g = h^* \cdot r_Y$ we have the equality $g \cdot f = h$; moreover, g is unique: if $g' : Y \to B$ is another morphism such that $g' \cdot f = h$, let $g^* : RY \to B$ be the morphism such that $g^* \cdot r_Y = g$. Then

$$g^* \cdot Rf \cdot r_X = g^* \cdot r_Y \cdot f = g \cdot f = h = h^* \cdot r_Y \cdot f = h^* \cdot Rf \cdot r_X;$$

this implies that $g^* \cdot Rf = h^* \cdot Rf$ and, since $Rf \in \mathcal{A}^{\perp_{\mathcal{B}}}$, it follows that $g^* = h^*$ and, consequently, $g = h$. $\square$

Using 2.12 and 2.10, we obtain the following

Corollary 2.13 *If $\mathcal{X}$ has connected colimits, then the orthogonal hull $\mathcal{O}(\mathcal{A})$ is a reflective hull of $\mathcal{A}$ in $\mathcal{X}$ if and only if there is some reflective subcategory $\mathcal{B}$ of $\mathcal{X}$ which contains $\mathcal{A}$ and such that the class of all $\mathcal{B}$-morphisms orthogonal to $\mathcal{A}$ satisfies the solution set condition.* $\square$

Let $\mathcal{X}$ be an $(\mathcal{E}, \mathbb{M})$-category and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. Then the $\mathcal{E}$-reflective hull of $\mathcal{A}$ in $\mathcal{X}$ exists and consists of all objects X in $\mathcal{X}$ such that the source $\mathcal{X}(X, \mathcal{A})$ belongs to $\mathbb{M}$ (see [2]).

Notation 2.14 If $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category and $\mathcal{A}$ is a subcategory of $\mathcal{X}$, then we denote by $\mathbb{M}(\mathcal{A})$ the $\mathcal{E}$-reflective hull of $\mathcal{A}$ in $\mathcal{X}$.

Next, we show that the class $\mathcal{A}^{\perp}$ has nice properties when considered in $\mathbb{M}(\mathcal{A})$.

Notation 2.15 Throughout, for an $(\mathcal{E}, \mathbb{M})$-category, $\mathcal{M}$ denotes the intersection $\mathbb{M} \cap Mor(\mathcal{X})$.

Definition 2.16 Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, an $\mathcal{X}$-morphism $X \xrightarrow{f} Y$ is said to be $\mathcal{A}$-*injective* provided that, for each $A \in Obj(\mathcal{A})$, the map

$$\mathcal{X}(f, A) : \mathcal{X}(Y, A) \longrightarrow \mathcal{X}(X, A)$$

is surjective, that is, each morphism with domain X and codomain in $\mathcal{A}$ is factorizable through f (not necessarily in a unique way).

We denote by $Inj(\mathcal{A})$ the class of all $\mathcal{A}$-injective morphisms in $\mathcal{X}$.

Lemma 2.17 *Let $\mathcal{X}$ be an $(\mathcal{E}, \mathbb{M})$-category with $\mathbb{M} \subseteq MonoSource(\mathcal{X})$ and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. Then we have that:*

1. *A morphism of $\mathcal{X}$ is $\mathcal{A}$-cancellable if and only if it is an epimorphism;*
2. *$\mathcal{A}^{\perp} = Inj(\mathcal{A}) \bigcap Epi(\mathcal{X})$;*
3. *$\mathcal{A}^{\perp} \subseteq Epi(\mathcal{X}) \bigcap \mathcal{M}$.*

Proof.

1. If $f : X \to Y$ is an $\mathcal{A}$-cancellable morphism and $a, b : Y \to Z$ are morphisms such that $a \cdot f = b \cdot f$, then for each $g \in \mathcal{X}(Z, \mathcal{A})$ $g \cdot a \cdot f = g \cdot b \cdot f$, thus $g \cdot a = g \cdot b$. As $\mathcal{X}(Z, \mathcal{A})$ is a monosource, it follows that $a = b$.
2. This is obvious by 1., since an $\mathcal{X}$-morphism is orthogonal to $\mathcal{A}$ iff it is $\mathcal{A}$-injective and $\mathcal{A}$-cancellable.
3. Due to 2., it is sufficient to show that $Inj(\mathcal{A}) \subseteq \mathcal{M}$.

Let $(f : X \to Y) \in Inj(\mathcal{A})$ and let $m \cdot e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of f. Let $(f_i)_I$ be an $\mathbb{M}$-source with the domain X and codomain in $\mathcal{A}$. For each $i \in I$, there is some f_i' such that $f_i' \cdot f = f_i$. Then we have that

$$(f_i' \cdot m) \cdot e = f_i \cdot 1_X,\ i \in I.$$

Since $(f_i)_I \in \mathcal{M}$ and $e \in \mathcal{E}$, there is a morphism d such that $d \cdot e = 1_X$. Since $e \in \mathcal{E}$, it is an isomorphism and so $f \in \mathcal{M}$. $\square$

Now, using 2.17 and 2.13, we obtain the following

Corollary 2.18 *Let $\mathcal{X}$ be an $(E, \mathbb{M})$-category with connected colimits, where $\mathbb{M} \subseteq MonoSource(\mathcal{X})$. Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A})$ is cowellpowered. Then the orthogonal hull of $\mathcal{A}$ is reflective and, thus, it is a reflective hull of $\mathcal{A}$ in $\mathcal{X}$.*

Proof. Under the above hypotheses, $\mathbb{M}(\mathcal{A})$ has connected colimits and, on the other hand, cowellpoweredness of $\mathbb{M}(\mathcal{A})$ and lemma 2.17 guarantee that, in $\mathbb{M}(\mathcal{A})$, the class $\mathcal{A}^{\perp}$ satisfies the solution set condition. Therefore, it follows from 2.13 that $\mathcal{O}(\mathcal{A})$ is reflective. $\square$

Let us recall here the following result due to R.-E. Hoffmann [33]:

If $\mathcal{X}$ is complete, wellpowered and cowellpowered and $\mathcal{A}$ is a subcategory of $\mathcal{X}$ whose epireflective hull in $\mathcal{X}$ is cowellpowered, then the limit-closure of $\mathcal{A}$ in $\mathcal{X}$ is its reflective hull.

This assertion remains true if "to have connected colimits" replaces "to be complete and wellpowered" and "orthogonal hull" replaces "limit-closure", as the next corollary shows.

Corollary 2.19 *If $\mathcal{X}$ has connected colimits and is cowellpowered and $\mathcal{A}$ is a subcategory of $\mathcal{X}$ whose epireflective hull in $\mathcal{X}$ is cowellpowered, then the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ is its reflective hull.*

Proof. The statement follows from 2.18 and the fact that, if $\mathcal{X}$ has connected colimits and is cowellpowered, then it is an $(Epi,\ ExtrMonoSource(\mathcal{X}))$-category (see 6.5 and 7.3 of [71] and 15.8 of [2]). $\square$

We point out that most of the categories $\mathcal{X}$ of Examples 1.5 have connected colimits and are cowellpowered. So, in these cases, every subcategory $\mathcal{A}$ such that $\mathbb{M}(A) = \mathcal{X}$ has its orthogonal hull as a reflective hull.

However, from lemma 2.17 it is clear that in the Corollary 2.18 instead of cowellpoweredness we may just assume cowellpoweredness with respect to the morphisms orthogonal to $\mathcal{A}$. And, as a matter of fact, cowellpoweredness and $\mathcal{A}^{\perp}$-cowellpoweredness can differ. This is shown by the following example.

Example 2.20 Let $\mathcal{X}$ and $\mathcal{A}$ be the categories considered in Example 2.5. Then the $\mathcal{E}$-reflective hull of $\mathcal{A}$ is just $\mathcal{X}$.

The category $\mathcal{X}$ is $\mathcal{A}^{\perp}$-cowellpowered since, as we have seen, $\mathcal{A}^{\perp} = Iso(\mathcal{X})$.

But $\mathcal{X}$ is not cowellpowered; in fact, it is not even $Epi(\mathcal{X}) \cap \mathcal{M}$-cowellpowered:

Let X be a set and let x be the empty map $\emptyset \to X$. For each ordinal i, consider the $\mathcal{X}$-object (Y_i, y_i) where

$$Y_i = X \mathbin{\dot\cup} \{j \in Ord \mid j \leq i\}$$

and $y_i : Ord \to Y_i$ is defined by

$$y_i(j) = \begin{cases} j & \text{if } j \leq i \\ i & \text{otherwise.} \end{cases}$$

Let $f_i : X \to Y$ be the inclusion of X into Y_i.

Then, the morphisms

$$f_i : (X, x) \longrightarrow (Y_i, y_i),\ i \in Ord$$

form a class of pairwise non-isomorphic epimorphic embeddings.

3 Firm classes of morphisms

In [12], which is somehow a refinement of the ideas introduced in [13], G. Brummer and E. Giuli presented the concept of firm classes of morphisms as an approach to the concept of completions of objects in arbitrary categories.

Let $\mathcal{E}$ be a class of morphisms of a category $\mathcal{X}$ closed under composition and under composition with isomorphisms on both sides. The class $\mathcal{E}$ is said to be *subfirm* if there is an $\mathcal{E}$-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$ such that

$$\mathcal{E} \subseteq \{f \in Mor(\mathcal{X}) : Rf \in Iso(\mathcal{X})\},$$

where $R : \mathcal{X} \to \mathcal{A}$ is a reflector. If, moreover, we have that

$$\mathcal{E} = \{f \in Mor(\mathcal{X}) : Rf \in Iso(\mathcal{X})\} \tag{3}$$

then $\mathcal{E}$ is said to be *firm*.

The corresponding subcategory $\mathcal{A}$ is said to be *subfirmly* (respectively, *firmly*) $\mathcal{E}$-reflective in $\mathcal{X}$.

For a subfirm class $\mathcal{E}$ of $\mathcal{X}$, the fulfilment of the equality (3) is equivalent to the following: each morphism in $\mathcal{E}$ with codomain in $\mathcal{A}$ is a reflection. This translates into a general categorical setting the behaviour of completions (where $\mathcal{E}$ is in addition a class of monomorphisms).

A classical example is the usual completion of a metric space: each metric space X has a reflection $r_X : X \to RX$ into the subcategory $\mathcal{A}$ of complete metric spaces with r_X a dense embedding; moreover, if $f : X \to A$ is another dense embedding in $\mathcal{M}et$ with $A \in Obj(\mathcal{A})$, then there is an isomorphism f^* such that $f^* \cdot r_X = f$, that is, f is also a reflection. Consequently, in $\mathcal{M}et$, the class of all dense embeddings is firm.

We point out that, since for any reflective subcategory $\mathcal{A}$ we have that

$$\mathcal{A}^{\perp} = \{f \in Mor(\mathcal{X}) : Rf \in Iso(\mathcal{X})\},$$

we may rewrite the above definitions as follows:

> $\mathcal{E}$ is said to be *subfirm* (*firm*) provided that there is some $\mathcal{E}$-reflective subcategory $\mathcal{A}$ such that $\mathcal{E} \subseteq \mathcal{A}^{\perp}$ (respectively, $\mathcal{E} = \mathcal{A}^{\perp}$).

Hence, it is clear that, to each reflective subcategory $\mathcal{A}$, there corresponds a unique firm class, namely $\mathcal{A}^{\perp}$. Now, we may restate Theorem 1.4 of [12] as follows:

Proposition 3.1 *If $\mathcal{A}$ is $\mathcal{E}$-reflective in $\mathcal{X}$, then $\mathcal{A}$ is subfirmly $\mathcal{E}$-reflective in $\mathcal{X}$ if and only if $\mathcal{A} = \mathcal{E}_{\perp}$.* □

Corollary 3.2 *$\mathcal{E}$ is subfirm if and only if $\mathcal{E}_{\perp}$ is $\mathcal{E}$-reflective.* □

Thus, from 2.9, it follows that:

Corollary 3.3 *If $\mathcal{X}$ is a category with multiple coequalizers and $\mathcal{E}$ is a class of $\mathcal{X}$-morphisms which is closed under composition and satisfies the coequalizer, the fill-in and the pseudoreflectivity conditions, then $\mathcal{E}$ is a subfirm class of $\mathcal{X}$.*

In fact, we may obtain a more complete result by using the following lemma.

Lemma 3.4 *If $\mathcal{E}$ is a class of morphisms which contains all isomorphisms, is closed under composition and such that $\mathcal{E}_\perp$ is $\mathcal{E}$-reflective, then $\mathcal{E}$ is left-cancellable iff $\mathcal{E} = (\mathcal{E}_\perp)^\perp$.*

Proof. If $\mathcal{E} = (\mathcal{E}_\perp)^\perp$, then it is clear that $\mathcal{E}$ is left-cancellable, see 1.4. Let $\mathcal{E}$ be left-cancellable, let $f : X \to Y$ belong to $(\mathcal{E}_\perp)^\perp$ and consider the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r_X"] & Y \arrow[d, "r_Y"] \\ RX \arrow[r, "Rf"] & RY \end{tikzcd} .$$

Then Rf is an isomorphism and, from the conditions on $\mathcal{E}$, $Rf.r_X \in \mathcal{E}$. Thus, since r_Y also belongs to $\mathcal{E}$ and $\mathcal{E}$ is left-cancellable, it follows that $f \in \mathcal{E}$. $\square$

Now, we have the following characterization of firm classes:

Theorem 3.5 *Let $\mathcal{X}$ be a category with connected colimits and let $\mathcal{E}$ be a class of $\mathcal{X}$-morphisms. Then $\mathcal{E}$ is a firm class of $\mathcal{X}$ if and only if the following conditions are satisfied:*

1) $Iso(\mathcal{X}) \subseteq \mathcal{E}$.

2) $\mathcal{E}$ is closed under composition.

3) $\mathcal{E}$ is left-cancellable.

4) $\mathcal{E}$ fulfils the coequalizer condition.

5) $\mathcal{E}$ is closed under the formation of pushouts and multiple pushouts.

6) $\mathcal{E}$ satisfies the solution set condition.

Proof. If $\mathcal{E}$ is firm, there is a reflective subcategory $\mathcal{A}$ of $\mathcal{X}$ such that $\mathcal{E} = \mathcal{A}^{\perp}$ and, then, $\mathcal{A}^{\perp}$ fulfils all conditions 1)-6).

Conversely: It is clear that 5) and 6) imply that $\mathcal{E}$ fulfils the fill-in and the pseudoreflectivity conditions. This fact with 2) and 4) implies, due to 2.9, that $\mathcal{E}_{\perp}$ is $\mathcal{E}$-reflective. From lemma 3.4, it follows that $\mathcal{E} = (\mathcal{E}_{\perp})^{\perp}$. □

Remark 3.6 Theorem 3.5 of [12] also gives a characterization of firm classes. Our Theorem [12] imposes weaker conditions on the category $\mathcal{X}$ than that theorem. However, concerning to the conditions on $\mathcal{E}$, I could not compare our "solution set condition" with the "solution set condition" on factorization of morphisms of 3.5 in [12].

In Theorem 3.5, we have just proved that for a category $\mathcal{X}$ with connected colimits, the maps $(-)_{\perp}$ and $(-)^{\perp}$ yield a bijection between the collection of all reflective subcategories of $\mathcal{X}$ and the collection of all classes $\mathcal{E}$ of morphisms which satisfy conditions 1) to 6).

Therefore, a reflective hull of a subcategory $\mathcal{A}$ in $\mathcal{X}$ exists if and only if the conglomerate of such classes of morphisms which, furthermore, are contained in $\mathcal{A}^{\perp}$, has a greatest element. Thus, Theorem 2.10 gives a necessary and sufficient condition for $\mathcal{A}^{\perp}$ to be the greatest element.

The classical examples of completions correspond to firm classes of monomorphisms which are also epimorphisms. The firm classes in 1.5 are all subclasses of this type. Since, for these classes, the coequalizer condition trivially holds, we have the following

Corollary 3.7 *Let $\mathcal{X}$ have connected colimits. If $\mathcal{E}$ is a class of epimorphisms containing $Iso(\mathcal{X})$, then $\mathcal{E}$ is firm if and only if it is closed under composition, is left-cancellable, is closed under pushouts and under multiple pushouts and satisfies the solution set condition.*
□

Remark 3.8 For all examples of 1.5, we have that $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category for $\mathcal{E}$ the class of all surjections and a suitable $\mathbb{M}$ which is easily determined.

In all cases, except when $\mathcal{X}$ is the category $\mathcal{T}ych$ of Tychonoff spaces, $\mathcal{A}^{\perp} = Epi(\mathcal{X}) \cap \mathcal{M}$; therefore, since we have that $\mathcal{O}(\mathcal{A})^{\perp} = \mathcal{A}^{\perp}$ and $\mathcal{O}(\mathcal{A})$ is reflective, the class $Epi(\mathcal{X}) \cap$

$\mathcal{M}$ is firm in $\mathcal{X}$. Moreover, by 2.17, it is the greatest firm class of $\mathcal{M}$-morphisms in $\mathcal{X}$ and, consequently, $\mathcal{O}(\mathcal{A})$ is the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{X}$.

So, we may ask if $Epi(\mathcal{X}) \cap \mathcal{M}$ is also firm when $\mathcal{X}$ is the category of Tychonoff spaces and continuous maps, that is, if there exists some reflective subcategory $\mathcal{A}$ of $\mathcal{T}ych$ such that $\mathcal{A}^{\perp} = Epi(\mathcal{X}) \cap \mathcal{M}$. The answer is negative. Furthermore, the answer remains the same for any epireflective subcategory $\mathcal{X}$ of $\mathcal{T}op$ consisting of Hausdorff spaces and having a space with more than one point (considering always the class $\mathcal{M}$ of all embeddings) as follows from 1.8(2) of [13].

From 9.5.3 and 9.6 below, it follows that the class of C^*-embeddings is just the greatest firm class of embeddings in $\mathcal{T}ych$. In Section 9 we shall also study the greatest firm classes for several other categories, including some of the ones referred above.

Chapter II

The orthogonal closure operator

For the category $\mathcal{M}et$ and its subcategory $\mathcal{A}$ of complete metric spaces, we have that:

- $\mathcal{A}^{\perp}$ consists of all dense embeddings;
- $\mathcal{O}(\mathcal{A})$ consists of all "strongly closed" spaces X, i.e. such that any embedding of X in a metric space is closed.

In this chapter we shall define a closure operator, called the orthogonal closure operator, which encompasses this example as well as the other examples of 1.5 and 2.5 and many others. In fact, the orthogonal closure operator in a category $\mathcal{X}$ with respect to a convenient class of morphisms $\mathcal{M}$ and induced by a subcategory $\mathcal{A}$ of $\mathcal{X}$ gives us means to characterize $\mathcal{A}^{\perp}$ and $\mathcal{O}(\mathcal{A})$ in terms of denseness and closedness, as in the above example. Furthermore, it allows us to find sufficient conditions for the orthogonal hull to be a reflective hull.

We also present interesting relationship between this closure operator and the regular closure operator.

Finally, we pay special attention to a particular class of morphisms, namely the greatest pushout-stable subclass of a given class of monomorphisms. This class is closely related to the study of the orthogonal closure operator which is developed below.

4 Closure operators

In this section we give a brief account of closure operators in the sense of [18] and [20]. For a detailed information on this subject, the reader is referred to [18], [20] or [21].

Throughout this chapter, $\mathcal{M}$ will always denote a class of monomorphisms in $\mathcal{X}$ which contains all isomorphisms, is closed under composition and is left-cancellable. We shall also consider $\mathcal{M}$ as a full subcategory of the category $\mathcal{X}^2$ (of all $\mathcal{X}$-morphisms).

Let

$$u : \mathcal{M} \to \mathcal{X}$$

be the codomain functor, i.e., the functor which assigns, to each morphism in $\mathcal{M}$ $(r, s) : (X \overset{m}{\to} Y) \longrightarrow (Z \overset{n}{\to} W)$, the $\mathcal{X}$-morphism $s : Y \to W$.

Definition 4.1 A *closure operator* on $\mathcal{X}$ with respect to $\mathcal{M}$ consists of a functor

$$c : \mathcal{M} \to \mathcal{M}$$

such that $u \cdot c = u$, together with a natural transformation

$$\delta : Id_{\mathcal{M}} \to c$$

such that $u \cdot \delta = Id_u$.

So, the closure operator (c, δ) determines, for each $m : X \to Y$ in $\mathcal{M}$, morphisms $c(m)$ and $d(m)$ and a commutative square

$$\begin{tikzcd} X \arrow[r, "d(m)"] \arrow[d, "m"'] & \overline{X} \arrow[d, "c(m)"] \\ Y \arrow[r, "1_Y"] & Y \end{tikzcd} \tag{4}$$

where $\delta_m = (d(m), 1_Y) : m \to c(m)$.

Since $c(m)$ is a monomorphism, δ is uniquely determined by c; consequently, we usually write just c instead of (c, δ).

Definitions 4.2 If c is a closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$, then a morphism $(m : X \to Y) \in \mathcal{M}$ is called *c-dense* if $c(m) \cong 1_Y$. It is called *c-closed* if $c(m) \cong m$.

The closure operator c is said to be *weakly hereditary in a subclass* $\mathcal{N}$ of $\mathcal{M}$ if, for each $n \in \mathcal{N}$, $d(m)$ is c-dense; if $\mathcal{N} = \mathcal{M}$, we simply say that c is *weakly hereditary.*

The closure operator c is said to be *idempotent* provided that $c(m)$ is c-closed for all $m \in \mathcal{M}$.

The class of all c-dense morphisms in $\mathcal{M}$ is denoted by $\mathcal{E}_c^{\mathcal{M}}$ and the class of all c-closed morphisms in $\mathcal{M}$ is denoted by $\mathcal{M}_c$.

If c and c' are two closure operators on $\mathcal{X}$ with respect to the same $\mathcal{M}$, then we say that c *is smaller than* c', written $c \le c'$, provided that $c(m) \le c'(m)$ for all $m \in \mathcal{M}$, that is, provided that, for each $m \in \mathcal{M}$, there is a morphism t such that $c(m) = c'(m) \cdot t$; this morphism t is obviously unique.

We write $c = c'$ if $c \le c'$ and $c' \le c$, that is, if for each $m \in \mathcal{M}$, $c(m) \cong c'(m)$.

Recall that, using terminology of [20], a *factorization system* on $\mathcal{X}$ with respect to $\mathcal{M}$ gives, for each m in $\mathcal{M}$, a pair of morphisms (d_m, c_m) in $\mathcal{M} \times \mathcal{M}$ such that

- $m = c_m \cdot d_m$
- for all m, n in $\mathcal{M}$ and all u, v with $v \cdot m = n \cdot u$, there is a unique morphism t such that $t \cdot d_m = d_n \cdot u$ and $c_n \cdot t = v \cdot c_m$.

$$\begin{tikzcd}
\bullet \arrow[r, "d_m"] \arrow[d, "u"] & \bullet \arrow[r, "c_m"] \arrow[d, "t"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "d_n"] & \bullet \arrow[r, "c_n"] & \bullet
\end{tikzcd}$$

Proposition 4.3 [20] *There is a one-to-one correspondence between closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$ and factorization systems on $\mathcal{X}$ with respect to $\mathcal{M}$.* $\square$

In fact, a closure operator c of $\mathcal{X}$ with respect to $\mathcal{M}$ induces for each m in $\mathcal{M}$ a factorization

$$X \xrightarrow{d(m)} \overline{X} \xrightarrow{c(m)} Y\,,$$

illustrated by the diagram (4). Furthermore, these factorizations form a factorization system on $\mathcal{X}$ with respect to $\mathcal{M}$. The mentioned one-to-one correspondence assigns to each closure operator c this factorization system.

Proposition 4.4 ([18] and [20]) *For a closure operator c of $\mathcal{X}$ with respect to $\mathcal{M}$, the following assertions are equivalent:*

(i) c is weakly hereditary and idempotent;

(ii) c is weakly hereditary and $\mathcal{E}_c^{\mathcal{M}}$ is closed under composition;

(iii) c is idempotent and $\mathcal{M}_c$ is closed under composition. □

Let $\mathcal{M}$ be a class of morphisms in $\mathcal{X}$ which contains all isomorphisms and is closed under composition. We recall that $\mathcal{X}$ is said to be $\mathcal{M}$*-complete* provided that pullbacks of $\mathcal{M}$-morphisms along arbitrary morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$. The pullback of an $\mathcal{M}$-morphism m along a morphism f is called the *inverse image of m under f* and it is denoted by $f^{-1}(m)$.

If $\mathcal{X}$ is a $\mathcal{M}$-complete category then every morphism in $\mathcal{M}$ is a monomorphism and $\mathcal{M}$ is left-cancellable. Furthermore, for each object X in $\mathcal{X}$, the preordered class $\mathcal{M}_X$ of all $\mathcal{M}$-morphisms with codomain X is large-complete. Furthermore, there is a (uniquely determined) class of morphisms $\mathcal{E}$ in $\mathcal{X}$ such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure for morphisms in $\mathcal{X}$.

If $\mathcal{X}$ is $\mathcal{M}$-complete, a closure operator $c : \mathcal{M} \to \mathcal{M}$ may be equivalently described by a family of maps

$$(c_X : \mathcal{M}_X \to \mathcal{M}_X)_{X \in \mathcal{X}},$$

where $c_X(m) = c(m)$ for each m, satisfying the conditions:

1. $m \leq c_X(m)$, $m \in \mathcal{M}_X$ (c is *extensive*);

2. if $m \leq n$, then $c_X(m) \leq c_X(n)$, $m, n \in \mathcal{M}_X$ (c is *monotone*);

3. $c_X(f^{-1}(m)) \leq f^{-1}(c_Y(m))$, for each morphism $(f : X \to Y) \in Mor(\mathcal{X})$ and $m \in \mathcal{M}_Y$ (every morphim $f : X \to Y$ is c-*continuous*).

As mentioned above, the $\mathcal{M}$-completeness of $\mathcal{X}$ determines a factorization structure $(\mathcal{E}, \mathcal{M})$ in $\mathcal{X}$. In this case, we may extend the notion of c-density to all morphisms in $\mathcal{X}$. Actually, an $\mathcal{X}$-morphism is said to be *c-dense* if the $\mathcal{M}$-part of its $(\mathcal{E}, \mathcal{M})$-factorization is c-dense.

Let us recall the concept of dominion of J. Isbell [40]. Let $\mathcal{X}$ be a category of universal algebras. Given an algebra B in $\mathcal{X}$ and a subalgebra A of B, the *dominion of A in B* is the set

$$Dom_B(A) = \{b \in B \mid \text{for all } f, g : B \to C \text{ in } \mathcal{X},\ f|_A = g|_A \Rightarrow g(b) = h(b)\}.$$

J. Isbell used this notion for characterizing the epimorphisms of a subcategory closed under subobjects. His famous Zig-Zag Theorem characterizes the elements of $Dom_B(A)$ when the coproduct of two copies of B exists.

In [60], S. Salbany introduced the regular closure operators for the category of topological spaces; namely, for a subcategory $\mathcal{A}$ of $\mathcal{T}op$, given a space Y and a subspace $X \subseteq Y$, the regular closure of X in Y induced by $\mathcal{A}$ is the subspace

$$[X] = \cap\{Z \subseteq Y \mid Z = \text{eq}(f, g) \text{ for some } f,\ g \in \mathcal{T}op(Y, \mathcal{A}) \text{ such that } f|_X = g|_X\}.$$

In fact, regular closure operators may be defined in a categorical setting (see [18]) and have been widely investigated. In particular, they are a useful tool in the investigation of cowellpoweredeness of some categories, since they provide a characterization of epimorphisms in terms of denseness.

The dominion of a subalgebra in the sense of Isbell and the regular closure of a subspace in the sense of Salbany turn out to be examples of regular closure operators in categories.

Definition 4.5 Let $\mathcal{X}$ be $\mathcal{M}$-complete with $\mathcal{M}$ containing all regular monomorphisms of $\mathcal{X}$, and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$.

The *regular closure operator* in $\mathcal{X}$ with respect to $\mathcal{M}$ and induced by $\mathcal{A}$, which we shall denote by

$$r_{\mathcal{A}} : \mathcal{M} \to \mathcal{M},$$

assigns to each $m \in \mathcal{M}_X$ the intersection of all $n \in \mathcal{M}_X$ such that $m \leq n$ and n is the equalizer of a pair of morphisms with codomain in $\mathcal{A}$.

It is easy to check that $r_{\mathcal{A}}$ actually is a closure operator in the sense defined above. Moreover, it is idempotent.

Remark 4.6 We recall here two well-known and important properties of the regular closure operator:

1. ([18, 20]) *If $\mathcal{X}$ has equalizers and $r_{\mathcal{A}}$ is a regular closure operator on $\mathcal{X}$ induced by a subcategory $\mathcal{A}$, then the $r_{\mathcal{A}}$-dense morphisms of $\mathcal{X}$ are just the $\mathcal{A}$-cancellable morphisms. In particular, the epimorphisms of $\mathcal{A}$ are the $r_{\mathcal{A}}$-dense $\mathcal{A}$-morphisms.*

2. (cf. [15]) *Let $\mathcal{X}$ be an $(\mathcal{E}, \mathbb{M})$-category with equalizers, let $RegMono(\mathcal{X}) \subseteq \mathbb{M} \cap Mor(\mathcal{X})$ and let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{B}) = \mathbb{M}(\mathcal{A})$. Then, the regular closure operator with respect to $\mathcal{M}$ fulfils*

$$r_{\mathcal{A}} = r_{\mathcal{B}}.$$

As a consequence of 1. and 2., if $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, then the $\mathcal{X}$-epimorphisms are just the $r_{\mathcal{A}}$-dense $\mathcal{X}$-morphisms.

5 The orthogonal closure operator

For the rest of this chapter, unless explicitely stated, $\mathcal{X}$ is an $\mathcal{M}$-complete category with pushouts, where $\mathcal{M}$ contains all isomorphisms and is closed under composition and $(\mathcal{E}, \mathcal{M})$ is the corresponding factorization structure for morphisms in $\mathcal{X}$.

In this section, we introduce the orthogonal closure operator which will be the main tool in the remainder of this chapter.

Definition 5.1 Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. For each $m : X \to Y$ in $\mathcal{M}$, we denote by $c_{\mathcal{A}}(m) : \overline{X} \to Y$ the $\mathcal{M}$-morphism defined as follows:

(C) For each $g : X \to A$ with A in $\mathcal{A}$, we form a pushout $(\overline{m}, g')$ of (m, g) in $\mathcal{X}$. Let $m' \cdot e$ be an $(\mathcal{E}, \mathcal{M})$-factorization of $\overline{m}$ and let (m_g, g^*) be a pullback of (m', g').

$$
\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dr] \arrow[dd, "g"'] & & Y \arrow[dd, "g'"] \\
& X_g \arrow[ur, "m_g"'] \arrow[dd, "g^*"] & \\
A \arrow[rr, "\overline{m}"'] \arrow[dr, "e"'] & & \bullet \\
& \bullet \arrow[ur, "m'"'] &
\end{tikzcd}
$$

Let $P_{\mathcal{A}}(m) = \{m_g \,|\, g : X \to A, A \in \mathcal{A}\}$. The morphism $c_{\mathcal{A}}(m) : \overline{X} \to Y$ is the intersection of $P_{\mathcal{A}}(m)$.

It is clear that the morphism $c_{\mathcal{A}}(m)$ belongs to $\mathcal{M}$. We prove now that $c_{\mathcal{A}}$ is a closure operator.

Proposition 5.2 *For each subcategory $\mathcal{A}$ of $\mathcal{X}$, the map $c_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ is a closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$.*

Proof. We first show that $c_{\mathcal{A}}$ is functorial. Let

$$(p, f) : (m : X \to Y) \to (n : Z \to W)$$

be a morphism in the category $\mathcal{M}$. We are going to define $c_{\mathcal{A}}(p, f)$. For every $(h : Z \to A) \in \mathcal{X}(Z, \mathcal{A})$, let $(\overline{n}, h')$ be a pushout of (n, h), let $n' \cdot q$ be an $(\mathcal{E}, \mathcal{M})$-factorization of $\overline{n}$ and let (n_h, h^*) be a pullback of (n', h'). For $g = h \cdot p$, let the morphisms $\overline{m}$, g', m', e, m_g and g^* be as in (C). Since

$$(h' \cdot f) \cdot m = h' \cdot n \cdot p = \overline{n} \cdot h \cdot p = \overline{n} \cdot g$$

and $(\overline{m}, g')$ is a pushout of (m, g), there is a unique morphism d such that $h' \cdot f = d \cdot g'$ and $\overline{n} = d \cdot \overline{m}$. From the last equality we get $n' \cdot q = d \cdot m' \cdot e$ and, by the diagonal fill-in property, there is a unique morphism k such that $k \cdot e = q$ and $n' \cdot k = d \cdot m'$.

$$\begin{tikzcd}
X \arrow[rrrr, "m"] \arrow[ddd, "g"] \arrow[ddr, "p"] & & & & Y \arrow[ddr, "f"] \arrow[ddd, "g'"] & \\
& & X_g \arrow[urr, "m_g"] \arrow[dd, "g^*"] & & & \\
& Z \arrow[rrrr, "n"] \arrow[dd, "h"] & & & & W \arrow[dd, "h'"] \\
A \arrow[rr, "e"] \arrow[dr, "1_A"] & & \bullet \arrow[rr, "m'"] \arrow[dr, "k"] & & \bullet \arrow[dr, "d"] & \\
& A \arrow[rr, "q"] & & \bullet \arrow[rr, "n'"] & & \bullet
\end{tikzcd}$$

Then we have

$$n' \cdot (k \cdot g^*) = d \cdot m' \cdot g^* = d \cdot g' \cdot m_g = h' \cdot (f \cdot m_g).$$

Since (n_h, h^*) is a pullback of (n', h'), there exists a morphism r_h such that

$$f \cdot m_{h \cdot p} = f \cdot m_g = n_h \cdot r_h.$$

Now, for each $h \in \mathcal{X}(Z, \mathcal{A})$, let t_h be the unique morphism which fulfils $m_{h \cdot p} \cdot t_h = c_{\mathcal{A}}(m)$. Then,

$$n_h \cdot (r_h \cdot t_h) = f \cdot m_{h \cdot p} \cdot t_h = f \cdot c_{\mathcal{A}}(m).$$

Consequently, since $c_{\mathcal{A}}(n) : \overline{Z} \to W$ is the intersection of $P_{\mathcal{A}}(n)$, there is a unique morphism $u : \overline{X} \to \overline{Z}$ such that $f \cdot c_{\mathcal{A}}(m) = c_{\mathcal{A}}(n) \cdot u$.

Taking

$$c_{\mathcal{A}}(p, f) = (u, f) : c_{\mathcal{A}}(m) \to c_{\mathcal{A}}(n),$$

it is easy to see that $c_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ is a functor for which $u \cdot c = u$, where u is the codomain functor from $\mathcal{M}$ to $\mathcal{X}$.

At last, we show that there is a natural transformation $\delta : Id_u \to c$ such that $u \cdot \delta = Id_u$. Let $(m : X \to Y) \in \mathcal{M}$. For each $g \in \mathcal{X}(X, \mathcal{A})$, there is a unique morphism $d_g : X \to X_g$ such that

$$m_g \cdot d_g = m \ \text{ and } \ g^* \cdot d_g = e \cdot g.$$

Then, since $c_{\mathcal{A}}(m) : \overline{X} \to Y$ is the intersection of $P_{\mathcal{A}}(m)$, from the first equality, there is a unique morphism $d_{\mathcal{A}}(m) : X \to \overline{X}$ such that $c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m) = m$. The family of morphisms

$$\delta_m = ((d_{\mathcal{A}}(m), 1_Y) : m \to_{\mathcal{A}} (m)), \ m \in \mathcal{M},$$

defines a natural transformation $\delta : Id_{\mathcal{M}} \to c_{\mathcal{A}}$ such that $u\delta = Id_u$. $\square$

Definition 5.3 The closure operator $c_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ of 5.1 and 5.2 will be called the *orthogonal closure operator* on $\mathcal{X}$ with respect to $\mathcal{M}$ induced by $\mathcal{A}$.

Notation 5.4 As in the above proof, throughout this dissertation, $d_{\mathcal{A}}(m)$ always denotes the unique morphism such that $m = c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m)$.

Remark 5.5 In order to define the orthogonal closure operator, we can assume that $\mathcal{X}$, instead of being $\mathcal{M}$-complete, fulfils the following weaker conditions:

- If $m \in \mathcal{M}$, the pullback of m along an arbitrary morphism g with the same codomain as m exists whenever there is a commutative diagram in $\mathcal{X}$ of the form

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "m"'] & {} \arrow[d, "g"] \\ {} \arrow[r] & {} \end{tikzcd} .$$

- If $(X_i \xrightarrow{m_i} Y)_I$ is a family of morphisms of $\mathcal{M}$ such that for some $\mathcal{X}$-object X there are morphisms $X \xrightarrow{d_i} X_i$, $i \in I$, such that $m_i \cdot d_i = m_{i'} \cdot d_{i'}$ for all $i, i' \in I$, then the intersection of $(m_i)_I$ exists and belongs to $\mathcal{M}$.

For instance, let $\mathcal{M}et_*$ be the category obtained from $\mathcal{M}et$ by removing the empty space. Then $\mathcal{M}et_*$ has pushouts (although $\mathcal{M}et$ does not have pushouts) and fulfils the above two conditions for $\mathcal{M}$ the class of all embeddings, but it is not $\mathcal{M}$-complete. A similar behaviour has the category $\mathcal{N}orm_*$ which is obtained from $\mathcal{N}orm$ by removing the empty space.

Remark 5.6 The following properties are immediate:

1. The orthogonal closure operator induced by a subcategory $\mathcal{A}$ of $\mathcal{X}$ is discrete in the subclass of morphisms with domains in $\mathcal{A}$, that is, for all $m \in \mathcal{M}$ with domain in $\mathcal{A}$, $c_{\mathcal{A}}(m) = m$.

2. If $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{X}$ such that $\mathcal{A} \subseteq \mathcal{B}$, then $c_{\mathcal{B}} \le c_{\mathcal{A}}$.

The following lemma will be used in the proofs of Propositions 5.8 and 5.9 and below.

Lemma 5.7 *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. Under the assumption $SplitMono(\mathcal{X}) \subseteq \mathcal{M}$, if $a, b : Y \to A$ is a pair of morphisms with $A \in Obj(\mathcal{A})$ and $(m : X \to Y) \in \mathcal{M}$, then*

$$a \cdot m = b \cdot m \implies a \cdot c_{\mathcal{A}}(m) = b \cdot c_{\mathcal{A}}(m).$$

Proof. Let $g = a \cdot m = b \cdot m$ and let $(\overline{m}, g')$, $m' \cdot e$ and (m_g, g^*) be as in (C). We are going to show that $a \cdot m_g = b \cdot m_g$. The equality $1_A \cdot g = a \cdot m$ implies the existence of a unique morphism t such that $t \cdot \overline{m} = 1_A$ and $t \cdot g' = a$; hence $t \cdot m' \cdot e = t \cdot \overline{m} = 1_A$ and so, since $e \in \mathcal{E} \cap \mathcal{M}$, e is an isomorphism. Analogously, there is a unique morphism t' such that $t' \cdot \overline{m} = 1_A$ and $t' \cdot g' = b$. Then

$$\begin{aligned} a \cdot m_g = \ & t \cdot g' \cdot m_g = t \cdot m' \cdot g^* = t \cdot \overline{m} \cdot e^{-1} \cdot g^* = e^{-1} \cdot g^* = \\ = \ & t' \cdot \overline{m} \cdot e^{-1} \cdot g^* = t' \cdot m' \cdot g^* = t' \cdot g' \cdot m_g \\ = \ & b \cdot m_g. \end{aligned}$$

Let t_g be the morphism that fulfils the equality $m_g \cdot t_g = c_{\mathcal{A}}(m)$. Hence

$$a \cdot c_{\mathcal{A}}(m) = a \cdot m_g \cdot t_g = b \cdot m_g \cdot t_g = b \cdot c_{\mathcal{A}}(m).$$ □

Next we relate the closure operator $c_{\mathcal{A}}$ to the regular closure operator induced by the same category:

Proposition 5.8 *If $RegMono(\mathcal{X}) \subseteq \mathcal{M}$, then for each subcategory $\mathcal{A}$ of $\mathcal{X}$ we have that $c_{\mathcal{A}} \leq r_{\mathcal{A}}$.*

Proof. Given $m : X \to Y$ in $\mathcal{M}$, let n be a morphism in $\mathcal{M}$ with codomain in Y such that $m \leq n$ and $n = eq(a, b)$ where $a, b : Y \to A$ is a pair of morphisms with codomain in $\mathcal{A}$. Hence, we have that $a \cdot m = b \cdot m$ and, by 5.7, $a \cdot c_{\mathcal{A}}(m) = b \cdot c_{\mathcal{A}}(m)$. But, then, $c_{\mathcal{A}}(m) \leq n$.

Therefore, by definition of $r_{\mathcal{A}}$, $c_{\mathcal{A}}(m) \leq r_{\mathcal{A}}(m)$. □

It is easy to see that the orthogonal closure operator is, in general, distinct from the regular one. It suffices to notice that if $\mathcal{A} = \mathcal{X}$, then the closure operator $c_{\mathcal{A}}$ is discrete and so the $c_{\mathcal{A}}$-dense morphisms are just the $\mathcal{E}$-morphisms. But $\mathcal{E}$ can obviously be different from the class of all epimorphisms, which, under mild conditions, is just the class of $r_{\mathcal{A}}$-dense morphisms, as mentioned in 4.6. Furthermore, several examples with $\mathcal{A} \neq \mathcal{X}$ and such that $\mathcal{A}$-cancellable morphisms are not necessarily $c_{\mathcal{A}}$-dense will be given below.

We are going to see that $c_{\mathcal{A}}$-dense morphisms play an important rôle in characterizing $\mathcal{A}^{\perp}$-morphisms, for suitable subcategories $\mathcal{A}$.

Proposition 5.9 *Under the assumption $SplitMono(\mathcal{X}) \subseteq \mathcal{M}$, every $c_{\mathcal{A}}$-dense morphism in $\mathcal{M}$ is $\mathcal{A}$-cancellable.*

Proof. Let $a \cdot m = b \cdot m$, where a and b are morphisms with codomain in $\mathcal{A}$ and $m : X \to Y$ is a dense morphism in $\mathcal{M}$. Then $c_{\mathcal{A}}(m) \cong 1_Y$ and from 5.7 it follows that $a = b$. $\square$

Corollary 5.10 *Assuming that $\mathcal{E}$ is a class of epimorphisms, every $c_{\mathcal{A}}$-dense morphism is $\mathcal{A}$-cancellable.*

6 Dense morphisms and $\mathcal{A}^{\perp}$-morphisms

From now on we assume that $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category, with $\mathbb{M}$ a conglomerate of monosources and $\mathcal{M} = \mathbb{M} \cap Mor(\mathcal{X})$. It follows that $\mathcal{E}$ is a class of epimorphisms (cf. [2], [72]). As before we assume that $\mathcal{X}$ has pushouts.

Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. From 2.12, the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ coincides with its orthogonal hull in $\mathbb{M}(\mathcal{A})$. Hence, taking into account 1.2.2, it is clear that the orthogonal hull of $\mathcal{A}$ is a reflective hull of $\mathcal{A}$ in $\mathbb{M}(\mathcal{A})$ if and only if it is a reflective hull of $\mathcal{A}$ in $\mathcal{X}$. Consequently, for characterizing the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ as well as for finding conditions under which $\mathcal{O}(\mathcal{A})$ is the reflective hull, we can assume, without loss of generality, that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. This is often assumed for the rest of this chapter. We also restrict to $\mathcal{X} = \mathbb{M}(\mathcal{A})$ for characterizing the class $\mathcal{A}^{\perp}$. But, even in this case, the condition $\mathcal{X} = \mathbb{M}(\mathcal{A})$ is not too restrictive, since we saw in 2.12.1 that, in general, an

$\mathcal{X}$-morphism f is orthogonal to $\mathcal{A}$ if and only if its reflection in $\mathbb{M}(\mathcal{A})$ is orthogonal to $\mathcal{A}$.

Notation 6.1 We denote by $PS(\mathcal{M})$ the subclass of $\mathcal{M}$ consisting of all morphisms for which the pushout along any morphism belongs to $\mathcal{M}$.

Thus, $PS(\mathcal{M})$ is the greatest pushout-stable subclass of $\mathcal{M}$. Furthermore, since $\mathcal{M}$ is closed under composition and left cancellable, the same holds for $PS(\mathcal{M})$.

The class $PS(\mathcal{M})$ plays a crucial rôle in almost all the results presented in this chapter. The second part of Lemma 6.3 will give a reason for that.

The following two lemmas will be useful in the sequel.

Lemma 6.2 *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. An $\mathcal{X}$-morphism f belongs to $PS(\mathcal{M})$ if and only if*

(**P**) *the pushout of f along any morphism with codomain in $\mathcal{A}$ lies in $\mathcal{M}$.*

Proof. Clearly, condition (**P**) is necessary for f belonging to $PS(\mathcal{M})$. To show that it is also sufficient, consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "\overline{g}"] \\ Z \arrow[r, "\overline{f}"] \arrow[d, "h_i"] & \bullet \arrow[d, "h_i'"] \\ A_i \arrow[r, "\overline{h_i}"] & \bullet \end{tikzcd}$$

where both of the inner squares are pushouts, $(h_i)_I$ is in $\mathbb{M}$ and $A_i \in Obj(\mathcal{A})$. Since condition (**P**) is fulfilled, $\overline{h_i}$ belongs to $\mathcal{M}$. Then the source $(\overline{h_i} \cdot h_i)_I$ is also in $\mathbb{M}$. Consequently, since $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category, the equalities

$$\overline{h_i} \cdot h_i = h_i' \cdot \overline{f}, \quad i \in I$$

imply that $\overline{f} \in \mathcal{M}$. $\square$

Lemma 6.3 *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$.*

1. *$Inj(\mathcal{A})$ consists of precisely all $m \in PS(\mathcal{M})$ such that every pushout of m along a morphism with codomain in $\mathcal{A}$ is a split monomorphism.*

2. *$\mathcal{A}^{\perp}$ consists of precisely all $m \in PS(\mathcal{M})$ such that every pushout of m along a morphism with codomain in $\mathcal{A}$ is an isomorphism.*

Proof.

1. It is clear that if m is a $PS(\mathcal{M})$-morphism such that every pushout of m along a morphism with codomain in $\mathcal{A}$ is a split monomorphism, then it is $\mathcal{A}$-injective. Conversely, let $f : X \to Y$ belong to $Inj(\mathcal{A})$. Then, as it was shown in the proof of 2.17, $f \in \mathcal{M}$. Let $g : X \to A$ be a morphism with codomain in $\mathcal{A}$ and let $(\overline{f}, \overline{g})$ be the pushout of (f, g). Since f is $\mathcal{A}$-injective, there is some morphism $g' : Y \to A$ such that $g' \cdot f = 1_A \cdot g$. Hence, there is a unique morphism t such that $t \cdot \overline{g} = g'$ and $t \cdot \overline{f} = 1_A$; so, $\overline{f}$ is a split monomorphism; furthermore, it follows that $\overline{f} \in \mathcal{M}$, and, therefore, from 6.2, we have that $f \in PS(\mathcal{M})$.

2. Since $\mathcal{A}^{\perp} \subseteq Inj(\mathcal{A})$, it is clear that $\mathcal{A}^{\perp} \subseteq PS(\mathcal{M})$. On the other hand, if $f \in \mathcal{A}^{\perp}$, then f is $\mathcal{A}$-cancellable and so, by 2.17, it is an epimorphism. Hence, using the fact that epimorphisms are stable under pushout, it is easily seen that a morphism f belongs to $\mathcal{A}^{\perp}$ if and only if the pushout of m along a morphism with codomain in $\mathcal{A}$ is an isomorphism. $\square$

Now, we have the following characterization of the class $\mathcal{A}^{\perp}$:

Theorem 6.4 *For a subcategory $\mathcal{A}$ of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $\mathcal{A}^{\perp}$ consists of precisely all $c_{\mathcal{A}}$-dense morphisms in $PS(\mathcal{M})$.*

Proof. Let $m \in \mathcal{A}^{\perp}$. Then, by 6.3.2, $m \in PS(\mathcal{M})$ and every $m_g \in P_{\mathcal{A}}(m)$ is an isomorphism; this implies that $c_{\mathcal{A}}(m) \cong 1_Y$, i. e., m is dense.

Conversely, let $m : X \to Y$ belong to $PS(\mathcal{M})$ and be such that $c_{\mathcal{A}}(m) \cong 1_Y$. Hence, every $m_g \in P_{\mathcal{A}}(m)$ must be an isomorphism. Now, let us recall that every pullback of a pushout is a pushout, i. e., if (m', g') is a pushout of (m, g) and (m^*, g^*) is a pullback of (m', g') then (m', g') is a pushout of (m^*, g^*). Then, for each $g \in \mathcal{X}(X, \mathcal{A})$, a pushout of

(m, g) is a pushout of m_g along a certain morphism, thus it is an isomorphism. Therefore, by 6.3.2, $m \in \mathcal{A}^{\perp}$. □

As we have seen, and in contrast to what happens with regular closure operators when $\mathcal{X} = \mathbb{M}(\mathcal{A})$, an epimorphism need not be $c_{\mathcal{A}}$-dense. The next proposition shows that inside a particular subclass of morphisms of $\mathcal{X}$ the epimorphisms are just the $c_{\mathcal{A}}$-dense morphisms.

Proposition 6.5 *Let $\mathcal{D}$ be the subclass of $PS(\mathcal{M})$ given by*

$$\mathcal{D} = \{n \in \mathcal{M} \mid n \cong d_{\mathcal{A}}(m) \text{ for some } m \in PS(\mathcal{M})\}.$$

Then $\mathcal{D} \subseteq Inj(\mathcal{A})$ and, whenever $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, a $\mathcal{D}$-morphism is $c_{\mathcal{A}}$-dense if and only if it is an epimorphism.

Proof. We first show that $\mathcal{D} \subseteq Inj(\mathcal{A})$. For $(d_{\mathcal{A}}(m) : X \to \overline{X})$ in $\mathcal{D}$, with $m \in PS(\mathcal{M})$,if $g : X \to A$ is a morphism with codomain in $\mathcal{A}$, let (m', g') be a pushout of (m, g) and let (m_g, g^*) be the pullback of (m', g'). Then, for the unique morphism d_g such that $m_g \cdot d_g = c_{\mathcal{A}}(m)$, we have that

$$\begin{aligned} m' \cdot g \quad &= g' \cdot c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m) = g' \cdot m_g \cdot d_g \cdot d_{\mathcal{A}}(m) \\ &= m' \cdot g^* \cdot d_g \cdot d_{\mathcal{A}}(m). \end{aligned}$$

Since m' is a monomorphism, it follows that $g = (g^* \cdot d_g) \cdot d_{\mathcal{A}}(m)$. Thus $d_{\mathcal{A}}(m) \in Inj(\mathcal{A})$.

Now, on the one hand, by 2.17.1 and 5.10, every epimorphism in $\mathcal{D}$ is $c_{\mathcal{A}}$-dense. On the other hand, since $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $\mathcal{A}^{\perp}$=$Inj(\mathcal{A}) \cap Epi(\mathcal{X})$ (using 2.17) and, from the above theorem, $\mathcal{A}^{\perp} \subseteq \mathcal{D}$. Therefore, we have that

$$\mathcal{A}^{\perp} = (\mathcal{D} \cap Inj(\mathcal{A})) \cap Epi(\mathcal{X}) = \mathcal{D} \cap Epi(\mathcal{X}).$$

Consequently, using 6.4, every epimorphism belonging to $\mathcal{D}$ is $c_{\mathcal{A}}$-dense. □

7 Strongly closed objects and the orthogonal hull

In [40], J. Isbell defined an algebra A to be *absolutely closed* if, for every embedding $A \subseteq B$, the dominion of A in B (see Section 4) is just A. In [19], D. Dikranjan and E. Giuli extended this concept to the general setting of regular closure operators. Enlarging this concept to all closure operators, we have the following definition.

Definition 7.1 For a closure operator $c : \mathcal{M} \to \mathcal{M}$ of $\mathcal{X}$, an object $X \in \mathcal{X}$ is said to be *c-absolutely closed* if every morphism in $\mathcal{M}$ with domain X is c-closed.

The c-absolutely closed objects were studied in [19] and [64], when c is a regular closure operator.

In this section, we shall see that:

- The subcategory of all $c_{\mathcal{A}}$-absolutely closed objects is always contained in the orthogonal hull $\mathcal{O}(\mathcal{A})$.
- If $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms (thus, it is also a closure operator when restricted to $PS(\mathcal{M})$), the subcategory of all absolutely closed objects with respect to
$$c_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$$
coincides with the orthogonal hull of $\mathcal{A}$.

We remark that, in contrast, the subcategory of all absolutely closed objects with respect to the regular closure operator induced by $\mathcal{A}$ has a very irregular behaviour with respect to the orthogonal hull, even when $\mathcal{O}(\mathcal{A})$ is reflective (see [64] and remark in 8.8 below).

In order to characterize the orthogonal hull of a subcategory of $\mathcal{X}$ by means of the orthogonal closure operator, let us consider the following

Definition 7.2 An object $X \in \mathcal{X}$ is said to be *$\mathcal{A}$-strongly closed* provided that each morphism in $PS(\mathcal{M})$ with domain X is $c_{\mathcal{A}}$-closed.

We denote by $\mathcal{SCl}(\mathcal{A})$ the subcategory of all $\mathcal{A}$-strongly closed objects .

We shall prove that, under convenient assumptions, the subcategory $\mathcal{SC}l(\mathcal{A})$ and the orthogonal hull $\mathcal{O}(\mathcal{A})$ coincide.

It is obvious that every object in $\mathcal{A}$ is $c_{\mathcal{A}}$-absolutely closed and that $\mathcal{SC}l(\mathcal{A})$ contains all $c_{\mathcal{A}}$-absolutely closed objects. Next we show that, when $\mathcal{X}$ is the $\mathcal{E}$-reflective hull of $\mathcal{A}$, the inclusion $\mathcal{SC}l(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A})$ also holds.

Proposition 7.3 *For each subcategory $\mathcal{A}$ of $\mathcal{X}$, if $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, then $\mathcal{SC}l(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A})$.*

Proof. Let $\mathbb{M}(\mathcal{A}) = \mathcal{X}$ and let X be an $\mathcal{A}$-strongly closed object of $\mathcal{X}$. In order to show that $X \in \mathcal{O}(\mathcal{A})$, we consider $(m : Y \to Z) \in \mathcal{A}^{\perp}$, an $\mathcal{X}$-morphism $f : Y \to X$ and the pushout $(n : X \to W,\ f' : Z \to W)$ of (m, f). Then $(n : X \to W) \in \mathcal{A}^{\perp}$ and so, by 6.4, n is $c_{\mathcal{A}}$-dense, thus $c_{\mathcal{A}}(n) \cong 1_W$. On the other hand, since X is $\mathcal{A}$-strongly closed, $c_{\mathcal{A}}(n) \cong n$. Hence we have $n \cong 1_W$ and $(n^{-1} \cdot f') \cdot m = f$. Therefore, X is m-injective. But, by 2.17, $m \in Epi(\mathcal{X})$ and so $X \in \mathcal{O}(\mathcal{A})$. □

Remark 7.4 Both the inclusion of $\mathcal{A}$ in the subcategory of all $c_{\mathcal{A}}$-absolutely closed objects and the inclusion of the latter in $\mathcal{SC}l(\mathcal{A})$ may be strict (see 8.8 below). But we do not know any example with $\mathcal{O}(\mathcal{A}) \neq \mathcal{SC}l(\mathcal{A})$.

In what follows we show that the assumption that $c_{\mathcal{A}}$ *preserves $PS(\mathcal{M})$-morphisms*, that is, $c_{\mathcal{A}}(m) \in PS(\mathcal{M})$ whenever $m \in PS(\mathcal{M})$, has very relevant consequences.

Theorem 7.5 *Assuming that $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, the orthogonal closure operator $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms if and only if it is weakly hereditary in $PS(\mathcal{M})$. In this case, $c_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ is an idempotent, weakly hereditary closure operator and $\mathcal{O}(\mathcal{A}) = \mathcal{SC}l(\mathcal{A})$.*

Proof.

I. If $c_{\mathcal{A}}$ is weakly hereditary in $PS(\mathcal{M})$, consider a morphism $(m : X \to Y) \in PS(\mathcal{M})$, its factorization $X \xrightarrow{d_{\mathcal{A}}(m)} \overline{X} \xrightarrow{c_{\mathcal{A}}(m)} Y$ and a morphism $f : \overline{X} \to Z$. Let $(m^{\sharp}, f^{\sharp})$ be a pushout of $(c_{\mathcal{A}}(m), f)$. From 6.5, the fact that the morphism $d_{\mathcal{A}}(m) : X \to \overline{X}$ is $c_{\mathcal{A}}$-dense implies that it is an epimorphism. Since $(m^{\sharp}, f^{\sharp})$ is the pushout of $(c_{\mathcal{A}}(m), f)$, it is easy to see that $(m^\sharp, f^\sharp)$ is also the pushout of $(m, f \cdot d_{\mathcal{A}}(m))$. Then $m^\sharp \in \mathcal{M}$. Therefore, $c_{\mathcal{A}}(m) \in PS(\mathcal{M})$.

II. Suppose $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms. We prove that it is weakly hereditary. Given $(m : X \to Y) \in PS(\mathcal{M})$, we want to show that $c_{\mathcal{A}}(d_{\mathcal{A}}(m)) \cong 1_{\overline{X}}$. For each $g \in \mathcal{X}(X, \mathcal{A})$, let $(n, \hat{g})$ be a pushout of $(d_{\mathcal{A}}(m), g)$. Since $PS(\mathcal{M})$ is left-cancellable, $d_{\mathcal{A}}(m) \in PS(\mathcal{M})$ and, therefore, $n \in \mathcal{M}$. Let $(\hat{m}_g, g^\sharp)$ be a pullback of $(n, \hat{g})$, let (r, g') be a pushout of $(c_{\mathcal{A}}(m), \hat{g})$, let (s, h) be a pullback of (r, g') and let (d, g^*) be a pulback of (n, h) as illustrated in the following diagram.

$$
\begin{tikzcd}
X \arrow[rr, "d_{\mathcal{A}}(m)"] \arrow[ddd, "g"'] & & \overline{X} \arrow[rr, "c_{\mathcal{A}}(m)"] \arrow[ddd, "\hat{g}"] & & Y \arrow[ddd, "g'"] \\
& \hat{X}_g \arrow[ur, "\hat{m}_g"] \arrow[ddl, "g^\sharp"'] & & \bullet \arrow[ur, "s"] \arrow[ddl, "h"] & \\
& X_g \arrow[urr, "d"] \arrow[dl, "g^*"] & & & \\
\bullet \arrow[rr, "n"] & & \bullet \arrow[rr, "r"] & & \bullet
\end{tikzcd}
$$

Then $(s \cdot d, g^*)$ is a pullback of the pushout of (m, g). So $s \cdot d \in P_{\mathcal{A}}(m)$ and then, there exists some morphism $t_g : \overline{X} \to X_g$ such that $s \cdot d \cdot t_g = c_{\mathcal{A}}(m)$. Thus, we have

$$r \cdot \hat{g} = g' \cdot c_{\mathcal{A}}(m) = g' \cdot s \cdot d \cdot t_g = r \cdot h \cdot d \cdot t_g = r \cdot n \cdot g^* \cdot t_g.$$

It follows that $\hat{g} \cdot 1_{\overline{X}} = n \cdot g^* \cdot t_g$, because $r \in \mathcal{M}$. Since $(\hat{m}_g, g^\sharp)$ is the pullback of $(n, \hat{g})$, there is a morphism $w : \overline{X} \to \hat{X}_g$ such that $\hat{m}_g \cdot w = 1_{\overline{X}}$. Thus, for each $g \in \mathcal{X}(X, \mathcal{A})$ we have that $\hat{m}_g \cong 1_{\overline{X}}$, so $c_{\mathcal{A}}(d_{\mathcal{A}}(m)) \cong 1_{\overline{X}}$.

III. Now, let $c_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ be a weakly hereditary closure operator. By 6.4 and taking into account that $\mathcal{A}^\perp$ is closed under composition, we have that the class of all $c_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphisms are closed under composition. Together with the fact that $c_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ is weakly hereditary, this implies, by 4.4, that $c_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ is idempotent.

Finally, we want to show that $\mathcal{O}(\mathcal{A}) \subseteq \mathcal{SCl}(\mathcal{A})$. Let $X \in \mathcal{O}(\mathcal{A})$. If $(m : X \to Y) \in PS(\mathcal{M})$, then $d_{\mathcal{A}}(m)$ is a $c_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphism and, then, by 6.4, $d_{\mathcal{A}}(m)$ belongs to $\mathcal{A}^\perp$. The fact that $X \in \mathcal{O}(\mathcal{A})$ and $(d_{\mathcal{A}}(m) : X \to \overline{X}) \in \mathcal{A}^\perp$ implies that $d_{\mathcal{A}}(m)$ is an isomorphism and, then, $c_{\mathcal{A}}(m) \cong m$. Thus X is $\mathcal{A}$-strongly closed. Therefore, from 7.3, we have that $\mathcal{O}(\mathcal{A}) = \mathcal{SC}l(\mathcal{A})$. $\square$

Corollary 7.6 *If $\mathcal{M} = PS(\mathcal{M})$ and $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, then $c_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ is an idempotent, weakly hereditary closure operator and $\mathcal{O}(\mathcal{A}) = \mathcal{SC}l(\mathcal{A})$.* $\square$

Remark 7.7 For some $(\mathcal{E}, \mathbb{M})$-categories, the equality $\mathcal{M} = PS(\mathcal{M})$ holds. This is the case, for instance, of the categories $\mathcal{T}op$, $\mathcal{T}op_0$, $\mathcal{M}et_*$, $\mathcal{N}orm_*$ and $\mathcal{T}f\mathcal{A}b$, when $\mathbb{M}$ is the conglomerate of initial monosources.

However, there are several examples of categories for which $\mathcal{M} \neq PS(\mathcal{M})$. In the last section of the present chapter, we study the class $PS(\mathcal{M})$ for several categories.

The following corollary gives conditions under which the orthogonal closure operator and the regular one coincide.

Corollary 7.8 *Let $\mathcal{X}$ have equalizers, $RegMono(\mathcal{X}) \subseteq \mathcal{M}$, $\mathcal{M} = PS(\mathcal{M})$ and let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. If the regular closure operator $r_{\mathcal{A}}$ is weakly hereditary and all $\mathcal{X}$-epimorphisms are $c_{\mathcal{A}}$-dense, then $r_{\mathcal{A}} = c_{\mathcal{A}}$.*

Proof. Under the above conditions, the $r_{\mathcal{A}}$-dense morphisms are just the $\mathcal{X}$-epimorphisms and, then, by 4.6, they are just the $c_{\mathcal{A}}$-dense morphisms. Let $m : X \to Y$ belong to $\mathcal{M}$. From 5.8, there is a morphism d such that $c_{\mathcal{A}}(m) = r_{\mathcal{A}}(m) \cdot d$, and, since $r_{\mathcal{A}}$ is weakly hereditary, the morphism $d \cdot d_{\mathcal{A}}(m)$ is $r_{\mathcal{A}}$-dense, therefore, by hyphotesis, it is also $c_{\mathcal{A}}$-dense. On the other hand, from the fact that $c_{\mathcal{A}}$ is an idempotent, weakly hereditary closure operator (see 7.6), it follows that $\mathcal{X}$ has an ($c_{\mathcal{A}}$-dense, $c_{\mathcal{A}}$-closed)-factorization system with respect to $\mathcal{M}$ (see 4.3) and, thus, the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "d_{\mathcal{A}}(m)"] \arrow[d, "1_X"] & \bullet \arrow[r, "d"] & \bullet \arrow[dr, "r_{\mathcal{A}}(m)"] & \\ X \arrow[rr, "d_{\mathcal{A}}(m)"] & & \bullet \arrow[r, "c_{\mathcal{A}}(m)"] & Y \end{tikzcd}$$

implies the existence of a morphism t such that $t \cdot d \cdot d_{\mathcal{A}}(m) = d_{\mathcal{A}}(m)$. So, one derives that d is an isomorphism and, consequently, $r_{\mathcal{A}}(m) \cong c_{\mathcal{A}}(m)$. $\square$

Concerning regular closure operators, we have by Remark 4.6 that if $\mathcal{X}$ has equalizers and $RegMono(\mathcal{X}) \subseteq \mathcal{M}$, then, for any two subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$ with the same $\mathcal{E}$-reflective hull, the equality $r_{\mathcal{A}} = r_{\mathcal{B}}$ holds. The following proposition and corollary show that, under suitable conditions, a similar result holds if *orthogonal* replaces *regular* and *the orthogonal hull* of $\mathcal{A}$ replaces *the $\mathcal{E}$-reflective hull.*

Proposition 7.9 *Let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of $\mathcal{X}$ with $\mathbb{M}(\mathcal{A}) = \mathcal{X}$.*

1. *If $c_{\mathcal{A}}(m) \leq c_{\mathcal{B}}(m)$ for all $m \in PS(\mathcal{M})$, then $\mathcal{O}(\mathcal{B}) \subseteq \mathcal{O}(\mathcal{A})$.*

2. *If $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms and $\mathcal{B} \subseteq \mathcal{O}(\mathcal{A})$, then*

$$(c_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})) \leq (c_{\mathcal{B}} : PS(\mathcal{M}) \to PS(\mathcal{M})).$$

Proof.

1. If $c_{\mathcal{A}} \leq c_{\mathcal{B}}$ in $PS(\mathcal{M})$, then every $c_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphism is $c_{\mathcal{B}}$-dense. Thus, using 6.4, 6.5 and 5.9, we have that

$$\begin{aligned} \mathcal{A}^{\perp} &= \{m \in PS(\mathcal{M}) \,|\, m \text{ is } c_{\mathcal{A}}\text{-dense}\} \\ &\subseteq \{m \in PS(\mathcal{M}) \,|\, m \text{ is } c_{\mathcal{B}}\text{-dense}\} \subseteq B^{\perp} \end{aligned}$$

and so $\mathcal{O}(\mathcal{B}) \subseteq \mathcal{O}(\mathcal{A})$.

2. Given $(m : X \to Y) \in PS(\mathcal{M})$, we show that $c_{\mathcal{A}}(m) \leq m_h$ for all $m_h \in P_{\mathcal{B}}(m)$ and then, since $c_{\mathcal{B}}(m) = \wedge P_{\mathcal{B}}(m)$, it follows that $c_{\mathcal{A}}(m) \leq c_{\mathcal{B}}(m)$. Let $h : X \to B$ be a morphism with codomain in $\mathcal{B}$, let (m', h') be a pushout of (m, h) and let (m_h, h^*) be a pullback of (m', h'). Since $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms and $PS(\mathcal{M})$ is left-cancellable, $d_{\mathcal{A}}(m) \in PS(\mathcal{M})$ and, by 7.5 and 6.4, $d_{\mathcal{A}}(m) \in \mathcal{A}^{\perp}$. Since $B \in \mathcal{O}(\mathcal{A})$, there is a morphism $h^{\sharp}$ such that $h^{\sharp} \cdot d_{\mathcal{A}}(m) = h$.

$$
\begin{tikzcd}
\overline{X} \arrow[drrr, "c_{\mathcal{A}}(m)"] \arrow[dddr, "h^{\sharp}"'] & & & \\
& X \arrow[ul, "d_{\mathcal{A}}(m)"'] \arrow[rr, "m"] \arrow[dd, "h"] & & Y \arrow[dd, "h'"] \\
& & \bullet \arrow[ur, "m_h"] \arrow[dl, "h^*"] & \\
& B \arrow[rr, "m'"] & & \bullet
\end{tikzcd}
$$

Thus,

$$h' \cdot c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m) = m' \cdot h = m' \cdot h^{\sharp} \cdot d_{\mathcal{A}}(m)$$

and, from the fact that $d_{\mathcal{A}}(m)$ is an epimorphism (by 6.5), it follows that $h' \cdot c_{\mathcal{A}}(m) = m' \cdot h^{\sharp}$. As (m_h, h^*) is the pullback of (m', h'), there exists a morphism t such that $m_h \cdot t = c_{\mathcal{A}}(m)$, that is, $c_{\mathcal{A}}(m) \leq m_h$. $\square$

Corollary 7.10 *If $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathbb{M}(\mathcal{B}) = \mathcal{X}$ and $c_{\mathcal{A}}$ and $c_{\mathcal{B}}$ preserve $PS(\mathcal{M})$-morphisms (in particular, if $\mathcal{M}$ is pushout-stable), then $c_{\mathcal{A}} = c_{\mathcal{B}}$ with respect to $PS(\mathcal{M})$ if and only if $\mathcal{O}(\mathcal{A}) = \mathcal{O}(\mathcal{B})$.*

8 The orthogonal closure operator versus reflectivity

Let $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. It is clear that if $\mathcal{O}(\mathcal{A})$ is reflective in $\mathcal{X}$ then, for each $X \in \mathcal{X}$, the reflection of X in $\mathcal{O}(\mathcal{A})$ is a morphism of $PS(\mathcal{M})$ with codomain in $\mathcal{O}(\mathcal{A})$. The next theorem, which is the main result of this section, states that, if $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms and every $\mathcal{X}$-object is a $PS(\mathcal{M})$-subobject of an object in $\mathcal{O}(\mathcal{A})$ then $\mathcal{O}(\mathcal{A})$ is reflective.

Theorem 8.1 *If $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms and for every $X \in \mathcal{X}$ there is a morphism in $PS(\mathcal{M})$ with domain X and codomain in $\mathcal{O}(\mathcal{A})$, then $\mathcal{O}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{X}$.*

Proof. It is clear that, under the above assumptions, $\mathbb{M}(\mathcal{A}) = \mathcal{X}$.

First, we prove that if Y is an $\mathcal{A}$-strongly closed object and $(m : X \to Y) \in PS(\mathcal{M})$ then the domain $\overline{X}$ of $c_{\mathcal{A}}(m)$ is an $\mathcal{A}$-strongly closed object too. Consider such Y and m, and let $n : \overline{X} \to Z$ be a morphism in $PS(\mathcal{M})$. We want to show that $c_{\mathcal{A}}(n) \cong n$. Let the diagram

$$\begin{tikzcd}
\overline{X} \arrow[r, "n"] \arrow[d, "c_{\mathcal{A}}(m)"'] & Z \arrow[d, "u"] \\
Y \arrow[r, "n'"] & \bullet
\end{tikzcd}$$

be a pushout of $(n, c_{\mathcal{A}}(m))$. Then $n' \in PS(\mathcal{M})$ and $c_{\mathcal{A}}(n') \cong n'$. The morphism $(c_{\mathcal{A}}(m), u) : n \to n'$ in the category $\mathcal{X}^2$ is a morphism in the category $PS(\mathcal{M})$. Let $c_{\mathcal{A}}((c_{\mathcal{A}}(m), u)) = (t, u) : c_{\mathcal{A}}(n) \to c_{\mathcal{A}}(n')$. Since $c_{\mathcal{A}}(n') \cong n'$, we have that, for a suitable t', the following diagram is commutative

$$\begin{tikzcd}
\bullet \arrow[r, "c_{\mathcal{A}}(n)"] \arrow[d, "t'"'] & Z \arrow[d, "u"] \\
Y \arrow[r, "n'"] & \bullet
\end{tikzcd}$$

Hence

$$n' \cdot c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m) = u \cdot n \cdot d_{\mathcal{A}}(m) = u \cdot c_{\mathcal{A}}(n) \cdot d_{\mathcal{A}}(n) \cdot d_{\mathcal{A}}(m) = n' \cdot t' \cdot d_{\mathcal{A}}(n) \cdot d_{\mathcal{A}}(m). \quad (5)$$

Since n' is a monomorphism, it follows that

$$c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m) = t' \cdot d_{\mathcal{A}}(n) \cdot d_{\mathcal{A}}(m). \quad (6)$$

By 7.5, $c_{\mathcal{A}} : PS(\mathcal{M}) \to \mathcal{PS}(\mathcal{M})$ is an idempotent, weakly hereditary closure operator and this fact implies that:

- $d_{\mathcal{A}}(n)$ and $d_{\mathcal{A}}(m)$ are $c_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphisms; in particular, from 6.5, $d_{\mathcal{A}}(m)$ is an epimorphism;
- $c_{\mathcal{A}}(m)$ is $c_{\mathcal{A}}$-closed.

Consequently, on the one hand, from the equality (6), we get that

$$c_{\mathcal{A}}(m) \cdot 1_{\overline{X}} = t' \cdot d_{\mathcal{A}}(n);$$

on the other hand, this equality and the fact that $\mathcal{X}$ has an ($c_{\mathcal{A}}$-dense, $c_{\mathcal{A}}$-closed)-factorization system with respect to $PS(\mathcal{M})$ (see 4.3 and 7.5) imply the existence of a morphism s such that $s \cdot d_{\mathcal{A}}(n) = 1_{\overline{X}}$. Therefore, $d_{\mathcal{A}}(n)$ is an isomorphism and, consequently, $c_{\mathcal{A}}(n) \cong n$.

Now, let $X \in \mathcal{X}$ and $(m : X \to Y) \in PS(\mathcal{M})$ with $Y \in \mathcal{O}(\mathcal{A})$. Then from 6.4 and 7.5 it follows that $d_{\mathcal{A}}(m) : X \to \overline{X}$ is a reflection of X in $\mathcal{O}(\mathcal{A})$. □

Corollary 8.2 *Let $\mathcal{A}$ be a subcategory of an $\mathcal{M}$-complete $(\mathcal{E}, \mathbb{M})$-category with pushouts. If $\mathcal{M}$ is pushout-stable in $\mathbb{M}(\mathcal{A})$ and each $X \in \mathbb{M}(\mathcal{A})$ is an $\mathcal{M}$-subobject of some object in $\mathcal{O}(\mathcal{A})$, then the orthogonal hull of $\mathcal{A}$ is its reflective hull.* □

The study of the reflectors which preserve morphisms of $\mathcal{M}$ has been performed, for instance, in [54]. In the following proposition we show that under the above conditions the reflectors always preserve $PS(\mathcal{M})$-morphisms.

Proposition 8.3 *If $\mathcal{A}$ is $\mathcal{M}$-reflective in $\mathcal{X}$, then a corresponding reflector preserves morphisms of $PS(\mathcal{M})$.*

Proof. It is clear that, since $\mathcal{A}$ is $\mathcal{M}$-reflective in $\mathcal{X}$, we have that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. Let $R : \mathcal{X} \to \mathcal{A}$ be the corresponding reflector, let $m : X \to Y$ belong to $PS(\mathcal{M})$ and let us consider the following diagram

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dd, "r_X"'] & & Y \arrow[dd, "r_Y"] \arrow[dl, "r'"] \\
& \bullet \arrow[dr, "t"] & \\
RX \arrow[rr, "Rm"'] \arrow[ur, "m'"] & & RY
\end{tikzcd}$$

where (m', r') is a pushout of (m, r_X) and t is the unique morphism which turns the two smaller triangles commutative. The fact that $r_X \in \mathcal{A}^{\perp}$ implies that $r' \in \mathcal{A}^{\perp}$, by 1.4.4, and then, since $r_Y \in \mathcal{A}^\perp$, it follows from 1.4.2 that $t \in \mathcal{A}^\perp$, so that $t \in PS(\mathcal{M})$. Now, the morphism Rm is the composition of two morphisms of $PS(\mathcal{M})$, hence it belongs to $PS(\mathcal{M})$. $\square$

Remark 8.4 From 8.3, it follows that if $\mathcal{M}$ is stable under pushouts, then, for every $\mathcal{M}$-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, a corresponding reflector preserves $\mathcal{M}$-morphisms. In fact, examining the proof of 8.3, it is clear that, instead of the stability of $\mathcal{M}$ under pushouts, it suffices that $\mathcal{X}$ has $\mathcal{M}$-amalgamations, that is, that the pushout of a pair of morphisms in $\mathcal{M}$ is a pair of morphisms in $\mathcal{M}$.

Proposition 8.5 *If $\mathcal{A}$ is an $\mathcal{M}$-reflective subcategory of $\mathcal{X}$, then:*

1. *For each $X \overset{m}{\to} Y \in \mathcal{M}$ and each reflection $X \overset{r_X}{\to} RX$ of X in $\mathcal{A}$, we have*

$$c_{\mathcal{A}}(m) = m_{r_X},$$

that is, $c_{\mathcal{A}}(m)$ is obtained by forming a pushout $(\overline{m}, r')$ of (m, r_X) and taking a pullback of the $\mathcal{M}$-part of $\overline{m}$ along r'.

2. *If $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms then each reflection of a $PS(\mathcal{M})$-morphism is $c_{\mathcal{A}}$-closed.*

Proof.

1. Given a morphism $g : X \to A$ with codomain in $\mathcal{A}$, let $\overline{g} : RX \to A$ be such that $\overline{g} \cdot r_X = g$. Then, we obtain the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dd, "r_X"'] & & Y \arrow[dd, "r'"] & X_g \arrow[l, "m_g"'] \arrow[dddd, "g^*"] \\
& \bullet \arrow[ur, "m_{r_X}"] \arrow[dd, "r^*"] & & \\
RX \arrow[rr, "\overline{m}"] \arrow[dr, "e"] \arrow[dd, "\overline{g}"'] & & \bullet \arrow[dd, "g'"] & \\
& \bullet \arrow[ur, "m'"] \arrow[dd, "t"] & & \\
A \arrow[rr, "\overline{\overline{m}}"] \arrow[dr, "e'"] & & \bullet & \bullet \arrow[l, "m''"] \\
& \bullet \arrow[ur, "m''"] & &
\end{tikzcd}$$

where $(\overline{m}, r')$ and $(\overline{\overline{m}}, g')$ are pushouts of (m, r_X) and $(\overline{m}, \overline{g})$, respectively; $m' \cdot e$ and $m'' \cdot e'$ are $(\mathcal{E}, \mathcal{M})$-factorizations of $\overline{m}$ and $\overline{\overline{m}}$, respectively; t is the unique morphism such that $t \cdot e = e' \cdot \overline{g}$ and $m'' \cdot t = g' \cdot m'$; (m_{r_X}, r^*) and (m_g, g^*) are pullbacks of (m', r') and $(m'', g' \cdot r')$, respectively.

Thus, there is a unique morphism n such that $m_g \cdot n = m_{r_X}$. Consequently, for each $m_g \in P_{\mathcal{A}}(m)$ we have that $m_{r_X} \leq m_g$ and, since m_{r_X} also belongs to $P_{\mathcal{A}}(m)$, $c_{\mathcal{A}}(m) = m_{r_X}$.

2. Let $c_{\mathcal{A}}$ preserve $PS(\mathcal{M})$-morphisms and let $m \in PS(\mathcal{M})$. We want to prove that Rm is $c_{\mathcal{A}}$-closed, where R is a reflector from $\mathcal{X}$ to $\mathcal{A}$. First we consider a morphism $(m : X \to A) \in PS(\mathcal{M})$ with $A \in \mathcal{A}$. Let $m^\sharp$ be the unique morphism such that $m^\sharp \cdot r_X = m$, where $r_X : X \to RX$ is a reflection of X in $\mathcal{A}$. Since $\mathcal{A}$ is reflective, the equality $\mathcal{A} = \mathcal{O}(\mathcal{A})$ holds and then, following the proof of 8.1, we have that $r_X \cong d_{\mathcal{A}}(m)$ and, thus, $(c_{\mathcal{A}}(m) : \overline{X} \to A) \cong (m^\sharp : RX \to A)$. Now, let $(m : X \to Y) \in PS(\mathcal{M})$ and let $r_Y : Y \to RY$ be a reflection of Y in $\mathcal{A}$. Since $r_Y \in \mathcal{A}^\perp$, $r_Y \in PS(\mathcal{M})$ and so $r_Y \cdot m \in PS(\mathcal{M})$. Then, we have that $Rm \cong c_{\mathcal{A}}(r_Y \cdot m)$ and as, by 7.5, $c_{\mathcal{A}}$ is idempotent, Rm is $c_{\mathcal{A}}$-closed. □

From 7.5 and 8.1 it is clear that conditions under which the orthogonal closure operator $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms are important. The following proposition gives us a way to obtain some positive examples.

Proposition 8.6 *Let $\mathcal{M}$ satisfy the following condition:*

> ($\mathbf{D}$) *If the composite $c \cdot d$ of two $\mathcal{M}$-morphisms is an epimorphism, then the first one, d, is also an epimorphism.*

Then, for every $\mathcal{M}$-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms.

Proof. Let $(m : X \to Y) \in PS(\mathcal{M})$. Then by 8.5.1 $c_{\mathcal{A}}(m)$ is just the pullback of the pushout of m along r_X, as illustrated by the following diagram

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dd, "r_X"'] \arrow[dr, "d_{\mathcal{A}}(m)"] & & Y \arrow[dd, "r'"] \\
& \bullet \arrow[ur, "c_{\mathcal{A}}(m)"'] \arrow[dl, "r^*"] & \\
RX \arrow[rr, "m'"'] & & \bullet
\end{tikzcd}$$

Since $r_X \in PS(\mathcal{M})$, $r' \in \mathcal{M}$ and so $r^* \in \mathcal{M}$. Hence, using the property ($\mathbf{D}$) and the fact that $r_X \in Epi(\mathcal{X})$ (by 2.17), it follows that $d_{\mathcal{A}}(m) \in Epi(\mathcal{X})$, thus, by 6.5, $d_{\mathcal{A}}(m)$ is $c_{\mathcal{A}}$-dense. $\square$

Examples 8.7 Next, we list some examples of categories $\mathcal{X}$ and classes of monomorphisms $\mathcal{M}$ for which the property ($\mathbf{D}$) holds.

1. $\mathcal{X} = \mathcal{S}et$ and $\mathcal{M}$ is the class of all monomorphisms;

2. $\mathcal{X} = \mathcal{T}op$ or $\mathcal{X} = \mathcal{T}op_0$ or $\mathcal{X} = \mathcal{T}ych$ and $\mathcal{M}$ is the class of all embeddings;

3. $\mathcal{X} = \mathcal{T}f\mathcal{A}b$ and $\mathcal{M}$ is the class of all monomorphisms.

Examples 8.8

1. Let $\mathcal{X}$=$\mathcal{T}op_0$ and let $\mathbb{M}$ be the conglomerate of all initial monosources. Then $\mathcal{M}$ is the class of all embeddings and it coincides with $PS(\mathcal{M})$. We can consider every $(m : X \to Y) \in \mathcal{M}$ as the inclusion of a subspace X in Y. Thus, if $c : \mathcal{M} \to \mathcal{M}$ is a closure operator, we identify $c(m)$ with the corresponding subspace of Y which we denote by $c(X)$.

   If $\mathcal{S}$ is the subcategory of all Sierpiński spaces, then $\mathbb{M}(\mathcal{S}) = \mathcal{X}$. It was shown in [60] that the corresponding regular closure operator $r_{\mathcal{S}} : \mathcal{M} \to \mathcal{M}$ is the b-closure, i.e., given $Y \in \mathcal{T}op_0$, for every subspace X of Y,

   $$r_{\mathcal{S}}(X) = \{y \in Y \mid \overline{\{y\}} \cap H \cap X \neq \emptyset \text{ for every open neighborhood } H \text{ of } y \text{ in } Y\}.$$

   As we proved in 5.8, $c_{\mathcal{S}}(X) \subseteq r_{\mathcal{S}}(X)$ for every subspace X of Y. In fact, as we shall see, we have that $c_{\mathcal{S}}(X) = r_{\mathcal{S}}(X)$.

   By 8.1, we have that $\mathcal{SCl}(\mathcal{S}) = \mathcal{O}(\mathcal{S})$ and that $\mathcal{SCl}(\mathcal{S})$ is the reflective hull of $\mathcal{S}$ in $\mathcal{X}$, that is, the subcategory of all sober spaces.

2. As a matter of fact, the examples from 4., 5., 6. and 8. of 1.5 provide a situation similar to the last one, that is, for the corresponding $\mathcal{M}$, it holds that $\mathcal{M} = PS(\mathcal{M})$, $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $c_{\mathcal{A}} = r_{\mathcal{A}}$ and $\mathcal{SCl}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{X}$. We shall see in the next section why the orthogonal and regular closure operators coincide in these cases.

3. Let $\mathcal{X}$ and $\mathbb{M}$ be as in 1. and let $\mathcal{N}$ be the subcategory of $\mathcal{T}op_0$ having as objects those spaces which are isomorphic to $\mathbf{N}$, where $\mathbf{N}$ is the set $\mathbb{N} = \{1, 2, ...\}$ with the upper topology with respect to the natural order. (That is, the non-empty open sets are just all $\uparrow n = \{m \in \mathbb{N} \mid m \geq n\}$ for a natural number n.)

   Since $\mathbb{M}(\mathcal{N}) = \mathcal{X}$, we have that $r_{\mathcal{N}} = r_{\mathcal{X}}$ (by 4.6.2), i.e., $r_{\mathcal{N}}$ is the b-closure. But the inequality $c_{\mathcal{N}} \leq r_{\mathcal{N}}$ is strict; in fact, let Y be the set $\mathbb{N} \cup \{\infty\}$ endowed with the topology whose non-empty open sets are all $\uparrow n \cup \{\infty\}$, $n \in \mathbb{N}$. Thus $\mathbf{N}$ is a subspace of Y; it is clear that $r_{\mathcal{N}}(N) = Y$ and, on the other hand, $c_{\mathcal{N}}(N) = N$, since $N \in \mathcal{N}$ (by 5.6.1).

   We have again that the subcategory $\mathcal{SCl}(\mathcal{N})$ is the reflective hull of $\mathcal{N}$ in $\mathcal{T}op_0$.

4. Let $\mathcal{X}$ and $\mathcal{A}$ be the categories of Example 2.5. Let $\mathbb{M}$ be the conglomerate of all initial monosources of $\mathcal{X}$; clearly, $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. In this case we have that $\mathcal{M} \neq PS(\mathcal{M})$; in fact, let X be a set, let $Y = X \dot{\cup} \{a\}$ and let $y = (y_i)_{i \in Ord}$ with $y_i = a$ for every i. Then $m : (X, x) \to (Y, y)$, where m is the inclusion of X in Y and x is the empty map, is a morphism of $\mathcal{M} \backslash PS(\mathcal{M})$.

   It is easy to see that the closure operators $c_{\mathcal{A}}$ and $r_{\mathcal{A}}$ coincide in $\mathcal{M}$, that $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms and that the $c_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphisms are just the isomorphisms, hence $\mathcal{SC}l(\mathcal{A}) = \mathcal{X}$.

Remark 8.9 Obviously, in the examples in 8.8.1 and 8.8.3, the notions of $\mathcal{A}$-strongly closed and $c_{\mathcal{A}}$-absolutely closed object coincide. We emphasize that, in some sense, this concept of closedness for objects has a better behaviour when we deal with orthogonal closure operators than when we deal with regular ones. Indeed, as we have seen, under mild conditions, $\mathcal{A} \subseteq \mathcal{SC}l(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A})$ and, adding the assumption that $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms, $\mathcal{O}(\mathcal{A}) = \mathcal{SC}l(\mathcal{A})$; whereas, with respect to the regular closure operator, we have that, for instance, $\mathbf{N}$ is not $r_{\mathcal{N}}$-absolutely closed although it belongs to $\mathcal{N}$ as observed by M. Sobral in [64].

9 Pushout-stable $\mathcal{M}$-morphisms

As we have seen, for certain classes $\mathcal{M}$ of monomorphisms, the class $PS(\mathcal{M})$ plays an important rôle in the characterization of $\mathcal{A}^{\perp}$, $\mathcal{O}(\mathcal{A})$ and the reflectivity of $\mathcal{O}(\mathcal{A})$. We thus want to study the class $PS(\mathcal{M})$ in everyday categories.

Let us remark that in the literature pushout-stability of a class $\mathcal{M}$ or just the existence of $\mathcal{M}$-amalgamations has been studied (see [45] and references there). We are interested in a more general study of this subject.

The dual question, that is, the determination, for a given class of epimorphisms $\mathcal{E}$, of its subclass

$$\mathcal{E}' = \{e \in \mathcal{E} \mid \text{any pulback of } e \text{ along an arbitrary morphism belongs to } \mathcal{E}\}$$

has been investigated by several authors. In [16], Day and Kelly characterized $\mathcal{E}'$ for $\mathcal{X} = \mathcal{T}op$ and $\mathcal{E}$ the class of all quotients. This has been important in *Descent Theory*; in fact, in $\mathcal{T}op$ the above class $\mathcal{E}'$ is just the class of descent morphisms (see [41], [53] and [65]).

In 6.3.1, we have shown that, if $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, then $Inj(\mathcal{A}) \subseteq PS(\mathcal{M})$. We are going to see that there are several examples for which the equality $Inj(\mathcal{A}) = PS(\mathcal{M})$ holds.

Proposition 9.1 *For any $(\mathcal{E}, \mathbb{M})$-category $\mathcal{X}$ with pushouts, if $\mathcal{X}$ has enough $\mathcal{M}$-injectives, then $\mathcal{M}$ is pushout-stable.*

Proof. For $\mathcal{A} = Inj(\mathcal{M})$, we have $\mathbb{M}(\mathcal{A}) = \mathcal{X}$ and, consequently, $Inj(\mathcal{A}) \subseteq PS(\mathcal{M})$. On the other hand, we clearly have that $\mathcal{M} \subseteq Inj(\mathcal{A})$. Therefore,

$$Inj(\mathcal{A}) = \mathcal{M} = PS(\mathcal{M})$$

and, thus, $\mathcal{M}$ is stable under pushouts. $\square$

Examples 9.2

1. Here we list some examples of a category $\mathcal{X}$ and a subcategory $\mathcal{A}$ such that, for the class $\mathcal{M}$ of all initial monomorphisms,

$$Inj(\mathcal{A}) = PS(\mathcal{M}) = \mathcal{M}.$$

   (a) $\mathcal{X} = \mathcal{M}et$ and $\mathcal{A}$ is the subcategory of all complete metric spaces;

   (b) $\mathcal{X} = \mathcal{N}orm$ and $\mathcal{A} = \mathcal{B}an$;

   (c) $\mathcal{X} = \mathcal{T}f\mathcal{A}b$ and $\mathcal{A}$ is the subcategory of all divisible torsion-free abelian groups.

2. The examples listed next are of the same type, that is, the equalities $Inj(\mathcal{A}) = PS(\mathcal{M}) = \mathcal{M}$ also hold, but now $\mathcal{A}$ is a subcategory generated by a unique object A. In fact, the following categories $\mathcal{X}$ are simple epireflective subcategories of $\mathcal{T}op$ (i.e., $\mathcal{X}$ is the epireflective hull of a topological space A in $\mathcal{T}op$).

   (a) In $\mathcal{T}op_0$, $\mathcal{M} = Inj(S)$, where S denotes the Sierpinski space.

   (b) For the subcategory $\mathcal{I}nd$ of all indiscrete spaces, we have that $\mathcal{M} = Inj(I_2)$, where I_2 denotes the indiscrete spaces of cardinality 2.

(c) For $\mathcal{T}op$, we have that $\mathcal{M} = Inj(C_1)$, where C_1 is the space with underlying set $\{0, 1, 2\}$ and whose only non trivial open set is $\{0\}$.

Remark 9.3 It was stated in [45] without proof that, in $\mathcal{T}op_1$, each diagram of the form

$$\begin{tikzcd} {} \arrow[r, "m"] \arrow[d, "g"'] & {} \\ {} & \end{tikzcd}$$

with m an embedding, can be "completed" by a pair of morphisms (m', g') such that m' is an embedding and $g' \cdot m = m' \cdot g$. Since $\mathcal{T}op_1$ has pushouts, this is equivalent to say that $\mathcal{M} = PS(\mathcal{M})$ for $\mathcal{M}$ the class of all embeddings. But this equality does not hold in $\mathcal{T}op_1$. Moreover, the equality $\mathcal{M} = PS(\mathcal{M})$ is not true for any epireflective subcategory $\mathcal{X}$ of $\mathcal{T}op$ contained in $\mathcal{T}op_1$ and having a space with more than one point as we now show[1]: We first recall that such a subcategory $\mathcal{X}$ has to contain all the 0-dimensional Hausdorff spaces. Now, let $X = [0, 1] \cap \mathbb{Q}$ (with the euclidean topology), and consider the embedding $m : X \backslash \{\frac{1}{2}\} \to X$. Let $D = \{0, 1\}$ be discrete, and let $f : X \backslash \{\frac{1}{2}\} \to D$ be defined by $f(x) = 0$ for all $x < \frac{1}{2}$ and $f(x) = 1$ for all $x > \frac{1}{2}$. Then the pushout of m along f in $\mathcal{X}$ is $D \to \{*\}$. Therefore, $m \notin PS(\mathcal{M})$.

In the following examples we characterize the class $PS(\mathcal{M})$ for some of such epireflective subcategories of $\mathcal{T}op$.

The following lemma will be useful to characterize the class $PS(\mathcal{M})$ in the next group of examples.

Lemma 9.4 *If $\mathcal{X} = \mathbb{M}(\mathcal{A})$ and*

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "g"'] & Y \arrow[d, "\overline{g}"] \\ Z \arrow[r, "\overline{m}"] & W \end{tikzcd}$$

is a pushout diagram in $\mathcal{X}$, then $\overline{m} \in \mathcal{M}$ iff there exist sources $(m_i : Z \to A_i)_I \in \mathbb{M}$ and $(f_i : Y \to A_i)_I$, with $A_i \in Obj(\mathcal{A})$, $i \in I$, such that $f_i \cdot m = m_i \cdot g$ for each $i \in I$.

[1]M. M. Clementino, private communication

Proof. Let $\overline{m} \in \mathcal{M}$; since $W \in \mathbb{M}(\mathcal{A})$, there is some $(h_i : W \to A_i)_I \in \mathbb{M}$ with $A_i \in Obj(\mathcal{A})$, $i \in I$. Then $(m_i)_I = (h_i \cdot \overline{m})_I$ and $(f_i)_I = (h_i \cdot \overline{g})_I$ satisfy the required condition.

Conversely, let $(m_i)_I$ and $(f_i)_I$ fulfil the above condition condition. Hence, for each $i \in I$, there is t_i such that $t_i \cdot \overline{m} = m_i$ and $t_i \cdot \overline{g} = f_i$. Now, the equality $(t_i)_I \cdot \overline{m} = (m_i)_I$ with $(m_i)_I \in \mathbb{M}$ implies that $\overline{m} \in \mathbb{M}$. □

Examples 9.5 In the following examples of epireflective subcategories of $\mathcal{T}op$, $\mathbb{M}$ always denote the class of all embeddings. We characterize the class $PS(\mathcal{M})$ and we show that, in these cases, we have again the equality $PS(\mathcal{M}) = Inj(A)$ for a convenient topological space A. For all of the examples below, the equality $\mathbb{M}(\mathcal{A}) = \mathcal{X}$ was proved in [32].

1. *Let $\mathcal{X}$ be the category 0-dim$\mathcal{H}$aus of all* 0*-dimensional Hausdorff spaces and continuous maps and let D_2 denote the set* $\{0, 1\}$ *with the discrete topology. Then*

$$PS(\mathcal{M}) = Inj(D_2)$$
$$= \{X \overset{m}{\to} Y \in \mathcal{M} \mid A \text{ is clopen in } X \Rightarrow A = m^{-1}(B) \text{ for some clopen set } B \text{ in } Y\}.$$

First, let us show that $PS(\mathcal{M}) \subseteq Inj(D_2)$, so that $PS(\mathcal{M}) = Inj(D_2)$. Let $m : X \to Y$ belong to $PS(\mathcal{M})$ and consider a morphism $f : X \to D_2$. If f is constant, it is clear that there is $\overline{f} : Y \to D_2$ such that $\overline{f} \cdot m = f$. If f is not constant, from the above lemma it follows that there are some $(n_i : D_2 \to D_2)_I$ and $(g_i : Y \to D_2)_I$ such that $n_i \cdot f = g_i \cdot m$, $i \in I$. Hence, since $(n_i)_I$ is a monosource, there exists $n : D_2 \to D_2$ and $g : Y \to D_2$ such that $g \cdot m = n \cdot f$ and $n(0) \neq n(1)$. If $n = 1_{D_2}$, then $\overline{f} = g$ fulfils the required equality; otherwise, $n \cdot n = 1_{D_2}$ and then $f = n \cdot n \cdot f = n \cdot g \cdot m$ and so we may choose $\overline{f} = n \cdot g$.

Now, let us show that $Inj(D_2)$ is as described above. Let $m : X \to Y$ belong to $Inj(D_2)$ and let G be a clopen set in X. Then $\chi_G : X \to D_2$, where $\chi_G(x) = 0$ iff $x \in G$, is a continuous map. Let $g : Y \to D_2$ be such that $g \cdot m = \chi_G$. Hence $g^{-1}(\{0\})$ is a clopen set in Y and $G = g^{-1}(\{0\}) \cap X$ (assuming that m is an inclusion).

Conversely, let $m : X \to Y$ satisfy the above condition and consider the morphism

$f : X \to D_2$. Let H be a clopen in Y such that $X \cap H = f^{-1}(\{0\})$. Then $\chi_H : Y \to D_2$ fulfils $\chi_H \cdot m = f$.

2. *Let $\mathcal{X}$ be the category 0-dim$\mathcal{T}$op of all* 0*-dimensional topological spaces and continuous maps and let C_0 be the set* $\{0, 1, 2\}$ *with the topology generated by* $\{0\}$ *and* $\{1, 2\}$. *Then*

$$PS(\mathcal{M}) = Inj(C_0)$$

$$= \{X \overset{m}{\to} Y \in \mathcal{M} \mid A \textit{ is clopen in } X \Rightarrow A = m^{-1}(B) \textit{ for some clopen set } B \textit{ in } Y \}.$$

In fact, analogously to the above example, we can prove that if $m : X \to Y$ belongs to $Inj(C_0)$, then every clopen set in X is the inverse image by m of some clopen set in Y.

Conversely, suppose that $m : X \to Y$ fulfils the above condition. We show that $m \in Inj(C_0)$. Given $f : X \to C_0$, let H be an open set in Y such that $H \cap X = f^{-1}(\{0\})$. Then, for $\overline{f} : Y \to C_0$ defined by

$$\overline{f}(y) = \begin{cases} 0 & \text{if} \quad y \in H \\ 1 & \text{if} \quad y \in f^{-1}(\{1\}) \\ 2 & \text{otherwise} \end{cases},$$

we have that $\overline{f} \cdot m = f$.

It remains to show that $PS(\mathcal{M}) = Inj(C_0)$ and, since the inclusion $Inj(C_0) \subseteq PS(\mathcal{M})$ holds by 6.3.1, we have just to prove that $PS(\mathcal{M}) \subseteq Inj(C_0)$. In order to prove this inclusion, we are going to show that every $m : X \to Y$ in $PS(\mathcal{M})$ fulfils the following condition: every clopen set in X is the intersection of X with some clopen set in Y (assuming that m is an inclusion). Let G be clopen in X. Define $f : X \to C_0$ by $f(x) = 0$ if $x \in G$, $f(x) = 1$, otherwise. From Lemma 9.4, there are sources $(m_i : C_0 \to C_0)_I$ in $\mathbb{M}$ and $(f_i : Y \to C_0)_I$ such that $m_i \cdot f = f_i \cdot m$, $i \in I$. Hence, since $(m_i)_I$ is initial, there are j_1, ..., $j_k \in I$ and open sets G_{j_1}, ..., G_{j_K} in C_0 such that $\{0\} = \cap_{k=1}^{n} m_{j_k}^{-1}(G_{j_k})$. This implies that $m_i^{-1}(G_i) = \{0\}$ for some $i \in \{j_1, ..., j_k\}$. Hence, either $G_i = \{0\}$ or $G_i = \{1, 2\}$. In both cases we have that $f_i^{-1}(G_i)$ is clopen in Y and $G = f^{-1}(\{0\}) = f^{-1}(m_i^{-1}(G_i)) = X \cap f_i^{-1}(G_i)$.

3. *Let* $\mathcal{X} = \mathcal{T}ych$ *and let* $\mathbf{I}$ *be the closed unit interval* $[0,1]$ *with the euclidean topology. Then*

$$PS(\mathcal{M}) = Inj(\mathbf{I}) = \{C^*\text{-}embeddings\}.$$

In fact, the $Inj(\mathbf{I})$-morphisms are just the C^*-embeddings and an embedding $X \hookrightarrow Y$ is a C^*-embedding iff each pair of completely separated subsets of X is also completely separated in Y (see 1.5.2). Thus, from the Tietze-Uryshon Extension Theorem, it follows that an embedding of a subspace X into a space Y is a C^*-embedding iff for each continuous map $f : X \to \mathbf{I}$ there is a continuous map $g : Y \to \mathbf{I}$ which carries all elements of $f^{-1}(\{0\})$ into 0 and all elements of $f^{-1}(\{1\})$ into 1. We use this characterization of the C^*-embeddings to show that $PS(\mathcal{M}) \subseteq \{\mathcal{C}^*\text{-embeddings}\}$. Let $m : X \to Y$ belong to $PS(\mathcal{M})$ and let $f : X \to I$ be an arbitrary continuous map. Then, from Lemma 9.4, there are sources $(m_j : I \to I)_J$ in $\mathbb{M}$ and $(f_j : Y \to I)_J$ such that $f_j \cdot m = m_j \cdot f$, $j \in J$. Since $\mathbb{M} \subseteq MonoSource(\mathcal{X})$, there is some $j \in J$ such that $m_j(0) = a \neq b = m_j(1)$. Let $h : I \to I$ be a continuous map such that $h(a) = 0$ and $h(b) = 1$ (which always exists). Then for $g = h \cdot f_j$ we have that for each $x \in f^{-1}(\{0\})$ and each $y \in f^{-1}(\{1\})$,

$$g \cdot m(x) = h \cdot f_j \cdot m(x) = h \cdot m_j \cdot f(x) = h(a) = 0$$

and, analogously, $g \cdot m(y) = 1$. Consequently, $m \in Inj(I)$.

We point out that in most of the above examples, $\mathcal{O}(\mathcal{A})$ is precisely the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{X}$. The following proposition gives an explanation of this fact.

Proposition 9.6 *If* $\mathcal{A}$ *is a subcategory of* $\mathcal{X}$ *such that* $\mathbb{M}(\mathcal{A}) = \mathcal{X}$ *and* $\mathcal{O}(\mathcal{A})$ *is reflective, then the equality* $PS(\mathcal{M}) = Inj(\mathcal{A})$ *implies that* $\mathcal{O}(\mathcal{A})$ *is the smallest* $\mathcal{M}$*-reflective subacategory of* $\mathcal{X}$.

Proof. From 2.17 and 6.3, we have that

$$Inj(\mathcal{A}) \cap Epi(\mathcal{X}) = \mathcal{A}^{\perp} \subseteq PS(\mathcal{M}) \cap Epi(\mathcal{X}).$$

For $PS(\mathcal{M}) = Inj(\mathcal{A})$ it follows that $\mathcal{A}^{\perp} = PS(\mathcal{M}) \cap Epi(\mathcal{X})$, that is, $\mathcal{A}^{\perp}$ is the largest of all classes $\mathcal{B}^{\perp}$ with $\mathcal{B}$ $\mathcal{M}$-reflective. Consequently, $\mathcal{O}(\mathcal{A})$ is the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{X}$. □

All examples of 1.5, except the third one, and example 2.5 satisfy the conditions of the above proposition. So, in each of them, $\mathcal{O}(\mathcal{A})$ is the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{X} = \mathbb{M}(\mathcal{A})$.

We have seen that, in general, the orthogonal closure operator induced by a given subcategory is smaller than the regular one induced by the same subcategory. The following proposition shows that there is at most one $\mathcal{M}$-reflective subcategory of $\mathcal{X}$ for which these two closure operators agree.

Proposition 9.7 *Let $\mathcal{X}$ have equalizers and let $RegMono(\mathcal{X}) \subseteq \mathcal{M}$. If $r_{\mathcal{A}} = c_{\mathcal{A}}$ for some $\mathcal{M}$-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, then $\mathcal{A}$ is the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{X}$.*

Proof. The fact that $\mathcal{A}$ is $\mathcal{M}$-reflective implies that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. So we have that

$$\begin{aligned} r_{\mathcal{A}} = c_{\mathcal{A}} &\implies \{r_{\mathcal{A}}\text{-dense } PS(\mathcal{M})\text{-morphisms}\} = \{c_{\mathcal{A}}\text{-dense } PS(\mathcal{M})\text{-morphisms}\} \\ &\implies Epi(\mathcal{X}) \cap PS(\mathcal{M}) = \mathcal{A}^{\perp}, \quad \text{by 4.6 and 6.4} \end{aligned}$$

Hence, by 2.17 and 6.3, we conclude that $\mathcal{A} = \mathcal{O}(\mathcal{A})$ is the smallest $\mathcal{M}$-reflective subcategory of $\mathcal{X}$. □

The next proposition is, in a certain way, a partial converse of the above one.

Proposition 9.8 *Let $\mathcal{X}$ have equalizers, let $RegMono(\mathcal{X}) \subseteq \mathcal{M}$ and let $\mathcal{A} = (PS(\mathcal{M}) \cap Epi(\mathcal{X}))_{\perp}$. If $r_{\mathcal{A}}$ is weakly hereditary and $c_{\mathcal{A}}$ preserves $PS(\mathcal{M})$-morphisms, then $r_{\mathcal{A}} = c_{\mathcal{A}}$ with respect to $PS(\mathcal{M})$.*

Proof. We have that

$$\mathcal{A}^{\perp} \subseteq PS(\mathcal{M}) \cap Epi(\mathcal{X}) \subseteq ((PS(\mathcal{M}) \cap Epi(\mathcal{X}))_{\perp})^{\perp} = \mathcal{A}^{\perp},$$

so that

$$\mathcal{A}^{\perp} = PS(\mathcal{M}) \cap Epi(\mathcal{X}).$$

Consequently, it follows from 4.6.1 and 6.4 that a $PS(\mathcal{M})$-morphism is $r_{\mathcal{A}}$-dense iff it is $c_{\mathcal{A}}$-dense. Let $m : X \to Y$ be a $PS(\mathcal{M})$-morphism. From 5.8, there is a morphism d such that $c_{\mathcal{A}}(m) = r_{\mathcal{A}}(m) \cdot d$, and, since $r_{\mathcal{A}}$ is weakly hereditary, the morphism $d \cdot d_{\mathcal{A}}(m)$ is $r_{\mathcal{A}}$-dense, therefore it is also $c_{\mathcal{A}}$-dense. Since $c_{\mathcal{A}}$ is an idempotent, weakly hereditary closure operator (by 7.5), it follows that $\mathcal{X}$ has an ($c_{\mathcal{A}}$-dense, $c_{\mathcal{A}}$-closed)-factorization system with respect to $\mathcal{M}$ (see 4.3) and, thus, the equality

$$r_{\mathcal{A}}(m) \cdot (d \cdot d_{\mathcal{A}}(m)) = (c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m)) \cdot 1_X$$

implies the existence of a morphism t such that $t \cdot d \cdot d_{\mathcal{A}}(m) = d_{\mathcal{A}}(m)$. Consequently d is an isomorphism and $r_{\mathcal{A}}(m) \cong c_{\mathcal{A}}(m)$. $\square$

It is clear that the equality $PS(\mathcal{M}) = Inj(\mathcal{A})$ depends on the choice of the subcategory $\mathcal{A}$. For instance, let $\mathcal{S}$ and $\mathcal{N}$ be the subcategories of $\mathcal{T}op_0$ defined in 8.8.1 and 8.8.2, respectively. Then $\mathcal{M} = Inj(\mathcal{S}) \neq Inj(\mathcal{N})$.

In fact, if $\mathcal{A}$ and $\mathcal{B}$ are subcategories of a category $\mathcal{X}$ with $\mathbb{M}(\mathcal{A}) = \mathbb{M}(\mathcal{B})$, then the equality $Inj(\mathcal{A}) = Inj(\mathcal{B})$ implies that $\mathcal{A}^{\perp} = \mathcal{B}^{\perp}$ (by 2.17.2) and, hence, $\mathcal{O}(\mathcal{A}) = \mathcal{O}(\mathcal{B})$.

Next, we characterize the class $PS(\mathcal{M})$ for $\mathcal{M}$ the class of all embeddings, in another epireflective subcategory of $\mathcal{T}op$, the subcategory $\mathcal{T}op_1$ of all T_1-spaces.

Proposition 9.9 *In $\mathcal{T}op_1$, the embedding of a subspace X into a space Y belongs to $PS(\mathcal{M})$ for $\mathcal{M}$ the class of all embeddings if and only if it fulfils the following condition*

$$\textbf{(S)} \qquad (A,\ B \subseteq X \text{ and } \overline{A}^X \cap \overline{B}^X = \emptyset) \Rightarrow \overline{A}^Y \cap \overline{B}^Y = \emptyset.$$

Proof.

I. Let the diagram

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "g"'] & Y \arrow[d, "\overline{g}"] \\ Z \arrow[r, "\overline{m}"] & W \end{tikzcd} \tag{7}$$

be a pushout in $\mathcal{T}op$ with $X, Y, Z \in \mathcal{T}op_1$ and m an embedding. We may assume that m and $\overline{m}$ are inclusions, $W = Z \mathbin{\dot\cup} (Y \setminus X)$ and

$$\overline{g}(y) = \begin{cases} y & \text{if} \quad y \in Y \setminus X \\ g(y) & \text{if} \quad y \in X \end{cases}.$$

The set W is endowed with the final topology induced by $\overline{m}$ and $\overline{g}$, that is, a subset U of W is open iff both $\overline{m}^{-1}(U)$ and $\overline{g}^{-1}(U)$ are open in Z and Y, respectively. Then, a pushout of m along g in $\mathcal{T}op_1$ is $r_W \cdot \overline{m}$ where $r_W : W \to RW$ is a reflection of W in $\mathcal{T}op_1$.

Let $q : W \to \widetilde{W}$ be the quotient of W determined by the smallest equivalence relation $\sim$ in W such that

$$w \in \overline{\{w'\}}^W \implies w \sim w'.$$

It is clear that a reflection $r_W : W \to RW$ is factorizable through q.

II. Let us prove that **(S)** is necessary. If it fails to be true, then there exist two closed subsets of X, say F_1 and F_2, such that $F_1 \cap F_2 = \emptyset$ but $y \in \overline{F_1}^Y \cap \overline{F_2}^Y$ for some $y \in Y \setminus X$. Define a map $g : X \to Z = X \setminus (F_1 \cup F_2) \dot{\cup} \{1,\ 2\}$ by

$$g(x) = \begin{cases} x & \text{if} \quad x \notin F_1 \cup F_2 \\ 1 & \text{if} \quad x \in F_1 \\ 2 & \text{if} \quad x \in F_2 \end{cases}$$

and let us consider Z with the quotient topology induced by g. The space Z is clearly T_1. But a pushout in $\mathcal{T}op_1$ of m along g is not one-to-one; in fact, given an open set U in W such that $y \in U$, then, since $y \in \overline{F_1}^Y \cap \overline{F_2}^Y$, we have that $\overline{g}^{-1}(U) \cap F_i \neq \emptyset$, $i = 1, 2$, and, consequently, $1, 2 \in U$; therefore, $y \in \overline{\{1\}}^W \cap \overline{\{2\}}^W$ and $q(1) = q(2)$. It turns out that $r_W \cdot \overline{m}$ is not one-to-one either.

III. To prove that the condition (**S**) is also sufficient, we first verify the following two properties of the pushout (7) as above:

(i) If $y \in Y \setminus X$ and $z \in Z$, then $y \in \overline{\{z\}}^W$ iff $y \in \overline{g^{-1}(z)}^Y$;

(ii) If $w,\ w' \in W$, $w \neq w'$ and $w \in \overline{\{w'\}}^W$, then $w \in Y \setminus X$ and $w' \in Z$.

Proof of (*i*): If $y \in \overline{g^{-1}(z)}^Y$ and H is an open set in W which contains y, then $\overline{g}^{-1}(H) \cap g^{-1}(z) \neq \emptyset$ and this implies that $z \in H$; therefore, $y \in \overline{\{z\}}^Y$.

Conversely, if $y \notin \overline{g^{-1}(z)}^Y$, then there exists an open set A in Y such that $y \in A$ but $A \cap g^{-1}(z) = \emptyset$. Let B be an open set in Y such that $B \cap X = X \setminus g^{-1}(z)$. Then $H = A \cup B$ is an open subset of Y such that $y \in H$ and $H \cap X = X \setminus g^{-1}(z)$. Put $U = (H \setminus X) \cup (Z \setminus \{z\})$. Hence, U is an open set in W such that $y \in U$ but $z \notin U$, thus

$y \notin \overline{\{z\}}^W$.

Proof of (ii): On the one hand, if $w' \in Y \setminus X$, the set $W \setminus \{w'\}$ is open in W, contains w and does not contain w', then $w \notin \overline{\{w'\}}^W$. On the other hand, if $w,\ w' \in Z$, let V be an open set in Z such that $w \in V$ but $w' \notin V$; then $H = V \cup B \setminus X$, where B is an open set in Y such that $g^{-1}(V) = B \cap X$, is open in W and contains w but not w'; consequently, $w \notin \overline{\{w'\}}^W$.

Therefore, if $w \in \overline{\{w'\}}^W$, one must have $w \in Y \setminus X$ and $w' \in Z$.

Now, let condition (**S**) holds. We show that, then, for each morphism $g : X \to Z$ with $Z \in \mathcal{T}op_1$, the map $q \cdot \overline{m}$ is an embedding and $\widetilde{W}$ is a T_1-space. The fact that $\widetilde{W}$ is a T_1-space implies that q is a reflection from W to $\mathcal{T}op_1$ and thus $q \cdot \overline{m}$ is a pushout of m along g in $\mathcal{T}op_1$. Consequently, if $\overline{m}$ is an embedding we conclude that m belongs to $PS(\mathcal{M})$.

- $q \cdot \overline{m}$ is one-to-one:

$$\begin{aligned} q \cdot \overline{m}(z) = q \cdot \overline{m}(z') \ &\Leftrightarrow q(z) = q(z') \\ &\Leftrightarrow \exists y \in Y \setminus X : y \in \overline{\{z\}}^W \cap \overline{\{z'\}}^W, \quad \text{by } (ii), \\ &\Leftrightarrow \exists y \in Y \setminus X : y \in \overline{g^{-1}(z)}^Y \cap \overline{g^{-1}(z')}^Y, \quad \text{by } (i). \end{aligned}$$

  This implies that $z = z'$, since, otherwise, $g^{-1}(z)$ and $g^{-1}(z')$ would be disjoint closed subsets of X and then, by (**S**) $\overline{g^{-1}(z)}^Y$ and $\overline{g^{-1}(z')}^Y$ would be disjoint too.

- $\widetilde{W}$ is a T_1-space:

  We show that for each $b \in \widetilde{W}$, $q^{-1}(b)$ is closed in W, and, hence, $\{b\}$ is closed in $\widetilde{W}$. Indeed, from (ii) and the fact that $q \cdot \overline{m}$ is one-to-one it follows that

$$q^{-1}(b) = \{y\} \quad \text{with} \quad y \in Y \setminus X \quad \text{or} \quad q^{-1}(b) = \{z\} \cup \{y \in Y \setminus X \mid y \in \overline{\{z\}}\}^W.$$

  Well, $\{y\}$ is clearly closed in W; concerning the other case, we have that

$$\begin{aligned} \overline{m}^{-1}(q^{-1}(b)) \ &= \{z\}\,, \text{ wich is closed in } Z, \qquad \text{and} \\ \overline{g}^{-1}(q^{-1}(b)) \ &= g^{-1}(z) \cup \{y \in Y \setminus X \mid y \in \overline{\{z\}}\}^W \\ &= g^{-1}(z) \cup \{y \in Y \setminus X \mid y \in \overline{g^{-1}(z)}\}^Y, \text{ by } (ii) \\ &= \overline{g^{-1}(z)}^Y \end{aligned}$$

  Thus, $q^{-1}(b)$ is closed in W.

- $q \cdot \overline{m}$ is initial:

  We show that, for each closed set F of Z, there exists $E \subseteq W$ such that $q^{-1}(q(E))$ is closed in W (so that $q(E)$ is closed in $\widetilde{W}$) and $F = Z \cap q^{-1}(q(E)) = (q \cdot m)^{-1}(q(E))$. Given F closed in Z, put $E = (\overline{g^{-1}(F)}^Y \setminus X) \cup F$. The set E is closed in W. Moreover, we prove that $q^{-1}(q(E)) = E$, so $q(E)$ is closed. Indeed, $q^{-1}(q(E)) = E \cup E_1 \cup E_2$ where

  $$E_1 = \{z \in Z \mid y \in \overline{\{z\}}^W, \text{ for some } y \in E \cap (Y \setminus X)\}$$

  and

  $$E_2 = \{y \in Y \setminus X \mid y \in \overline{\{z\}}^W, \text{ for some } z \in (E \cup E_1) \cap Z\}.$$

  Concerning $z \in E_1$, we have $y \in \overline{g^{-1}(F)}^Y \cap \overline{g^{-1}(z)}^Y$, thus $g^{-1}(F) \cap g^{-1}(z) \neq \emptyset$, from condition (**S**), therefore $z \in F \subseteq E$.

  Now, concerning $y \in E_2$, the fact that $z \in E \cup E_1 = E$ and E is closed implies that $\overline{\{z\}}^W \subseteq E$, thus, since $y \in \overline{\{z\}}^W$, we conclude $y \in E$. $\square$

Remark 9.10 From 9.9 it is clear that in $\mathcal{T}op_1$ the class of all closed embeddings is contained in $PS(\mathcal{M})$ for $\mathcal{M}$ the class of all embeddings. But this inclusion is strict. Indeed, let Y be a T_1-space which has an infinite subspace X such that X has the cofinite topology. Then the inclusion of X in Y fulfils the condition (**S**) but it is not closed.

Chapter III

α-sober spaces

It is well-known that the conglomerate of all $\mathcal{E}$-reflective subcategories of an $(\mathcal{E}, \mathbb{M})$-category is a complete "lattice" with respect to the inclusion order. Several authors have contributed to the study of the "lattice" of epireflective subcategories of "everyday" categories (see, e.g., [24] and references there). In particular, as observed by H. Herrlich [24], it follows from results in [76], [46] and [43] that the "lattice " of epireflective subcategories of $\mathcal{H}aus$ contains a well-ordered proper class and that, if we assume the non existence of measurable cardinals, the same holds for $\mathcal{HC}omp$.

As far as epireflective subcategories of $\mathcal{T}op_0$ are concerned, we refer to [50], [31], [39] and [48].

In this chapter, we use results of the last one to show that the "lattice" of epireflective subcategories of $\mathcal{T}op_0$ also contains a well-ordered proper class.

Every ordinal α equipped with the Alexandrov topology is a T_0-space. It is well known that for $\alpha = 2$ the reflective hull of α in $\mathcal{T}op_0$ is the subcategory of *sober spaces*. Here, we characterize the orthogonal closure operator induced in $\mathcal{T}op_0$ by the category whose only object is α (which for $\alpha = 2$ coincides with the b-closure). Then we define *α-sober space* for each $\alpha \geq 2$ in such a way that the reflective hull of α in $\mathcal{T}op_0$ is the subcategory of α-sober spaces. Moreover, we obtain an order-preserving bijective correspondence between a proper class of ordinals and the corresponding (epi)reflective hulls, which gives us the claimed well-ordered proper class of epireflective subcategories of $\mathcal{T}op_0$. Our main tool is the concept of orthogonal closure operator.

10 The orthogonal closure operators c_α in $\mathcal{T}op_0$

Let $\mathcal{A}$ be a subcategory of $\mathcal{T}op_0$ and let $\mathcal{M}$ be the class of all embeddings in $\mathcal{T}op_0$. For the sake of simplicity, we usually deal with embeddings as inclusions of subspaces. Thus, since in $\mathcal{T}op_0$ embeddings are pushout stable, we have that the orthogonal closure operator in $\mathcal{T}op_0$ with respect to $\mathcal{M}$ and induced by $\mathcal{A}$ assigns, to each subspace X of a space Y, another subspace $c_{\mathcal{A}}(X)$ which is the intersection of all subspaces X_g of Y which are pullbacks of some pushout of m along some $g \in \mathcal{T}op_0(X, \mathcal{A})$.

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dd, "g"'] \arrow[dr] & & Y \arrow[dd] \\
& X_g \arrow[ur] \arrow[dl] & \\
A \arrow[rr] & & \bullet
\end{tikzcd}$$

Of course, for a space in $\mathcal{T}op_0$, to be $\mathcal{A}$-strongly closed means just to be $c_{\mathcal{A}}$-absolutely closed, that is, each of its embeddings into some other space is $c_{\mathcal{A}}$-closed.

Now, it is easy to deduce the following

Proposition 10.1 *If $\mathcal{T}op_0$ is the epireflective hull of $\mathcal{A}$ in $\mathcal{T}op$, then the closure operator $c_{\mathcal{A}}$ is idempotent and weakly hereditary, and the (epi)reflective hull of $\mathcal{A}$ in $\mathcal{T}op_0$ consists of precisely all $\mathcal{A}$-strongly closed spaces.*

Proof. If $\mathcal{T}op_0$ is the epireflective hull of $\mathcal{A}$ in $\mathcal{T}op$, then, for each T_0-space X, there is some small initial monosource $(X \stackrel{f_i}{\to} A_i)_I$, with codomain in $\mathcal{A}$. Thus, the morphism $< f_i >: X \to \Pi_{i \in I} A_i$ is an embedding with codomain in $\mathcal{O}(\mathcal{A})$. Therefore, from 7.6 and 8.1, we conclude that the reflective hull of $\mathcal{A}$ in $\mathcal{T}op_0$ is the subcategory of all $\mathcal{A}$-strongly closed spaces. This coincides with the epireflective hull since in $\mathcal{T}op_0$ the class $\mathcal{A}^{\perp}$ consists of epimorphisms. $\square$

We are going to study the orthogonal closure operator $c_{\mathcal{A}}$ for a particular kind of subcategories $\mathcal{A}$ of $\mathcal{T}op_0$.

It is well known that, for each T_0-space X, we may define a partial order in X, the *specialization order*, by $x \leq y$ iff $x \in \overline{\{y\}}$. Furthermore, given a poset $(X, \leq)$, there are two canonical ways of defining a T_0 topology in X for which $\leq$ is the specialization order, namely:

- the *Alexandrov topology*, which consists of all upper sets, i.e., sets U such that if $x \in U$ and $x \leq y$ then $y \in U$;
- the *upper-interval topology*, which is the smallest topology containing all sets of the form
$$X \backslash \downarrow x$$
where $\downarrow x = \{y \in X \mid y \leq x\}$.

The first topology above is the maximal topology, and the second one is the minimal topology, for which $(X, \leq)$ is the specialization order.

Let $\alpha > 0$ be an ordinal. We consider α as a topological space endowed with the Alexandrov topology. Therefore, since non-trivial open sets of α are all upper sets $\uparrow \beta = \{\delta \in \alpha \mid \delta \geq \beta\}$ (with $\beta \in \alpha$), α is a T_0-space. We point out that proper closed subsets of α are precisely the ordinals smaller than α, that is, a set $\gamma \subset \alpha$ is closed in α iff $\gamma \in \alpha$.

Of course, the ordinal 2 is the Sierpinski space. Furthermore, for $\alpha \geq 2$, we have an embedding $2 \hookrightarrow \alpha$ in $\mathcal{T}op_0$ and, then, since $\mathcal{T}op_0$ is the epireflective hull of 2 in $\mathcal{T}op$, it is also the epireflective hull of α in $\mathcal{T}op$. If $\mathcal{A}$ is the full and replete subcategory of $\mathcal{T}op_0$ generated by α, we denote by c_α the respective orthogonal closure operator and, analogously, we use the term *α-strongly closed space*.

Along this chapter, the set of all open sets of a space X will be denoted by $\Omega(X)$.

As we observed in 8.8.1, the closure c_2 is just the b-closure, first used by Baron in [9] for characterizing the epimorphisms in $\mathcal{T}op_0$. We recall that if X is a subspace of Y then $y \in Y$ belongs to $c_2(X)$ iff

(b) For each $H \in \Omega(Y)$ with $y \in H$ we have that $\overline{\{y\}} \cap H \cap X \neq \emptyset$.

It is known that (b) is equivalent to the condition

(b') For arbitrary open sets H and H' in Y such that $H \cap X = H' \cap X$, we have that $y \in H$ iff $y \in H'$.

Moreover, it is easy to see that (b') is also equivalent to the condition

(b_2) For each open set G of X, there is an ordinal $\beta_0 < 2$ such that, given open sets H_0, H_1 and H_2 of Y such that $H_0 \cap X = X$, $H_1 \cap X = G$ and $H_2 \cap X = \emptyset$, we have $y \in H_\delta$ iff $\delta \leq \beta_0$.

In order to generalize this characterization of the c_2-closure to all c_α-closures, with α an ordinal, let us say that a family $(G_\delta)_{\delta<\alpha}$ of open sets of X is a *continuous α-sequence* provided that for every $x \in X$ there exists an ordinal $\beta_x < \alpha$ such that

$$x \in G_\delta \text{ iff } \delta \leq \beta_x.$$

It is clear that all continuous sequences are decreasing, i.e., $G_{\delta_1} \supseteq G_{\delta_2}$ whenever $\delta_1 \leq \delta_2$. Moreover, $G_0 = X$.

Now, for an ordinal $\alpha \geq 1$ and a subspace X of a T_0-space Y we consider the following assumption on a given $y \in Y$:

(b_α) For each continuous α-sequence $(G_\delta)_{\delta<\alpha}$ of open sets of X, there is an ordinal $\beta_0 < \alpha$ such that for each family $(H_\delta)_{\delta\leq\alpha}$ of open sets of Y with $H_\delta \cap X = G_\delta$ for all $\delta < \alpha$ and $H_\alpha \cap X = \emptyset$, we have that $y \in H_\delta$ iff $\delta \leq \beta_0$.

Next we show that we may characterize the c_α-closure of a subspace in $\mathcal{T}op_0$, for $\alpha \geq 1$, by means of the condition (b_α). For that, we use the following

Lemma 10.2 *If X is a subspace of Y in $\mathcal{T}op_0$ and $y \in Y$ then, for each $\alpha \geq 1$, condition (b_α) is equivalent to the condition*

(b'_α) *for each continuous map $g : X \to \alpha$ there exists an ordinal $\beta_0 < \alpha$ such that for every $H \in \Omega(Y)$ and every $\beta \leq \alpha$ with $H \cap X = g^{-1}(\uparrow \beta)$, one has $y \in H$ iff $\beta \leq \beta_0$*[2].

Proof. It is immediate by taking into account that the function which assigns to each continuous map $g : X \to \alpha$ the family

$$(g^{-1}(\uparrow \delta))_{\delta<\alpha}$$

[2]For $g : X \to \alpha$, $g^{-1}(\uparrow \alpha)$ is the empty set. For $\alpha \geq 2$, "$\beta \in \alpha$" may, equivalently, replace "$\beta \in \alpha + 1$" in (b_α).

is a bijection from $hom(X, \alpha)$ to the set of all continuous α-sequences of open sets of X.
□

Proposition 10.3 *If X is a subspace of Y in $\mathcal{T}op_0$ and $y \in Y$, then $y \in c_\alpha(X)$ if and only if fulfils condition (b_α).*

Proof. Taking into account the above lemma, we show that $y \in c_\alpha(X)$ if and only if fulfils condition (b'_α). Since for $y \in X$ the result is trivial, we assume that $y \in Y \backslash X$.

Let y satisfy condition (b'_α). Firstly, we show that $y \in \overline{X}$ (where $\overline{X}$ denotes the usual closure of X in Y). In fact, let $H \in \Omega(Y)$ be such that $H \cap X = \emptyset$; define $g : X \to \alpha$ by $g(x) = 0$, $x \in X$. Since $g^{-1}(\uparrow 1) = \emptyset = \emptyset \cap X$, the ordinal β_0 required by (b'_α) must be smaller than 1, thus $\beta_0 = 0$, and the equality $H \cap X = g^{-1}(\uparrow 1)$ implies that $y \notin H$.

Now, let $g : X \to \alpha$ be an arbitrary continuous map and let us consider the following pushout in $\mathcal{T}op$, where $m : X \to Y$ is the embedding of X in Y.

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ \alpha \arrow[r, "m'"] & W \end{tikzcd} \tag{8}$$

We assume that $m' : \alpha \to W$ is the inclusion of α into $\alpha \,\dot{\cup}\, (Y \backslash X)$. So, the pushout of m along g in $\mathcal{T}op_0$ is the pair $(r_W \cdot m', r_W \cdot g')$, where r_W is the reflection of W in $\mathcal{T}op_0$. Let β_0 be the ordinal whose existence is guaranteed by (b'_α). We show that, for every $U \in \Omega(W)$, $y \in U$ iff $\beta_0 \in U$, so that $r_W(y) = r_W(\beta_0)$ and, consequently, $y \in X_g$. Let $U \in \Omega(W)$, i.e., $(g')^{-1}(U) \in \Omega(Y)$ and $(m')^{-1}(U) \in \Omega(\alpha)$. If $y \in U$, then $y \in (g')^{-1}(U)$ and, hence, since $y \in \overline{X}$, $(g')^{-1}(U) \cap X \neq \emptyset$. Thus, since $(g')^{-1}(U) \cap X = g^{-1}((m')^{-1}(U))$, the set $(m')^{-1}(U)$ is non empty and, then, $(m')^{-1}(U) = \uparrow \beta$ for some $\beta < \alpha$; moreover, since y satisfies (b'_α), one has $\beta \leq \beta_0$. So we have that

$$\begin{aligned} y \in U \quad & \text{iff } y \in (g')^{-1}(U) \\ & \text{iff } (m')^{-1}(U) = \uparrow \beta \text{ for some } \beta \leq \beta_0 \\ & \text{iff } \beta_0 \in (m')^{-1}(U) = \uparrow \beta \text{ for some } \beta \\ & \text{iff } \beta_0 \in U. \end{aligned}$$

Therefore, since $y \in X_g$ for each $g \in hom(X, \alpha)$, it follows that $y \in c_\alpha(X)$.

Conversely, let $y \in c_\alpha(X)$ and let $g \in hom(X, \alpha)$. Let us consider the corresponding pushout as in (8) and let X_g be the pullback of $r_W \cdot m'$ along $r_W \cdot g'$. Then $y \in X_g$, which is equivalent to saying that there exists an ordinal $\beta_0 < \alpha$ such that $r_W(y) = r_W(\beta_0)$; furthermore, this β_0 is unique since the pushout-stability of embeddings in $\mathcal{T}op_0$ assures that $r_W \cdot m'$ is one-to-one. Let $H \in \Omega(Y)$ and $\beta \leq \alpha$ be such that $g^{-1}(\uparrow \beta) = H \cap X$. Put $U = (H \backslash X) \dot{\cup} \uparrow \beta$; then $U \in \Omega(W)$. Well, as it is well-known, $r_W(y) = r_W(\beta_0)$ iff, for every $G \in \Omega(W)$, $y \in G$ iff $\beta_0 \in G$; hence, for the open set U considered, we have that $y \in (g')^{-1}(U)$ iff $\beta_0 \in \uparrow \beta$, i.e., $y \in H$ iff $\beta \leq \beta_0$. □

Corollary 10.4 *If α and β are ordinals such that $\alpha \leq \beta$ then $c_\beta \leq c_\alpha$.*

Proof. Let X be a subspace of Y in $\mathcal{T}op_0$ and $y \in c_\beta(X)$. Let $g : X \to \alpha$ be a continuous map and let $e : \alpha \hookrightarrow \beta$ be the inclusion of α in β. Then, since y satisfies condition (b'_β), there is an ordinal $\beta_0 < \beta$ such that for every $H \in \Omega(Y)$ and every $\delta \leq \beta$ which fulfil the equality $H \cap X = (e \cdot g)^{-1}(\uparrow \delta)$, $y \in H$ iff $\delta \leq \beta_0$. Since $(e \cdot g)^{-1}(\uparrow \delta) = \emptyset$ for $\delta \geq \alpha$, it must be $\beta_0 < \alpha$ and we have that this β_0 fulfils also the condition (b'_α) for $g : X \to \alpha$. Consequently, $y \in c_\alpha(X)$. □

Remark 10.5 The closure operator c_n coincides with the b-closure for all finite $n > 1$. Indeed, if X is a subspace of Y and $y \in c_2(X)$, let

$$X = G_0 \supseteq G_1 \supseteq ... \supseteq G_{n-1}$$

be a continuous n-sequence. Then, for each $k = 1, ..., n-1$,

$$X = G_0 \supseteq G_k$$

is a continuous 2-sequence. Consequently, from (b_2), it determines an ordinal $\delta_k < 2$ such that for each family $(H_\delta)_{\delta \leq 2}$ of open sets of Y with $H_0 \cap X = G_0$, $H_1 \cap X = G_k$ and $H_2 \cap X = \emptyset$, $y \in H_\delta$ iff $\delta \leq \delta_k$. It is easy to check that the ordinal

$$\beta_0 = \sum_{k=1}^{n-1} \delta_k$$

fulfils condition (b_n) for y and the given continuous n-sequence.

Consequently, $y \in c_n(X)$. Now, from 10.4, it follows that $c_2 = c_n$, for $n \in \omega_0 \setminus 2$.

On the other hand, c_2 is strictly smaller than c_1. In order to show this, let us consider the embedding $m : 2 \to 3$ defined by $m(0) = 0$ and $m(1) = 2$. Then $c_2(m) = m$, since the domain of m is 2 (by 5.6.1). But $c_1(m) = 1_3$; this is easily verified by taking into account that, for each subspace X of Y, we can characterize $c_1(X)$ as follows:

> $y \in c_1(X)$ *if and only if* $y \in \overline{X}$ *and, for every* $H \in \Omega(Y)$*, if* $X \subseteq H$ *then that* $y \in H$*.*

In fact, a continuous 1-sequence consists of the set X only and, in this case, β_0 must be equal to 0. Thus, on the one hand, if $H \cap X = \emptyset$, then $y \notin H$ and, on the other hand, if $H \cap X = X$, that is, $X \subseteq H$, then $y \in H$.

11 α-sober spaces

Definitions 11.1 1. Let $X \in \mathcal{T}op_0$ and let $\alpha \geq 1$. A closed subset F of X is said to be *α-irreducible* if it satisfies the following conditions:

(i_0) F is irreducible, i.e., for arbitrary open sets G_1 and G_2, if $F \cap G_1 \cap G_2 = \emptyset$ then $F \cap G_1 = \emptyset$ or $F \cap G_2 = \emptyset$.

(i_α) For each continuous α-sequence $(G_\delta)_{\delta<\alpha}$ of open sets in X such that $F \cap (\cap_{\delta\in\alpha} G_\delta) = \emptyset$, the set $\{\delta < \alpha \mid F \cap G_\delta \neq \emptyset\}$ has a maximum.

2. A T_0-space X is said to be *α-sober* if each of its α-irreducible closed set is the closure of a single point.

Remarks 11.2

1. We note that condition (i_α) implies that an α-irreducible closed set is non-empty.

   It is clear that, for every finite ordinal $n \neq 0$, a non-empty closed set is n-irreducible iff it satisfies the condition (i_0) (since (i_n) trivially holds). Consequently, to be an n-sober space means to be a sober space. However, we introduced α-sober for finite ordinals α because of the characterizations 11.3 and 11.4 below. They "work" well for all $\alpha \geq 2$, but not for $\alpha = 1$, as we will see. We point out that $\mathcal{T}op_0$ is the (epi)reflective hull of α in $\mathcal{T}op$ only if $\alpha > 1$.

2. Combining (i_0) with (i_α) we obtain the following condition which is equivalent to the conjunction of the two above ones:

   (I_α) For each continuous α-sequence $(G_\delta)_{\delta<\alpha}$ with $G_\delta = A_1^\delta \cap A_2^\delta$ and all A_1^δ, A_2^δ in $\Omega(X)$, such that $F \cap (\cap_{\delta<\alpha} G_\delta) = \emptyset$, there exists an ordinal $\delta_0 < \alpha$ such that $F \cap G_{\delta_0} \neq \emptyset$ and $F \cap A_j^{\delta_0+1} = \emptyset$ for $j = 1$ or $j = 2$.

3. It is well-known that the above condition (i_0) on F is equivalent to

   (i'_0) If F is the union of two closed sets then F is equal to one of them.

   By using Lemma 10.2, it is easy to verify that (i_α) is equivalent to

   (i'_α) For every continuous map $g : X \to \alpha$, the set $g(F)$ has a maximum.

   The formulation (i'_α) of condition (i_α) will be very useful in the sequel.

Proposition 11.3 *For an ordinal $\alpha \geq 2$, a T_0-space X is α-sober if and only if it is α-strongly closed.*

Proof. Let us assume that X is not an α-sober space. This means that X has an α-irreducible closed set F which is not the closure of a singleton. Let us define a space Y as follows:

$$Y = X \,\dot{\cup}\, \{a\}$$
$$\Omega(Y) = \{\emptyset\} \bigcup \{H \mid H \in \Omega(X) \text{ and } H \cap F = \emptyset\} \bigcup \{H \cup \{a\} \mid H \in \Omega(X) \text{ and } H \cap F \neq \emptyset\}.$$

It is clear that $\Omega(Y)$ is closed under arbitrary unions. For finite intersections, we use the irreducibility of F to conclude that $\bigcap_{i=1}^{2} (H_i \,\dot{\cup}\, \{a\}) \in \Omega(Y)$ whenever $H_i \,\dot{\cup}\, \{a\} \in \Omega(Y)$ for $i = 1, 2$.

Let us show that $Y \in \mathcal{T}op_0$. It is clear that arbitrary two distinct points of X are "separated" by some open set of Y; further, if x is a point of X and $x \notin F$, there is some $H \in \Omega(X)$ such that $x \in H$ and $H \cap F = \emptyset$ and, then, H is an open set of Y which separates x from a. Now, let us consider the point a and some $x \in F$. If $x \in F$, we have that $\overline{\{x\}} \neq F$, by hypothesis on F and by using 11.2.3. Then there is some $x' \in F$ and $G \in \Omega(X)$ such that $x' \in G$ but $x \notin G$. Hence $G \dot{\cup} \{a\} \in \Omega(Y)$ "separates" a from x.

It is obvious that X is a subspace of Y. Now, let us show that $a \in c_\alpha(X)$ by showing that a and α fulfil condition (b'_α); so that X is not α-strongly closed. Let $g : X \to \alpha$ be a continuous map. By hypothesis on F and by 11.2.3, there exists an ordinal $\beta_0 \in \alpha$ such that $\beta_0 = max\, g(F)$. In order to show that β_0 fulfils the requirement of (b'_α) of 10.2, let $H \in \Omega(Y)$ and $\beta \leq \alpha$ be such that $H \cap X = g^{-1}(\uparrow \beta)$. Hence, on the one hand, if $a \in H$, then $H = g^{-1}(\uparrow \beta) \dot{\cup} \{a\}$ with $g^{-1}(\uparrow \beta) \cap F \neq \emptyset$ and, by definition of β_0, we have that $\beta \leq \beta_0$. On the other hand, if $a \notin H$, then $H = g^{-1}(\uparrow \beta)$ and $g^{-1}(\uparrow \beta) \cap F = \emptyset$; thus, $\beta_0 \notin \uparrow \beta$, i.e., $\beta_0 < \beta$.

Conversely, let us assume that X is α-sober. Let X be a subspace of a T_0-space Y and let $y \in Y$ be such that $y \in c_\alpha(X)$. We want to show that y must be a point of X.

Let $\overline{\{y\}}$ be the closure of $\{y\}$ in Y. Firstly, let us notice that, from Corollary 10.4, $y \in c_2(X)$ and, then, since c_2 is the b-closure operator, it easily follows that $\overline{\{y\}} \cap X$ is a closed set of X which satisfies condition (i'_0) (which is equivalent to (i_0), by 11.2.3). On the other hand, $\overline{\{y\}} \cap X$ satisfies condition (i_α); to show that, we prove that it fulfils the equivalent condition (i'_α) (see 11.2.3). In fact, since $y \in c_\alpha(X)$, for each continuous map $g : X \to \alpha$, let $\beta_0 \in \alpha$ be the ordinal whose existence is guaranteed in condition (b_α). We are going to show that $\beta_0 = max\, g(\overline{\{y\}} \cap X)$. Let $x \in \overline{\{y\}} \cap X$; then, for some $H \in \Omega(Y)$, $g^{-1}(\uparrow g(x)) = H \cap X$, and, since $x \in H \cap \overline{\{y\}}$, y must belong to H, hence $g(x) \leq \beta_0$. Now, let $H \in \Omega(Y)$ be such that $H \cap X = g^{-1}(\uparrow \beta_0)$; then $y \in H$ and, since $y \in c_2(X)$, $\overline{\{y\}} \cap X \cap H \neq \emptyset$, that is, there is some $x \in \overline{\{y\}} \cap X$ such that $g(x) \in \uparrow \beta_0$. But, as we have seen, $g(x) \leq \beta_0$; then $g(x) = \beta_0$ and β_0 is the desired maximum.

Therefore, since X is an α-sober space and $\overline{\{y\}} \cap X$ is α-irreducible, $\overline{\{y\}} \cap X = \overline{\{x\}} \cap X$ for some $x \in X$. Thus, on the one hand, $\overline{\{x\}} \subseteq \overline{\{y\}}$; on the other hand, given $H \in \Omega(Y)$ with $y \in H$, we have that $\overline{\{x\}} \cap H \neq \emptyset$, since $\overline{\{x\}} \cap H \cap X = \overline{\{y\}} \cap H \cap X \neq \emptyset$, and, then, $x \in H$; consequently, we also have the inclusion $\overline{\{y\}} \subseteq \overline{\{x\}}$. Now, since $\overline{\{y\}} = \overline{\{x\}}$ and Y is a T_0-space, it follows that $y = x$. $\square$

Corollary 11.4 *For each ordinal $\alpha \in Ord \backslash 2$, the (epi)reflective hull of α in $\mathcal{T}op_0$ is the full subcategory of all α-sober spaces.*

Proof. It is an immediate consequence of the above proposition and 10.1. $\square$

From now on, for each $\alpha \in Ord\backslash 2$, we denote the full subcategory of α-sober spaces by $Sob(\alpha)$.

Corollary 11.5 *For α, $\beta \in Ord\backslash 2$ such that $\alpha \leq \beta$, $Sob(\alpha) \subseteq Sob(\beta)$.*

Proof. It is a consequence of the above proposition and of Corollary 10.4. $\square$

12 The chain of subcategories $Sob(\alpha)$

As we have seen, for $n \in \omega_0\backslash 2$, $Sob(n) = Sob(2)$. Next we deal with the question for which ordinals $\alpha < \beta$ we have that $Sob(\alpha)$ is strictly contained in $Sob(\beta)$.

First, we recall some definitions and facts about cardinals, essentially collected from [49], and which will be very useful in what follows.

A *cardinal* is just an ordinal which is not equipotent with any of its elements.

A cardinal λ is said to be *regular* if it is not a sum of a smaller number of smaller ordinals. In other words, λ is a regular cardinal if, for all sets $\Gamma \subseteq \lambda$ with cardinality smaller than λ, we have that $\bigcup \Gamma < \lambda$. For example, ω_0 and ω_1 are regular; moreover, for any infinite cardinal α, α^+ is regular, where α^+ is the smallest cardinal which is larger than α. But, for instance, ω_ω is not regular since it is the union of all ω_i with $i \in \omega$.

If α and β are ordinals, we say that α *is cofinal with* β if there is a strictly increasing function f with domain β such that

$$\bigcup_{\gamma<\beta} (f(\gamma) + 1) = \alpha.$$

If α is a limit ordinal and α is cofinal with β, then β is also a limit ordinal and the cofinality of α with β means precisely that there is a strictly increasing function $f : \beta \to \alpha$ such that

$$\bigcup_{\gamma<\beta} f(\gamma) = \alpha.$$

Let us also recall that an infinite ordinal α is a regular cardinal iff it is not cofinal with any ordinal smaller than α.

For any ordinal α, the *cofinality character* of α, denoted by $cf(\alpha)$, is the least ordinal β such that α is cofinal with β. If α is a limit ordinal, then $cf(\alpha)$ is a regular cardinal.

Let $\mathcal{O}rd$ denote the category whose objects are all ordinals and whose morphisms are all order-preserving maps. The following lemma, which establishes that, up to a concrete isomorphism, $\mathcal{O}rd$ is a full subcategory of $\mathcal{T}op_0$, will be very useful in the sequel.

Lemma 12.1 *The function which transforms each ordinal α into a T_0-space by equipping it with the Alexandrov tolology is a concrete full embedding of $\mathcal{O}rd$ in $\mathcal{T}op_0$.*

Proof. We have to show that a map $f : \alpha \to \beta$ between two ordinals is order-preserving iff it is continuous with respect to the Alexandrov topologies.

In fact, the specialization order for these topologies coincide with the usual order and it is well-known that, then, for T_0-spaces X and Y, every continuous map $f : X \to Y$ preserves the specialization order.

Conversely, if $f : \alpha \to \beta$ preserves order, given $\delta \in \beta$, let

$$\gamma_0 = min\{\gamma \in \alpha \,|\, f(\gamma) \in \uparrow \delta\};$$

hence, $f^{-1}(\uparrow \delta) = \uparrow \gamma_0$. Consequently, f is continuous. □

The following theorem enables us to conclude that there exists a well-ordered proper class of subcategories $Sob(\alpha)$ with $\alpha \in Ord$.

Theorem 12.2 *Given ordinals $\beta > \alpha \geq 2$, then $Sob(\alpha)$ is strictly contained in $Sob(\beta)$ if and only if there is some infinite regular cardinal λ such that $\alpha \leq \lambda \leq \beta$.*

Proof. Let α, β, $\lambda \in Ord\backslash 2$ be such that $\alpha < \beta$ and $\alpha \leq \lambda \leq \beta$ with λ an infinite regular cardinal. The closed set λ of β trivially satisfies (i'_0); we shall show that it also satisfies (i'_α), so that λ is α-irreducible.

Let $g : \beta \to \alpha$ be a continuous map.

If $\lambda < \beta$, let $\delta \in \alpha$ be such that $g(\lambda) = \delta$; hence, since the continuity of g is equivalent to the preservation of order (by 12.1), it follows that $\theta \in \lambda \Rightarrow g(\theta) \leq g(\lambda) = \delta$ and, consequently, $\lambda \subseteq g^{-1}(\downarrow \delta)$.

If $\lambda = \beta$, since $\alpha = \bigcup\limits_{\delta\in\alpha} \downarrow \delta$, we have that $\lambda \subseteq \bigcup\limits_{\delta\in\alpha} g^{-1}(\downarrow \delta)$ and, then, as $\alpha < \lambda$ and λ is regular, $\lambda \subseteq g^{-1}(\downarrow \delta)$ for some $\delta \in \alpha$.

Thus, there exists $\delta_0 = min\,\{\delta \in \alpha \,|\, \lambda \subseteq g^{-1}(\downarrow \delta)\}$. Moreover, $\lambda \cap g^{-1}(\{\delta_0\}) \neq \emptyset$, so that $\delta_0 = max\, g(\lambda)$. Indeed, if $\lambda \cap g^{-1}(\{\delta_0\}) = \emptyset$, then $\lambda \subseteq \bigcup_{\delta\in\delta_0} g^{-1}(\downarrow \delta)$; but, since λ is regular, it follows that $\lambda \subseteq g^{-1}(\downarrow \delta)$ for some $\delta \in \delta_0$, which contradicts to the definition of δ_0. Hence, $\delta_0 \in g(\lambda)$.

Therefore, we have shown that λ is an α-irreducible closed set of β. But λ is not the closure of a single point; in fact, a set is the closure of a singleton iff it is a successor ordinal. Thus β is not an α-sober space, so the inclusion $Sob(\alpha) \subseteq Sob(\beta)$ is strict.

Conversely, let us assume that there exists no infinite regular cardinal between α and β. The only closed subsets of β which are not the closure of a single point are the limit ordinals. We shall show that they are not α-irreducible, so thus $\beta \in Sob(\alpha)$ and $Sob(\alpha) = Sob(\beta)$. Let γ be a limit ordinal in β, let λ be its cofinality character (which is an infinite regular cardinal) and let

$$f : \lambda \to \gamma$$

be a strictly increasing function such that

$$\gamma = \bigcup_{\delta\in\lambda} f(\delta).$$

By hypothesis, λ must be smaller than α, and, according to the assumptions on f, for each $\phi \in \gamma$ there is some $\delta \in \lambda$ such that $\phi < f(\delta)$ and, so, the set $\{\delta \in \lambda \,|\, \phi \leq f(\delta)\}$ is not empty. Thus, let

$$g : \beta \to \alpha$$

be defined as follows:

$$g(\phi) = \begin{cases} min\{\delta \in \lambda \,|\, \phi \leq f(\delta)\}, & \text{if } \phi \in \gamma, \\ \lambda, \text{ otherwise.} & \end{cases}$$

It is obvious that g is nondecreasing, hence it is continuous, by 12.1. But γ fails (i'_α) with respect to g; indeed, we have that $g(\gamma) = \lambda$, since the definition of g and the fact that f is strictly increasing imply that, for each $\delta \in \lambda$, $g(f(\delta)) = \delta$. $\square$

Corollary 12.3 *The family $(Sob(\alpha))$, such that α is an infinite cardinal, is a well-ordered proper class which is contained in the "lattice" of epireflective subcategories of $\mathcal{T}op_0$.*

Proof. If α and β are infinite cardinals and $\alpha < \beta$ then there is some infinite regular cardinal between them, since, for every infinite cardinal α, the cardinal α^+ is regular. Thus, the inequality $Sob(\alpha) \neq Sob(\beta)$ follows. Now, using 11.5, we get the claimed result. □

Remark 12.4 Indeed, by 12.2 and 12.3, the ordinals $\alpha > 2$ for which $(Sob(\alpha))$ strictly contains $(Sob(\beta))$ for all $\beta < \alpha$ are precisely all infinite cardinals and all ordinals which are the successor of an infinite regular cardinal. Thus, we have that

$$Sob(2) \subset Sob(\omega_0) \subset Sob(\omega_0 + 1) = Sob(\omega_0 + \omega_0) = \cdots = Sob(\omega_0 \cdot \omega_0) = \cdots$$

$$\cdots \subset Sob(\omega_1) \subset Sob(\omega_1 + 1) = \cdots \subset Sob(\omega_\omega) = Sob(\omega_\omega + 1) = \cdots.$$

Remark 12.5 In the last section we characterized the epireflective hull of each ordinal endowed with the Alexandrov topology in $\mathcal{T}op_0$. In [48], S. Mantovani considered each ordinal α equipped with the upper-interval topology, i.e., the non trivial open sets are of the form $\{\delta\in\alpha \mid \delta > \beta\}$, $\beta \in \alpha$, and characterized the epireflective hulls of these spaces in $\mathcal{T}op_0$. It is obvious that, for each ordinal α, the upper-interval topology and the Alexandrov topology coincide iff $\alpha \leq \omega_0$: for $\alpha > \omega_0$, each limit ordinal in α is closed for the Alexandrov topology, but not for the upper-interval one. For $\alpha > \omega_0$, the epireflective hulls obtained in this chapter are different from Mantovani's hulls. In fact, each successor ordinal with the upper-interval topology is a sober space. More generally, it is proved in [48] that for α and β with the upper-interval topology the corrresponding epireflective hulls coincide if and only if $cf(\alpha) = cf(\beta)$. Moreover, S. Mantovani showed that these epireflective hulls are not comparable in the "lattice" of epireflective subcategories of $\mathcal{T}op_0$. Thus, our definition of α-sober space provides a more natural generalization of the concept of sober space. Namely, and in contrast with Mantovani's epireflective hulls, we have that:

1. The function
$$Ord\backslash 2 \to \mathcal{L}(\mathcal{T}op_0)$$
where $\mathcal{L}(\mathcal{T}op_0)$ denotes the "lattice" of epireflective subcategories of $\mathcal{T}op_0$, which assigns, to each ordinal α, the subcategory $Sob(\alpha)$ is order-preserving (from Corollary 11.5).

2. As we showed in Lemma 12.1, the class of all ordinals and all order-preserving maps may be considered as a full concrete subcategory of $\mathcal{T}op_0$, by equipping each ordinal with the Alexandrov topology. This fails to hold if the upper-interval topology replaces the Alexandrov one. Indeed, let

$$f : \omega + 1 \longrightarrow \omega + 1$$

be defined by

$$f(\delta) = 0 \quad \text{for all} \;\; \delta \in \omega;$$
$$f(\omega) = \omega.$$

Then f is order-preserving, 1 is a closed set for both topologies, the Alexandrov one and the upper-interval one, but $f^{-1}(1) = \omega$ is not closed for the upper-interval topology.

Chapter IV

Solid hulls

Solid categories are concrete categories in which every structured sink has a semi-final lift. These categories, introduced, under different names, by V. Trnková [75] and R.-E. Hoffmann [35, 36], are known to retain properties of the base category, such as completeness, cocompletness and other convenient ones, and yet to be broad enough to encompass all "well-behaved" categories in Topology and Algebra; see [2] for more details.

One property is, however, less satisfactory: there seems to be no general procedure for a construction of a solid extension as small as possible, i.e., a *solid hull*, of an arbitrary concrete category. This contrasts with the situation of *topological categories*, i.e., categories in which every structured sink has a final lift: the *topological hull*, the so-called *MacNeille completion*, introduced by H. Herrlich [27], was constructed generally by J. Adámek, H. Herrlich and G. E. Strecker [1] in the sense that, whenever that construction is legitimate, is the topological hull, and, whenever it is not legitimate, a topological hull fails to exist.

In the present chapter we study conditions under which a given concrete category has a solid hull. This continues the research initiated by J. Rosický [55, 56, 57] who presented, inter alia, a concrete category over $\mathcal{S}et$ which does not have a solid hull, although it has a finally dense, solid extension (see 13.11 below). In [57], Rosický shows that, under the set axiom (M) of the non-existence of a proper class of measurable cardinals, there is a concrete category over $\mathcal{S}et$ with a small finally dense subcategory which does not have a solid hull. Based on results of J. Adámek, J. Rosický and V. Trnková ([5], [7], [57]), we show that, furthermore, the existence of solid hulls for concrete categories over $\mathcal{S}et$ with a small finally dense subcategory is equivalent to the large-cardinal Weak Vopěnka's Principle. ((M) implies the negation of *Weak Vopěnka's Principle*, see [7].)

The existence of solid hulls and of reflective hulls are closely related, as we shall see in this chapter.

13 Solid hull

We recall that a *concrete category* over a category $\mathcal{X}$ is a pair $(\mathcal{A}, U)$, where $\mathcal{A}$ is a category and $U : \mathcal{A} \to \mathcal{X}$ is a faithful functor; furthermore, a *concrete functor* from $(\mathcal{A}, U)$ to another concrete category $(\mathcal{B}, V)$ over $\mathcal{X}$, denoted by $F : (\mathcal{A}, U) \to (\mathcal{B}, V)$, is a functor $F : \mathcal{A} \to \mathcal{B}$ such that $U = V \cdot F$.

A convenient reference for background information on concrete categories is [2].

Throughout this chapter, for all concrete categories $(\mathcal{A}, U)$, we assume that U is *amnestic*, i.e., every $\mathcal{A}$-isomorphism whose U-image is an identity must be an identity.

A well-known concrete category is $\mathcal{T}op$ endowed with the natural forgetful functor over $\mathcal{S}et$. An important property of $\mathcal{T}op$ is the following:

(1) If (X_i, τ_i) are topological spaces, $i \in I$, and $(f_i : X_i \to X)_I$ is a family of maps, then there is a unique topology τ in X, the final topology with respect to $(f_i)_I$, such that, if (Y, υ) is a topological space and $g : X \to Y$ is a map for which $g \cdot f_i$ is continuous for all $i \in I$, then $g : (X, \tau) \to (Y, \upsilon)$ is a continuous map.

In fact, a number of properties of $\mathcal{T}op$ may be derived from (1).

Several known concrete categories fulfil the above condition and they are just said topological. More precisely:

We recall that if $(\mathcal{A}, U)$ is a concrete category, then an $\mathcal{A}$-sink $(f_i : A_i \to A)_I$ is *U-final* provided that each $\mathcal{X}$-morphism $g : UA \to UB$ carries an $\mathcal{A}$-morphism whenever $g \cdot f_i$ carries an $\mathcal{A}$-morphism for all $i \in I$. The dual notion is *U-initial* source.

A concrete category $(\mathcal{A}, U)$ is called *topological* provided that every U-structured sink $(x_i : UA_i \to X)_I$ has a U-final lift $(f_i : A_i \to A)_I$, i.e., $UA = X$ and $(f_i : A_i \to A)_I$ is U-final. We may equivalently define a topological category as a concrete category $(\mathcal{A}, U)$

for which every U-structured source $(x_i : X \to UA_i)_I$ has a U-initial lift. The faithfulness and amnesticity of U assures the unicity of each U-final (or U-initial) lift.

Topological categories have very good properties (see [2]); we recall here that, in particular, if $(\mathcal{A}, U)$ is a topological category over $\mathcal{X}$, then

(p_1) U is a right adjoint;

(p_2) $\mathcal{A}$ is (co)complete whenever $\mathcal{X}$ is (co)complete;

(p_3) If $(\mathcal{A}, U)$ is fibre-small (i.e., for each $\mathcal{X}$-object X the collection of all $\mathcal{A}$-objects A for which $UA = X$ is a set), then $\mathcal{A}$ is (co)wellpowered whenever $\mathcal{X}$ is (co)wellpowered.

Several examples of everyday topological categories may be found in [2]. Now, we describe an example of concrete categories which are topological, in spite of their algebraic origin, and which will be very useful in the sequel.

Example 13.1 Using terminology of [6], let Σ be a λ-*ary relational signature*, that is, Σ is a set of relation symbols, such that, for each $\sigma \in \Sigma$, we are given an arity $ar(\sigma)$ where $ar(\sigma)$ is a set with $card(ar(\sigma)) < \lambda$. A *relational structure* A of type Σ consists of an underlying set X_A and of relations $\sigma_A \subseteq X_A^{ar(\sigma)}$ for each σ. The category $\mathcal{R}el(\Sigma)$ has, as objects, all relational structures of type Σ and, as morphisms, all homomorphisms, i.e., maps preserving the corresponding relations.

The category $\mathcal{R}el(\Sigma)$, with the usual underlying functor over $\mathcal{S}et$, is topological. In fact, given relational structures A_i, $i \in I$, and a sink $(f_i : X_{A_i} \to X)_I$ in $\mathcal{S}et$, we define a final lift by taking the relational structure A defined by

$$X_A = X \quad \text{and, for each } \sigma \in \Sigma, \;\; \sigma_A = \bigcup_{i \in I} \{(f_i(a_t))_{t \in ar(\sigma)} \mid (a_t)_{t \in ar(\sigma)} \in \sigma_{A_i}\}.$$

The notion of solid category, which we recall next, generalizes topological category (as well as topologically algebraic category, see [2]). Thus, solid categories arise in abundance in Topology and Algebra.

Definition 13.2 A concrete category $(\mathcal{A}, U)$ is *solid* if, for each U-sink $S = (UA_i \xrightarrow{x_i} X)_I$, there exists a U-morphism $X \xrightarrow{y} UB$ such that:

(i) $y \cdot x_i$ carries an $\mathcal{A}$-morphism $A_i \to B$ for each $i \in I$;

(*ii*) whenever a U-morphism $X \xrightarrow{z} UC$ is such that $z \cdot x_i$ carries an $\mathcal{A}$-morphism for all $i \in I$, then there is a unique $\mathcal{A}$-morphism $B \xrightarrow{f} C$ such that $Uf \cdot y = z$.

Remarks 13.3 (cf.[71])

1. This concept is substantially weaker than that of topological category. Nevertheless, it retains some of the most significant properties, e.g. (p_1) and (p_2) mentioned above.

2. Other relevant properties concerning solidness are the following:

   (a) Solid functors, i.e., faithful functors $U : \mathcal{A} \to \mathcal{X}$ such that $(\mathcal{A}, U)$ is solid, are closed under composition.

   (b) If $\mathcal{A}$ is a reflective subcategory of a category $\mathcal{B}$, then the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ is solid.

The following problem has been studied by several authors (cf, for instance, [62] and references there): Given a concrete category $(\mathcal{A}, U)$, is there an extension of $(\mathcal{A}, U)$ with good enough properties, e.g., a topological or a solid extension? And, if so, is there a smallest one?

Here, we are just interested in the existence of a smallest solid extension.

To make the terminology more precise, we recall that:

A full concrete embedding $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ is called an *extension of* $(\mathcal{A}, U)$. We also say that $(\mathcal{B}, V)$ is an extension of $(\mathcal{A}, U)$.

An extension $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ of $(\mathcal{A}, U)$ is *finally dense* if for every $\mathcal{B}$-object B there exists a V-final sink $(f_i : EA_i \to B)_I$ with each A_i in $\mathcal{A}$.

Dually, we have the notion of *initially dense* extension.

Definition 13.4 If $E_1 : (\mathcal{A}, U) \to (\mathcal{B}_1, V_1)$ and $E_2 : (\mathcal{A}, U) \to (\mathcal{B}_2, V_2)$ are finally dense extensions of $(\mathcal{A}, U)$, we say that E_1 *is smaller or equal than* E_2 provided that there exists a full concrete embedding $F : (\mathcal{B}_1, V_1) \to (\mathcal{B}_2, V_2)$ such that $F \cdot E_1 = E_2$.

It is obvious that this relation "smaller or equal than" is reflexive and transitive; furthermore, it is "almost" antisymmetric: if E_1 is smaller or equal than E_2 and E_2 is smaller or equal than E_1, then the two extensions of $(\mathcal{A}, U)$ are isomorphic, that is, there is a concrete isomorphism F such that $F \cdot E_1 = E_2$. That is a consequence of the following lemma.

Lemma 13.5 *Given finally dense full concrete embeddings $E_i : (\mathcal{A}, U) \to (\mathcal{B}_i, V_i)$, $i = 1, 2$, there exists at most one full concrete embedding $F : (\mathcal{B}_1, V_1) \to (\mathcal{B}_2, V_2)$ with $F \cdot E_1 = E_2$.*

Proof. Let F and F' be full concrete embeddings such that $F \cdot E_1 = F' \cdot E_1 = E_2$. For each $B \in Obj(\mathcal{B}_1)$, we have that $(f_i : E_1 A_i \to B)_I$ is the sink of all morphisms with codomain B and domain in $E_1(\mathcal{A})$ if and only if $(F f_i : E_2 A_i \to FB)_I$ and $(F' f_i : E_2 A_i \to F' B)_I$ are the sinks of all morphisms with codomain FB and $F'B$, respectively, and domain in $E_2(\mathcal{A})$. Since E_2 is finally dense, both of the sinks $(F f_i : E_2 A_i \to FB)_I$ and $(F' f_i : E_2 A_i \to F' B)_I$ are final. Therefore, from the concretness of F and F' and the fact that V_2 is amnestic, we conclude that $FB = F'B$. Since $V_2 \cdot F = V_2 \cdot F'$ and V_2 is faithful, it turns out that F and F' coincide on morphisms too. $\square$

Let us recall that, given a concrete category $(\mathcal{A}, U)$ over $\mathcal{X}$, a U-sink $S = (UA_i \overset{f_i}{\to} X)_I$ is said to be *closed* provided that it contains all morphisms $g : UB \to X$ such that for each $h : X \to UA$, the $\mathcal{X}$-morphism $h \cdot g$ carries an $\mathcal{A}$-morphism whenever all $h \cdot f_i$ carry an $\mathcal{A}$-morphism.

We may consider the quasicategory of all closed U-sinks by taking as morphisms from $S = (UA_i \overset{f_i}{\to} X)_I$ to $S' = (UA_j \overset{g_j}{\to} Y)_J$ all $\mathcal{X}$-morphisms $f : X \to Y$ such that $f \cdot f_i$ belong to S', for all $i \in I$. As it was shown by J. Adámek, H. Herrlich and G. Strecker in [1], a concrete category $(\mathcal{A}, U)$ has a smallest topological extension if and only if the conglomerate of closed U-sinks is legitimate and, in this case, the category of closed U-sinks is a smallest topological extension, usually called *the MacNeille completion.* It is just the unique (up to isomorphism) initially and finally dense, topological extension of the concrete category.

The following result, due to Hoffmann and Tholen, is very important for this chapter. (Naturally, if $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ is an extension, then we say that $(\mathcal{A}, U)$, or simply $\mathcal{A}$, is reflective in $(\mathcal{B}, V)$, or $\mathcal{B}$, provided that $E(\mathcal{A})$ is reflective in $\mathcal{B}$.)

Proposition 13.6 ([37, 71]) *A concrete category is olid if and only if it has a MacNeille completion and is reflective in it.* $\square$

Now, let us define solid hull of a concrete category.

Definition 13.7 An extension $E^s : (\mathcal{A}, U) \to (\mathcal{A}^s, U^s)$ of a concrete category $(\mathcal{A}, U)$ is a *solid hull* of $(\mathcal{A}, U)$ provided that:

(i) it is a finally dense, solid extension of $(\mathcal{A}, U)$;

(ii) it is smaller or equal than any other finally dense, solid extension of $(\mathcal{A}, U)$.

Since, by 13.5, a solid hull, when it exists, is unique up to isomorphism, it will often be called *the solid hull*.

In this section, we will show that, if a concrete category has a solid hull, it is its reflective hull in any finally dense, solid extension.

We shall make use of the following

Lemma 13.8 *A solid category is reflective in each of its finally dense extensions.*

Proof. Let $(\mathcal{A}, U)$ be solid and let $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ be a finally dense extension. If $(f_i : EA_i \to B)_I$ is the sink of all morphisms with domain in $E(\mathcal{A})$ and codomain B, let $p : VB \to UA$ be the semi-final lift of the U-structered sink $(Vf_i : UA_i \to VB)_I$. Since $(f_i)_I$ is V-final, $p : VB \to VEA$ carries a $\mathcal{B}$-morphism, i.e., there is a $\mathcal{B}$-morphism $\overline{p} : B \to EA$ such that $V\overline{p} = p$. Now, it is easy to show that the morphism $\overline{p} : B \to EA$ is a reflection of B to E. $\square$

Let us remark that the problem of the existence of a solid hull or, even, of a solid extension, makes sense only for concrete categories which have a MacNeille completion. Indeed, if $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ is a solid extension, the MacNeille completion of $(\mathcal{B}, V)$ exists (by 13.6) and it is a topological extension of $(\mathcal{A}, U)$ which guarantees that $(\mathcal{A}, U)$ has a MacNeille completion ([1]).

Consequently, from now on, we shall always assume that

the concrete categories considered have a MacNeille completion.

Theorem 13.9 *A concrete category $(\mathcal{A}, U)$ has a solid hull if and only if there exists the reflective hull of $\mathcal{A}$ in any finally dense, solid extension of $(\mathcal{A}, U)$. Furthermore, if the solid hull exists, it is concretely isomorphic to each of those reflective hulls.*

Proof. Let

$$E^s : (\mathcal{A}, U) \to (\mathcal{A}^s, U^s)$$

be the solid hull of $(\mathcal{A}, U)$ and let $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ be a finally dense, solid extension. From 13.5 and 13.7, there exists a unique full concrete embedding $F : (\mathcal{A}^s, U^s) \to (\mathcal{B}, V)$ such that $F \cdot E^s = E$. Since E is finally dense, F is finally dense and, from 13.8, $F(\mathcal{A}^s)$ is reflective in $\mathcal{B}$, because $(\mathcal{A}^s, U^s)$ is solid. Now, we show that $F(\mathcal{A}^s)$ is the reflective hull of $E(\mathcal{A})$ in $\mathcal{B}$. Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{B}$ which contains $E(\mathcal{A})$. Then $(\mathcal{C}, V')$, where V' is the restriction of V to $\mathcal{C}$, is a finally dense solid extension of $\mathcal{A}$, because a reflective concrete subcategory of a solid category is solid, by 13.3.2. Therefore, from 13.5 and 13.7, $F(\mathcal{A}^s)$ is a subcategory of $\mathcal{C}$.

Conversely, let

$$E^t : (\mathcal{A}, U) \to (\mathcal{A}^t, U^t)$$

be the MacNeille completion of $(\mathcal{A}, U)$ and let A^r be the reflective hull of $E^t(\mathcal{A})$ in $\mathcal{A}^t$. Hence, $(\mathcal{A}^r, U^r)$, where U^r is the restriction of U^t to A^r, is a finally dense, solid extension of $(\mathcal{A}, U)$. We show that, moreover, $(\mathcal{A}^r, U^r)$ is a solid hull of $(\mathcal{A}, U)$.

For a finally dense, solid extension $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$, let $F^t : (\mathcal{B}, V) \to (\mathcal{B}^t, V^t)$ be the MacNeille completion of $(\mathcal{B}, V)$. Then, $F^t \cdot E : (\mathcal{A}, U) \to (\mathcal{B}^t, V^t)$ is a topological extension of $(\mathcal{A}, U)$ and so there is a full concrete embedding $G : (\mathcal{A}^t, U^t) \to (\mathcal{B}^t, V^t)$ such that $G \cdot E^t = F^t \cdot E$.

$$\begin{tikzcd}
(\mathcal{A}, U) \arrow[rr, "E^t"] \arrow[dr, "E^r"] \arrow[dd, "E"'] & & (\mathcal{A}^t, U^t) \arrow[dd, "G"] \\
& (\mathcal{A}^r, U^r) \arrow[ur] & \\
(\mathcal{B}, V) \arrow[rr, "F^t"] & & (\mathcal{B}^t, V^t)
\end{tikzcd}$$

It is clear that G is finally dense and, since $(\mathcal{A}^t, U^t)$ is solid, it follows from 13.8, that $G(\mathcal{A}^t)$ is reflective in $\mathcal{B}^t$. But, by hypothesis, $G \cdot E^t(\mathcal{A})$ has a reflective hull in $\mathcal{B}^t$. Hence, that hull must coincide with the reflective hull of $G \cdot E^t(\mathcal{A})$ in $G(\mathcal{A}^t)$ which, of course, is $G(\mathcal{A}^r)$. Consequently, the reflective hull of $G \cdot E^t(\mathcal{A})$ in $\mathcal{B}^t$ is concretely isomorphic to $\mathcal{A}^r$. Analogously, F^t is finally dense, $(\mathcal{B}, V)$ is solid and, thus, $F^t(\mathcal{B})$ is reflective in $\mathcal{B}^t$. Then the reflective hull of $F^t \cdot E(\mathcal{A}) = G \cdot E^t(\mathcal{A})$ in $\mathcal{B}^t$ coincides with the reflective hull of $F^t \cdot E(\mathcal{A})$ in $F^t(\mathcal{B})$. Thus, $\mathcal{A}^r$ is concretely isomorphic to the reflective hull of $E(\mathcal{A})$ in $\mathcal{B}$. Therefore, the extension $(\mathcal{A}^r, U^r)$ of $(\mathcal{A}, U)$ is smaller or equal than the extension $(\mathcal{B}, V)$. $\square$

Remarks 13.10

1. As we have just shown, the existence of a solid hull of a given concrete category depends on the existence of a convenient reflective hull. Let us point out now that the converse is also true: the existence of the reflective hull of a given subcategory depends on the existence of the solid hull of a convenient concrete category. Indeed, let $\mathcal{A}$ be a subcategory of a category $\mathcal{X}$. Then $(\mathcal{X}, 1_{\mathcal{X}})$ and $(\mathcal{A}, E)$, where E is the inclusion of $\mathcal{A}$ in $\mathcal{X}$, are concrete categories over $\mathcal{X}$; furthermore, $(\mathcal{A}, E) \hookrightarrow (\mathcal{X}, 1_{\mathcal{X}})$ is the MacNeille completion of $(\mathcal{A}, E)$ and, thus, the reflective hull of $\mathcal{A}$ in $\mathcal{X}$, if it exists, is the solid hull of $(\mathcal{A}, E)$.

2. For a given property P on concrete categories, an extension $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ is called a P-*extension* provided that $(\mathcal{B}, V)$ satisfies the property P. A P-hull of $(\mathcal{A}, U)$ is a finally dense P-extension of $(\mathcal{A}, U)$ which is smaller or equal than any other finally dense P-extension. For several properties P and for some classes E of morphisms, the E-reflective hull on every finally dense P-extension of $(\mathcal{A}, U)$ is a P-hull of $(\mathcal{A}, U)$ ([62]). But there is an important difference between the solid hull and several other P-hulls: Indeed, the existence of the P-hulls considered in [62] is guaranteed by that of finally dense P-extensions. However, the same does not hold for the solid hull, even if the base category is just $\mathcal{S}et$ as we are going to see in Example 13.11.

The following example of a concrete category over $\mathcal{S}et$ which has a finally dense solid extension but does not have a solid hull was presented by Rosický in 1.2 of [56], using a model-theoretic language. Next, we describe this example using a different aproach which stresses the relation between the problems of the existence of a solid hull and of a reflective hull.

Example 13.11 (cf [56]) Let $\mathcal{C}_1$ be the category obtained from the coproduct of the category of sets with the category of complete-join-semilattices by adding the following morphisms: for each set X and each complete semilattice A, $\mathcal{C}_1(X, A)$ consists of all maps from X to the underlying set of A. Let $\mathcal{C}_2$ be defined in an analogous way by replacing the category of complete-join-semilattices by that of algebras on one unary operation. For the usual forgetful functors $U_i : \mathcal{C}_i \to \mathcal{S}et$, the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are solid. Let $(\mathcal{A}, U) = (\mathcal{C}_1, U_1) \times (\mathcal{C}_2, U_2)$ be the product of $(\mathcal{C}_1, U_1)$ and $(\mathcal{C}_2, U_2)$ in the quasicategory $CAT(\mathcal{S}et)$ of concrete categories over $\mathcal{S}et$ and concrete functors between them. We recall that $\mathcal{A}$ is the subcategory of the product category $\mathcal{C}_1 \times \mathcal{C}_2$ with objects all pairs (C_1, C_2) such that C_1 and C_2 have the same underlying set and with morphisms all $f : (C_1, C_2) \to (D_1, D_2)$, where $f : C_i \to D_i$ is a $\mathcal{C}_i$-morphism, $i = 1, 2$. The functor U is defined by $U(C_1, C_2) = U_1 C_1 = U_2 C_2$ and $Uf = f$. We are going to show that $(\mathcal{A}, U)$ does not have a solid hull. Let $E_i^t : (\mathcal{C}_i, U_i) \to (\mathcal{C}_i^t, U_i^t)$ be the MacNeille completion of $(\mathcal{C}_i, U_i)$, $i = 1, 2$, and $(\mathcal{T}, V) = (\mathcal{C}_1^t, U_1^t) \times (\mathcal{C}_2^t, U_1^t)$. The concrete category $(\mathcal{T}, V)$ is solid, since it is topological. Furthermore, $(\mathcal{T}, V)$ is cocomplete, since it is solid over a cocomplete category (by 13.3.1). Consider the categories $(\mathcal{A}_1, V_1) = (\mathcal{C}_1, U_1) \times (\mathcal{C}_2^t, U_2^t)$ and $(\mathcal{A}_2, V_2) = ((\mathcal{C}_1^t, U_1^t) \times (\mathcal{C}_2, U_2)$. It is clear that there are full concrete embeddings $G_i : (\mathcal{A}, U) \to (\mathcal{A}_i, V_i)$ and $F_i : (\mathcal{A}_i, V_i) \to (\mathcal{T}, V)$, $i = 1, 2$, such that $F_1 \cdot G_1 = F_2 \cdot G_2$ and $F_1(\mathcal{A}_1) \cap F_2(\mathcal{A}_2) = F_1 \cdot G_1(\mathcal{A}) = F_2 \cdot G_2(\mathcal{A})$.

$$
\begin{tikzcd}
 & (\mathcal{A}_1, V_1) = (\mathcal{C}_1, U_1) \times (\mathcal{C}_2^t, U_2^t) \arrow[dr, "F_1"] & \\
(\mathcal{C}_1, U_1) \times (\mathcal{C}_2, U_2) = (\mathcal{A}, U) \arrow[ur, "G_1"] \arrow[dr, "G_2"'] & & (\mathcal{T}, V) = (\mathcal{C}_1^t, U_1^t) \times (\mathcal{C}_2^t, U_2^t) \\
 & (\mathcal{A}_2, V_2) = (\mathcal{C}_1^t, U_1^t) \times (\mathcal{C}_2, U_2) \arrow[ur, "F_2"'] &
\end{tikzcd}
$$

Moreover, the categories $F_i(\mathcal{A}_i)$ are reflective in $\mathcal{T}$ because $\mathcal{C}_i$ is reflective in $\mathcal{C}_i^t$ and $\mathcal{C}_i^t$ is topological, $i = 1, 2$. Then, as $F_1 \cdot G_1(\mathcal{A})$ is the intersection of two reflective subcategories, in order to conclude that it does not have a reflective hull, it suffices to show that it is not reflective. In fact, $F_1 \cdot G_1(\mathcal{A})$ is not cocomplete: let C_1 be the free complete semi-lattice generated by the set of all natural numbers, let C_2 be the underlying set of C_1, let D_2 be the unary algebra generated by a singleton set and let D_1 be the underlying set of D_2; then, the coproduct of (C_1, C_2) and (D_1, D_2) does not exist in $\mathcal{A}$. Hence, $F_1 \cdot G_1(\mathcal{A})$ is not reflective in the cocomplete category $\mathcal{T}$. On the other hand, $F_1 \cdot G_1 : (\mathcal{A}, U) \to (\mathcal{T}, V)$ is finally dense; it is a consequence of the fact that E_1^t and E_2^t are finally dense and that, for $i = 1, 2$, C_i has discrete structures which are preserved by E_i^t (see [2]). Therefore, from 13.9, $(\mathcal{A}, U)$ does not have a solid hull.

14 Orthogonal and solid hulls of a concrete category

In view of the two first chapters and the last section, an important candidate for being the solid hull of a concrete category is the orthogonal hull in the MacNeille completion. So, in the sequel, we use the following notion.

Definition 14.1 By *the orthogonal hull of a concrete category* $(\mathcal{A}, U)$ we shall mean the extension of $(\mathcal{A}, U)$ to the orthogonal hull of its image in the MacNeille completion.

We will see that, under suitable conditions, the orthogonal hull is a solid hull.

Proposition 14.2 *The orthogonal hull of a concrete category* $(\mathcal{A}, U)$ *is smaller or equal than any finally dense, solid extension of* $(\mathcal{A}, U)$*.*

Proof. Let $(\mathcal{A}, U)$ be a concrete category with the MacNeille completion $E^t : (\mathcal{A}, U) \to (\mathcal{A}^t, U^t)$, let $\mathcal{A}^o$ be the orthogonal hull of $E^t(\mathcal{A})$ in $\mathcal{A}^t$, U^o the restriction of U^t to $\mathcal{A}^o$ and $E^o : (\mathcal{A}, U) \to (\mathcal{A}^o, U^o)$ the corestriction of E^t to $(\mathcal{A}^o, U^o)$. If

$$E : (\mathcal{A}, U) \to (\mathcal{B}, V)$$

is a finally dense, solid extension, let $\mathcal{A}^1$ be the orthogonal hull of $E(\mathcal{A})$ in $\mathcal{B}$, U^1 be the restriction of V to $\mathcal{A}^1$ and $E^1 : (\mathcal{A}, U) \to (\mathcal{A}^1, U^1)$ be the corresponding extension. Let $F^t : (\mathcal{B}, V) \to (\mathcal{B}^t, V^t)$ be the MacNeille completion of $(\mathcal{B}, V)$; then, there is a full concrete embedding $G : (\mathcal{A}^t, U^t) \to (\mathcal{B}^t, V^t)$ such that $G \cdot E^t = F^t \cdot E$.

$$\begin{tikzcd}
(\mathcal{A},U) \arrow[rr, "E^t"] \arrow[d, "E^1"'] \arrow[dr, "E^o"] \arrow[dd, bend right=60, "E"'] & & (\mathcal{A}^t,U^t) \arrow[dd, "G"] \\
(\mathcal{A}^1,U^1) \arrow[d] & (\mathcal{A}^o,U^o) \arrow[ur] & \\
(\mathcal{B},V) \arrow[rr, "F^t"] & & (\mathcal{B}^t,V^t)
\end{tikzcd}$$

Hence, from 2.12.2 and 13.8, we have that G yields a concrete isomorphism between $(\mathcal{A}^o, U^o)$ and the orthogonal hull of $F^t \cdot E(\mathcal{A})$ in $\mathcal{B}^t$, which, through F^t, is concretely isomorphic to $(\mathcal{A}^1, U^1)$; thus the two finally dense extensions E^o and E^1 are isomorphic. Therefore, it is clear that E^o is smaller or equal than E. □

Corollary 14.3 *If the orthogonal hull of a concrete category $(\mathcal{A}, U)$ is solid then it is the solid hull of (A, U) .* □

For the particular case of concrete categories over $\mathcal{S}et$ with a fibre-small MacNeille completion, the above proposition is stated in [56] as Theorem 1.1 (see also [57], where the translation from model-theoretic terms to categorical ones is mentioned).

Remarks 14.4

1. The proof of 14.2 shows that we obtain an equivalent definition of the orthogonal hull of a concrete category if, in 14.1, we replace "the MacNeille completion" by "some finally dense, solid extension".

2. Whenever $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ is a finally dense, solid extension of $(\mathcal{A}, U)$, the orthogonal hull $E^1 : (\mathcal{A}, U) \to (\mathcal{A}^1, U^1)$ as described above is a solid hull if and only if $\mathcal{A}^1$ is reflective in $\mathcal{B}$, as we can conclude using the above remark 1., 13.8 and the fact that a reflective subcategory of a solid category is solid.

3. Let $(\mathcal{A}, U)$ be a concrete category over a category with connected colimits and let $E : (\mathcal{A}, U) \to (\mathcal{B}, V)$ be a finally dense solid extension of $(\mathcal{A}, U)$. Then, from 2.10 and the fact that a solid category has all colimits which exist in the base category

(see [71]), the orthogonal hull of $(\mathcal{A}, U)$ is its solid hull if and only if the class $[E(\mathcal{A})]^{\perp_{\mathcal{B}}}$ satisfies the solution set condition in $\mathcal{B}$.

Proposition 14.5 *Let $(\mathcal{A}, U)$ be a concrete category over a complete, wellpowered base category. If $(\mathcal{A}, U)$ has a fibre-small MacNeille completion and $\mathcal{A}$ has a cogenerating set, then $(\mathcal{A}, U)$ has a solid hull.*

Proof. Let $E^t : (\mathcal{A}, U) \to (\mathcal{A}^t, U^t)$ be a fibre-small MacNeille completion of $(\mathcal{A}, U)$. Then, from the hypothesis over the base category, it follows that $\mathcal{A}^t$ is complete and wellpowered. Now, the proof follows from the Special Adjoint Functor Theorem: Let $\mathcal{A}^l$ be the closure under limits of $E^t(\mathcal{A})$ in $\mathcal{A}^t$. Then, $\mathcal{A}^l$ is complete and wellpowered and it has a cogenerating set. Consequently, $\mathcal{A}^l$ is reflective in $\mathcal{A}^t$ and, therefore, it is a solid hull of $\mathcal{A}$. $\square$

In [57] it was shown that any small concrete category over $\mathcal{S}et$ has a solid hull. From the above proposition we have the following more general result:

Corollary 14.6 *Every small concrete category over a complete and wellpowered category has a solid hull.* $\square$

A concrete category over an $(\mathcal{E}, \mathbb{M})$-category is said to be $\mathbb{M}$-*topological* if every structured source in $\mathbb{M}$ has an initial lift. It is well-known that, for a concrete category $(\mathcal{A}, U)$ over an $(\mathcal{E}, \mathbb{M})$-category, the following implications hold:

$$(\mathcal{A}, U) \textit{ is topological } \implies (\mathcal{A}, U) \textit{ is } \mathbb{M}\textit{-topological } \implies (\mathcal{A}, U) \textit{ is solid.}$$

The $\mathbb{M}$-topological hull of a concrete category over an $(\mathcal{E}, \mathbb{M})$-category is the smallest finally dense $\mathbb{M}$-topological extension. If it exists, it is the $\mathcal{E}$-reflective hull in the MacNeille completion (see, e.g., [62]).

Theorem 14.7 *Let $(\mathcal{A}, U)$ be a concrete category over a cocomplete $(\mathcal{E}, \mathbb{M})$-category $\mathcal{X}$, with $\mathbb{M} \subseteq MonoSource(\mathcal{X})$, and let $E^m : (\mathcal{A}, U) \to (\mathcal{A}^m, U^m)$ be the $\mathbb{M}$-topological hull of $(\mathcal{A}, U)$.*

1. *If $\mathcal{A}^m$ is cowellpowered with respect to U^m-initial bimorphisms then $(\mathcal{A}, U)$ has a solid hull.*

2. *If in $\mathcal{X}$ every epimorphism is split and $Epi(\mathcal{A}^m) = (U^m)^{-1}(Epi(\mathcal{X}))$, then $E^m : (\mathcal{A}, U) \to (\mathcal{A}^m, U^m)$ is the solid hull of $(\mathcal{A}, U)$.*

Proof.

1. If $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category, then $\mathcal{A}^m$ is an $(\mathcal{E}', \mathbb{M}')$-category with $\mathcal{E}' = (U^m)^{-1}(\mathcal{E}$ and $\mathbb{M})' = (U^m)^{-1}(\mathbb{M}) \cap InitialSource(U^m)$. Hence, from 2.17.3 and taking into account 14.4.3, the orthogonal hull of $(\mathcal{A}, U)$ is a solid hull.

2. Let $g : B \to C$ be an initial bimorphism in $\mathcal{A}^m$. Since it is initial and $U^m g$ is a split epimorphism in $\mathcal{X}$, it follows that g is a split epimorphism in $\mathcal{A}^m$. Hence, g is an $\mathcal{A}^m$-isomorphism. Then, by 2.17.3, $[E^m(\mathcal{A})]^{\perp_{\mathcal{A}^m}}$ consists of isomorphisms only. Therefore, the orthogonal hull of $E^m(\mathcal{A})$ in $\mathcal{A}^m$ is $\mathcal{A}^m$, and, from 14.3 and 14.4.1, $E^m : (\mathcal{A}, U) \to (\mathcal{A}^m, U^m)$ is the solid hull of $(\mathcal{A}, U)$. □

Corollary 14.8 *If $(\mathcal{A}, U)$ is a concrete category over $\mathcal{S}et$ with a monotopological hull $(\mathcal{B}, U)$ in which every epimorphism is a surjection then $(\mathcal{B}, U)$ is also the solid hull of $(\mathcal{A}, U)$.* □

Corollary 14.9 *Let $(\mathcal{A}, U)$ be a concrete category over a cocomplete $(\mathcal{E}, \mathbb{M})$-category $\mathcal{X}$, with $\mathbb{M} \subseteq MonoSource(\mathcal{X})$. If in $\mathcal{X}$ every epimorphism is split and the MacNeille completion of $(\mathcal{A}, U)$ is the $\mathbb{M}$-topological hull of $(\mathcal{A}, U)$, then it is also the solid hull of $(\mathcal{A}, U)$.*

Proof. It follows from 14.7.2 and the fact that, if $E^t : (\mathcal{A}, U) \to (\mathcal{A}^t, U^t)$ is the MacNeille completion, then $Epi(\mathcal{A}^t) = (U^t)^{-1}(Epi(\mathcal{X}))$. □

Examples 14.10 In the following examples, for each category $\mathcal{A}$ equipped with the obvious forgetful functor, we describe the MacNeille completion, the monotopological hull, the solid hull and the orthogonal hull of $\mathcal{A}$, which are denoted by $\mathcal{A}^t$, $\mathcal{A}^m$, $\mathcal{A}^s$ and $\mathcal{A}^o$, respectively. By $\mathcal{A}^l$ we denote the limit closure of $\mathcal{A}$ in $\mathcal{A}^t$. We also describe the classes $\mathcal{A}^{\perp_{\mathcal{A}^t}}$ and $\mathcal{A}^{\perp_{\mathcal{A}^m}}$.

1. For the examples (a)-(c) below, we have that $\mathcal{A}^{\perp_{\mathcal{A}^t}} = Iso(\mathcal{A}^t)$, $\mathcal{A}^{\perp_{\mathcal{A}^m}} = Iso(\mathcal{A}^m)$ and $\mathcal{A}^l = \mathcal{A}^o = \mathcal{A}^s = \mathcal{A}^m = \mathcal{A}^t$.

(a) (see [27]) $\mathcal{A}$ is a partially ordered set $(P, \leq)$ considered as a concrete category over the one-morphism category. The MacNeille completion of $(P, \leq)$ yields a MacNeille completion of the concrete category $\mathcal{A}$.

(b) (see [27]) $\mathcal{A}$ is the concrete category over $\mathcal{S}et$ consisting of all finite topological spaces and continuous maps. Then $\mathcal{A}^t$ is the category $\mathcal{F}in\mathcal{G}en$ of finitely generated spaces and continuous maps.

(c) (see [1]) $\mathcal{A}$ is the concrete category over $\mathcal{S}et$ consisting of all compact topological spaces and continuous maps. Then $\mathcal{A}^t$ is the category $\mathcal{C}omp\mathcal{G}en$ of compactly generated spaces.

2. (a) (see [34]) A *quasi-metric space* is a pair (X, d) where X is a set and d is a map $d : X \times X \to [0, \infty]$ such that, for any $x, y, z \in X$,

$$\begin{aligned} d(x,y) &= d(y,x), \\ d(x,x) &= 0 \\ d(x,z) &\leq d(x,y) + d(y,z). \end{aligned}$$

A map $f : (X, d) \to (Y, e)$ is called *non-expansive* if $e(f(x), f(y)) \leq d(x, y)$ for any $x, y \in X$. A quasi-metric space (X, d) is called *separated* if $d(x, y) = 0$ implies $x = y$ for any $x, y \in X$. A separated quasi-metric space (X, d) is called *complete* if every Cauchy sequence converges.

Let $\mathcal{A}$ be the concrete category over $\mathcal{S}et$ of complete metric spaces and non-expansive maps. Then $\mathcal{A}^t$ is the category $\mathcal{QM}et$ of quasi-metric spaces and non-expansive maps, $\mathcal{A}^m$ is its full subcategory of separated quasi-metric spaces and $\mathcal{A}^s$ is its full subcategory of complete separated quasi-metric spaces.

(b) (see [38]) Let $\mathcal{V}ec$ be the category of vector spaces over $\mathbb{K}$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and linear maps. A *quasi-normed space* over $\mathbb{K}$ is a pair $(X, ||.||)$ where $X \in \mathcal{V}ec$ and $||.||$ is a map from X to $[0, \infty]$ such that, for all $x, y \in X$ and $\lambda \in \mathbb{K}$

$$||\lambda x|| = |\lambda| ||x|| \quad \text{and}$$
$$||x + y|| \leq ||x|| + ||y||.$$

A map $f : (X, ||.||) \to (Y, ||.||)$ is *non-expansive* if $||f(x)|| \leq ||x||$ for any $x \in X$. A quasi-normed space is said to be *separated* if $||x|| = 0$ only if $x = 0$ and it is said to be *complete* if Cauchy sequences converge.

Let $\mathcal{A}$ be the category $\mathcal{B}an$ of Banach spaces over $\mathbb{K}$ and non-expansive maps. $\mathcal{B}an$ is a concrete category over the category $\mathcal{V}ec$. In this case, $\mathcal{A}^t$ coincides with the category $\mathcal{QN}orm$ of quasi-normed spaces and non-expansive maps, $\mathcal{A}^m$ is its full subcategory of the separated quasi-normed spaces and $\mathcal{A}^s$ is its full subcategory of the complete separated quasi-normed spaces.

In the examples (a) and (b) above, we have that $\mathcal{A}^{\perp_{\mathcal{A}^t}}$ consists of all initial $\mathcal{A}$-cancellable $\mathcal{A}^t$-morphisms, i.e., of all initial dense $\mathcal{A}^t$-morphisms, and $\mathcal{A}^{\perp_{\mathcal{A}^m}}$ consists of all dense embeddings. Furthermore, $\mathcal{A}^l = \mathcal{A}^o = \mathcal{A}^s \neq \mathcal{A}^m \neq \mathcal{A}^t$.

3. It is well known that the category $\mathcal{T}op$ is the MacNeille completion of its full subcategory $\mathcal{A}$ which consist of the Sierpinski space alone. In this case, $\mathcal{T}op_0$ is the monotopological hull of $\mathcal{A}$ and $\mathcal{S}ob$ is its solid hull. We have already seen that $\mathcal{A}^{\perp_m}$ consists of all b-dense embeddings. It is easy to see that $\mathcal{A}^{\perp_t}$ consists of all initial b-dense morphisms

4. Let $\mathcal{A}$ be the category described in 2.5. With the obvious forgetful functor, $\mathcal{A}$ is a concrete category over $\mathcal{S}et$. This category $\mathcal{A}$ was introduced by Rosický in [56] with the aim of showing that a concrete category over $\mathcal{S}et$ may be complete and simultaneously have a solid hull different from itself. In that paper, he describes the orthogonal hull of $\mathcal{A}$ as a category of models of a first-order theory and concludes that it is the solid hull. Here, we get the same conclusion by begining with the presentation of the MacNeille completion of $\mathcal{A}$.

   The MacNeille completion $\mathcal{A}^t$ of $\mathcal{A}$ can be described as the following category:

   - Objects are pairs
   $$(X, x)$$
   with X a set and $x = (X_i)_{i \in Ord}$ a collection of subsets of X such that either all X_i are empty or, for all $i \in Ord$, $X_i \neq \emptyset$ and,
   $$\text{if } X_i \cap X_k \neq \emptyset \text{ for some pair } (i, k) \text{ with } i < k, \text{ then, for all } j \geq i,\ X_j = X_i.$$
   - A morphism
   $$f : (X, x) \to (Y, y)$$
   is a function $f : X \to Y$ such that $f(X_i) \subseteq Y_i$ for every i.

To show that $\mathcal{A}^t$ is the MacNeille completion of $\mathcal{A}$ we prove that:

(a) $\mathcal{A}^t$ is a topological category over $\mathcal{S}et$;

(b) $\mathcal{A}$ is initially and finally dense in $\mathcal{A}^t$.

(a): It is clear that $\mathcal{A}^t$ is a category. Furthermore, with the obvious forgetful functor, it is a concrete category over $\mathcal{S}et$. In order to prove that it is topological, let $((X^k, x^k))_K$, where $x^k = (X_i^k)_{i\in Ord}$, be a family of $\mathcal{A}^t$-objects and let $(f^k : X^k \to X)_K$ be a family of morphisms in $\mathcal{S}et$. We show that there is a collection $x = (X_i)_{i\in Ord}$ of subsets of X such that the sink $((X^k, x^k) \xrightarrow{f^k} (X, x))_K$ is final. Since the case where $X_i^k = \emptyset$ for all $i \in Ord$ and $k \in K$ is trivial, let us assume that $X_i^k \neq \emptyset$ for $i \in Ord$ and some $k \in K$. We define $x = (X_i)_{i\in Ord}$ as follows:

Let $\hat{X}_i = \cup_K f^k(X_i^k)$ for all $i \in Ord$ and let us consider the class

$$C = \{i \in Ord,\ \hat{X}_i \cap \hat{X}_j \neq \emptyset \text{ for some } j \neq i\}. \tag{9}$$

It is clear that this class is non-empty. Let i_o be its minimum. We put

$$X_i = \begin{cases} \hat{X}_i, & \text{if } i < i_0 \\ \cup_{j\geq i_0}\hat{X}_j, & \text{if } i \geq i_0 \end{cases}$$

It is easy to verify that $((X^k, x^k) \xrightarrow{f^k} (X, x))_K$ is a final sink.

(b): If $X_i = \emptyset$ for all $i \in Ord$, then, on the one hand, the source $((X, x) \xrightarrow{1_X} (X, \overline{a}))_{a\in X}$ where $\overline{a}$ is the function from Ord to X defined by $\overline{a}(i) = a$ for all $i \in Ord$ is initial with codomain in $\mathcal{A}$. On the other hand, the empty sink with codomain (X, x) is final.

If $X_i \neq \emptyset$ for $i \in Ord$, let $(\tilde{X}, \tilde{x})$ be the $\mathcal{A}$-object obtained from (X, x) by merging for each $i \in Ord$ all elements of X_i to one denoted by x_i. Then we get a quotient $q : X \to \tilde{X}$. It is easy to see that the $\mathcal{A}^t$-morphism

$$(X, x) \xrightarrow{q} (\tilde{X}, \tilde{x}) \tag{10}$$

is initial. To see that there is a final sink with codomain (X, x) and domain in $\mathcal{A}$, let us consider the subclass C of Ord defined above, let i_0 be the minimum of C

and let $\overline{X}_{i_0} = \cup_{i \geq i_0} X_i$. Let E be the collection of all $e = (e_i)_{i \in Ord}$ such that

$$\begin{aligned} &e_i \in X_i, && \text{if } i < i_0; \\ &e_i = z, && \text{if } i \geq i_0, \text{ where } z \in \overline{X}_{i_0}. \end{aligned}$$

It is easy to verify that $((X, e) \xrightarrow{1_X} (X, x))_{e \in E}$ is a final sink.

It is straightforward to conclude that an $\mathcal{A}^t$-object (X, x) is the domain of some initial monosource with codomain in $\mathcal{A}$ iff $(X, x) \in Obj(\mathcal{A})$ or $x = (\emptyset)_{i \in Ord}$. Consequently, the monotopological hull $\mathcal{A}^m$ is the full subcategory of $\mathcal{A}^t$ consisting of all objects of $\mathcal{A}$ and the objects $(X, x) \in \mathcal{A}^t$ such that $X_i = \emptyset$, $i \in Ord$. It is clear that $\mathcal{A}^m$ is, up to concrete isomorphism, the category $\mathcal{X}$ described in 2.5.

The class $\mathcal{A}^\perp$ in the category $\mathcal{A}^t$ consists of all $\mathcal{A}^t$-morphisms $f : (X, x) \to (Y, y)$ such that

$$\begin{aligned} &(i) && f(a) \in Y_i \cap f(X) \Rightarrow a \in X_i, \text{ for all } a \in X,\ i \in Ord; \\ &(ii) && Y \setminus \textstyle\bigcup_i Y_i \subseteq f(X); \\ &(iii) && \text{if } f(a) = f(b) \text{ then } a = b \text{ or } a, b \in X_i \text{ for some } i. \end{aligned}$$

In fact, let $f : (X, x) \to (Y, y)$ be an $\mathcal{A}^t$-morphism orthogonal to $\mathcal{A}$. The fact that $\mathcal{A}$ is initially dense in $\mathcal{A}^t$ and there is a bijection between the families $\mathcal{A}^t(X, \mathcal{A})$ and $\mathcal{A}^t(Y, \mathcal{A})$ of all morphisms with domain X and Y, respectively, and codomain in $\mathcal{A}$, implies that f is initial. This means that f satisfies (i). It is easy to check that the $\mathcal{A}$-cancellability of f is equivalent to (ii). Finally, let $a, b \in X$ be such that $f(a) = f(b)$ and there is no $i \in Ord$ for which $a, b \in X_i$. If $x = (\emptyset)_{i \in Ord}$, let $X \xrightarrow{q} X \dot{\cup} \{a\}$ be the inclusion of X into $X \dot{\cup} \{a\}$ and let $\overline{x} = (\overline{x}_i)_{i \in Ord}$ be such that $\overline{x}_i = a$ for all $i \in Ord$. Then $(X, x) \xrightarrow{q} (X \dot{\cup} \{a\}, \overline{x})$ is an $\mathcal{A}^t$-morphism with domain in $\mathcal{A}$. If $x \neq (\emptyset)_{i \in Ord}$, let $q : (X, x) \to (\tilde{X}, \tilde{x})$ be as defined above (10). In both cases, there is an $\mathcal{A}^t$-morphism $\overline{q}$ such that $\overline{q} \cdot f = q$ and, since $q(a) \neq q(b)$, we have that $f(a) \neq f(b)$. Conversely, let f satisfy conditions (i), (ii) and (iii) and let $(X, x) \xrightarrow{g} (Z, z)$ be an $\mathcal{A}^t$-morphism with codomain in $\mathcal{A}$. Then the morphism $\overline{g} : (Y, y) \to (Z, z)$ defined by $\overline{g}(c) = z_i$, if $c \in Y_i$ and $\overline{g}(c) = d$ such that $f(d) = c$, if $c \in Y \setminus \cup_{i \in Ord} Y_i$ is the unique one such that $\overline{g} \cdot f = g$.

On the other hand, as we have seen in 2.5, the class $\mathcal{A}^\perp$ in $\mathcal{A}^m$ consists of all

$\mathcal{A}^m$-isomorphisms.

Thus we have $A^l \neq A^o = A^s = A^m \neq A^t$.

5. Taking into account Remark 13.10.1., Example 2.2 provides an example of a concrete category with the solid hull different from the orthogonal hull.

Th following proposition states that in several categories the reflective hull of each subcategory, if it exists, must coincide with the orthogonal hull . As a consequence, for several concrete categories, if they have a solid hull, it coincides with the orthogonal hull (Corollary 14.12).

Proposition 14.11 *The reflective hull of a subcategory in a fibre-small topological category over $\mathcal{S}et$, if it exists, coincides with the orthogonal hull.*

Proof. From Theorem 4.1.3 and Proposition 3.1.2 of [22], it follows that if $\mathcal{X}$ satisfies the following conditions

it is complete, cocomplete and cowellpowered,

it has a factorization structure $(\mathcal{E}, \mathcal{M})$ for morphisms with $\mathcal{E} = Epi(\mathcal{X})$,

it has a separator

for each numerable family $(C_i \xrightarrow{m_i} B)_{i \in \omega}$ of $\mathcal{M}$-subobjects of an arbitrary $\mathcal{X}$-object B the union of all pullbacks of m_i along a given morphism g is equal to the pullback of the union of all m_i along g (i.e., $\vee_{i\in\omega} g^{-1}(m_i) = g^{-1}(\vee_{i\in\omega} m_i)$),

then for each $\mathcal{X}$-morphisms f the subcategory $\{f\}_\perp$ is reflective.

Let $\mathcal{X}$ be a fibre-small topological category over $\mathcal{S}et$. Since $\mathcal{S}et$ satisfies all those conditions for $\mathcal{E}$ the class of all epimorphisms and $\mathcal{M}$ the class of all monomorphisms, it follows that $\mathcal{X}$ satisfies all those conditions for $\mathcal{E}$ the class of all epimorphisms and $\mathcal{M}$ the class of all initial monomorphisms (see 21.16 and 21.17 of [2]).

Consequently, for each $\mathcal{X}$-morphism f the subcategory $\{f\}_\perp$ is reflective. Therefore, by 1.2, if a subcategory $\mathcal{A}$ of $\mathcal{X}$ has a reflective hull in $\mathcal{X}$ it must coincide with the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$. □

We recall that Adámek, Herrlich and Strecker ([1]) characterized concrete categories which have a fibre-small MacNeille completion; they are the so-called *strongly fibre-small* concrete categories.

Corollary 14.12 *If a strongly fibre-small concrete category over $\mathcal{S}et$ has a solid hull, it must coincide with the orthogonal hull.* $\square$

Remark 14.13 We point out that, however, the solid hull of a strongly fibre-small concrete category over $\mathcal{S}et$ may not coincide with the limit-closure in its MacNeille completion, as it is shown by the category $\mathcal{A}$ of 14.10.4 above which, in fact, is a strongly fibre-small concrete category over $\mathcal{S}et$.

15 Solid hulls and Vopěnka's Principle

Let us begin by recalling the notion of locally presentable category. Let λ be a regular cardinal and let X be an object of a given category; we say that X is λ-presentable if its hom-functor $hom(X, -)$ preserves λ-directed colimits. A *locally presentable category* is a cocomplete category which, for some regular cardinal λ, has a set S of λ-presentable objects such that every object is a λ-directed colimit of objects from S.

For a detailed account of locally presentable categories the reader is referred to the book [6] of J. Adámek and J. Rosický.

All categories of structures of a given signature of operation and relation symbols are locally presentable ([6]). In particular, the category $Rel(\Sigma)$ of relational structures of type Σ as described in 13.1 is locally presentable.

An example of such a category is the category of graphs $\mathcal{G}ra$, i.e., the category of sets with a binary relation and homomorphisms between them.

We are going to consider the following three large-cardinal axioms of set theory:

Vopěnka's Principle: $\mathcal{G}ra$ does not have a large, discrete, full subcategory;

Weak Vopěnka's Principle: Ord^{op} cannot be fully embedded into $\mathcal{G}ra$ (where Ord is the large poset of all ordinals considered as a category and Ord^{op} is its dual category);

(M): There do not exist arbitrarily large measurable cardinals.

As it is shown in [7], Vopěnka's Principle implies Weak Vopěnka's Principle and it also implies the negation of (M).

Under Vopěnka's Principle, locally presentable categories are precisely the cocomplete categories with a dense subcategory (see 6.14 in [6]).

The following important result, which was proved by J.Adámek, J.Rosický and V.Trnková in [7], shows that the existence of a reflective hull may depend on set theory.

Theorem 15.1 ([7]) *Let $\mathcal{B}$ be a locally presentable category. Assuming Weak Vopěnka's Principle, the limit closure of each subcategory of $\mathcal{B}$ is reflective.* □

The idea that, for concrete categories over $\mathcal{S}et$, the existence of a solid hull depends on a large-cardinal principle is due to J. Rosický who showed, in [57], that, under the axiom (M) there is a concrete category over $\mathcal{S}et$ with a small finally dense subcategory, which does not have a solid hull. Now, we prove a refinement of this result: for concrete categories over $\mathcal{S}et$ with a small finally dense subcategory, the existence of solid hulls is equivalent to Weak Vopěnka's Principle.

Theorem 15.2 *The following assertions are equivalent:*

(a) Every concrete category over Set with a small, finally dense subcategory has a solid hull.

(b) Weak Vopěnka's Principle holds.

Proof.

$(b) \Rightarrow (a)$: Let $(\mathcal{A}, U)$ be a concrete category over $\mathcal{S}et$ and let $\mathcal{C}$ be a small, finally dense subcategory of $\mathcal{A}$. We define a category $\mathcal{A}_{\mathcal{C}}$ as follows:

- Objects are pairs
$$(X, \alpha)$$
where X is a set and α is a U-structured sink with domain in $\mathcal{C}$, codomain X and such that, for all morphisms $c : C' \to C$ in $\mathcal{C}$ and $g : UC \to X$ in α, we have that $g \cdot Uc$ belong to α.

- Morphisms

$$f : (X, \alpha) \to (Y, \beta)$$

are maps $f : X \to Y$ such that, for each $g \in \alpha$, $f \cdot g \in \beta$.

The pair

$$(\mathcal{A}_\mathcal{C}, U_\mathcal{C}),$$

where $U_\mathcal{C}$ is defined by $U_\mathcal{C}(X, \alpha) = X$ and $U_\mathcal{C}(f) = f$, is a concrete category over $\mathcal{S}et$.

Furthermore, let

$$E_\mathcal{C} : \mathcal{A} \to \mathcal{A}_\mathcal{C}$$

be the functor such that, for each $A \in \mathcal{A}$, $E_\mathcal{C}(A) = (UA, \alpha)$ where α is the sink of all morphisms Ug with $(g : C \to A) \in Mor(\mathcal{A})$ and $C \in \mathcal{C}$, and, for each $f \in Mor(\mathcal{A})$, $E_\mathcal{C}(f) = Uf$.

Then, we have the following two properties:

1. (see [38]) $E_\mathcal{C} : (\mathcal{A}, U) \to (\mathcal{A}_\mathcal{C}, U_\mathcal{C})$ is a finally dense, topological extension of $(\mathcal{A}, U)$.

2. (see [55, 6]) $\mathcal{A}_\mathcal{C}$ is a locally presentable category.

From property 2. above and 15.1, we have that, under Weak Vopěnka's Principle, the limit closure of $E_\mathcal{C}(\mathcal{A})$ in $\mathcal{A}_\mathcal{C}$ is its reflective hull, and, by 1.2.2, 14.4.2 and property 1. above, it yields the solid hull of $(\mathcal{A}, U)$.

$(a) \Rightarrow (b)$: Conversely, we are going to show that, under the negation of Weak Vopěnka's Principle, there is a concrete category over $\mathcal{S}et$ with a small finally dense subcategory which does not have a solid hull. Our main tool is a construction given in [5] I.13. Assuming the negation of Weak Vopěnka's Principle, there exist:

(i) a class of graphs $L_i = (Y_i, \beta_i)$, $i \in Ord$, such that

$$hom(L_i, L_j) = \begin{cases} \emptyset & if \quad i < j \\ \{l_{ij} : L_i \to L_j\} & if \quad i \geq j \end{cases}$$

and, since the negation of Vopěnka's Principle follows,

(ii) a class of graphs $K_i = (X_i, \alpha_i)$, $i \in Ord$, such that

$$hom(K_i, K_j) = \begin{cases} \emptyset & if \quad i \neq j \\ \{1_{K_i}\} & if \quad i = j \end{cases} .$$

In the category $\mathcal{R}el(2,2,1)$ of structures with two binary and one unary relation, consider the following objects

$$A_i = (X_i \,\dot\cup\, Y_i \,\dot\cup\, \{t_i\},\ \alpha_i \cup Y_i \times \{t_i\},\ \beta_i \cup X_i \times \{t_i\},\ \{t_i\})$$

for all $i \in Ord$. For each ordinal i, put

$$\overline{A}_i = \amalg_{k \le i} A_k$$

and

$$\mathcal{M} = \{v_i : A_0 \to \overline{A}_i \mid i \in Ord\}$$

where $v_i : A_0 \to \overline{A}_i$ is the coproduct injection. We want to show that $\mathcal{M}_\perp$ is not reflective in $\mathcal{R}el(2,2,1)$.

For each $j \in Ord$, let B_j be the object obtained from $\overline{A}_j$ by merging all points of X_j to one denoted by s_j, i.e.,

$$\begin{aligned} B_j &= \\ &= (Q_j, \gamma_j, \delta_j, \epsilon_j) \\ &= \amalg_{k<j} A_j \amalg (\{s_j\} \,\dot\cup\, Y_j \,\dot\cup\, \{t_j\},\ \{(s_j,s_j)\} \cup Y_j \times \{t_j\},\ \beta_j \cup \{(s_j,t_j)\},\ \{t_j\}). \end{aligned}$$

We show that all B_j belong to $\mathcal{M}_\perp$. To conclude this, we first note that, for arbitrary $i, j \in Ord$, the cardinality of $hom(A_i, B_j)$ is 1. In fact:

$$\begin{aligned} \text{If } i \ge j, \text{ let} \quad & f_{ij} : A_i \to B_j \text{ be defined by} \\ & f(x) = s_j,\ x \in X_i \\ & f(y) = l_{ij}(y),\ y \in Y_i \\ & f(t_i) = t_j. \end{aligned}$$

It is obvious that f_{ij} is a homomorphism. Furthermore, it is the only one from A_i to B_j. In fact, let $g : A_i \to B_j$ be such that $g(t_i) = t_k$ with $k < i$. Then $f(Y_i) \times \{f(t_i)\} = F(Y_i) \times \{t_k\}$ must be contained in γ_j and, similarly, $f(X_i) \times \{t_k\}$ must be contained in δ_j. This implies, respectively, that $f(Y_i) \subseteq Y_K$ and $f(X_i) \subseteq X_K$. But this would determine an homomorphism from A_i to A_k which, by hypothesis does not exist! A similar argument shows that if $g(t_i) = t_j$ then $g = f_{ij}$ above.

$$\begin{aligned} \text{If } i < j, \text{ let} \quad & f_{ij} : A_i \to B_j \text{ be defined by} \\ & f(x) = x,\ x \in X_i \\ & f(y) = y,\ y \in Y_i \\ & f(t_i) = t_i. \end{aligned}$$

Again, it is easy to see that this is the only homomorphism from A_i to B_j.

Now, it is clear that, for any $i, j \in Ord$, there is a unique homomorphism from $\overline{A}_i$ to B_j, say

$$g_{ij} : \overline{A}_i \to B_j;$$

furthermore, the diagrams

$$\begin{tikzcd} A_0 \arrow[r, "v_i"] \arrow[d, "f_{0j}"'] & \overline{A}_i \arrow[dl, "g_{ij}"] \\ B_j & \end{tikzcd}$$

are commutative. Consequently, all objects B_j are orthogonal to $\mathcal{M}$.

Now, we show that A_0 does not have a reflection in $\mathcal{M}_\perp$.

If, to the contrary,

$$A_0 \xrightarrow{r} A_0^*$$

is a reflection in $\mathcal{M}_\perp$, then, since $A_0^* \in \mathcal{M}_\perp$, for each $i \in Ord$, we obtain a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "v_i"] \arrow[d, "r"'] & \overline{A}_i \arrow[dl, "p_i"] \\ A_0^* & \end{tikzcd} .$$

We show that for $i \neq i'$, $p_i(i) \neq p_{i'}(i')$, which is obviously false. In fact, for $i \neq i'$, let j be an ordinal larger than i and i' and let $f_j^* : A_0^* \to B_j$ be the unique morphism which makes the diagram

$$\begin{tikzcd} A_0 \arrow[r, "r"] \arrow[d, "f_{0j}"'] & A_0^* \arrow[dl, "f_j^*"] \\ B_j & \end{tikzcd} .$$

commutative.

Hence, one must have

$$f_j^* \cdot p_i(t_i) = g_{ij}(t_i) = t_i \quad \text{and} \quad f_j^* \cdot p_{i'}(t_{i'}) = g_{i'j}(t_{i'}) = t_{i'};$$

consequently, $p_i(t_i) \neq p_{i'}(t_{i'})$.

Now, put

$$C_1 = (\{0,1\}, \{(0,1)\}, \emptyset, \emptyset), \quad C_2 = (\{0,1\}, \emptyset, \{(0,1)\}, \emptyset) \quad \text{and} \quad C_3 = (\{0\}, \emptyset, \emptyset, \{0\}).$$

It is clear that the set $\mathcal{C} = \{C_1, C_2, C_3\}$ is finally dense in $\mathcal{R}el(2,2,1)$. On the other hand, since the unary relation in C_1 and C_2 is empty, there is no homomorphisms from A_0 to C_1 or C_2; since the subcategory $\{K_i, i \in Ord\}$ is discrete in $\mathcal{G}ra$, we conclude $\alpha_0 \neq \emptyset$ and again $hom(A_0, C_3) = \emptyset$. Hence, it follows that C_1, C_2 and C_3 belong to $\mathcal{M}_\perp$ and, thus, $\mathcal{C}$ is a finally dense set of $\mathcal{M}_\perp$.

Furthermore $\mathcal{R}el(2,2,1)$ is topological, by 13.1, thus it is a finally dense, solid extension of $\mathcal{M}_\perp$.

By 1.2, the orthogonal hull of $\mathcal{M}_\perp$ in $\mathcal{R}el(2,2,1)$ is $\mathcal{M}_\perp$ and, since $\mathcal{M}_\perp$ is not reflective and $\mathcal{R}el(2,2,1)$ is locally presentable, it follows from 2.4 that $\mathcal{M}_\perp$ does not have a reflective hull in $\mathcal{S}tr(2,2,1)$. Therefore, using 13.9, we conclude that the concrete category $\mathcal{M}_\perp$ does not have a solid hull. $\square$

Chapter V

Multireflectivity and multicolimits

Kaput's paper [44] led to the study of generalizations of the concept of reflectivity. One of these generalizations, multireflectivity, which has been investigated by several authors (e.g., [11, 17, 74, 10, 61, 8]), has very relevant consequences such as, closedness under connected limits and existence of multicolimits.

In this chapter, we study the interplay between multireflectivity, multicolimits, connected limits and multisolidness, and we generalize some known results on colimits, limits and solidness to the above corresponding concepts. Namely, we give conditions under which a multicocomplete category has connected limits and we prove that a cowellpowered concrete category $(\mathcal{A}, U)$ over a multicocomplete category is multisolid if and only if $\mathcal{A}$ is multicocomplete and U is a right multi-adjoint.

16 Multireflectivity

Definition 16.1

1. Let $U : \mathcal{A} \to \mathcal{X}$ be a functor. A *universal source* from X to U is a U-source $(X \overset{\eta_j}{\to} UA_j)_J$ such that for each U-morphism $X \overset{x}{\to} UB$ there is a unique pair $(j,\ f)$ with $j \in J$ and $f : A_j \to B$ fulfilling the equality $Uf \cdot \eta_j = x$. The functor U is said to be a *right multi-adjoint* if, for each $X \in Obj(\mathcal{X})$, there is a universal source from X to U.

2. A subcategory $\mathcal{A}$ of a category $\mathcal{X}$ is *multireflective* if the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{X}$ is a right multi-adjoint. In this case, a universal source from X to the inclusion functor is said to be a *multireflection of* X *in* $\mathcal{A}$.

   If, for some class $\mathcal{E} \subseteq Mor(\mathcal{X})$, all multireflections are formed by $\mathcal{E}$-morphisms, then the subcategory $\mathcal{A}$ is said to be *multi-$\mathcal{E}$-reflective* in $\mathcal{X}$. If, for some conglomerate $\mathbb{M} \subseteq Source(\mathcal{X})$, each multireflection belongs to $\mathbb{M}$, we say that $\mathcal{A}$ is $\mathbb{M}$-*multireflective*.

Examples 16.2 ([17, 74])

1. The category $\mathcal{F}ld$ of fields is a multireflective subcategory of the category $\mathcal{R}ng$ of commutative unitary rings. Given a commutative unitary ring X, let $\mathcal{I}$ be the set of all maximal ideals of X and, for each $I \in \mathcal{I}$, let f_I be the quotient map of X into X/I. Then, the source
$$(X \xrightarrow{f_I} X/I)_{I \in \mathcal{I}}$$
   is a multireflection.

2. The category $\mathcal{L}ord$ of linearly ordered sets is a subcategory of the category of posets and strictly increasing maps. Given a poset $(X, \leq)$, the family of all morphisms
$$(X, \leq) \xrightarrow{q} (X', \leq') \xrightarrow{id_{X'}} (X', \prec)$$
   where q is a quotient morphism and $\prec$ is a linear ordering in X' containing $\leq$, is a multireflection of $(X, \leq)$ in $\mathcal{L}ord$.

3. Let $\mathcal{C}on$ be the category of non-empty connected topological spaces. Then its dual category $\mathcal{C}on^{\mathrm{op}}$ is multireflective in $\mathcal{T}op^{\mathrm{op}}$, i.e. $\mathcal{C}on$ is multicoreflective in $\mathcal{T}op$. For each topological space X, a multicoreflection consists of the inclusions of all connected components of X.

   Analogously, the category of pathwise connected spaces is multicoreflective in $\mathcal{T}op$ and the category of connected graphs is multicoreflective in $\mathcal{G}ra$.

4. A ring $X \in \mathcal{R}ng$ is called *connected* provided that its prime spectrum is connected with respect to the Zariski topology. Equivalently, X is connected if its only

idempotents are 0 and 1. The category $\mathcal{A}$ of connected rings is a multireflective subcategory of $\mathcal{R}ng$.

In order to show that, let $X \in \mathcal{R}ng$ and let $\mathbf{J}$ be the set of all proper ideals J of X such that

$$x^2 - x \in J \Rightarrow (x \in J \ \text{ or } \ x - 1 \in J).$$

Given an ideal I of X, it is obvious that X/I is connected iff $I \in \mathbf{J}$. Let $\mathbf{K}$ be the set of all minimal elements of $\mathbf{J}$. Then for each $J \in \mathbf{J}$ there is a unique $K \in \mathbf{K}$ such that $K \subseteq J$. Consequently, the source

$$(X \xrightarrow{q} X/K)_{K \in \mathbf{K}}$$

is a multireflection of X in $\mathcal{A}$.

Definitions 16.3 A *multicolimit* of a diagram $D : I \to \mathcal{X}$ is a family of natural sinks $((Di \overset{l_i^k}{\to} L_k)_I)_K$ from D such that for each natural sink $(Di \overset{u_i}{\to} X)_I$ from D there is a unique pair $(k,\ L_k \overset{t}{\to} X)$ such that $k \in K$ and $u_i = t \cdot l_i^k$ for all $i \in I$. Each of the natural sinks $(Di \overset{l_i^k}{\to} L_k)_I$ is said to be a *component of the multicolimit.*

A category $\mathcal{X}$ is *multicocomplete* provided that each small diagram in $\mathcal{X}$ has a multicolimit.

In general, we use the colimits terminology with respect to multicolimits, adding the prefix "multi". For instance:

- A *multipushout* is a multicolimit of the diagram whose scheme is

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \quad .$$

- A *multiple multipushout* is a multicolimit with scheme

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[d] \arrow[dr] & \\ {} & i & {} \end{tikzcd} \cdots \quad , i \in I$$

  where I is a set or a class.

- A *multiple multicoequalizer* is a multicolimit of a scheme of the form

$$\bullet \quad \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow\end{array} \quad \bullet$$

where the family of all morphisms is a set or a class.

Remark 16.4 Some well-known properties of colimits can be generalized to multicolimits. For example, it is easy to prove the following assertions:

1. Each component of a multicolimit is an epi-sink.
2. Each component of a multipushout of an epimorphism along another morphism is an epimorphism.
3. If $\mathcal{X}$ has a factorization structure $(\mathcal{E}, \mathcal{M})$ for morphisms, then:
   (a) each component of a multipushout of a morphism in $\mathcal{E}$ along any morphism belongs to $\mathcal{E}$;
   (b) if $(X \xrightarrow{d_k} Z_k, (Y_i \xrightarrow{d_{ik}} Z_k)_I)_K$ is a multiple multipushout of a family $(X \xrightarrow{e_i} Y_i)_I$ of $\mathcal{E}$-morphisms, then each morphism d_k belongs to $\mathcal{E}$.

Examples 16.5 In [17], Diers presents a great variety of examples of multicocomplete categories which are not cocomplete. This is the case, for instance, of the categories $\mathcal{F}ld$, $\mathcal{L}ord$ and $\mathcal{C}on^{\mathrm{op}}$.

It is well-known that the notions of reflectivity and cocompleteness may be interpreted in terms of the existence of initial objects for convenient categories. Now, we consider a generalization of initial object which leads to a similar interpretation concerning multireflectivity and multicolimits.

Definition 16.6 A family $(A_i)_I$ of objects of a category $\mathcal{A}$ is said to be *initial* in $\mathcal{A}$, if for each $\mathcal{A}$-object A there is a unique pair (i, f) with $i \in I$ and $f : A_i \to A$.

Remarks 16.7

1. From Definition 16.6, it is clear that, if $(A_i)_I$ is an initial family in $\mathcal{A}$ and B_1, B_2, B_3 are $\mathcal{A}$-objects for which there is a diagram of the form

$$B_1 \to B_2 \leftarrow B_3,$$

then the unique $i_j \in I$ such that $\mathcal{A}(A_{i_j}, B_j)\neq\emptyset$ is the same for all $j = 1, 2, 3$. Consequently, it follows that the family $(A_i)_I$ is initial iff, for each connected component $\mathcal{C}$ of $\mathcal{X}$, there exists a unique $i \in I$ such that A_i is an initial object in $\mathcal{C}$.

Thus, we have that:

(a) A functor $U : \mathcal{A} \to \mathcal{X}$ is a right multi-adjoint iff for each $\mathcal{X}$-object X the comma category $X \downarrow U$ has an initial family, or, equivalently, each connected component of the comma category $X \downarrow U$ has an initial object. Of course, such a initial family forms the corresponding universal source from X to U. Each initial object which is part of the initial family is said to be a *component of the universal source.*

As a consequence we have that, whenever two morphisms $X \xrightarrow{x} UB$ and $X \xrightarrow{y} UC$ belong to the same connected component of $X \downarrow U$, then they are factorized through the same component of the universal source from X to U. Thus, a right multi-adjoint is a right adjoint iff, for each $X \in Obj(\mathcal{X})$, the comma category $X \downarrow U$ is connected.

(b) Given a category $\mathcal{X}$ and a diagram $D : I \to \mathcal{X}$ in $\mathcal{X}$, let $D \downarrow \mathcal{X}$ denote the quasicategory of natural sinks from D, that is, objects of $D : I \to \mathcal{X}$ are all natural sinks from D and morphisms are all

$$h : (Di \xrightarrow{f_i} X)_{i\in Obj(I)} \longrightarrow (Di \xrightarrow{g_i} Y)_{i\in Obj(I)}$$

where $h : X \to Y$ is an $\mathcal{X}$-morphism such that $h \cdot f_i = g_i$ for all $i \in Obj(I)$.

The diagram $D : I \to \mathcal{X}$ has a multicolimit iff the quasicategory $D \downarrow \mathcal{X}$ has an initial family. The elements of this family are just the components of the multicolimit and, of course, each component is an initial object in its connected component in $D \downarrow \mathcal{X}$.

2. By definition, it is clear that each initial family is unique up to isomorphism, i.e., if $(A_i)_I$ and $(B_j)_J$ are initial families of a given category, then there are a bijection $\phi : I \to J$ and isomorphisms $h_i : A_i \to B_{\phi(i)}$ for all $i \in I$.

Consequently, a universal source is unique up to isomorphism (and so is a multicolimit).

3. In the definition of initial family given above, I is empty whenever $\mathcal{A}$ is an empty category. Furthermore, in contrast to the definition of initial family introduced by Diers ([17]), we allow I to be a class. In fact, the main results which we obtain in the sequel are true independently from accepting classes or not in the definition of initial family. This stresses the fact that multireflectivity is a local notion and that only the "local smallness" plays a rôle here. To illustrate the rôle of the requirement that I be a set, we point out that, for instance, a small discrete category has an initial family, in both senses, whereas a large discrete category has an initial family only if we admit the index family to be a class. Similarly, given a subcategory $\mathcal{A}$ of a category $\mathcal{X}$, a multireflection of an $\mathcal{X}$-object X in $\mathcal{A}$ in the "large" sense is a multireflection from X to $\mathcal{A}$ in the "small" sense iff the family of all connected components of the comma category $X \downarrow \mathcal{A}$ is a set.

The following two propositions generalize well-known results on adjoint functors to multi-adjoint functors.

Proposition 16.8 (c.f. [17])

1. *Right multi-adjoint functors preserve connected limits.*

2. *If $\mathcal{X}$ is a category with connected limits, then a functor $U : \mathcal{A} \to \mathcal{X}$ is a right multi-adjoint if and only if it preserves connected limits and, for each $\mathcal{X}$-object X, each connected component of $X \downarrow U$ has a weakly initial set.* □

We point out that if we consider right multi-adjoints in the sense of Diers (i.e., universal sources are indexed by sets), then in 16.8.2 we may replace "for each $\mathcal{X}$-object X, each connected component of $X \downarrow U$ has a weakly initial set" by "U satisfies the solution set condition". This result was, in fact, proved in [17]. Assertion 16.8.1 was also proved by Diers for the case where universal sources are indexed by sets. An obvious adaptation of the proofs in [17] provides 16.8 for the present definition of right multi-adjoint, i.e., for the case where universal sources may be indexed by proper classes.

Proposition 16.9 ([17, 74]) *If $\mathcal{A}$ is a multireflective subcategory of a multicocomplete category, then $\mathcal{A}$ is multicocomplete.* □

The following proposition shows the rôle which the fact that an initial family may be empty can play.

Proposition 16.10 *Let $\mathcal{X}$ be a category with the following property:*

(**T**) *For all morphisms f, g with common domain there are morphisms u, v for which the square*

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g"'] & {} \arrow[d, "u"] \\ {} \arrow[r, "v"'] & {} \end{tikzcd}$$

commutes.

Then every multireflective subcategory $\mathcal{A}$ of $\mathcal{X}$, such that $\mathcal{X}(X, \mathcal{A}) \neq \emptyset$ for all X in $\mathcal{X}$, is reflective in $\mathcal{X}$.

Proof. Let X be an $\mathcal{X}$-object and let $(r_i : X \to A_i)_I$ be the multireflection of X in $\mathcal{A}$ which is non-empty, since $\mathcal{X}(X, \mathcal{A}) \neq \emptyset$. For each pair $i, j \in I$, there is some pair of morphisms $(u : A_i \to W, v : A_j \to W)$ such that $u \cdot r_i = v \cdot r_j$. Let $s : W \to A$ be a morphism with codomain in $\mathcal{A}$; hence $s \cdot u$ and $s \cdot v$ are $\mathcal{A}$-morphisms and $s \cdot u \cdot r_i = s \cdot v \cdot r_j$. Thus r_i and r_j belong to the same connected component of $X \downarrow \mathcal{A}$ and, from 16.7.1(a), we conclude $i = j$. Therefore, I is a singleton and, thus, X has a reflection in $\mathcal{A}$. $\square$

It is clear that each of the following conditions on $\mathcal{X}$ implies condition (**T**):

- $\mathcal{X}$ has pushouts;
- $\mathcal{X}$ has non-empty multipushouts;
- $\mathcal{X}$ has a terminal object.

Definition 16.11 Let $\mathcal{A}$ be a subcategory of the category $\mathcal{X}$. A subcategory $\mathcal{B}$ of $\mathcal{X}$ is said to be a *multireflective hull of $\mathcal{A}$ in $\mathcal{X}$* provided that it is multireflective in $\mathcal{X}$, contains $\mathcal{A}$ and is contained in every multireflective subcategory of $\mathcal{X}$ which contains $\mathcal{A}$.

If, in the above definition, we replace “multireflective” by “multi-$\mathcal{E}$-reflective” (“$\mathbb{M}$-multireflective, respectively), we obtain the definition of *multi-$\mathcal{E}$-reflective hull of $\mathcal{A}$ in $\mathcal{X}$* ($\mathbb{M}$-*multireflective hull of $\mathcal{A}$ in $\mathcal{X}$*, respectively).

Let $\mathcal{X}$ be an $(\mathcal{E},\mathbb{M})$-category. Then every subcategory $\mathcal{A}$ of $\mathcal{X}$ has an $\mathcal{E}$-reflective hull in $\mathcal{X}$ which consists of all $\mathcal{X}$-objects which are domains of $\mathbb{M}$-sources with codomains in $\mathcal{A}$. Now, we prove that, if $\mathcal{X}$ is an $\mathcal{E}$-cowellpowered $(\mathcal{E},\mathbb{M})$-category, then every subcategory has a multi-$\mathcal{E}$-reflective hull.

We shall make use of the following definition and lemma.

Definition 16.12 If $G:\mathcal{A}\to\mathcal{X}$ is a functor, a source $(X\xrightarrow{f_i}GA_i)_I$ is said to be *G-connected* provided that the subcategory of $X\downarrow G$ which consists of all f_i is connected.

If $\mathcal{A}$ is a subcategory of $\mathcal{X}$, an $\mathcal{X}$-source $(X\stackrel{f_i}{\to}A_i)_I$ is said to be *$\mathcal{A}$-connected* if it is connected with respect to the inclusion functor.

An $\mathcal{X}$-source $(X\xrightarrow{f_i}X_i)_I$ which is $\mathcal{X}$-connected is simply said to be *connected* .

Lemma 16.13 *If $\mathcal{X}$ is an $(\mathcal{E},\mathbb{M})$-category and $\mathcal{A}$ is a multi-$\mathcal{E}$-reflective subcategory of $\mathcal{X}$, then an $\mathcal{X}$-object X belongs to $\mathcal{A}$ if and only if it is the domain of some $\mathcal{A}$-connected source in $\mathbb{M}$.*

Proof. If X belongs to $\mathcal{A}$, then the source of all morphisms with domain X and codomain in $\mathcal{A}$ contains the identity 1_X and, consequently, it is an $\mathcal{A}$-connected source which belongs to $\mathbb{M}$.

Conversely, let $(X\stackrel{f_i}{\to}A_i)_I$ be an $\mathcal{A}$-connected $\mathbb{M}$-source and let $(X\stackrel{r_j}{\to}B_j)_J$ be a multi-$\mathcal{E}$-reflection of X in $\mathcal{A}$. Then, since $(f_i)_I$ is $\mathcal{A}$-connected, all morphisms f_i are factorizable through the same r_j for some $j\in J$. Consequently, the fact that $r_j\in\mathcal{E}$ and $(f_i)_I\in\mathbb{M}$ implies that r_j is an isomorphism and, thus, X belongs to $\mathcal{A}$. $\square$

Given an $(\mathcal{E},\mathbb{M})$-category $\mathcal{X}$ and a subcategory $\mathcal{A}$ of $\mathcal{X}$, let us consider a chain $(\mathcal{A}_\alpha)_{\alpha\in Ord}$ of subcategories of $\mathcal{X}$ defined as follows:

- The category $\mathcal{A}_0$ is just $\mathcal{A}$.
- For each $\alpha\in Ord$,

$$\mathcal{A}_{\alpha+1}$$

consists of all $\mathcal{X}$-objects X such that X is the domain of some $\mathcal{A}_\alpha$-connected source which belongs to $\mathbb{M}$.

- For each limit ordinal λ,
$$\mathcal{A}_\lambda = \cup_{\alpha<\lambda}\mathcal{A}_\alpha.$$

Let us denote the union of all these subcategories, $\cup_{\alpha\in Ord}\mathcal{A}_\alpha$, by
$$c\mathbb{M}(\mathcal{A}).$$

Proposition 16.14 *If $\mathcal{A}$ is a subcategory of an $\mathcal{E}$-cowellpowered $(\mathcal{E},\mathbb{M})$-category $\mathcal{X}$, then $c\mathbb{M}(\mathcal{A})$ is the multi-$\mathcal{E}$-reflective hull of $\mathcal{A}$ in $\mathcal{X}$.*

Proof.

We are going to use the following two results of Salicrup [61]:

I . If $\mathcal{X}$ is an $\mathcal{E}$-cowellpowered $(\mathcal{E},\mathbb{M})$-category then, for each source $(m_i : X \to Y_i)_I$ belonging to $\mathbb{M}$, there exists a set $J \subseteq I$ such that $(m_i : X \to Y_i)_J$ belongs to $\mathbb{M}$.

II . If X is an $\mathcal{E}$-cowellpowered $(\mathcal{E},\mathbb{M})$-category and $\mathcal{A}$ is a subcategory of $\mathcal{X}$, then the following assertions are equivalent:

(i) $\mathcal{A}$ is multi-$\mathcal{E}$-reflective in $\mathcal{X}$.

(ii) If the following diagram

$$\begin{tikzcd} X \arrow[r, "m_i"] \arrow[dr, "g"'] & A_i \arrow[d, "f_i"] \\ & A \end{tikzcd}$$

commutes in $\mathcal{X}$ for each $i \in I$, f_i is an $\mathcal{A}$-morphism for each $i \in I$ and $(m_i : X \to A_i)_I$ belongs to $\mathbb{M}$, then $X \in Obj(\mathcal{A})$.

Let $\mathcal{A}$ be a subcategory of an $\mathcal{E}$-cowellpowered $(\mathcal{E},\mathbb{M})$-category $\mathcal{X}$. We show that $c\mathbb{M}(\mathcal{A})$ fulfils condition **(ii)**.

Let the diagram

$$\begin{tikzcd} X \arrow[r, "m_i"] \arrow[dr, "g"'] & Y_i \arrow[d, "f_i"] \\ & Y \end{tikzcd}$$

commute in $\mathcal{X}$, for each $i \in I$, let f_i be an $c\mathbb{M}(\mathcal{A})$-morphism, $i \in I$, and let $(m_i : X \to Y_i)_I$ belong to $\mathbb{M}$. Hence, using condition **I**, there is a set $J \subseteq I$ such that $(m_i : X \to Y_i)_J$ belongs to $\mathbb{M}$. Since J is a set, there is some $\alpha \in Ord$ such that $f_i \in \mathcal{A}_\alpha$, $i \in J$. Thus, $X \in Obj(\mathcal{A}_{\alpha+1})$ and, therefore, $X \in Obj(c\mathbb{M}(\mathcal{A}))$. In order to show that $c\mathbb{M}(\mathcal{A})$ is the smallest multi-$\mathcal{E}$-reflective subcategory containing $\mathcal{A}$, let $\mathcal{B}$ be another multi-$\mathcal{E}$-reflective subcategory of $\mathcal{X}$ which contains $\mathcal{A}$. Therefore we have that $\mathcal{A} \subseteq \mathcal{B}$ and, for each $\alpha \in Ord$, if $\mathcal{A}_\alpha \subseteq \mathcal{B}$ then, from Lemma 16.13, $\mathcal{A}_{\alpha+1} \subseteq \mathcal{B}$. Consequently, by the construction of the subcategories $\mathcal{A}_\lambda$, it follows that, for each ordinal λ, $\mathcal{A}_\lambda \subseteq \mathcal{B}$ and, thus, $c\mathbb{M}(\mathcal{A}) \subseteq \mathcal{B}$. □

Remark 16.15 In fact, according to Salicrup's proof of the result **II**, under the assumptions of the above theorem, the multi-$\mathcal{E}$-reflections of objects of $\mathcal{X}$ into $c\mathbb{M}(\mathcal{A})$ are indexed by a set.

17 Multicocompleteness and connected limits

As we have seen, sometimes the rôle played by colimits and limits when we deal with the concept of relectivity turns out to be played by, respectively, multicolimits and connected limits if we deal with multireflectivity. In the present sequel we explore some other similarities between the pairs colimits/limits and multicolimits/connected limits. Our main inspiration is [3] (as well as section 12 of [2]) where the authors give conditions under which a cocomplete category is complete. Thus we want to study the question of when a multicocomplete category has connected limits.

Let $D : I \to \mathcal{A}$ be a small diagram in $\mathcal{A}$. As in [2], we denote by $\mathcal{S}^D$ the category whose objects are all natural sources $(A, (f_i)_{Obj(I)})$ for D, whose morphisms $(A, (f_i)_{Obj(I)}) \xrightarrow{h} (A', (f'_i)_{Obj(I)})$ are all those $\mathcal{A}$-morphisms $h : A \to A'$ such that $f'_i \cdot h = f_i$ for all $i \in Obj(I)$, and whose identities and composition law are as in $\mathcal{A}$. We also denote by

$D^* : \mathcal{S}^D \to \mathcal{A}$ the forgetful functor given by

$$D^*((A, (f_i)_{Obj(I)}) \xrightarrow{h} (A', (f'_i)_{Obj(I)})) = (A \xrightarrow{h} A').$$

Dually, we define S_D to be the category of natural sinks from D and by $D_* : S_D \to \mathcal{A}$ to be the corresponding forgetful functor.

Lemma 17.1 *If $D : I \to \mathcal{A}$ is a small connected diagram in a multicocomplete category $\mathcal{A}$, then S^D is cocomplete.*

Proof. Let $D_o : J \to \mathcal{S}^D$ be a small diagram such that, for each $j \in Obj(J)$, $D_o(j) = (A_j, (f_{ji})_I)$. Let $((A_j \overset{c_j^k}{\to} L^k)_{j \in Obj J})_{k \in K}$ be a multicolimit of the composite diagram $J \xrightarrow{D_o} \mathcal{S}^D \xrightarrow{D^*} \mathcal{A}$. It is easily checked that, for each object i of I, $(A_j \xrightarrow{f_{ji}} Di)_{j \in Obj(J)}$ is a natural sink for $D^* \cdot D_o$; then there is a unique $k \in K$ and a unique morphism $g_i : L^k \to Di$ such that $g_i \cdot c_j^k = f_{ji}$, for all object j of J. But the fact that I is connected implies that all the natural sinks $(A_j \overset{f_{ji}}{\to} Di)_{j \in Obj(J)}$ belong to the same connected component of $\mathcal{S}_{D^* \cdot D_o}$. Thus, the existing k in K is the same for all i in $Obj(I)$. It is now easy to check that $(g_i : L^k \to Di)_{i \in Obj(I)}$ is a natural source for D and
$((A_j, (f_{ji})_{Obj(I)}) \xrightarrow{c_j^k} (L^k, (g_i)_{Obj(I)})_{Obj(J)}$ is a colimit of D_o. □

We recall that a subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is said to be *colimit-dense* in $\mathcal{A}$ provided that each $\mathcal{A}$-object is the colimit of some small diagram with codomain in $\mathcal{B}$. Dually, we define *limit-dense* subcategory.

We introduce the following definition:

Definition 17.2 A subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be *multicolimit-dense* in $\mathcal{A}$ if for every object A in $\mathcal{A}$ there is a small diagram $D : I \to \mathcal{B}$ and a natural sink $(l_i : Di \to A)_{i \in I}$ which is a component of a multicolimit of D.

Proposition 17.3 *Every multicocomplete category with a small multicolimit-dense subcategory has connected limits.*

Proof. Let $\mathcal{B}$ be a small multicolimit-dense subcategory of a multicocomplete category $\mathcal{A}$. Let $D : I \to \mathcal{A}$ be a small connected diagram. To show that it has a limit in $\mathcal{A}$

is equivalent to showing that S^D has a terminal object. Furthermore, since, by Lemma 17.1, S^D is cocomplete, to show that S^D has a terminal object it suffices to show that it has a weakly terminal object (see [2]). Let I^* be the subcategory of S^D of all natural sources for D with domain in $\mathcal{B}$. Since $\mathcal{B}$ is small, so is I^* and the inclusion functor $I^* \hookrightarrow \mathcal{S}^D$ has a colimit, since $\mathcal{S}^D$ is cocomplete. Let this colimit be

$$(S \xrightarrow{c_S} R)_{S \in I^*}, \quad \text{with } R = (C, (p_i)_{Obj(I)}).$$

We claim that $R = (C, (p_i)_{ObJ(I)})$ is a weakly terminal object of $\mathcal{S}^D$. Indeed, let $S = (f_i : A \to Di)_{ObJ(I)}$ belong to $\mathcal{S}^D$. By hypothesis, there is a small diagram $\overline{D} : N \to \mathcal{B}$ and a natural sink $(t_n : \overline{D}n \to A)_{n \in Obj(N)}$ for $\overline{D}$ which is a component of a multicolimit of $N \xrightarrow{\overline{D}} \mathcal{B} \hookrightarrow \mathcal{A}$. For each object n of N, the source $S_n =$
$(\overline{D}n \xrightarrow{t_n} A \xrightarrow{f_i} Di)_{i \in Obj(I)}$ belongs to I^*. We claim that $(\overline{D}_n \xrightarrow{c_{S_n}} C)_{n \in Obj(N)}$ is a natural sink from $\overline{D}$, that is, $(\overline{D}n \xrightarrow{c_{S_n}} C)_{n \in Obj(N)}$ belongs to $S_{\overline{D}}$. Indeed, let $d : m \to n$ be a N-morphism. Since $(t_n)_{n \in Obj(N)}$ is a natural sink from $\overline{D}$, we have $t_m = t_n \cdot \overline{D}d$ and thus $f_i \cdot t_m = (f_i \cdot t_n) \cdot \overline{D}d$ for all $i \in Obj(I)$. This means that $\overline{D}d$ is an $S_{\overline{D}}$-morphism from S_m to S_n. Hence $c_{S_m} = \overline{D}d \cdot c_{S_n}$. Therefore, there is a component of the multicolimit of $N \xrightarrow{\overline{D}} \mathcal{B} \hookrightarrow \mathcal{A}$ which is just the initial object of the connected component of $\mathcal{S}_{\overline{D}}$ which contains $(\overline{D}n \xrightarrow{c_{S_n}} C)_{n \in Obj(N)}$. But, $(\overline{D}n \xrightarrow{c_{S_n}} C)_{n \in Obj(N)}$ and $(\overline{D}n \xrightarrow{t_n} A)_{n \in Obj(N)}$ belong to the same connected component of $\mathcal{S}_{\overline{D}}$, since, given $i \in Obj(I)$, we have the following morphisms in $\mathcal{S}_{\overline{D}}$:

$$(\overline{D}n \xrightarrow{t_n} A)_{Obj(N)} \xrightarrow{\quad f_i \quad} (\overline{D}n \xrightarrow{t_n} A \xrightarrow{f_i} Di)_{Obj(N)} \xleftarrow{\quad p_i \quad} (\overline{D}n \xrightarrow{c_{S_n}} C)_{Obj(N)}\ .$$

Consequently, $(\overline{D}n \xrightarrow{t_n} A)_{Obj(N)}$ is the component of the multicolimit of $\overline{D}$ referred above. Thus, there is a morphism $w : A \to C$ such that $w \cdot t_n = c_{S_n}$ for all objects n of N. From the fact that, for each object $n \in Obj(N)$ and each object $i \in Obj(I)$, we have that $p_i \cdot w \cdot t_n = f_i \cdot t_n$ and that $(t_n)_{Obj(N)}$ is an epi-sink, it follows that $p_i \cdot w = f_i$ for all $i \in Obj(I)$. Therefore, w is an $\mathcal{S}^D$-morphism from $S = (A \xrightarrow{f_i} Di)_{i \in Obj(I)}$ to $R = (C \xrightarrow{p_i} Di)_{i \in Obj(I)}$. □

The following proposition gives conditions under which a category with connected limits is multicocomplete.

Proposition 17.4 *Every category with connected limits and such that each of its connected components has a small limit-dense subcategory is multicocomplete.*

Proof. Let $\mathcal{A}$ fulfil the hypotheses and let $D : I \to \mathcal{A}$ be a small diagram in $\mathcal{A}$. We want to show that D has a multicolimit in $\mathcal{A}$. Let $(\mathcal{C}_k)_{k\in K}$ be the family of all connected components of the category S_D of all natural sinks from D. To show that D has a multicolimit is equivalent to showing that each connected component $\mathcal{C}_k$ has an initial object. Thus, it suffices to prove that

(i) $\mathcal{C}_k$ has connected limits (then, in particular, $\mathcal{C}_k$ has equalizers)

and

(ii) $\mathcal{C}_k$ has a weakly initial object.

Proof of **(i)**: Let $\overline{D} : J \to \mathcal{C}_k$ be a small connected diagram in $\mathcal{C}_k$ such that $\overline{D}j = (Di \xrightarrow{f_{ij}} A_j)_{Obj(I)}$ and let us consider the diagram

$$J \xrightarrow{\overline{D}} \mathcal{C}_k \overset{E_k}{\hookrightarrow} S_D \xrightarrow{D_*} \mathcal{A}$$

where E_k is the inclusion functor. Since $\mathcal{A}$ has connected limits, the functor $D_* \cdot E_k \cdot \overline{D}$ has a limit in $\mathcal{A}$, let it be

$$(L \xrightarrow{l_j} A_j)_{j\in Obj(J)} \ .$$

It is easy to see that the fact that $\overline{D}$ is a functor implies that, for each $i \in Obj(I)$, $(Di \xrightarrow{f_{ij}} A_j)_{j\in Obj(J)}$ is a natural source for $D_* \cdot E_k \cdot \overline{D}$. Then, there is a unique morphism $t_i : D_i \to L$ such that $l_j \cdot t_i = f_{ij}$ for all $j \in Obj(J)$. The sink $(D_i \xrightarrow{t_i} L)_{i\in Obj(I)}$ is natural from D, since, given an I-morphism $d : i \to i'$, the equalities

$$l_j \cdot t_{i'} \cdot Dd = f_{i'j} \cdot Dd = f_{ij} = l_j \cdot t_i \ \text{ for all } \ j \in Obj(J)$$

imply that $t_{i'} \cdot Dd = t_i$. Furthermore, the S_D-source

$$((Di \xrightarrow{t_i} L)_{i\in Obj(I)} \xrightarrow{l_j} (Di \xrightarrow{f_{ij}} A_j)_{i\in Obj(I)})_{j\in OBj(J)}$$

is a limit for $\overline{D}$. The naturality of this source for $\overline{D}$ is a consequence of the naturality of $(L \xrightarrow{l_j} A_j)_{Obj(J)}$ for $D_* \cdot E_k \cdot \overline{D}$. To show that it is a limit of $\overline{D}$, let

$$((Di \xrightarrow{v_i} V)_{i\in Obj(I)} \xrightarrow{u_j} (Di \xrightarrow{f_{ij}} A_j)_{i\in Obj(I)})_{j\in OBj(J)}$$

be another natural source for $\overline{D}$. The naturality of this source implies that the source $(u_j : V \to A_j)_{j\in Obj(J)}$ is natural for $D_* \cdot E_k \cdot \overline{D}$. Since $(L \xrightarrow{l_j} A_j)_{Obj(J)}$ is a limit of

$D_* \cdot E_k \cdot \overline{D}$, there is a unique morphism $t : V \to L$ such that $l_j \cdot t = u_j$ for all $j \in J$. Furthermore, t is an S_D-morphism from $(D_i \xrightarrow{v_i} V)_{Obj(I)}$ to $(D_i \xrightarrow{t_i} L)_{Obj(I)}$. In fact, for each $i \in Obj(I)$, since

$$l_j \cdot t \cdot v_i = u_j \cdot v_i = f_{ij} = l_j \cdot t_i \text{ for all } j \in Obj(J),$$

we have $t \cdot v_i = t_i$. Consequently, t is the unique S_D-morphism from $(D_i \xrightarrow{v_i} V)_{Obj(I)}$ to $(D_i \xrightarrow{t_i} L)_{Obj(I)}$ such that $l_j \cdot t = u_j$ for all $j \in Obj(J)$.

Proof of **(ii)**: Let $\mathcal{A}_k$ be the subcategory of $\mathcal{A}$ which consists of all codomains of natural sinks from D which lie in $\mathcal{C}_k$. Then, $\mathcal{A}_k$ is connected, since $\mathcal{C}_k$ is. By hypothesis, the connected component of $\mathcal{A}$ which contains $\mathcal{A}_k$ has a small limit-dense subcategory $\mathcal{B}$. Let I_* be the subcategory of $\mathcal{C}_k$ of all natural sinks from D with codomain in $\mathcal{B}$. Since $\mathcal{C}_k$ is connected, for each pair of $\mathcal{C}_k$-objects S and S' we may choose a finite set of $\mathcal{C}_k$-objects $I_{(S,S')} = \{S_r = (Di \xrightarrow{g_{ri}} A_r)_{i \in Obj(I)}\, , \, r = 1, ..., m\}$ for which there is a diagram of the form

$$S \longrightarrow S_1 \longleftarrow S_2 \longrightarrow ... \longleftarrow S_m \longrightarrow S'\, .$$

Let I_{**} be the subcategory of $\mathcal{C}_k$ which consists of all objects in $I_* \cup (\cup_{S,S' \in I_*} I_{(S,S')})$. Then I_{**} is clearly a small connected subcategory of $\mathcal{C}_k$. Consequently, by **(i)**, the inclusion functor $I_{**} \hookrightarrow \mathcal{C}_k$ has a limit in $\mathcal{C}_k$. Let it be

$$(S_o \xrightarrow{p_S} S)_{S \in I_{**}}$$

with $S_o = (D_i \xrightarrow{l_i} A)_{i \in Obj(I)}$. We show that S_o is a weakly initial object of $\mathcal{C}_k$. Indeed, let $\hat{S} = (Di \xrightarrow{h_i} \hat{A})_{i \in Obj(I)}$ belong to $\mathcal{C}_k$. Then, there is a small diagram $\hat{D} : N \to \mathcal{B}$ which has as limit a source with domain $\hat{A}$, say, $(\hat{A} \xrightarrow{t_n} B_n)_{n \in Obj(N)}$. It is clear that, for each $n \in Obj(N)$, the sink $S_n = (Di \xrightarrow{h_i} \hat{A} \xrightarrow{t_n} B_n)_{i \in Obj(I)}$ belongs to I_{**}. On the other hand, the source $(A \xrightarrow{p_{S_n}} B_n)_{n \in Obj(N)}$ is natural for $\hat{D}$. In fact, let $n \xrightarrow{d} n'$ be a N-morphism. The naturality of $(t_n)_{Obj(N)}$ implies that $\hat{D}d \cdot t_n = t_{n'}$ and thus, that $\hat{D}d \cdot (t_n \cdot h_i) = t_{n'} \cdot h_i$ for all $i \in Obj(I)$. That is, $\hat{D}d$ is a I_{**}-morphism from S_n to $S_{n'}$. Consequently, as $(S_o \xrightarrow{p_S} S)_{S \in I_{**}}$ is a limit, we have that $\hat{D}d \cdot p_{S_n} = p_{S_{n'}}$. Therefore, there exists a unique morphism $A \xrightarrow{w} \hat{A}$ such that $t_n \cdot w = p_{s_n}$ for all $n \in Obj(N)$. Now, for each $i \in Obj(I)$, we have $t_n \cdot w \cdot l_i = p_{S_n} \cdot l_i = t_n \cdot h_i$ $(n \in Obj(N))$. Hence, since $(t_n)_{n \in Obj(N)}$ is a limit, it follows that $w \cdot l_i = h_i$ for al $i \in Obj(I)$, that is, w is a morphism in $\mathcal{C}_k$ from S_o to $\hat{S}$. □

We recall that a monosource $(m_i)_I$ is said to be *extremal* provided that it fulfils the following condition

(E) every epimorphism e through which all m_i factors is an isomorphism.

It is well-known that every cocomplete and cowellpowered category is an $(Epi, ExtrMonoSource)$-category (this follows, for instance, from 6.5 and 7.3 of [71]).

Lemma 17.5 *If $\mathcal{A}$ is a multicocomplete and cowellpowered category, then:*

(i) *Each connected source in $\mathcal{A}$ has an $(Epi, ExtrMonoSource)$-factorization.*

(ii) *If $(B \xrightarrow{m_i} B_i)_I$ is a connected extremal monosource, $A \xrightarrow{e} C$ is an epimorphism, $A \xrightarrow{h} B$ is a morphism and $(C \xrightarrow{h_i} B_i)_I$ is a source such that $m_i \cdot h = h_i \cdot e$ for all $i \in I$, then there is a unique morphism $t : C \to B$ such that $t \cdot e = h$ and $m_i \cdot t = h_i$ for all $i \in I$.*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "h"'] & C \arrow[d, "h_i"] \arrow[dl, "t"] \\ B \arrow[r, "m_i"'] & B_i \end{tikzcd}$$

Proof. **(i)** Let $(f_i : A \to A_i)_I$ be a connected source and let us consider the family indexed by K of all pairs $(e_k, (m_{ki})_I)$ where $e_k : A \to E_k$ is an epimorphism and, for all $i \in I$, $m_{ki} : E_k \to A_i$ are morphisms such that

$$m_{ki} \cdot e_k = f_i. \tag{11}$$

Now, let us form a multicointersection of the family $(e_k)_{k \in K}$. From (11) it follows that there are a unique component of the multicointersection, say

$$(e : A \to B; (g_k : E_k \to B)_K),$$

and a unique morphism $m_i : B \to A_i$ such that

$$m_i \cdot e = f_i \text{ and } m_i \cdot g_k = m_{ki} \text{ for all } k \in K.$$

Since the $\mathcal{A}$-source $(f_i)_I$ is connected and all e_k are epimorphisms, we have that all $(f_i, (m_{ki})_K)$ with $i \in I$ belong to the same connected component of the quasicategory of natural sinks for the diagram $A \xrightarrow{e_k} E_k,\ k \in K$. Thus, the existing component $(e, (g_k)_K)$ of the multicointersection is the same for all $i \in I$. Hence, $(A \xrightarrow{e} B \xrightarrow{m_i} A_i)_I$ is a factorization of $(f_i)_I$, with $e \in Epi(\mathcal{A})$. Since $(f_i)_I$ is connected and e is an epimorphism, it is clear that $(m_i)_I$ is connected.

Now, we first show that $(m_i)_I$ fulfils the above condition (E). Let d be an epimorphism and let $(l_i)_I$ be a source such that $m_i = l_i \cdot d$ for all $i \in I$. Then $(f_i)_I = (l_i)_I \cdot (d \cdot e)$ and, thus, $d \cdot e = e_k$ for some $k \in K$. Hence, we have the equality $g_k \cdot d \cdot e = e$, which implies that $g_k \cdot d = 1$. thus, d is an isomorphism. To show that $(m_i)_I$ is a monosource, let a and b be morphisms with codomain in B and such that $m_i \cdot a = m_i \cdot b$ for all $i \in I$. Then for each $i \in I$ there are a component $B \xrightarrow{c} C$ of the multicoequalizer of (a, b) and a morphism $r_i : C \to A_i$ such that $r_i \cdot c = m_i$. Since $(m_i)_I$ is connected, the component $B \xrightarrow{c} C$ is the same for all m_i. Consequently, the equality $(m_i)_I = (r_i)_I \cdot c$, with c an epimorphism, implies that c is an isomorphism, since $(m_i)_I$ fulfils condition (E). Thus, $a = b$.

(ii) Let us form a multipushout of e along h. For each $i \in I$, the equality $h_i \cdot e = m_i \cdot h$ implies the existence of a unique component $(\hat{e}, \hat{h})$ of the multipushout of (e, h) and of a unique morphism t_i such that $t_i \cdot \hat{e} = m_i$ and $t_i \cdot \hat{h} = h_i$. Since $(m_i)_I$ is connected and $e \in Epi(\mathcal{A})$, all the pairs (m_i, h_i) belong to the same connected component of the category of natural sinks from the diagram (e, h) and, thus, the same pair $(\hat{e}, \hat{h})$ corresponds to each one of them. Consequently, we have that $(m_i)_I = (t_i)_I \cdot \hat{e}$ and, since $\hat{e}$ is an epimorphism (by 16.4.2) and $(m_i)_I$ fulfils condition (E), $\hat{e}$ is an isomorphism. Therefore, $t = \hat{e}^{-1} \cdot \hat{h}$ is the required morphism. □

We recall that, given a category $\mathcal{A}$, an $\mathcal{A}$-object S is said to be a *separator* in $\mathcal{A}$ provided that for each pair of morphisms $f, g : A \to B$ with $f \neq g$, there is some morphism $S \xrightarrow{h} A$ such that $f \cdot h \neq g \cdot h$.

Theorem 17.6 *Every cowellpowered, multicocomplete category with a separator has connected limits.*

Proof. Let $\mathcal{A}$ be a cowellpowered, multicocomplete category and let S be a separator of

$\mathcal{A}$.

I. We show that every object B is a quotient of some component of a multicoproduct of S indexed by $\mathcal{A}(S,B)$. In fact, let

$$(\,(S \xrightarrow{\sigma_g^t} C^t)_{g\in\mathcal{A}(S,B)}\,)_{\,t\in T} \tag{12}$$

be a multicoproduct of S indexed by $\mathcal{A}(S,B)$. Then, there is a unique pair $(t_o, C^{t_o} \xrightarrow{w} B)$ such that the triangles

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "\sigma_g^{t_o}"'] & B \\ C^{t_o} \arrow[ur, "w"] & \end{tikzcd} \tag{13}$$

are commutative for all $g \in \mathcal{A}(S,B)$. Furthermore, the fact that S is a separator implies that w is an epimorphism. Of course, if $\mathcal{A}(S,B) = \emptyset$, then the multicoproduct of S indexed by $\mathcal{A}(S,B)$ is just an initial family of $\mathcal{A}$.

II. We prove that if $(B \xrightarrow{m_i} A_i)_I$ is a small non-empty monosource with $\mathcal{A}(S,B) \neq \emptyset$ then the domain B is the quotient of some component of a multicoproduct of S indexed by $\prod_{i\in I}\mathcal{A}(S,A_i)$. By I. it suffices to show that each component of the multicoproduct of S indexed by $\mathcal{A}(S,B)$ is a quotient of some component of the multicoproduct of S indexed by $\prod_{i\in I}\mathcal{A}(S,A_i)$. The fact that $(B \xrightarrow{m_i} A_i)_I$ is a monosource implies that the map
$\varphi : \mathcal{A}(S,B) \to \prod_{i\in I}\mathcal{A}(S,A_i)$ which assigns $(m_i \cdot g)_I$ to each $g \in \mathcal{A}(S,B)$ is one-to-one. Then, it suffices to prove the following general result: If N and M are nonempty sets such that $N \subseteq M$, and B is an $\mathcal{A}$-object, then each component of a multicoproduct of B indexed by N is a quotient of some component of a multicoproduct of B indexed by M.

So, let $(\nu_n : B \to C)_{n\in N}$ be a component of the multicoproduct of B indexed by N. Fix n_o in N and put, for each $m \in M$,

$$\delta_m = \begin{cases} \nu_m & \text{if } m \in N \\ \nu_{n_o} & \text{if } m \notin N \end{cases}$$

Hence, there are a unique component of the multicoproduct of B indexed by M, say, $(\theta_m : B \to L)_{m\in M}$, and a unique morphism $u : L \to C$ such that $u \cdot \theta_m = \delta_m$, $m \in M$.

Thus, we have that $(\theta_n : B \to L)_{n\in N}$ and $(\nu_n : B \to C)_{n\in N}$ belong to the same connected component of the category of all sinks $(g_n : B \to X)_{n\in N}$, since

$$(\theta_n : B \to L)_{n\in N} \xrightarrow{u} (\nu_n : B \to C)_{n\in N}$$

is a morphism in that category. This proves the existence of a unique morphism $v : C \to L$ such that $v \cdot \nu_n = \theta_n$ for all $n \in N$. Then $u \cdot v \cdot \nu_n = u \cdot \theta_n = \delta_n = \nu_n$ for all $n \in N$; hence, $u \cdot v = 1_C$ and, thus, $u : L \to C$ is a split epi.

III. We prove that, given a small non-empty family $(A_i)_I$ of objects in $\mathcal{A}$, there is a set $\mathbf{F}_{(A_i)_I}$ of $\mathcal{A}$-objects such that every domain B of a monosource with codomain $(A_i)_I$, i.e., of the form $(B \xrightarrow{m_i} A_i)_I$, is a quotient of some object in $\mathbf{F}_{(A_i)_I}$. The set $\mathbf{F}_{(A_i)_I}$ is the union of $\{C\}$ and $\mathbf{F}$, for C and $\mathbf{F}$ as follows:

(a) It is clear that all objects B which are the domain of a monosource with the codomain $(A_i)_I$ belong to the same connected component of $\mathcal{A}$; consequently, all such objects B which, furthermore, fulfil $\mathcal{A}(S, B) = \emptyset$ are quotients of the initial object of the connected component of $\mathcal{A}$ which contains them.

(b) We show that there is a set $\mathbf{F}$ of components of the multicoproduct of S indexed by $\prod_{i\in I} \mathcal{A}(S, A_i)$ such that every domain B of a monosource of the form $(B \xrightarrow{m_i} A_i)_I$ with $\mathcal{A}(S, B) \neq \emptyset$ is a quotient of some object in $\mathbf{F}$.

For each monosource $(B \xrightarrow{m_i} A_i)_I$ with $\mathcal{A}(S, B) \neq \emptyset$, let

$$\mathbf{G}_{(B,(m_i)_I)} = \{(m_i \cdot g)_I \in \textstyle\prod_{i\in I} \mathcal{A}(S, A_i) \mid g \in \mathcal{A}(S, B)\}.$$

Since $\{\mathbf{G}_{(B,(m_i)_I)} \mid (B \xrightarrow{m_i} A_i)_I \text{ is a monosource}\}$ is contained in the set of all subsets of $\prod_{i\in I} \mathcal{A}(S, A_i)$, it is a set. Let $\{(B^j, (m_i^j)_I),\ j \in J\}$ be a set of monosources with the codomain $(A_i)_I$ such that for each monosource $(B, (m_i)_I)$ with the codomain $(A_i)_I$ there is one and only one $j \in J$ such that $\mathbf{G}_{(B,(m_i)_I)} = \mathbf{G}_{(B^j,(m_i^j)_I)}$. By II., there exists a set $\mathbf{F} = \{C^j, j \in J\}$ of components of a multicoproduct of S indexed by $\prod_{i\in I} \mathcal{A}(S, A_i)$ such that B^j is a quotient of C^j. We show now that for each monosource $(B \xrightarrow{m_i} A_i)_I$ with $\mathcal{A}(S, B) \neq \emptyset$ the domain B is a quotient of some C^j.

Let $(B \xrightarrow{m_i} A_i)_I$ and $(B^j \xrightarrow{m_i^j} A_i)_I$ be two monosources such that $\mathbf{G}_{(B,(m_i)_I)} = \mathbf{G}_{(B^j,(m_i^j)_I)}$. Then we may define an isomorphism ϕ between $\mathcal{A}(S, B)$ and $\mathcal{A}(S, B^j)$

by putting, for each $g \in \mathcal{A}(S, B)$, $\phi(g) = h$ such that $(m_i \cdot g)_I = (m_i^j \cdot h)_I$. Consequently, it is clear that the multicoproduct (12) of S indexed by $\mathcal{A}(B, S)$ is also a multicoproduct of S indexed by $\mathcal{A}(S, B^j)$. Hence, there is a unique pair $(t', C^{t'} \overset{w'}{\to} B^j)$ such that the triangles

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[d, "\sigma_g^{t'}"'] & B^j \\ C^{t'} \arrow[ur, "w'"'] & \end{tikzcd} \tag{14}$$

where $\phi(g) = h$, are commutative for all $g \in \mathcal{A}(S, B)$. As before, w' is an epimorphism and we may assume that $C^{t'} = C^j$. Consequently, from the commutativity of the diagrams (13) and (14), we get the equalities $(m_i^j \cdot w') \cdot \sigma_g^{t'} = (m_i \cdot w) \cdot \sigma_g^{t_o}$ for all $g \in \mathcal{A}(S, B)$, which imply that $t' = t_o$ and $m_i^j \cdot w' = m_i \cdot w$, since $((\sigma_g^t)_{\mathcal{A}(S,B)})_T$ is a multicoproduct. Thus B is also a quotient of C^j.

IV. Now, since, given $A \in Obj(\mathcal{A})$, there is a set $\mathbf{F}_{(A)}$ such that each subobject of A is a quotient of some object in $\mathbf{F}_{(A)}$, it turns out that $\mathcal{A}$ is wellpowered.

To show that $\mathcal{A}$ has connected limits, let $D : I \to \mathcal{A}$ be a small connected diagram in $\mathcal{A}$. From Lemma 17.1, the category S^D of natural sources for D is cocomplete. Hence, to show that S^D has a terminal object - which, then, is the limit of D - it suffices to show that it has a weakly terminal object. Let $\mathbf{F}_{(Di)_{Obj(I)}}$ be the set chosen above and let I^* be the set of all natural sources for D with domain in $\mathbf{F}_{(Di)_{Obj(I)}}$. Since I^* is small, the diagram $I^* \hookrightarrow S^D$ has a colimit in S^D, let it be

$$(B, (f_i)_{Obj(I)}) \overset{\mu_{(B,f_i)}}{\longrightarrow} (C, (w_i)_{Obj(I)}).$$

From Lemma 17.5, the connected source $(C \xrightarrow{w_i} Di)_{Obj(I)}$ has an $(Epi,$ $ExtrMonoSource)$-factorization, say $(C \overset{e}{\to} L \xrightarrow{l_i} Di)_{Obj(I)}$. We show that $(L \xrightarrow{l_i} Di)_{Obj(I)}$ is a weakly terminal object of S^D. Let $(A \xrightarrow{g_i} Di)_{Obj(I)}$ belong to S^D and let $(A \overset{d}{\to} B \xrightarrow{n_i} Di)_{Obj(I)}$ be an $(Epi, ExtrMonoSource)$-factorization of $(g_i)_{Obj(I)}$. Then there is some object E in $\mathbf{F}_{(Di)_{Obj(I)}}$ and some epimorphism $E \overset{q}{\to} B$. It is clear that $(E \overset{q}{\to} B \xrightarrow{n_i} Di)_{Obj(I)}$ is natural for D and, thus, it belongs to I^*. Consequently, we have the equality $n_i \cdot q = l_i \cdot (e \cdot \mu_{(E, n_i \cdot q)})$ for all $i \in Obj(I)$. Hence, again by Lemma 17.5, there is a unique $t : B \to L$ such that $t \cdot q = e \cdot \mu_{(E,n_i \cdot q)}$ and $l_i \cdot t = n_i$ for all $i \in Obj(I)$. Therefore, it is clear that $t \cdot d$ is an S^D-morphism from $(A, (g_i)_{Obj(I)})$ to $(L, (l_i)_{obj(I)})$. □

Remark 17.7 Let $\mathcal{A}$ be category with terminal object. Then trivially holds that if $\mathcal{A}$ is multicocomplete then it is cocomplete. Furthermore, if $\mathcal{A}$ has connected limits then it is complete. This follows from the fact that a product $\prod_{i\in I} A_i$ is the same as a limit of the cone-diagram of all morphisms from A_i to a terminal object.

18 Multisolid categories

The concept of a solid concrete category has turned out to be extremely useful in unifying "well-behaved" concrete categories from topology, algebra and other fields of mathematics. We recall that, for cowellpowered concrete categories $(\mathcal{A}, U)$ over a cocomplete category, solidness is equivalent to $\mathcal{A}$ being cocomplete and U having a left adjoint (see [71]). In the present section we study a generalization of solid concrete categories to multisolid ones, introduced by W. Tholen [74] under the name strongly locally semitopological. The main result is that a cowellpowered concrete category $(\mathcal{A}, U)$ over a multicocomplete category is multisolid if and only if $\mathcal{A}$ is multicocomplete and U is a right multi-adjoint. Thus, these categories include examples such as the category of strictly linearly ordered sets or the category of fields. This result improves Theorem 6.3 in [74], using a different approach which stresses the similarity between the behaviour of solid and multisolid categories.

Definition 18.1 A concrete category $(\mathcal{A}, U)$ is *multisolid* if for each U-sink $S = (UA_i \overset{x_i}{\to} X)_I$ there exists a U-source $(X \overset{y_j}{\to} UB_j)_J$ such that

(i) $y_j \cdot x_i$ carries an $\mathcal{A}$-morphism $A_i \to B_j$ for each $i \in I$, $j \in J$;

(ii) whenever a U-morphism $X \overset{y}{\to} UB$ is such that $y \cdot x_i$ carries an $\mathcal{A}$-morphism for all $i \in I$, then there is a unique pair (j, f) with $j \in J$ and $B_j \overset{f}{\to} B$ satisfying $Uf \cdot y_j = y$.

The U-source $(X \overset{y_j}{\to} UB_j)_J$ is called a *semifinal multilift of* S.

Examples 18.2 (cf. [73])

1. Solid categories (i.e., the case where J is a singular set).

2. The category of strictly linearly ordered sets is multisolid over the category of strictly ordered posets.

3. The category of fields is multisolid over the category of commutative unitary rings.

4. A functor $U : \mathcal{A} \to \mathcal{X}$ is said to be *locally full* if, for each commutative diagram of the form

$$\begin{tikzcd} UA \arrow[rr, "f"] \arrow[dr, "Ua"'] & & UB \arrow[dl, "Ub"] \\ & UC & \end{tikzcd}$$

f is the underlying $\mathcal{X}$-morphism of an $\mathcal{A}$-morphism from A to B.

If $U : \mathcal{A} \to \mathcal{X}$ is a faithful and locally full right multi-adjoint, then the concrete category $(\mathcal{A}, U)$ is multisolid. To show this, let $(UA_i \stackrel{x_i}{\to} X)_I$ be a U-sink and consider the U-source $(X \stackrel{z_t}{\to} UC_t)_T$ of all U-morphisms such that $z_t \cdot x_i$ carries an $\mathcal{A}$-morphism from A_i to C_t, for all $i \in I$. Let $(X \stackrel{\eta_k}{\to} UB_k)_K$ be the sub-source of the universal source from X to U of all morphisms $X \stackrel{\eta_k}{\to} UB_k$ such that $Ug_t \cdot \eta_k = z_t$ for some $t \in T$ and some $\mathcal{A}$-morphism $g_t : B_k \to C_t$. Thus, for each $i \in I$, we have the following commutative diagram

$$\begin{tikzcd} UA_i \arrow[r, "x_i"] \arrow[dr, "z_t \cdot x_i"'] & X \arrow[r, "\eta_k"] & UB_k \arrow[dl, "Ug_t"] \\ & UC_t & \end{tikzcd}$$

with $z_t \cdot x_i$ the underlying $\mathcal{X}$-morphism of a morphism from A_i to C_t. Consequently, since U is locally full, we have that $\eta_k \cdot x_i$ carries an $\mathcal{A}$-morphism from A_i to B_k. Therefore $(X \stackrel{\eta_k}{\to} UB_k)_K$ is a semifinal multilift of $(x_i)_I$.

Many other examples from topology, algebra and geometry can be found in [73].

Remarks 18.3

1. It is clear that a multisolid category is solid iff, for each $X \in Obj(\mathcal{X})$, the comma category $X \downarrow U$ is connected.

2. From the definition it follows immediately that if $(\mathcal{A}, U)$ is a multisolid category, then U is a right multi-adjoint, the universal source from X to U being the semifinal multilift of the empty source with domain X.

   Furthermore, as W. Tholen observed in [74], if $(\mathcal{A}, U)$ is a multisolid category over a multicocomplete category, then $\mathcal{A}$ is multicocomplete.

Theorem 18.4 *Let $\mathcal{X}$ be a multicocomplete category. Then a concrete category $(\mathcal{A}, U)$ over $\mathcal{X}$ with $\mathcal{A}$ cowellpowered is multisolid if and only if U is a right multi-adjoint and $\mathcal{A}$ is multicocomplete.*

Proof. By 18.3.2, we have to prove just the sufficiency. Let $U : \mathcal{A} \to \mathcal{X}$ be a faithful, right multi-adjoint functor. We show that every U-sink $(UA_i \overset{x_i}{\to} X)_I$ has a semifinal multilift. Consider the U-source $(X \overset{z_t}{\to} UC_t)_T$ of all U-morphisms z_t such that, for all $i \in I$, $z_t \cdot x_i$ is the underlying $\mathcal{X}$-morphism of an $\mathcal{A}$-morphism from A_i to C_t. Then

$$T = \dot{\bigcup}_{j \in J} T_j,$$

where, for each $j \in J$, $(X \overset{z_t}{\to} UC_t)_{T_j}$ is a connected component of the U-source $(z_t)_T$. Since U is a right multi-adjoint, for each $t \in T$ there is a unique pair (η, f) such that $\eta : X \to UD$ belongs to the universal source from X to U and $f : D \to C_t$ fulfils $Uf \cdot \eta = z_t$. For each $j \in J$, the U-connectedness of the source $(Z \overset{z_t}{\to} UC_t)_{T_j}$ implies that the U-morphism η is the same for all z_t with $t \in T_j$. For each $t \in T_j$ we denote the above pair by

$$(\eta_j : X \to UD_j,\ f_t : D_j \to C_t).$$

For each $j \in J$, let $(D_j \overset{d_j}{\to} B_j \overset{l_t}{\to} C_t)_{T_j}$ be an $(Epi, ExtrMonoSource)$-factorization of $(f_t)_{T_j}$ which exists by Lemma 17.5. We claim that

$$(X \overset{\eta_j}{\to} UD_j \overset{Ud_j}{\to} UB_j)_J$$

is a semifinal multilift of $(UA_i \xrightarrow{x_i} X)_I$. In fact, if $X \xrightarrow{y} UB$ is such that $y \cdot x_i$ carries an $\mathcal{A}$-morphism for all $i \in I$, then $B = C_t$ and$y = z_t$ for some $t \in T$. Let j be the unique element in J such that $t \in T_j$; hence

$$y = Ul_t \cdot (Ud_j \cdot \eta_j). \tag{15}$$

Furthermore, since $(\eta_j)_J$ is a sub-source of a universal source from X to U and d_j is an $\mathcal{A}$-epimorphism, it follows that (j, l_t) is the unique pair for which the equality (15) holds. $\square$

Chapter VI

Multireflectivity and multiorthogonality

In the two first chapters we studied relations between orthogonal and reflective hulls; in particular, conditions under which an orthogonal subcategory is reflective were given. In the the present chapter we study the existence and characterization of the multireflective hull of a given subcategory. Namely, we investigate a generalizaton of the results on orthogonality and reflectivity to the setting of multiorthogonality and multireflectivity. We relate multiorthogonality with orthogonality via *free large-product completions* and we obtain sufficient conditions for the multiorthogonal hull of a subcategory to be its multireflective hull.

Furthermore, we extend the notion of orthogonal closure operator to categories with multipushouts - instead of pushouts - and we use this closure operator to express multiorthogonal sources in terms of density and multireflective hulls in terms of closedness.

19 Multiorthogonality

The main goal of this section is to find conditions under which the multiorthogonal hull is a multireflective hull.

We begin by recalling the concept of multiorthogonality.

Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. When we pass from the notion of reflectivity to that of orthogonality, we enlarge the class of all reflections of some $\mathcal{X}$-object X in $\mathcal{A}$, that is, the class of all morphisms $X \xrightarrow{f} A$ with codomain in $\mathcal{A}$ such that

(o) each morphism $X \xrightarrow{g} A'$ with codomain in $\mathcal{A}$ is uniquely factorized through f,

by considering the class of all morphisms $X \xrightarrow{f} Y$, with codomain not necessarily in $\mathcal{A}$, which fulfil condition (o).

The notion of multiorthogonality is obtained in an analogous way from that of multireflectivity. This concept has been studied by some authors (see, for instance, [10], in the dual situation, [6] and references there).

Definitions 19.1 Let $X \in Obj(\mathcal{X})$ and let $S = (Y, (f_i : Y \to Z_i)_{i \in I})$ be a source in $\mathcal{X}$. We say that X *is multiorthogonal to* S, or S *is multiorthogonal to* X, written $X \underline{\perp} S$, provided that, for each morphism $Y \xrightarrow{g} X$, there is a unique pair $(i, \overline{g})$ with $i \in I$ and $\overline{g} : Z_i \to X$ a morphism such that $\overline{g} \cdot f_i = g$.

If $\mathcal{A}$ is a subcategory of $\mathcal{X}$, we denote by $\mathcal{A}^{\underline{\perp}}$ the conglomerate of all sources S such that, for each $A \in Obj(\mathcal{A})$, $A \underline{\perp} S$.

If $\mathcal{S}$ is a conglomerate of sources, we denote by $\mathcal{S}_{\underline{\perp}}$ the subcategory of all X in $\mathcal{X}$ such that, for each $S \in \mathcal{S}$, $X \underline{\perp} S$.

A subcategory $\mathcal{A}$ of $\mathcal{X}$ is said to be *multiorthogonal* if it coincides with $\mathcal{S}_{\underline{\perp}}$ for some conglomerate $\mathcal{S}$ of sources.

Remark 19.2 For each source $S = (X \xrightarrow{f_i} Y_i)_I$ in $\mathcal{A}^{\underline{\perp}}$, we have that, if $g : X \to A$ and $h : X \to A'$ belong to the same connected component of $X \downarrow \mathcal{A}$, then g and h are factorizable through the same f_i. Indeed, let g and h belong to the same connected component; this means that there is a commutative diagram of the following form

$$\begin{tikzcd}
A \arrow[d, "d_1"'] & & & X \arrow[lll, "g"'] \arrow[rrr, "h"] \arrow[ddll, "g_1"] \arrow[ddd, "g_2"] & & & A' \arrow[d, "d_n"] \\
A_1 & & & & & & A_n \\
& A_2 \arrow[ul, "d_2"] \arrow[dr, "d_3"'] & & & & A_{n-1} \arrow[ur, "d_{n-1}"'] & \\
& & A_3 & A_4 \arrow[l, "d_4"] & \cdots & &
\end{tikzcd}$$
.

with A_1, A_2 ..., A_n in $\mathcal{A}$. Let g and g_1 be factorizable through f_i and $f_{i'}$, respectively. Then, the morphism $d_1 \cdot g = d_2 \cdot g_1$ is factorizable by both f_i and $f_{i'}$; therefore $i = i'$. By using the same argument for g_1 and g_2, and so on, we conclude that h must also factorize through f_i.

As a consequence, it is clear that if $X \downarrow \mathcal{A}$ is connected for each X in $\mathcal{X}$, then $\mathcal{A}^{\underline{\perp}} = \mathcal{A}^{\perp}$.

The following propositions show that the interplay between the notions multireflectivity, multicocompleteness, multiorthogonality and connected limits is parallel to that between the notions of reflectivity, cocompleteness, orthogonality and completeness.

Proposition 19.3

1. *If $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{X}$ and $\mathcal{S}$ and $\mathcal{T}$ are conglomerates of $\mathcal{X}$-sources, then:*
   - $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow \mathcal{A}^{\underline{\perp}} \supseteq \mathcal{B}^{\underline{\perp}}$
   - $\mathcal{S} \subseteq \mathcal{T} \Longrightarrow \mathcal{S}_{\underline{\perp}} \supseteq \mathcal{T}_{\underline{\perp}}$
   - $\mathcal{A} \subseteq \mathcal{S}_{\underline{\perp}} \Longleftrightarrow \mathcal{S} \subseteq \mathcal{A}^{\underline{\perp}}$
2. *For every subcategory $\mathcal{A}$ of $\mathcal{X}$, each of the assertions (a)-(c) below implies the next one:*
   (a) *$\mathcal{A}$ is multireflective;*
   (b) *$\mathcal{A}$ is multiorthogonal;*
   (c) *$\mathcal{A}$ is closed under connected limits.*
3. *For every family $(\mathcal{S}_i)_I$ of conglomerates of sources, we have that*

$$\textstyle\bigcap_{i\in I}(\mathcal{S}_i)_{\underline{\perp}} = (\bigcup_{i\in I}\mathcal{S}_i)_{\underline{\perp}}.$$

Proof. The proof of 1. and 3. and of the implication $(a) \Rightarrow (b)$ of 2. are straightforward. The implication $(b) \Rightarrow (c)$ of 2. is proved in [6] for the case of multiorthogonality with respect to small sources only. But it also works for possibly large sources. $\square$

From 19.3.1, it follows that a subcategory $\mathcal{A}$ of $\mathcal{X}$ is multiorthogonal iff $\mathcal{A} = (\mathcal{A}^{\underline{\underline{\perp}}})_{\underline{\underline{\perp}}}$ and that the subcategory $(\mathcal{A}^{\underline{\underline{\perp}}})_{\underline{\underline{\perp}}}$ is the smallest multiorthogonal subcategory of $\mathcal{X}$ containing $\mathcal{A}$. Thus, we shall use the following

Definition 19.4 Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, the *multiorthogonal hull of* $\mathcal{A}$ in $\mathcal{X}$ is the subcategory $(\mathcal{A}^{\underline{\underline{\perp}}})_{\underline{\underline{\perp}}}$ and it will be denoted by $\underline{\underline{\mathcal{O}}}(\mathcal{A})$.

Definitions 19.5 A conglomerate $\mathcal{S}$ of sources in $\mathcal{X}$ is said to be

- *left-cancellable* provided that if $S = (X \xrightarrow{f_i} Y_i)_I$ and $S_i = (Y_i \xrightarrow{g_{ij}} Z_{ij})_{j \in J_i}$, $i \in I$, are sources such that S_i belong to $\mathcal{S}$ for all $i \in I$ and the composition $(S_i)_I \cdot S = (X \xrightarrow{g_{ij} \cdot f_i} Z_{ij})_{i \in I, j \in J_i}$ also belongs to $\mathcal{S}$, then S belongs to $\mathcal{S}$.

- *right-cancellable* provided that if $S = (X \xrightarrow{f_i} Y_i)_I$ and $S_i = (Y_i \xrightarrow{g_{ij}} Z_{ij})_{j \in J_i}$, $i \in I$, are sources such that the source S and the composition $(S_i)_I \cdot S =$ $(X \xrightarrow{g_{ij} \cdot f_i} Z_{ij})_{i \in I, j \in J_i}$ belong to $\mathcal{S}$, then S_i belongs to $\mathcal{S}$ for all $i \in I$.

Proposition 19.6 *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$.*

1. *If $(f_i)_I$ belongs to $\mathcal{A}^{\underline{\underline{\perp}}}$, then each f_i is $\mathcal{A}$-cancellable.*

2. *$\mathcal{A}^{\perp} \subseteq \mathcal{A}^{\underline{\underline{\perp}}}$ and $\underline{\underline{\mathcal{O}}}(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A})$.*

3. *$\mathcal{A}^{\underline{\underline{\perp}}}$ is closed under composition, that is, if the sources $S = (X \xrightarrow{f_i} Y_i)_I$ and $S_i = (Y_i \xrightarrow{g_{ij}} Z_{ij})_{j \in J_i}$, $i \in I$, belong to $\mathcal{A}^{\underline{\underline{\perp}}}$, then the composition $(S_i)_I \cdot S = (X \xrightarrow{g_{ij} \cdot f_i} Z_{ij})_{i \in I, j \in J_i}$ also belongs to $\mathcal{A}^{\underline{\underline{\perp}}}$.*

4. *$\mathcal{A}^{\underline{\underline{\perp}}}$ is left-cancellable and right-cancellable.*

Proof.

1. and 2. are obvious.

3. Let $(X \overset{f_i}{\to} Y_i)_I$ and $(Y_i \overset{g_{ij}}{\to} Z_{ij})_{J_i}$, $i \in I$, be sources in $\mathcal{A}^{\stackrel{\perp}{=}}$ and let $h : X \to A$ be a morphism with codomain in $\mathcal{A}$. Hence, there is a unique pair $(i, \overline{h})$ such that $\overline{h} \cdot f_i = h$; and then there is a unique pair $(j, \overline{\overline{h}})$ such that $j \in J_i$ and $\overline{\overline{h}} \cdot g_{ij} = \overline{h}$. It is clear that, thus, $((i,j), \overline{\overline{h}})$ is the unique pair with $(i,j) \in \amalg_{i\in I} J_i$, where
$$\amalg_{i\in I} J_i = \cup_{i \in I}\{(i,j) \mid j \in J_i\},$$
and such that $\overline{\overline{h}} \cdot (g_{ij} \cdot f_i) = h$.

4. In order to show that $\mathcal{A}^{\stackrel{\perp}{=}}$ is left-cancellable, let $S = (f_i : X \to Y_i)_{i\in I}$ and $S_i = (g_{ij} : Y_i \to Z_{ij})_{j \in J_i}$, $i \in I$, be sources such that S_i belongs to $\mathcal{A}^{\stackrel{\perp}{=}}$ for all $i \in I$ and the composition $(S_i)_I \cdot S$ also belongs to $\mathcal{A}^{\stackrel{\perp}{=}}$. Let $h : X \to A$ be a morphism with codomain in $\mathcal{A}$. Then there is a unique pair $((i,j), h')$ such that $(i,j) \in \amalg_{i\in I} J_i$ and $h' \cdot g_{ij} \cdot f_i = h$. Thus, the pair $(i, h' \cdot g_{ij})$ is such that $i \in I$ and
$$(h' \cdot g_{ij}) \cdot f_i = h. \tag{16}$$
To show that this pair is unique, let (i', h'') be a pair with $i' \in I$ and $h'' \cdot f_{i'} = h$. Then, since $S_{i'} \in \mathcal{A}^{\stackrel{\perp}{=}}$, there are $j' \in J_{i'}$ and $g : Z_{i'j'} \to A$ such that $g \cdot g_{i'j'} = h'$. Consequently, $g \cdot (g_{i'j'} \cdot f_{i'}) = h$ and, since $(S_i)_I \cdot S \in \mathcal{A}^{\stackrel{\perp}{=}}$, it follows that $i = i'$ and $j = j'$. Concerning the unicity of $h' \cdot g_{ij}$ in the equality (16), let u be a morphism such that $u \cdot f_i = h$. Then, since $S_i \in \mathcal{A}^{\stackrel{\perp}{=}}$, there is a unique pair $((i,j'), u')$ such that $u = u' \cdot g_{ij'}$ and, thus,
$$u' \cdot (g_{ij'} \cdot f_i) = u \cdot f_i = h = h' \cdot (g_{ij} \cdot f_i);$$
hence, $j' = j$ and $u' = h'$, from which it follows that $u = u' \cdot g_i^j = h' \cdot g_{ij}$.

Let us show that $\mathcal{A}^{\stackrel{\perp}{=}}$ is also right-cancellable. Let $S = (f_i : X \to Y_i)_{i \in I}$ and $S_i = (g_{ij} : Y_i \to Z_{ij})_{j\in J_i}$, $i \in I$, be sources such that S and the composition $(S_i)_I \cdot S$ belong to $\mathcal{A}^{\stackrel{\perp}{=}}$. Fix $i \in I$ and let $h : Y_i \to A$ be a morphism with codomain in $\mathcal{A}$. Hence, there is a unique pair $((i',j), h')$ such that $h' \cdot (g_{i'j} \cdot f_{i'}) = h \cdot f_i$. But, since $(f_i)I \in \mathcal{S}$, this equality guarantees that $i = i'$ and $h' \cdot g_{ij} = h$. Now, if $j' \in J_i$ is such that, for some morphism h'', we have that, $h' \cdot g_{ij} = h = h'' \cdot g_{ij'}$, then we also have $h' \cdot (g_{ij} \cdot f_i) = h'' \cdot (g_{ij'} \cdot f_i)$ and, since $(S_i)_I \cdot S \in \mathcal{A}^{\stackrel{\perp}{=}}$, $j = j'$ and $h = h''$. Therefore, (j, h') is the unique pair such that $j \in J_i$ and h' fulfils the equality $h' \cdot g_{ij} = h$. □

Next, we define the free large-product completion of a given category, which allows us to establish interesting relationships between the notions of multiorthogonality and orthogonality.

Definitions 19.7

1. Given a category $\mathcal{X}$, the *free large-product completion of* $\mathcal{X}$, denoted by $\Pi^l(\mathcal{X})$, is the quasicategory defined as follows:

   - objects are families (possibly large) $(X_i)_I$ of $\mathcal{X}$-objects;
   - morphisms are of the form

     $$(X_i)_I \xrightarrow{(\alpha,(f_j)_J)} (Y_j)_J$$

     where $\alpha : J \to I$ is a function and, for each j, $f_j : X_{\alpha(j)} \to Y_j$ is an $\mathcal{X}$-morphism;
   - composition and identity morphisms are obvious.

   It is clear that $\mathcal{X}$ is a subcategory of $\Pi^l(\mathcal{X})$, if objects of $\mathcal{X}$ are identified with singleton-indexed families.

2. If $U : \mathcal{A} \to \mathcal{X}$ is a functor between the categories $\mathcal{A}$ and $\mathcal{X}$, we define the functor $\Pi^l(U) : \Pi^l(\mathcal{A}) \to \Pi^l(\mathcal{X})$ by

   $$\Pi^l(U)((A_i)_I) = (UA_i)_I$$
   $$\Pi^l(U)(\alpha,(f_j)_J) = (\alpha,(Uf_j)_J).$$

Remark 19.8 Let $\Pi^s(\mathcal{X})$ denote the subcategory of the quasicategory $\Pi^l(\mathcal{X})$ which consists of all families $(X_i)_I$ such that I is a set; analogously, given a functor $U : \mathcal{A} \to \mathcal{X}$, we define the functor $\Pi^s(U) : \Pi^s(\mathcal{A}) \to \Pi^s(\mathcal{X})$. As observed by Diers [17], $\Pi^s(\mathcal{X})$ is a free product completion of $\mathcal{X}$, i.e.,

(i) $\Pi^s(\mathcal{X})$ has products:

(ii) For every functor $F : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y}$ is a category with products there exists a functor $F^* : \Pi^s(\mathcal{X}) \to \mathcal{Y}$ preserving products and extending F (i.e., $FX = F^*X$ and $Ff = F^*f$), unique up to isomorphism.

The analogous result holds for $\Pi^l(\mathcal{X})$, only here products must be substituted by large products (and $\mathcal{Y}$ is now an arbitrary quasicategory with large products).

Furthermore, Y. Diers ([17]) proved that

A. *$\mathcal{X}$ has connected limits iff $\Pi^s(\mathcal{X})$ is complete*;

B. *$\mathcal{X}$ is multicocomplete iff $\Pi^s(\mathcal{X})$ is cocomplete*;

C. *$U : \mathcal{A} \to \mathcal{X}$ is a right multi-adjoint iff $\Pi^s(U) : \Pi^s(\mathcal{A}) \to \Pi^s(\mathcal{X})$ is a right-adjoint* (see also [74]).

The following two lemmas will be useful in the sequel. We point out that they extend the assertions **B.** and **C.** in 19.8 to the case where multireflections and multicolimits are allowed to be indexed by a proper class.

Lemma 19.9

1. *If $\mathcal{A}$ is a subcategory of $\mathcal{X}$, then a $\Pi^l(\mathcal{X})$-object $(X_i)_I$ has a reflection in $\Pi^l(\mathcal{A})$ if and only if, for each $i \in I$, X_i has a multireflection in $\mathcal{A}$.*
2. *$U : \mathcal{A} \to \mathcal{X}$ is a right multi-adjoint if and only if $\Pi^l(U) : \Pi^l(\mathcal{A}) \to \Pi^l(\mathcal{X})$ is a right-adjoint.*

Proof.

1. Let

$$(X_i)_I \xrightarrow{(\alpha,(a_j)_J)} (A_j)_J$$

be a reflection of $(X_i)_I$ in $\Pi^l(\mathcal{A})$. We claim that for each $i_o \in I$ the source

$$(X_{i_o} \xrightarrow{a_j} A_j)_{\alpha(j)=i_o}$$

is a multireflection of X_{i_o} in $\mathcal{A}$. Indeed, if $X_{i_o} \xrightarrow{g} A$ is a morphism with codomain in $\mathcal{A}$, then $(X_i)_I \xrightarrow{(\beta,g)} A$, with $\beta(\bullet) = i_o$, is a $\Pi^l(\mathcal{X})$-morphism with codomain in $\Pi^l(\mathcal{A})$; hence, there is a unique $\Pi^l(\mathcal{A})$-morphism $(\overline{\beta},\overline{g}) : (A_j)_J \to A$ such that $(\overline{\beta},\overline{g}) \cdot (\alpha,(a_j)_J) = (\beta, g)$, that is, $\alpha \cdot \overline{\beta} = \beta$ and $\overline{g} \cdot a_{\alpha(\overline{\beta}(\bullet))} = g$. Therefore, taking $j = \overline{\beta}(\bullet)$, the pair $(j,\overline{g})$ is unique with $\alpha(j) = i$ and $\overline{g} \cdot a_j = g$.

Conversely, for each $i \in I$ let

$$(X_i \xrightarrow{r^i_j} A^i_j)_{j\in J_i}$$

be a multireflection of X_i in $\mathcal{A}$. Let

$$K = \dot{\cup}_{i\in I}\ J_i = \cup_{i\in I}\{(j,i) \,|\, j \in J_i\}$$

and let us define $\alpha : K \to I$ by $\alpha(j,i) = i$. Then, it is easy to see that

$$(X_i)_I \xrightarrow{\quad (\alpha,(r^i_j)_{(j,i)\in K}) \quad} (A^i_j)_{(j,i)\in K}$$

is a reflection of $(X_i)_I$ in $\Pi^l(\mathcal{A})$.

2. It is a consequence of 1. □

Lemma 19.10

1. *Let $\mathcal{D}$ be a quasicategory with a set of objects. For each category $\mathcal{X}$, every diagram $D : \mathcal{D} \to \mathcal{X}$ has a multicolimit if and only if every diagram $D : \mathcal{D} \to \Pi^l(\mathcal{X})$ has a colimit.*

2. *$\mathcal{X}$ is multicocomplete if and only if $\Pi^l(\mathcal{X})$ has colimits of all diagrams $D : \mathcal{D} \to \Pi^l(\mathcal{X})$ such that $\mathcal{D}$ is a quasicategory with a set of objects.*

Proof.

1. Let $\mathcal{D}$ be a quasicategory such that $Obj(\mathcal{D})$ is a set. Let $D : \mathcal{D} \to \mathcal{X}$ be a diagram in $\mathcal{X}$, and let $(Dd \xrightarrow{(l^i_d)_I} (X_i)_I)_{d\in Obj(\mathcal{D})}$ be a colimit of $\mathcal{D} \xrightarrow{D} \mathcal{X} \hookrightarrow \Pi^l(\mathcal{X})$ in $\Pi^l(\mathcal{X})$. Then it is immediate that $(Dd \xrightarrow{l^i_d} (X_i)_{d\in Obj(\mathcal{D}))_{i\in I}}$ is a multicolimit of D in $\mathcal{X}$.

   Conversely, let let $D : \mathcal{D} \to \Pi^l(\mathcal{X})$ be a diagram in $\Pi^l(\mathcal{X})$. For each object d of $\mathcal{D}$ put

$$Dd = (X_{i_d})_{I_d}$$

   and for each morphism $t : d \to d'$ put

$$D(d \xrightarrow{t} d') = (X_{i_d})_{I_d} \xrightarrow{\quad (\alpha^t,(f^t_i)_{I_{d'}}) \quad} (X_{i_{d'}})_{I_{d'}} \ .$$

Let I be the subclass of the class $J = \Pi_{d\in Obj(\mathcal{D})} I_d$ which consists of all $(i_d)_{d\in Obj(\mathcal{D})} \in J$ such that for each $\mathcal{D}$-morphism $d \xrightarrow{t} d'$ the map $I_{d'} \xrightarrow{\alpha^t} I_d$ fulfils $\alpha^t(i_{d'}) = i_d$. Thus, for each $i = (i_d)_{d\in Obj(\mathcal{D})} \in I$ we obtain a functor $D_i : \mathcal{D} \to \mathcal{X}$ defined by $D_i d = X_{i_d}$ and $D_i(d \xrightarrow{t} d') = (X_{i_d} \xrightarrow{f^t_{i_{d'}}} X_{i_{d'}})$. By hypothesis, for each $i \in I$, the functor $D_i : \mathcal{D} \to \mathcal{X}$ has a multicolimit in $\mathcal{X}$, let it be $((D_i d \xrightarrow{l_{ik}} L_{ik})_{d\in Obj(\mathcal{D})})_{k\in K_i}$. Let $K = \dot\cup_{i\in I} K_i = \cup_{i\in I}\{(k,i) \mid k \in K_i\}$ and let $\alpha : K \to I_d$ be defined by $\alpha(k,i) = \alpha(k,(i_d)_{d\in Obj(\mathcal{D})}) = i_d$. Then, it is easy to show that

$$\left((X_{i_d})_{I_d} \xrightarrow{(\alpha,(l_{ik})_{(k,i)\in K})} (L_{ik})_{(k,i)\in K} \right)_{d\in Obj(\mathcal{D})}$$

is a colimit of the functor $D : \mathcal{D} \to \Pi^l(\mathcal{X})$.

2. It follows from 1., since the fact that $\mathcal{X}$ has multicolimits of functors whose domain is a small category implies that $\mathcal{X}$ also has multicolimits of functors whose domain is a quasicategory with just a set of objects. In fact, this is a consequence of the following assertion which can be easily proved:

   Let $D : \mathcal{D} \to \mathcal{X}$ be a functor such that $\mathcal{D}$ is a quasicategory with a set of objects. Let $\widetilde{\mathcal{D}}$ be the quotient category obtained from $\mathcal{D}$ such that $Obj(\widetilde{\mathcal{D}}) = Obj(\mathcal{D})$, and, for each pair of objects $d, d' \in Obj(\mathcal{D})$, we define an equivalence relation $\sim$ in the class $\mathcal{D}(d,d')$ by $f \sim f'$ iff $Df = Df'$ and we define $\widetilde{\mathcal{D}}(d,d')$ to be the set of all equivalent classes for $\sim$ in $\mathcal{D}(d,d')$.
   Let $\widetilde{D} : \widetilde{\mathcal{D}} \to \mathcal{X}$ be the corresponding quotient functor.
   Then, if $\widetilde{D} : \widetilde{\mathcal{D}} \to \mathcal{X}$ has a multicolimit in $\mathcal{X}$, so has $D : \mathcal{D} \to \mathcal{X}$ and, moreover, the two multicolimits coincide. □

Remark 19.11 Although we have yet a nice relationship between the multicolimits in a category $\mathcal{X}$ and colimits when we pass from the category $\Pi^s(\mathcal{X})$ to the quasicategory $\Pi^l(\mathcal{X})$, the same is not true with respect to connected limits of $\mathcal{X}$ versus limits in $\Pi^l(\mathcal{X})$. Indeed, the fact that $\mathcal{X}$ has connected limits does not implies the existence of equalizers in $\Pi^l(\mathcal{X})$ as the following example shows:

Let $\mathcal{X} = \mathcal{S}et$ and let us consider in $\Pi^l(\mathcal{X})$ the following diagram

$$\begin{tikzcd} (A_i)_I \arrow[r, shift left, "{(\alpha,(f_i)_I)}"] \arrow[r, shift right, "{(\beta,(g_i)_I)}"'] & (B_i)_I \end{tikzcd}$$

where I is the class of all ordinals, the map $\alpha : I \to I$ assigns the zero to each ordinal, $\beta = id_I$ and, for each $i \in I$, $A_i = \{0, 1\}$ and $B_i = \{0\}$; $f_i : A_0 \to B_i$ and $g_i : A_i \to B_i$ are the constant map. We are going to show that this pair of morphisms does not have an equalizer in $\Pi^l(\mathcal{S}et)$. Let us assume that, to the contrary, $(c_k)_K \overset{(\gamma,(h_i)_I)}{\longrightarrow} (A_i)_I$ is an equalizer of that pair. Then, in particular, $\gamma \cdot \alpha = \gamma \cdot \beta$ and, consequently, for each $i \in I$, $\gamma(i) = \gamma(\beta(i)) = \gamma(\alpha(i)) = \gamma(0)$; hence, γ is a constant map. Furthermore, the $\Pi^l(\mathcal{S}et)$-morphism $C_{\gamma(0)} \overset{(h_i)_I}{\longrightarrow} (A_i)_I$ equalizes the pair $((\alpha,(f_i)_I),(\beta,(g_i)_I))$. But, then, on the one hand, $C_{\gamma(0)} \overset{(h_i)_I}{\longrightarrow} (A_i)_I$ equalizes the pair $((\alpha,(f_i)_I),(\beta,(g_i)_I))$ and, on the other hand, the triangle

$$\begin{tikzcd} (C_k)_K \arrow[r, "{(\gamma,(h_i)_I)}"] & (A_i)_I \arrow[dl, "(h_i)_I"] \\ C_{\gamma(0)} \arrow[u, "{(\delta, id_{C_{\gamma(0)}})}"] & \end{tikzcd}$$

where the map $\delta : \{\bullet\} \to K$ is defined by $\delta(\bullet) = \gamma(0)$, is commutative. Consequently, K must be singular and

$$\begin{tikzcd} C_{\gamma(0)} \arrow[r, "(h_i)_I"] & (A_i)_I \arrow[r, shift left, "{(\alpha,(f_i)_I)}"] \arrow[r, shift right, "{(\beta,(g_i)_I)}"'] & (B_i)_I \end{tikzcd}$$

is an equalizer diagram. Now, we show that, for each ordinal α, there is a one-to-one map from the product $\{0, 1\}^\alpha$ into $C_{\gamma(0)}$, which is absurd. Let α be an ordinal. Let $\pi_i : \{0, 1\}^\alpha \to \{0, 1\}$, $i \in \alpha$, be the corresponding projections and define $\{0, 1\}^\alpha \overset{r_i}{\longrightarrow} A_i$ by $r_i = \pi_i$ if $i \in \alpha$, $r_i = \pi_0$, otherwise. Then $\{0, 1\}^\alpha \overset{(r_i)_I}{\longrightarrow} (A_i)_I$ is a morphism in $\Pi^l(\mathcal{S}et)$ which equalizes the pair $((\alpha,(f_i)_I),(\beta,(g_i)_I))$. Then, there is a unique morphism $t : \{0,1\}^\alpha \to C_{\gamma(0)}$ such that $h_i \cdot t = r_i$, for all ordinal i. In fact, t is one-to-one:

$$\begin{aligned} t \cdot a = t \cdot b \ &\Rightarrow h_i \cdot t \cdot a = h_i \cdot t \cdot b \ \text{ for all } \ i \\ &\Rightarrow r_i \cdot a = r_i \cdot b \ \text{ for all } \ i \\ &\Rightarrow \pi_i \cdot a = \pi_i \cdot b \ \text{ for all } \ i \in \alpha \\ &\Rightarrow a = b. \end{aligned}$$

In order to relate multiorthogonality with orthogonality via free large-product completions, we use the following definition.

Notation 19.12 Given a conglomerate $\mathcal{S}$ of sources in $\mathcal{X}$, we denote by $\Pi^l(\mathcal{S})$ the conglomerate of all morphisms $(X_i)_I \overset{(\alpha,(f_j)_J)}{\longrightarrow} (Y_j)_J$ $\Pi^l(\mathcal{X})$) such that for every $i \in I$ the source $(X_i \overset{f_j}{\to} Y_j)_{\alpha(j)=i}$ belongs to $\mathcal{S}$

In the sequel, when we use the operators $\underline{\perp}$ and $\perp$, we always consider $\underline{\perp}$ in $\mathcal{X}$ and $\perp$ in $\Pi^l(\mathcal{X})$.

Proposition 19.13 *For a subcategory $\mathcal{A}$ of the category $\mathcal{X}$ and a conglomerate $\mathcal{S}$ of sources in $\mathcal{X}$ which contains all isomorphisms, we have the following properties:*

1. $\Pi^l(\mathcal{A}^{\underline{\perp}}) = (\Pi^l(\mathcal{A}))^{\perp}$;
2. $(\Pi^l(\mathcal{S}))_{\perp} = \Pi^l(\mathcal{S}_{\underline{\perp}})$;
3. $\Pi^l(\underline{\underline{\mathcal{O}}}(\mathcal{A})) = \mathcal{O}(\Pi^l(\mathcal{A}))$.

Proof.

1. This equality follows from the following equivalences which are easily checked:

   the source $\quad (X_i)_I \xrightarrow{\quad(\alpha,(f_j)_J)\quad} (Y_j)_J \quad$ belongs to $\Pi^l(\mathcal{A}^{\underline{\perp}})$

   iff, for each $i \in I$, $\quad (X_i \overset{f_j}{\longrightarrow} Y_j)_{\alpha(j)=i}$ belongs to $\mathcal{A}^{\underline{\perp}}$

   iff, for each $i \in I$, $\quad X_i \xrightarrow{\quad(f_j)_{\alpha(j)=i}\quad} (Y_j)_{\alpha(j)=i} \quad$ belongs to $(\Pi^l(\mathcal{A}))^{\perp}$

   iff $\quad (X_i)_I \xrightarrow{\quad(\alpha,(f_j)_J)\quad} (Y_j)_J \quad$ belongs to $(\Pi^l(\mathcal{A}))^{\perp}$.

2. Let $(B_k)_K \in (\Pi^l(\mathcal{S}))_{\perp}$; in order to prove that $(B_k)_K \in \Pi^l(\mathcal{S}_{\underline{\perp}})$, we show that, for each $k \in K$, $B_k \in \mathcal{S}_{\underline{\perp}}$. Let $(X \overset{f_i}{\to} Y_i)_I \in \mathcal{S}$. Fix $k_o \in K$ and let $h : X \to B_{k_o}$ be an $\mathcal{X}$-morphism. We may define a $\Pi^l(\mathcal{X})$-morphism

$$(Z_k)_K \xrightarrow{(\alpha,(g_j)_J)} (W_j)_J$$

as follows:

$$Z_k = \begin{cases} X & \text{if} \quad k = k_o \\ B_k & \text{if} \quad k \neq k_o \end{cases}$$

$$J = I \,\dot{\cup}\, (K \setminus \{k_o\})$$

$$W_j = \begin{cases} Y_j & \text{if} \quad j \in I \\ B_j & \text{if} \quad j \in K \setminus \{k_o\} \end{cases}$$

$$\alpha(j) = \begin{cases} k_o & \text{if} \quad j \in I \\ j & \text{if} \quad j \in K \setminus \{k_o\} \end{cases}$$

the $\mathcal{X}$-morphisms $g_j : Z_{\alpha(j)} \to W_j$ are defined by

$$g_j = \begin{cases} f_j & \text{if} \quad j \in I \\ 1_{B_j} & \text{if} \quad j \in K \setminus \{k_o\} \end{cases} .$$

Since $Iso(\mathcal{X}) \subseteq \mathcal{S}$, it is clear that the morphism $(Z_k)_K \xrightarrow{(\alpha,(g_j)_J)} (W_j)_J$ belongs to $\Pi^l(\mathcal{S})$.

On the other hand, we may define a $\Pi^l(\mathcal{X})$-morphism

$$(Z_k)_K \xrightarrow{(\beta,(h_k)_K)} (B_k)_K$$

by

$$\beta = 1_K$$

$$h_k = \begin{cases} h & \text{if} \quad k = k_o \\ 1_{B_k} & \text{if} \quad k \neq k_o \end{cases} .$$

Then, there is a unique $\Pi^l(\mathcal{X})$-morphism

$$(\overline{\beta}, (\overline{h}_k)_K) : (W_j)_J \to (B_k)_K$$

such that
$$(\overline{\beta},(\overline{h}_k)_K)\cdot(\alpha,(g_j)_J)=(\beta,(h_k)_K).$$

Therefore, it is easy to conclude that the pair
$$(\overline{\beta}(k_o),\overline{h}_{k_o})$$
is the unique one such that $\overline{\beta}(k_o)\in I$ and the morphism $\overline{h}_{k_o}:Y_{\beta(k_o)}\to B_{k_o}$ fulfils the equality $\overline{h}_{k_o}\cdot f_{\beta(k_o)}=h$.

Conversely, let $(B_k)_K\in\Pi^l(\mathcal{S}_{\underline{\perp}})$ and let $(X_i)_I \xrightarrow{(\alpha,(f_j)_J)} (Y_j)_J$ belong to $\Pi^l(\mathcal{S})$. To show that $(\alpha,(f_j)_J)\perp(B_k)_K$, let $(X_i)_I \xrightarrow{(\beta,(g_k)_K)} (B_k)_K$ be a $\Pi^l(\mathcal{X})$-morphism. For each $k\in K$, let us consider $i=\beta(k)$; the source $(X_i\xrightarrow{f_i}Y_j)_{\alpha(j)=i}$ belongs to $\mathcal{S}$; hence, there is a unique pair $(j_k,\overline{g}_k)$ with $\alpha(j_k)=i$ and $\overline{g}_k:Y_j\to B_k$ such that $\overline{g}_k\cdot f_{j_k}=g_k$. It is easy to see that the $\Pi^l(\mathcal{X})$-morphism
$$(\gamma,(\overline{g}_k)_K),$$
with $\gamma:K\to J$ defined by $\gamma(k)=j_k$, is the unique one such that
$$(\gamma,(\overline{g}_k)_K)\cdot(\alpha,(f_j)_J)=(\beta,(g_k)_K).$$

3. It follows from 1. and 2. In fact, we have that
$$\begin{aligned}\Pi^l(\underline{\underline{\mathcal{O}}}(\mathcal{A})) &= \Pi^l((\mathcal{A}^{\underline{\perp}})_{\underline{\perp}})=(\Pi^l(\mathcal{A}^{\underline{\perp}}))_{\perp}=((\Pi^l(\mathcal{A}))^{\perp})_{\perp}\\ &=\mathcal{O}(\Pi^l(\mathcal{A})).\end{aligned}$$

$\square$

20 Multiorthogonal and multireflective hulls

Now, we investigate conditions under which the multiorthogonal hull is multireflective. In particular, we are going to show that Theorem 2.10 for reflectivity has a parallel for multireflectivity.

Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$.

For each $X \in Obj(\mathcal{X})$, we consider the quasicategory $X/\mathcal{A}^{\underline{\underline{\perp}}}$ defined as follows:

- objects are all sources in $\mathcal{A}^{\underline{\underline{\perp}}}$ with domain X;
- morphisms are $\Pi^l(\mathcal{X})$-morphisms $(\alpha, (a_j)_J) : (Y_i)_I \to (Z_j)_J$ with $a_j.f_{\alpha(j)} = g_j$ for each $j \in J$;
- the units and the composition of morphisms are as expected.

As a matter of fact, when the comma category $X \downarrow \mathcal{A}$ is connected, the quasicategory $X/\mathcal{A}^{\underline{\underline{\perp}}}$ coincides with the category $X/\mathcal{A}^{\perp}$ as defined in the second section of Chapter I, before 2.6.

The following lemma is obvious.

Lemma 20.1 *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ and let $(X, (r_j : X \to A_j)_{j\in J})$ be a source in $\mathcal{X}$. Then:*

1. *$(r_j)_J$ is a multireflection of X in $\mathcal{A}$ iff it belongs to $\mathcal{A}^{\underline{\underline{\perp}}}$ and $A_j \in Obj(\mathcal{A})$, $j \in J$.*
2. *If $(r_j)_J$ is a multireflection of X in $\mathcal{A}$ then it is a terminal object of $X/\mathcal{A}^{\underline{\underline{\perp}}}$.*$\square$

The following theorem establishes conditions under which the second item of the above lemma has a converse.

Theorem 20.2 *If $\mathcal{X}$ is a category with connected multicolimits and $\mathcal{A}$ is a subcategory of $\mathcal{X}$, then the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{X}$, $\underline{\underline{\mathcal{O}}}(\mathcal{A})$, is multireflective in $\mathcal{X}$ if and only if for each $X \in Obj(\mathcal{X})$ the quasicategory $X/\mathcal{A}^{\underline{\underline{\perp}}}$ has a weakly terminal set.*

Proof. The necessity is clear. Conversely, let us assume that, for $X \in Obj(\mathcal{X})$, $X/\mathcal{A}^{\underline{\underline{\perp}}}$ has a weakly terminal set. We want to prove that X has a multireflection in $\underline{\underline{\mathcal{O}}}(\mathcal{A})$. From Lemma 19.9 and Proposition 19.13, it suffices to show that X has a reflection in $\mathcal{O}(\Pi^l(\mathcal{A}))$. But, in 2.9 and 2.10, we have proved the following:

If $\mathcal{X}$ has connected colimits, $\mathcal{A}$ is a subcategory of $\mathcal{X}$ and X is an $\mathcal{X}$-object such that $X/\mathcal{A}^{\perp}$ has a weakly terminal set, then $X/\mathcal{A}^{\perp}$ has a terminal object and it is exactly a reflection from X to $\mathcal{O}(\mathcal{A})$.

By 19.10.1, the fact that $\mathcal{X}$ has connected multicolimits implies that the quasicategory $\Pi^l(\mathcal{X})$ has multipushouts and multicoequalizers of possibly large families of morphisms.

Consequently, since $X/(\Pi^l(\mathcal{A}))^\perp$ has a weakly terminal set, it is easily checked, by using the same technique as in 2.9, that $X/(\Pi^l(\mathcal{A}))^\perp$ has a terminal object which is a reflection of X in $\mathcal{O}(\Pi^l(\mathcal{A}))$. Therefore, X has a multireflection in $\underline{\underline{\mathcal{O}}}(\mathcal{A})$. $\square$

Let us consider the categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{X}$ of Example 2.11. For each $\mathcal{X}$-object X, the category $X \downarrow \mathcal{A}$ is connected and this implies that the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{B}$ and the one in $\mathcal{X}$ coincide with the orthogonal hull of $\mathcal{A}$ in $\mathcal{B}$ and the one in $\mathcal{X}$, respectively. We remark that, consequently, this example allows us to conclude that we may have subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$ such that $\mathcal{A}$ is contained in $\mathcal{B}$ but the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ is different from the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{B}$, even when $\mathcal{B}$ is multiorthogonal in $\mathcal{X}$. We are going to show that the two multiortogonal hulls coincide when $\mathcal{B}$ is multireflective in $\mathcal{X}$.

Remark 20.3 Let $\mathcal{B}$ be a multireflective subcategory of $\mathcal{X}$ and, given a source $S = (X \xrightarrow{f_i} Y_i)_I$ in $\mathcal{X}$, consider the following commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "r_{\varepsilon_S(i,k)} = r_j"'] & Y_i \arrow[d, "r_{ik}"] \\ B_{\varepsilon_S(i,k)} = B_j \arrow[r, "g_{ik}"] & B_{ik} \end{tikzcd} \tag{17}$$

where $(r_j : X \to B_j)_{j \in J}$ is the multireflection of X in $\mathcal{B}$, $(r_{ik} : Y_i \to B_{ik})_{k \in K_i}$ is the multireflection of Y_i in $\mathcal{B}$, $i \in I$, and, for each (i, k) with $i \in I$ and $k \in K_i$, j is the unique element of J such that $r_{ik} \cdot f_i$ is factorizable through r_j and $g_{ik} : B_j \to B_{ik}$ is the unique morphism such that $g_{ik} \cdot r_j = r_{ik} \cdot f_i$. We put $j = \varepsilon_S(i, k)$.

We know that $\Pi^l(\mathcal{B})$ is reflective in $\Pi^l(\mathcal{X})$ and, from the above diagrams, we obtain the following diagram

$$\begin{tikzcd}
X \arrow[r, "(f_i)_I"] \arrow[d, "(r_j)_J"'] & (Y_i)_I \arrow[d, "(r_{ik})_{k\in K,\, i\in I}"] \\
(B_j)_J \arrow[r, "(g_{ik})"] & (B_{ik})_{k\in K,\, i\in I}
\end{tikzcd} \tag{18}$$

in $\Pi^l(\mathcal{X})$, where $(r_j)_J$ is the reflection from X to $\Pi^l(\mathcal{B})$, $(r_{ik})_{k\in K_i,\, i\in I}$ is the reflection from $(Y_i)_I$ to $\Pi^l(\mathcal{B})$ and $(g_{ik})_{k\in K_i,\, i\in I}$ is the image of $(f_i)_I$ in $\Pi^l(\mathcal{B})$ through the reflector.

Consequently, from 2.12.1 and 19.13 it follows that if $\mathcal{B}$ is a multireflective subcategory of $\mathcal{X}$ and $\mathcal{A}$ is a subcategory of $\mathcal{B}$, then $\mathcal{A}^{\underline{\perp}_{\mathcal{X}}}$ is the collection of all sources $S = (X \overset{f_i}{\to} Y_i)_I$ such that in the above diagram the source $(B_j \overset{g_{ik}}{\to} B_{ik})_{(i,k)\in\varepsilon_S^{-1}(\{j\})}$ belongs to $\mathcal{A}^{\underline{\perp}_{\mathcal{B}}}$ for all $j \in J$.

On the other hand, since for a reflective subcategory $\mathcal{A}$ of a category $\mathcal{X}$ we always have that $\mathcal{A}^{\perp}$ consists of all $\mathcal{X}$-morphisms whose image through the reflector is an isomorphism, it follows that: If $\mathcal{B}$ is a multireflective subcategory of $\mathcal{X}$, a source $S = (f_i : X \to Y_i)_{i\in I}$ belongs to $\mathcal{B}^{\underline{\perp}}$ if and only if $\varepsilon_S : \amalg_{i\in I} K_i \to J$ is a bijection and all morphisms $B_j \overset{g_{ik}}{\to} B_{ik}$ are isomorphisms.

Using again the relationship between multiorthogonality and orthogonality via the "operator" Π^l, we obtain the following

Proposition 20.4 *If $\mathcal{B}$ is a multireflective subcategory of $\mathcal{X}$ and $\mathcal{A}$ is a subcategory of $\mathcal{B}$, then the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{B}$ coincides with the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{X}$.*

Proof. Let $\mathcal{B}$ be a multireflective subcategory of $\mathcal{X}$ and let $\mathcal{A}$ be a subcategory of $\mathcal{B}$. Then $\Pi^l(\mathcal{B})$ is a reflective sub-quasicategory of $\Pi^l(\mathcal{X})$ and $\Pi^l(\mathcal{A})$ is a sub-quasicategory of $\Pi^l(\mathcal{B})$. From 2.12.2, we have that

$$((\Pi^l(\mathcal{A}))^{\perp_{\Pi^l(\mathcal{X})}})_{\perp_{\Pi^l(\mathcal{X})}} = ((\Pi^l(\mathcal{A}))^{\perp_{\Pi^l(\mathcal{B})}})_{\perp_{\Pi^l(\mathcal{B})}},$$

which implies by 19.13,

$$\Pi^l((\mathcal{A}^{\underline{\perp}_{\mathcal{X}}})_{\underline{\perp}_{\mathcal{X}}}) = \Pi^l((\mathcal{A}^{\underline{\perp}_{\mathcal{B}}})_{\underline{\perp}_{\mathcal{B}}});$$

hence,

$$\Pi^l((\mathcal{A}^{\perp\!\!\!=_{\mathcal{X}}})_{\perp\!\!\!=_{\mathcal{X}}}) \cap \mathcal{X} = \Pi^l((\mathcal{A}^{\perp\!\!\!=_{\mathcal{B}}})_{\perp\!\!\!=_{\mathcal{B}}}) \cap \mathcal{X}\,,$$

that is,

$$(\mathcal{A}^{\perp\!\!\!=_{\mathcal{X}}})_{\perp\!\!\!=_{\mathcal{X}}} = (\mathcal{A}^{\perp\!\!\!=_{\mathcal{B}}})_{\perp\!\!\!=_{\mathcal{B}}}.$$

□

The following lemma shows that, if $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category, the conglomerate $\mathcal{A}^{\perp\!\!\!=}$ has nice properties when considered in the $\mathcal{E}$-reflective hull $\mathbb{M}(\mathcal{A})$.

Lemma 20.5 *Let $\mathcal{X}$ be an $(\mathcal{E}, \mathbb{M})$-category with $\mathbb{M} \subseteq MonoSource(\mathcal{X})$. If $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$ and $(f_i : X \to Y_i)_{i\in I}$ belongs to $\mathcal{A}^{\perp\!\!\!=}$, then*

1. *$f_i \in Epi(\mathcal{X})$, $i \in I$;*
2. *$(f_i)_I$ belongs to $\mathbb{M}$.*

Proof.

1. For a fixed $i \in I$, let $g, h : Y_i \to Z$ be morphisms such that $g \cdot f_i = h \cdot f_i$. Since $Z \in \mathbb{M}(\mathcal{A})$, there is a source $(m_j : Z \to A_j)_J$ belonging to $\mathbb{M}$ and with $A_j \in Obj(\mathcal{A})$, $j \in J$. For each $j \in J$, we have that the equality $m_j \cdot g \cdot f_i = m_j \cdot h \cdot f_i$ implies that $m_j \cdot g = m_j \cdot h$, since f_i is $\mathcal{A}$-cancellable. Consequently, since $\mathbb{M} \subseteq MonoSource(\mathcal{X})$, $g = h$.

2. Let $f_i = n_i \cdot e$, $i \in I$, be an $(\mathcal{E}, \mathbb{M})$-factorization of $(f_i)_I$ and let $(m_k : X \to A_k)_K$ be a source in $\mathbb{M}$ with $A_k \in Obj(\mathcal{A})$. For each k, there exists a unique pair $(\alpha(k), s_k : Y_{\alpha(k)} \to A_k)$ such that $\alpha(k) \in I$ and $m_k = s_k \cdot f_{\alpha(k)}$. Then, from the equalities $m_k \cdot id_X = (s_k \cdot n_{\alpha(k)}) \cdot e$, $k \in K$, and from the diagonal property for $(\mathcal{E}, \mathbb{M})$, it follows that e is an isomorphism, so that $(f_i)_I$ belongs to $\mathbb{M}$. □

Now, combining this lemma with 20.4 and 20.2, we obtain the following

Theorem 20.6 *Let $\mathcal{X}$ be an $(E, \mathbb{M})$-category with $\mathbb{M} \subseteq MonoSource(\mathcal{X})$ and let $\mathcal{X}$ have connected multicolimits. If $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A})$ is cowellpowered, then the multiorthogonal hull of $\mathcal{A}$ is multireflective and, thus, it is the multireflective hull of $\mathcal{A}$ in $\mathcal{X}$.*

Proof. Under the above hypotheses, $\mathbb{M}(\mathcal{A})$ has connected multicolimits. On the other hand, the cowellpowerdness of $\mathbb{M}(\mathcal{A})$ guarantees, by 20.5.1., that $X/\mathcal{A}^{\underline{\underline{\perp}}}$ has a terminal weakly set. Therefore, from 20.2, it follows that $\underline{\underline{\mathcal{O}}}(\mathcal{A})$ is multireflective. $\square$

Remark 20.7 Under the conditions of the above theorem, we conclude that, moreover, every $\mathcal{A}$-multireflection is just a set.

In the last section of the first chapter we studied the concept of firm classes of morphisms. Next, we extend the concept of firmness to conglomerates of sources.

Definition 20.8 A conglomerate $\mathcal{S}$ of sources is said to be *subfirm* provided that there exists an $\mathcal{S}$-multireflective subcategory $\mathcal{A}$ such that $\mathcal{S} \subseteq \mathcal{A}^{\underline{\underline{\perp}}}$. If, moreover, $\mathcal{S} = \mathcal{A}^{\underline{\underline{\perp}}}$, $\mathcal{S}$ is said to be *firm*.

Such a subcategory $\mathcal{A}$ is said to be *subfirmly* (respectively, *firmly*) *$\mathcal{S}$-multireflective* in $\mathcal{X}$.

Proposition 20.9 *A conglomerate $\mathcal{S}$ of sources is subfirm if and only if $\mathcal{S}_{\underline{\underline{\perp}}}$ is $\mathcal{S}$-multireflective and, in this case, $\mathcal{S}_{\underline{\underline{\perp}}}$ is the unique subfirmly $\mathcal{S}$-multireflective subcategory of $\mathcal{X}$.*

Proof. If $\mathcal{S}_{\underline{\underline{\perp}}}$ is $\mathcal{S}$-multireflective, then, since $\mathcal{S} \subseteq (\mathcal{S}_{\underline{\underline{\perp}}})^{\underline{\underline{\perp}}}$, $\mathcal{S}$ is subfirm.

Conversely, let $\mathcal{S}$ be subfirm. This means that there exists a $\mathcal{S}$-multireflective subcategory $\mathcal{A}$ of $\mathcal{X}$ such that $\mathcal{S} \subseteq \mathcal{A}^{\underline{\underline{\perp}}}$. Hence $\mathcal{A} = (\mathcal{A}^{\underline{\underline{\perp}}})_{\underline{\underline{\perp}}} \subseteq \mathcal{S}_{\underline{\underline{\perp}}}$. Now we show that the converse inclusion also holds and, consequently, $\mathcal{S}_{\underline{\underline{\perp}}}$ is the unique subfirmly $\mathcal{S}$-multireflective subcategory. Let $X \in \mathcal{S}_{\underline{\underline{\perp}}}$ and let $(r_i : X \to A_i)_{i\in I}$ be the $\mathcal{S}$-multireflection of X in $\mathcal{A}$. Then, there is some $i \in I$ and some $t : A_i \to X$ such that $t \cdot r_i = id_X$, because $X \in \mathcal{S}_{\underline{\underline{\perp}}}$. This implies $r_i \cdot t \cdot r_i = r_i$ and, consequently, $r_i \cdot t = id_{A_i}$. Thus r_i is an isomorphism and $X \in \mathcal{A}$. $\square$

Definitions 20.10 Let $\mathcal{S}$ be a conglomerate of sources in a multicocomplete category $\mathcal{X}$.

(a) We say that $\mathcal{S}$ fulfils the *multicoequalizer condition* provided that, given sources $(X \overset{f_i}{\to} Y_i)_I$ and $(X \overset{g_j}{\to} Z_j)_J$ in $\mathcal{S}$ and a family of pairs $(\alpha^k, (h_j^k)_J)$, indexed by K, such that $\alpha^k : J \to I$ is a function and $h_j^k : Y_{\alpha^k(j)} \to Z_j$ is an $\mathcal{X}$-morphism with

$$h_j^k \cdot f_{\alpha^k(j)} = g_j,$$

then for each $(i,j) \in I \times J$ the multiple multicoequalizer of the family

$$\{h_j^k : Y_i \to Z_j \mid \alpha^k(j) = i\}$$

belongs to $\mathcal{S}$.

(b) We say that $\mathcal{S}$ is *closed under multipushouts* provided that, given a source $(X \overset{f_i}{\to} Y_i)_I$ in $\mathcal{S}$ and an $\mathcal{X}$-morphism $X \overset{g}{\to} Z$, if, for each $i \in I$, $(Z \overset{g_{ik}}{\to} W_{ik})_{k\in K_i}$ is the multipushout of f_i along g, then the source $(Z \overset{g_{ik}}{\to} W_{ik})_{k\in K_i,\, i\in I}$ belongs to $\mathcal{S}$.

(c) We say that $\mathcal{S}$ is *closed under multiple multipushouts* provided that, given a family $\{T_k,\, k \in K\}$ of sources in $\mathcal{S}$ indexed by a set K, with

$$T_k = (X \overset{f_i^k}{\to} Y_i^k)_{i\in I_k},$$

if, for each $i = (i_k)_{k\in K}$ in $I = \Pi_{k\in K} I_k$, the source $(X \xrightarrow{g_i^j} Z_i^j)_{j\in J_i}$ is the multiple multipushout of $(X \xrightarrow{f_{i_k}^k} Y_{i_k}^k)_{k\in K}$, then the source

$$(X \xrightarrow{g_i^j} Z_i^j)_{j\in J_i, i\in I}$$

belongs to $\mathcal{S}$.

Remark 20.11 Comparing the above definition with 2.6, we see immediatly that (a) is equivalent to saying that $\Pi^l(\mathcal{S})$ fulfils the coequalizer condition in $\Pi^l(\mathcal{X})$, (b) is equivalent to saying that $\Pi^l(\mathcal{S})$ is closed under pushouts in $\Pi^l(\mathcal{X})$ and (c) is equivalent to saying that $\Pi^l(\mathcal{S})$ is closed under multiple pushouts in $\Pi^l(\mathcal{X})$.

Thus, from 1.4.4-5, 2.7.1 and 19.13, it follows that, given a subcategory $\mathcal{A}$ of $\mathcal{X}$, the conglomerate $\mathcal{A}^{\underline{\perp}}$ fulfils conditions (a), (b) and (c).

It is easy to see that $\Pi^l(\mathcal{S})$ contains all $\Pi^l(\mathcal{X})$-isomorphisms, is closed under composition and left-cancellable if and only if, respectively, $\mathcal{S}$ contains all $\mathcal{X}$-isomorphims, is closed under composition and left-cancellable. Hence, using 3.5, we obtain the following

Theorem 20.12 *If $\mathcal{X}$ is a multicocomplete category and $\mathcal{S}$ is a conglomerate of sources in $\mathcal{X}$, then $\mathcal{S}$ is firm if and only if the following conditions are fulfilled:*

1. *$Iso(\mathcal{X}) \subseteq \mathcal{S}$.*
2. *$\mathcal{S}$ is closed under composition.*
3. *$\mathcal{S}$ is left-cancellable.*
4. *$\mathcal{S}$ fulfils the multicoequalizer condition.*
5. *$\mathcal{S}$ is closed under multipushouts and multiple multipushouts.*
6. *For each $X \in Obj(\mathcal{X})$, the quasicategory $X/\mathcal{S}$ has a weakly terminal set.* □

Example 20.13 Let $\mathcal{X} = \mathcal{T}op^{op}$ and let $\mathcal{A} = \mathcal{C}on^{op}$. Then the firm conglomerate $\mathcal{A}^{\underline{\perp}}$ consists of all sources which are dual of episinks $(Y_j \xrightarrow{f_j} X)_J$ in $\mathcal{T}op$ such that

(a) f_j is an embedding, for all $j \in J$;

(b) $Im(f_j)$ are pairwise disjoint $(j \in J)$;

(c) each embedding f_j preserves connected components, that is, if C is a connected component of Y_j then $f_j(C)$ is a connected component of X.

In fact, for the class $\mathcal{M}$ of all embeddings in $\mathcal{T}op$ and $\mathbb{E}$ the conglomerate of all episinks in $\mathcal{T}op$, we have that $\mathcal{T}op^{op}$ is an $(\mathcal{M}^{op}, \mathbb{E}^{op})$-category. Since $\mathcal{C}on^{op}$ is $\mathbb{E}^{op}$-multireflective in $\mathcal{T}op$ (see 16.2.3), it follows, from the dual of 20.5, that

(i) $\mathcal{A}^{\underline{\perp}} \subseteq \mathbb{E}^{op}$, that is, each source in $\mathcal{A}^{\underline{\perp}}$ is the dual of an episink $(X \xrightarrow{f_j} Y_j)_J$ in $\mathcal{T}op^{op}$ and

(ii) such an episink $(f_j)_J$ fulfils condition (a).

The condition (b) is clear since, if $Im(f_i) \cap Im(f_j) \neq \emptyset$, then each connected component of that intersection may be simultaneously factorized through f_i and f_j. To show that $(f_j)_J$ satisfies (c), let $C \overset{c}{\hookrightarrow} Im(f_j)$ be the embedding of a connected component of $Im(f_j)$ and let C' be the connected component of X which contains C. Then, since $(f_j)_J \in \mathcal{A}^{\underline{\perp}}$, there exists $i \in J$ such that $C' \subseteq Im\, f_i$. But then $f_j \cdot c$ is simultaneously factorized trough f_j and f_i, which implies that $i = j$ and, consequently, $C' \subseteq Im(f_j)$. Hence, $C = C'$.

Conversely, let $(Y_j \hookrightarrow X)_J$ be an episink where each Y_j is a subspace of X, the subspaces Y_j are pairwise disjoint and each connected component in Y_j is a connected component in X. Let $C \xrightarrow{g} X$ be a continuous map from a connected space C to X. Then, since $(Y_j \hookrightarrow X)_J$ is an episink, we have that $g(C) \subseteq \cup_{j\in J} Y_j$. Hence, $g(C) \cap Y_j \neq \emptyset$ for some $j \in J$. Let C' be the connected component of X which contains $g(C)$; then $C' \cap Y_j \neq \emptyset$ and, by (c), one must have $C' \subseteq Y_j$; consequently $g(C) \subseteq Y_j$. The condition (b) assures that this j is the unique one such that g is factorizable through Y_j.

Remark 20.14 We point out that all results which we obtained on multiorthogonality remain true if we consistently interpret multireflections and multicolimits as being just indexed by sets, $\mathcal{A}^{\stackrel{\perp}{=}}$ is defined as consisting just of all small sources which are multi-orthogonal to $\mathcal{A}$ and the conglomerate of sources $\mathcal{S}$ considered in this section is assumed to contain small sources only.

21 A generalization of the orthogonal closure operator

Next, we consider a generalization of the orthogonal closure operator defined in Chapter II. We shall show that the orthogonal closure operator is also a useful tool in the investigation of the multireflectivity of the multiorthogonal hull of a given subcategory.

From now on, $\mathcal{X}$ is a category with multipushouts and $\mathcal{M}$ is a class of $\mathcal{X}$-monomorphisms which contains all isomorphisms, is closed under composition and such that $\mathcal{X}$ is $\mathcal{M}$-complete. Furthermore, $(\mathcal{E}, \mathcal{M})$ is the factorization structure for morphisms determined by the $\mathcal{M}$-completeness of $\mathcal{X}$.

Notations 21.1 Given a subcategory $\mathcal{A}$ of a category $\mathcal{X}$, we denote by

$$\mathcal{X}_{\mathcal{A}}$$

the subcategory of $\mathcal{X}$ of all $\mathcal{X}$-objects X such that $\mathcal{X}(X, \mathcal{A}) \neq \emptyset$.

For each class $\mathcal{M}$ of $\mathcal{X}$-morphisms,

$$\mathcal{M}_{\mathcal{A}} = \mathcal{M} \cap Mor(\mathcal{X}_{\mathcal{A}}).$$

Analogously, for each conglomerate $\mathbb{M}$ of sources in $\mathcal{X}$,

$$\mathbb{M}_{\mathcal{A}} = \mathbb{M} \cap Source(\mathcal{X}_{\mathcal{A}}).$$

Proposition 21.2 *For any subcategory $\mathcal{A}$ of a category $\mathcal{X}$,*

1. *$\mathcal{X}_{\mathcal{A}}$ is multireflective in $\mathcal{X}$.*

2. *If $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category, is $\mathcal{M}$-complete and has multipushouts, then the subcategory $\mathcal{X}_{\mathcal{A}}$ is an $(\mathcal{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$-category, it is $\mathcal{M}_{\mathcal{A}}$-complete and has multipushouts.*

Proof.

1. It is clear that, given an $\mathcal{X}$-object X, if it belongs to $\mathcal{X}_{\mathcal{A}}$, then the multireflection consists just of the identity 1_X; otherwise, the multireflection of X in $\mathcal{X}_{\mathcal{A}}$ is the empty source with domain X.

2. The fact that the $\mathcal{M}$-completeness of $\mathcal{X}$ implies the $\mathcal{M}_{\mathcal{A}}$-completeness of $\mathcal{X}_{\mathcal{A}}$ and that $\mathcal{X}_{\mathcal{A}}$ is an $(\mathcal{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$-category whenever $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category is a consequence of the following obvious property of $\mathcal{X}_{\mathcal{A}}$: If $X \xrightarrow{f} Y$ is an $\mathcal{X}$-morphism such that Y belongs to $X_{\mathcal{A}}$, then X also belongs to $\mathcal{X}_{\mathcal{A}}$. $\square$

In the sequel, the category $\mathcal{X}_{\mathcal{A}}$ plays an important rôle. By the above proposition, the question of the existence of a multireflection of each $\mathcal{X}$-object in $\mathcal{A}$ reduces to the one of the existence of a multireflection of each $\mathcal{X}_{\mathcal{A}}$-object in $\mathcal{A}$. Moreover, by 20.4, the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ coincides with the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{X}_{\mathcal{A}}$, and, then, we may restrict the study of the multirefletivity of the multiorhogonal hull of $\mathcal{A}$ in $\mathcal{X}$ to category $\mathcal{X}_{\mathcal{A}}$.

Definitions 21.3 Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$.

For each morphism $m : X \to Y$ in $\mathcal{M}_{\mathcal{A}}$, let $P_{\mathcal{A}}(m)$ be the class of all morphisms $n : N \to Y$ such that there are some morphism $g : X \to A$, with $A \in Obj(\mathcal{A})$, and $\overline{m}$, g', m', e such that $(\overline{m}, g')$ is a component of the multipushout of (m, g) in $\mathcal{X}_{\mathcal{A}}$, $\overline{m} = m' \cdot e$ is the $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorization of $\overline{m}$ and $n : N \to Y$ is the pullback of m' along g'.

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dr] \arrow[ddd, "g"] & & Y \arrow[ddd, "g'"] \\
& N \arrow[ur, "n"] \arrow[d] & \\
& \bullet \arrow[dr, "m'"] & \\
A \arrow[ur, "e"] \arrow[rr, "\overline{m}"] & & \bullet
\end{tikzcd}$$

It is clear that each morphism n in $P_{\mathcal{A}}(m)$ is an $\mathcal{M}_{\mathcal{A}}$-subobject of Y which contains m. Let

$$c_{\mathcal{A}}(m) = \bigwedge P_{\mathcal{A}}(m).$$

We denote by $d_{\mathcal{A}}(m)$ the unique morphism of $\mathcal{M}_{\mathcal{A}}$ such that $m = c_{\mathcal{A}}(m) \cdot d_{\mathcal{A}}(m)$.

Proposition 21.4 *For each subcategory $\mathcal{A}$ of $\mathcal{X}$, $c_{\mathcal{A}} : \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}}$ is a closure operator on $\mathcal{X}_{\mathcal{A}}$.*

Proof. By 4.3, it suffices to prove that, for each commutative diagram

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "p"] & Y \arrow[d, "f"] \\
Z \arrow[r, "n"] & W
\end{tikzcd} \tag{19}$$

with $m, n \in \mathcal{M}_{\mathcal{A}}$, there is a unique morphism f' such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "d_{\mathcal{A}}(m)"] \arrow[d, "p"] & \bullet \arrow[r, "c_{\mathcal{A}}(m)"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\
Z \arrow[r, "d_{\mathcal{A}}(n)"] & \bullet \arrow[r, "c_{\mathcal{A}}(n)"] & W
\end{tikzcd} \tag{20}$$

is commutative.

Let $(\overline{n}, h')$ be a component of a multipushout of n along some morphism $h : Z \to A$ with A in $\mathcal{A}$. Thus, the multipushout of m along $h \cdot p$ is non-empty, since the multipushout of n along h is non-empty. Moreover, by universality, there are a unique component $(\overline{m}, g')$ of the multipushout of $(m, h \cdot p)$ and a unique morphism d such that $d \cdot \overline{m} = \overline{n}$ and $d \cdot g' = h' \cdot f$. Let $n' \cdot q$ and $m' \cdot e$ be $(\mathcal{E}, \mathcal{M})$-factorizations of $\overline{n}$ and $\overline{m}$, respectively, let (s, h^*) be a pullback of (n', h') and let (r, g^*) be a pullback of (m', g'). The equality $n' \cdot q = (d \cdot m') \cdot e$ implies the existence of a unique morphism l such that $n' \cdot l = d \cdot m'$ and $l \cdot e = q$. Consequently,

$$n' \cdot (l \cdot g^*) = d \cdot m' \cdot g^* = d \cdot g' \cdot r = h' \cdot (f \cdot r)$$

and, since (s, h^*) is a pullback of (n', h'), there is a uniqe morphism t such that $s \cdot t = f \cdot r$ and $h^* \cdot t = l \cdot g^*$. We conclude that, for each $s \in P_{\mathcal{A}}(n)$, there are some $r \in P_{\mathcal{A}}(m)$ and a morphism t such that the following diagram

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dr] \arrow[ddd, "p"'] & & Y \arrow[dl, "r"] \arrow[ddd, "f"] \\
& \bullet \arrow[d, "t"] & \\
& \bullet & \\
Z \arrow[ur] \arrow[rr, "n"] & & W \arrow[ul, "s"']
\end{tikzcd}$$

is commutative. Therefore, since $c_{\mathcal{A}}(m)$ and $c_{\mathcal{A}}(n)$ are the intersections of, respectively, $P_{\mathcal{A}}(m)$ and $P_{\mathcal{A}}(n)$, this proves the existence of a unique morphism f' such that the diagram (20) is commutative. $\square$

Definition 21.5 The closure operator $c_{\mathcal{A}} : \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}}$ defined as above will be called *the orthogonal closure operator on $\mathcal{X}_{\mathcal{A}}$ with respect to $\mathcal{M}_{\mathcal{A}}$ induced by $\mathcal{A}$.*

Remarks 21.6

1. If $\mathcal{X}$ has pushouts and $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathcal{X}_{\mathcal{A}} = \mathcal{X}$, then $c_{\mathcal{A}} : \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}}$ is just the orthogonal closure operator defined in Chapter II (see 5.1 and 5.3).

2. Analogously to the orthogonal closure operator defined in Chapter II, for the present orthogonal closure operator $c_{\mathcal{A}}$ we have that, for subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$:

   (a) The closure operator $c_{\mathcal{A}}$ is discrete in the subclass of morphisms with domain in $\mathcal{A}$.

   (b) If $\mathcal{A} \subseteq \mathcal{B}$ then $c_{\mathcal{B}} \leq c_{\mathcal{A}}$.

   (c) If $SplitMono(\mathcal{X}_{\mathcal{A}}) \subseteq \mathcal{M}_{\mathcal{A}}$, then, for each pair $a, b : Y \to A$ of morphisms, with $A \in Obj(\mathcal{A})$ and each $X \xrightarrow{m} Y$ in $\mathcal{M}_{\mathcal{A}}$

$$a \cdot m = b \cdot m \Longrightarrow a \cdot c_{\mathcal{A}}(m) = b \cdot c_{\mathcal{A}}(m).$$

3. We may also conclude that, analogously to the orthogonal closure operator defined in Chapter II, for the present definition of orthogonal closure operator, we have that, if $RegMono(\mathcal{X}_{\mathcal{A}}) \subseteq \mathcal{M}_{\mathcal{A}}$, then:

   (a) $c_{\mathcal{A}} \leq r_{\mathcal{A}}$, where $r_{\mathcal{A}}$ denotes the regular closure operator with respect to $\mathcal{M}_{\mathcal{A}}$ induced by $\mathcal{A}$;

   (b) every $c_{\mathcal{A}}$-dense morphism in $\mathcal{M}_{\mathcal{A}}$ is $\mathcal{A}$-cancellable.

22 Density and multiorthogonality

For the rest of this chapter, we assume that the category $\mathcal{X}$ (which is supposed to have multipushouts and be $\mathcal{M}$-complete) is, furthermore, an $(\mathcal{E}, \mathbb{M})$-category, with $\mathbb{M} \subseteq MonoSource(\mathcal{X})$ and $\mathcal{M} = \mathbb{M} \cap Mor(\mathcal{X})$.

Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ and let $\mathbb{M}(\mathcal{A})$ be the $\mathcal{E}$-reflective hull of $\mathcal{A}$ in $\mathcal{X}$.

Since, by 20.4, the multiorthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ coincides with the multiorthogonal hull of $\mathcal{A}$ in $\mathbb{M}(\mathcal{A})$, in the present sequel we often assume that $\mathcal{X} = \mathbb{M}(\mathcal{A})$, which clearly implies that $\mathcal{X}_{\mathcal{A}} = \mathbb{M}_{\mathcal{A}}(\mathcal{A})$.

On the other hand, we recall that in $\mathbb{M}(\mathcal{A})$ every $\mathcal{A}$-cancellable morphism is an epimorphism, by 2.17. Thus, from 21.6.3(b), it follows that:

Every $c_{\mathcal{A}}$-dense morphism in $\mathcal{X}_{\mathcal{A}}$ is an epimorphism.

This fact will be often used.

Definition 22.1 Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. An $\mathcal{X}$-source $(f_i : X \to Y_i)_{i\in I}$ is said to be $\mathcal{A}$-*disjoint* provided that, for each pair (i, j) in I^2 with $i \neq j$, there is no commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f_i} & Y_i \\ {\scriptstyle f_j}\downarrow & & \downarrow \\ Y_j & \longrightarrow & A \end{array}$$

with $A \in Obj(\mathcal{A})$.

Remark 22.2 It is clear that in Definition 22.1, we may equivalently replace "$A \in Obj(\mathcal{A})$" by "$A \in Obj(\mathcal{X}_\mathcal{A})$". Furthermore, if $(f_i : X \to Y_i)_{i\in I}$ is an $\mathcal{A}$-disjoint $\mathcal{X}_\mathcal{A}$-source, then, for each pair $(i, j) \in I^2$ with $i \neq j$, the pair (f_i, f_j) has an empty multipushout in $\mathcal{X}_\mathcal{A}$.

Definitions 22.3 Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, let us consider the following classes and conglomerates:

$PC(\mathcal{M}_\mathcal{A})$ consists of all $\mathcal{X}_\mathcal{A}$-morphisms f such that all components of a multipushout in $\mathcal{X}_\mathcal{A}$ of f along a morphism with codomain in $\mathcal{A}$ belong to $\mathcal{M}_\mathcal{A}$.

$PS(\mathcal{M}_\mathcal{A})$ is the intersection of $PC(\mathcal{M}_\mathcal{A})$ with $\mathcal{M}_\mathcal{A}$.

$PS(\mathbb{M}_\mathcal{A})$ consists of all sources $(X, (f_i)_I) \in \mathbb{M}_\mathcal{A}$ such that each f_i belongs to $PC(\mathcal{M}_\mathcal{A})$ and for each morphism g with domain X and codomain in $\mathcal{A}$ there is some $i \in I$ such that the multipushouts of f_i along g in $\mathcal{X}_\mathcal{A}$ is non-empty.

Remarks 22.4

1. It is clear that each morphism of a $PS(\mathbb{M}_\mathcal{A})$-source belongs to $PC(\mathcal{M}_\mathcal{A})$.
2. It is obvious that, if $\mathcal{X}_\mathcal{A}$ has pushouts, then $PS(\mathcal{M}_\mathcal{A})$ consists precisely of all disjoint $PS(\mathbb{M}_\mathcal{A})$-sources and this conglomerate is just the class $PS(\mathcal{M}_\mathcal{A})$ in the sense used in Chapter II.

The folllowing two lemmas collect some properties on multipushouts which will be very useful in the sequel.

Lemma 22.5

1. *If f, g, $\overline{f}$, $\overline{g}$, f' and g' are $\mathcal{X}$-morphisms such that $(\overline{f}, \overline{g})$ is a component of the multipushout of (f, g) and (f', g') is the pullback of $(\overline{f}, \overline{g})$, then $(\overline{f}, \overline{g})$ is also a component of the multipushout of (f', g').*

2. *Let the diagram*

$$\begin{tikzcd}
\bullet \arrow[rr, "f_1"] \arrow[dddd, "g"'] \arrow[dr, "e_1"] & & \bullet \arrow[dddd, "\overline{g}"] \\
& \bullet \arrow[ur, "m_1"'] \arrow[dd, "h"] & \\
& & \\
& \bullet \arrow[dr, "m_2"] & \\
\bullet \arrow[ur, "e_2"] \arrow[rr, "f_2"'] & & \bullet
\end{tikzcd}$$

be commutative and let e_1 and e_2 be epimorphisms. If $(f_2, \overline{g})$ is a component of the multipushout of (f_1, g), then $(m_2, \overline{g})$ is a component of the multipushout of (m_1, h); the converse is true if e_2 is an isomorphism.

Proof.

1. In the following diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "g"'] \arrow[dr, "d"] & & Y \arrow[dd, "\overline{g}"] \\
& P \arrow[ur, "f'"'] \arrow[dl, "g'"] & \\
Z \arrow[rr, "\overline{f}"'] & & W
\end{tikzcd}$$

let $(\overline{f}, \overline{g})$ be a component of the multipushout of (f, g), let (f', g') be the pullback of $(\overline{f}, \overline{g})$ and let d be the unique morphism which turns both the smaller triangles commutative. Then there is a unique component (f^*, g^*) of the multipushout of (f', g') and a unique morphism t such that $t \cdot g^* = \overline{g}$ and $t \cdot f^* = \overline{f}$. But then (f^*, g^*) belongs to the same connected component as $(\overline{f}, \overline{g})$ in the category of all natural sinks from the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \\ Z & & \end{array} .$$

Hence, there is a unique morphism t' such that $t' \cdot \overline{f} = f^*$ and $t' \cdot \overline{g} = g^*$. Now, since $(\overline{f}, \overline{g})$ and (f^*, g^*) are components of multipushouts, they are episinks and, consequently, from the equalities

$$t \cdot t' \cdot \overline{f} = t \cdot f^* = \overline{f} \quad \text{and} \quad t \cdot t' \cdot \overline{g} = t \cdot g^* = \overline{g}$$

$$t' \cdot t \cdot f^* = t' \cdot \overline{f} = f^* \quad \text{and} \quad t' \cdot t \cdot g^* = t' \cdot \overline{g} = g^*$$

it follows that t is an isomorphism.

2. Under the given conditions, let $(f_2, \overline{g})$ be a component of the multipushout of (f_1, g). We want to show that $(m_2, \overline{g})$ is a component of the multipushout of (m_1, h). Since $m_2 \cdot h = \overline{g} \cdot f_1$, there are a unique component $(\hat{m}, \hat{h})$ of the multipushout of (m_1, h) and a unique morphism t such that $t \cdot \hat{h} = \overline{g}$ and $t \cdot \hat{m} = m_2$. Since $\hat{m} \cdot h = \hat{h} \cdot m_1$, we have that

$$(\hat{m} \cdot e_2) \cdot g = \hat{m} \cdot h \cdot e_1 = \hat{h} \cdot m_1 \cdot e_1 = \hat{h} \cdot f_1.$$

On the other hand, since

$$t \cdot (\hat{m} \cdot e_2) = m_2 \cdot e_2 = f_2 \quad \text{and} \quad t \cdot \hat{h} = \overline{g},$$

it follows that $(\hat{m} \cdot e_2, \hat{h})$ and $(f_2, \overline{g})$ belong to the same conneted component of the category of all natural sinks from the diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{f_1} & Y \\ \downarrow{\scriptstyle g} & & \\ \bullet & & \end{array} .$$

Therefore, there is a unique u such that $u \cdot f_2 = \hat{m} \cdot e_2$ and $u \cdot \overline{g} = \hat{h}$. It follows that $u \cdot t = 1$ and $t \cdot u = 1$, so that t is an isomorphism as we wanted to prove.

Now, let e_2 be an isomorphism and let $(m_2, \overline{g})$ be a component of the multipushout of (m_1, h). Let $((\hat{f}, \hat{g}), s)$ be the unique pair such that $(\hat{f}, \hat{g})$ is a component of the multipushout of (f_1, g) and s fulfils the equalities $s \cdot \hat{f} = f_2$ and $s \cdot \hat{g} = \overline{g}$. Then, since e_1 is an epimorphism, it turns out that $(s \cdot f_2 \cdot e_2^{-1}) \cdot h = (s \cdot \hat{g}) \cdot m_1$. Now, it is easy to see that s is an isomorphism and, consequently, $(f_2, \overline{g})$ is a component of the multipushout of (f_1, g). □

Lemma 22.6 *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ and let $f : X \to Y$ be an $\mathcal{X}_{\mathcal{A}}$-morphism in $PC(\mathcal{M}_{\mathcal{A}})$. Then*

1. *Every component of a multipushout in $\mathcal{X}_{\mathcal{A}}$ of f along another morphism belongs to $PC(\mathcal{M}_{\mathcal{A}})$.*
2. *If $m \cdot e$ is an $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorization of f, then $m \in PS(\mathcal{M}_{\mathcal{A}})$.*
3. *If f is $c_{\mathcal{A}}$-dense, then every component of a multipushout in $\mathcal{X}_{\mathcal{A}}$ of f along another morphism is $c_{\mathcal{A}}$-dense.*

Proof.

1. Let $(\overline{f}, \overline{g})$ be a component of the multipushout of f along an $\mathcal{X}_{\mathcal{A}}$-morphism $g : X \to Z$. Let $(\overline{\overline{f}}, \overline{h})$ be a component of the multipushout of $\overline{f}$ along a morphism $h : Z \to A$ with codomain in $\mathcal{A}$. Then it is easily seen that $(\overline{\overline{f}}, \overline{h} \cdot \overline{g})$ is a component of the multipushout of f along $h \cdot g$. Consequently, $\overline{\overline{f}} \in \mathcal{M}_{\mathcal{A}}$.
2. Let $g : Z \to A$, where Z is the domain of m, be a morphism with codomain in $\mathcal{A}$. If $(\overline{m}, \overline{g})$ is a component of the multipushout of m along g then, using the fact that e is an epimorphism, it is easy to conclude that $(\overline{m}, \overline{g})$ is also a component of the multipushout of f along $g \cdot e$. Therefore, since $f \in PC(\mathcal{M}_{\mathcal{A}})$, we have that $\overline{m} \in \mathcal{M}_{\mathcal{A}}$.
3. Let $f : X \to Y$ be $c_{\mathcal{A}}$-dense and let $(\overline{f}, \overline{g})$ be a component of the multipushout of f along $g : X \to Z$. Let $X \xrightarrow{e} E \xrightarrow{m} Y$ and $Z \xrightarrow{d} E \xrightarrow{n} W$ be $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorizations of f and $\overline{f}$, respectively. Then there exists a unique h such that $n \cdot h = \overline{g} \cdot m$ and $d \cdot g = h \cdot e$. Thus, by 22.5.2, $(n, \overline{g})$ is a component of the multipushout of (m, h). We show that n is $c_{\mathcal{A}}$-dense. Let $l : D \to A$ be a morphism with codomain in $\mathcal{A}$, let $(\overline{n}, \overline{l})$ be a component of the multipushout of (n, l) and let $(\hat{n}, \hat{l})$ be the pullback of $(\overline{n}, \overline{l})$. We want to show that $\hat{n}$ is an isomorphism. Let $(\hat{m}, \hat{g})$ be the pullback of $(\hat{n}, \overline{g})$; then $(\hat{m}, \hat{l} \cdot \hat{g})$ is the pullback of $(\overline{n}, \overline{l} \cdot \overline{g})$. But then, since $(\overline{n}, \overline{l} \cdot \overline{g})$ is a component of the multipushout of $(m, l \cdot h)$ and m is $c_{\mathcal{A}}$-dense, $\hat{m}$ must be an isomorphism; hence, by 22.5.1, $\overline{n}$ must be an isomorphism, and so is $\hat{n}$. □

Definitions 22.7 Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$, let $(X \xrightarrow{f_i} Y_i)_I$ be a source in $\mathcal{X}_{\mathcal{A}}$ and let $m_i \cdot e_i$ be an $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorization of f_i, for each i.

1. The source $(X \xrightarrow{f_i} Y_i)_I$ is said to be $c_{\mathcal{A}}$-*dense* if each morphism m_i is $c_{\mathcal{A}}$-dense.

2. The orthogonal closure operator $c_{\mathcal{A}}$ is said to be *weakly hereditary with respect to a conglomerate* $\mathbb{N} \subseteq \mathbb{M}_{\mathcal{A}}$ whenever, for each $(f_i)_I \in \mathbb{N}$, the source $(d_{\mathcal{A}}(m_i) \cdot e_i)_I$ is $c_{\mathcal{A}}$-dense and belongs to $\mathbb{N}$.

Proposition 22.8 *If $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, then the class $\mathcal{A}^{\underline{\perp}}$ in $\mathcal{X}_{\mathcal{A}}$ consists precisely of all $c_{\mathcal{A}}$-dense sources in $PS(\mathbb{M}_{\mathcal{A}})$ which are $\mathcal{A}$-disjoint.*

Proof. Let $(f_i : X \to Y_i)_{i \in I}$ belong to $\mathcal{A}^{\underline{\perp}}$ in $\mathcal{X}_{\mathcal{A}}$. Then $(f_i)_I$ belongs to $\mathbb{M}_{\mathcal{A}}$ (by 20.5) and it is clearly $\mathcal{A}$-disjoint. To show that $(f_i)_I$ belongs to $PS(\mathbb{M}_{\mathcal{A}})$, let $i \in I$, let $X \xrightarrow{g} A$ be a morphism with codomain in $\mathcal{A}$ and let $(A \xrightarrow{\overline{f}} W, Y_i \xrightarrow{\overline{g}} W)$ be a component of the multipushout of (f_i, g) in $\mathcal{X}_{\mathcal{A}}$ for some $i \in I$. Then, since $W \in Obj(\mathcal{X}_{\mathcal{A}})$, there is some morphism $h : W \to A'$ with codomain in $\mathcal{A}$. But then g and $h \cdot \overline{g} \cdot f_i$ belong to the same connected component of $X \downarrow \mathcal{A}$; hence, since $h \cdot \overline{g} \cdot f_i$ is factorizable through f_i, the morphism g is also factorizable through f_i, by 19.2. This implies that there is a morphism $W \xrightarrow{t} A$ such that $t \cdot \overline{f} = 1_A$. But, on the other hand, f_i is $\mathcal{A}$-cancellable, then, by 2.17, it is an epimorphism and, thus, $\overline{f}$ is also an epimorphism. Therefore, $\overline{f}$ is an isomorphism. This shows that $(f_i)_I$ lies in $PS(\mathbb{M}_{\mathcal{A}})$. To show that each f_i is $c_{\mathcal{A}}$-dense, let $X \xrightarrow{e_i} X_i \xrightarrow{m_i} Y_i$ be an $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorization of f_i and let $X_i \xrightarrow{h} A$ be a morphism with codomain in $\mathcal{A}$. Let $(\overline{m}, \overline{h})$ be a component of the multipushout of m_i along h, in $\mathcal{X}_{\mathcal{A}}$. Then by 22.6.2, $\overline{m} \in \mathcal{M}_{\mathcal{A}}$. Consequently, by 22.5.2, $(\overline{m}, \overline{h})$ is also a component of the multipushout of f_i along $h \cdot e_i$. Hence, as we have shown above, $\overline{m}$ must be an isomorphism, and, thus, the pullback of $\overline{m}$ along $\overline{h}$ is also an isomorphism. Since $P_{\mathcal{A}}(m_i)$ consists of isomorphisms only, it follows that $c_{\mathcal{A}}(m_i)$ is an isomorphism, that is, m_i is $c_{\mathcal{A}}$-dense. Furthermore, since the multipushout of m_i along g is also the multipushout of m_i along $g \cdot e$ (by 22.5.2) and the pullback of an isomorphism is an isomorphism, it follows hat each f_i is $c_{\mathcal{A}}$-dense.

Conversely, let $(f_i : X \to Y_i)_{i \in I}$ be an $\mathcal{A}$-disjoint, $c_{\mathcal{A}}$-dense source in $PS(\mathbb{M}_{\mathcal{A}})$ and let $g : X \to A$ be a morphism with codomain in $\mathcal{A}$. Then there is some $i \in I$ such that the multipushout of f_i along g is non-empty and, furthermore, each of its components belongs to $\mathcal{M}_{\mathcal{A}}$. Let $(\overline{f}, \overline{g})$ be a component of this multipushout and let $X \xrightarrow{e_i} X_i \xrightarrow{m_i} Y_i$ be an $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorization of f_i. Then, since $\overline{f} \in \mathcal{M}_{\mathcal{A}}$, there is a morphism t such that $t \cdot e_i = g$ and $\overline{f} \cdot t = \overline{g} \cdot m_i$. Thus, from 22.5.2, $(\overline{f}, \overline{g})$ is a component of the multipushout of m_i along t. Consequently, since m_i is $c_{\mathcal{A}}$-dense, the pullback of $\overline{f}$ along $\overline{g}$ is an isomorphism and, hence, from 22.5.1, $\overline{f}$ is also an isomorphism. Consequently, g is factorizable through f_i. Moreover, $\overline{f}^{-1} \cdot \overline{g}$ is the unique morphism such that $g = (\overline{f}^{-1} \cdot \overline{g}) \cdot f_i$, since the fact that f_i is $c_{\mathcal{A}}$-dense implies that it is an epimorphism (by 21.6.3(b) and 2.17). On the other hand, the $\mathcal{A}$-disjointness of $(f_i)_I$ ensures that there is a unique i such that g is factorizable through f_i. $\square$

23 Closedness and multireflectivity

In this section, we find conditions for the multiorthogonal hull of a subcategory to be its multireflective hull and we characterize such a multireflective hull in terms of closedness via the orthogonal closure operator.

Definition 23.1 Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, an $\mathcal{X}_{\mathcal{A}}$-object X is said to be $\mathcal{A}$-*strongly multiclosed* provided that, for each source $(f_i : X \to Y_i)_{i \in I}$ in $PS(\mathbb{M}_{\mathcal{A}})$, all f_i are $c_{\mathcal{A}}$-closed $\mathcal{M}_{\mathcal{A}}$-morphisms.

We denote by $\underline{\underline{\underline{\mathcal{SC}l}}}(\mathcal{A})$ the subcategory of $\mathcal{X}_{\mathcal{A}}$ of all $\mathcal{A}$-strongly multiclosed objects.

Remark 23.2 If $\mathcal{X}_{\mathcal{A}}$ has pushouts, then $\underline{\underline{\underline{\mathcal{SC}l}}}(\mathcal{A})$ is the subcategory $\mathcal{SC}l(\mathcal{A})$ of all $\mathcal{A}$-strongly closed objects in $\mathcal{X}_{\mathcal{A}}$, as defined in Chapter II.

Proposition 23.3 *Given a subcategory $\mathcal{A}$ of $\mathcal{X}$, we have that:*

1. $\mathcal{A} \subseteq \underline{\underline{\underline{\mathcal{SC}l}}}(\mathcal{A})$;

2. *If* $\mathbb{M}_{\mathcal{A}}(\mathcal{A}) = \mathcal{X}_{\mathcal{A}}$, *then* $\underline{\underline{\underline{\mathcal{SC}l}}}(\mathcal{A}) \subseteq \underline{\underline{\mathcal{O}}}(\mathcal{A})$;

3. *If $\mathcal{A}$ is* $\mathbb{M}$-*multireflective in* $\mathcal{X}$, *then* $\mathcal{A} = \underline{\underline{\underline{\mathcal{SC}l}}}(\mathcal{A}) = \underline{\underline{\mathcal{O}}}(\mathcal{A})$.

Proof.

1. Let $A \in Obj(\mathcal{A})$ and let $(A \xrightarrow{f_i} Y_i)_I$ be a source in $PS(\mathbb{M}_\mathcal{A})$. Then, for some $i \in I$, the multipushout of f_i along 1_A is non-empty. Let $(\overline{m}, d)$ be a component of such a multipushout. If $A \xrightarrow{e_i} X_i \xrightarrow{m_i} Y_i$ is an $(\mathcal{E}_\mathcal{A}, \mathbb{M}_\mathcal{A})$-factorization of f_i, we get the equality
$$(d \cdot m_i) \cdot e_i = \overline{m} \cdot 1_A.$$
Consequently, there is a unique morphism t such that $t \cdot e_i = 1_A$ and, thus, e_i is an isomorphism. Hence, $f_i \in \mathcal{M}_\mathcal{A}$. From 21.6.2(a), it is clear that f_i is also $c_\mathcal{A}$-closed.

2. Let us consider, in $\mathcal{X}_\mathcal{A}$, an $\mathcal{A}$-strongly multiclosed object X, a source $(Y \xrightarrow{f_i} Z_i)_I$ multiorthogonal to $\mathcal{A}$ and a morphism $g : Y \to X$. For each $i \in I$, let $((h_i^k, d_i^k))_{k\in K_i}$ be the multipushout in $\mathcal{X}_\mathcal{A}$ of f_i along g. The family $(h_i^k)_{k\in K_i, i\in I}$ is non-empty, since $X \in \mathcal{X}_\mathcal{A}$. Moreover, the source $(h_i^k)_{k\in K_i, i\in I}$ belongs to $\mathcal{A}^{\underline{\perp}}$ (by 20.11), then it also belongs to $PS(\mathbb{M}_\mathcal{A})$ (by 22.8). Now, since X is $\mathcal{A}$-strongly multiclosed, we have that all h_i^k are $c_\mathcal{A}$-closed $\mathcal{M}_\mathcal{A}$-morphisms. On the other hand, since $(h_i^k)_{k\in K_i, i\in I} \in \mathcal{A}^{\underline{\perp}}$, every morphism h_i^k is $c_\mathcal{A}$-dense, by 22.8. Being $c_\mathcal{A}$-closed and $c_\mathcal{A}$-dense, h_i^k is an isomorphism and, consequently, $g : Y \to X$ is factorizable through f_i. It is clear that there is a unique such i, since $(f_i)_I$ is $\mathcal{A}$-disjoint. And the factorization is unique since f_i is $c_\mathcal{A}$-dense, thus it is an epimorphism.

3. If $\mathcal{A}$ is $\mathbb{M}$-multireflective, then $\mathcal{A} = \underline{\underline{\mathcal{O}}}(\mathcal{A})$ (by 19.3) and, on the other hand, from 1. and 2., we have that $\mathcal{A} \subseteq \underline{\underline{\mathcal{SCl}}}(\mathcal{A}) \subseteq \underline{\underline{\mathcal{O}}}(\mathcal{A})$. Therefore, it follows that $\mathcal{A} = \underline{\underline{\mathcal{SCl}}}(\mathcal{A}) = \underline{\underline{\mathcal{O}}}(\mathcal{A})$. □

Theorem 23.4 *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ such that:*

1. *$\mathbb{M}_\mathcal{A}(\mathcal{A}) = \mathcal{X}_\mathcal{A}$;*

2. *$c_\mathcal{A}$ is weakly hereditary with respect to $PS(\mathbb{M}_\mathcal{A})$;*

3. *for each object X in $\mathcal{X}_\mathcal{A}$, there is some source $(f_i : X \to Y_i)_I$ in $PS(\mathbb{M}_\mathcal{A})$ with $Y_i \in \underline{\underline{\mathcal{SCl}}}(\mathcal{A})$ for all $i \in I$.*

Then $\underline{\underline{\mathcal{O}}}(\mathcal{A})$ coincides with $\underline{\underline{\mathcal{SCl}}}(\mathcal{A})$ and it is a multireflective hull of $\mathcal{A}$ in $\mathcal{X}$.

Proof. First, we show that

(i) $c_\mathcal{A}$ is weakly hereditary with respect to $PS(\mathcal{M}_\mathcal{A})$,

(ii) $c_{\mathcal{A}}$ preserves $PS(\mathcal{M}_{\mathcal{A}})$- morphisms,

(iii) the orthogonal closure operator

$$c_{\mathcal{A}} : PS(\mathcal{M}_{\mathcal{A}}) \to PS(\mathcal{M}_{\mathcal{A}})$$

is idempotent and weakly hereditary.

In fact,

(i) Let $m : X \to Y$ be a $PS(\mathcal{M}_{\mathcal{A}})$-morphism, then the source consisting of the morphisms m and 1_X belongs to $PS(\mathbb{M}_{\mathcal{A}})$; hence, by hypothesis, the source $(X,\ (d_{\mathcal{A}}(m), 1_X))$ is $c_{\mathcal{A}}$-dense and, so, $d_{\mathcal{A}}(m)$ is $c_{\mathcal{A}}$-dense.

(ii) Let $m : X \to Y$ be a $PS(\mathcal{M}_{\mathcal{A}})$-morphism and let $X \overset{d_{\mathcal{A}}(m)}{\longrightarrow} \overline{X} \overset{c_{\mathcal{A}}(m)}{\longrightarrow} Y$ be the factorization determined by $c_{\mathcal{A}}$. Let $g : \overline{X} \to A$ be a morphism with codomain in $\mathcal{A}$ and let $(\overline{m}, \overline{g})$ be a component of the multipushout of $c_{\mathcal{A}}(m)$ along g. Since $d_{\mathcal{A}}(m)$ is $c_{\mathcal{A}}$-dense, it is an epimorphism and, then, from 22.5.2, $(\overline{m}, \overline{g})$ is a component of a multipushout of m along $g \cdot d_{\mathcal{A}}(m)$, hence $\overline{m} \in \mathcal{M}_{\mathcal{A}}$; consequently, $c_{\mathcal{A}}(m) \in PS(\mathcal{M}_{\mathcal{A}})$.

(iii) It remains to show that $c_{\mathcal{A}} : PS(\mathcal{M}_{\mathcal{A}}) \to PS(\mathcal{M}_{\mathcal{A}})$ is idempotent and, by 4.4, it suffices to show that the class of all $c_{\mathcal{A}}$-dense $PS(\mathcal{M}_{\mathcal{A}})$-morphisms is closed under composition. Let $X \overset{m}{\to} Y$ and $Y \overset{n}{\to} Z$ be $c_{\mathcal{A}}$-dense $PS(\mathcal{M}_{\mathcal{A}})$-morphisms and let $X \overset{g}{\to} A$ be a morphism with codomain in $\mathcal{A}$. Let (r, s) be a component of the multipushout of $(n \cdot m, g)$. We show that, on the one hand, $r \in \mathcal{M}_{\mathcal{A}}$, and, on the other hand, the pullback of r along s is an isomorphism. Let $((\overline{m}, \overline{g}),\ t)$ be the unique pair such that $(\overline{m}, \overline{g})$ is a component of the multipushout of (m, g) and t is a morphism such that $t \cdot \overline{m} = r$ and $t \cdot \overline{g} = s \cdot n$. Then, since $\overline{m} \in \mathcal{M}_{\mathcal{A}}$ and m is $c_{\mathcal{A}}$-dense, the pullback of $\overline{m}$ along $\overline{g}$ is an isomorphism and, hence, by 22.5.1, it follows that $\overline{m}$ is an isomorphism. Now, since m is an epimorphism, because it is $c_{\mathcal{A}}$-dense, by 22.5.2, it turns out that (r, s) is a component of the multipushout of $(n, \overline{m}^{-1} \cdot \overline{g})$. Thus, since the codomain of the morphism $\overline{m}^{-1} \cdot \overline{g}$ lies in $\mathcal{A}$ and $n \in PS(\mathcal{M}_{\mathcal{A}})$, we have that $r \in \mathcal{M}_{\mathcal{A}}$. On the other hand, the fact that n is $c_{\mathcal{A}}$-dense implies that the pullback of r along s is an isomorphism. Thus, $P_{\mathcal{A}}(n \cdot m)$ consists of isomorphisms only and, consequently, $n \cdot m$ is $c_{\mathcal{A}}$-dense.

The inclusion $\underline{\underline{\mathcal{SC}l}}(\mathcal{A}) \subseteq \underline{\underline{\mathcal{O}}}(\mathcal{A})$ is clear from 23.3.2.

We now show that $\underline{\underline{\mathcal{SC}l}}(\mathcal{A})$ is multireflective in $\mathcal{X}_{\mathcal{A}}$, from which it follows that $\underline{\underline{\mathcal{SC}l}}(\mathcal{A}) = \underline{\underline{\mathcal{O}}}(\mathcal{A})$ and $\underline{\underline{\mathcal{O}}}(\mathcal{A})$ is the multireflective hull of $\mathcal{A}$ in $\mathcal{X}$. Let X be an object in $\mathcal{X}_{\mathcal{A}}$ and let $(f_i : X \to Y_i)_I$ belong to $PS(\mathbb{M}_{\mathcal{A}})$ with $Y_i \in \underline{\underline{\mathcal{SC}l}}(\mathcal{A})$ for all $i \in I$. Let $X \overset{e_i}{\to} X_i \overset{m_i}{\to} Y_i$ be an $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorization of each f_i and let $m_i = X_i \overset{d_{\mathcal{A}}(m_i)}{\longrightarrow} \overline{X}_i \overset{c_{\mathcal{A}}(m_i)}{\longrightarrow} Y_i$.

First, we show that, for each $i \in I$, $\overline{X}_i$ is an object of $\underline{\underline{\mathcal{SC}l}}(\mathcal{A})$. Given $i \in I$, let $(\overline{X}_i \overset{g_j}{\to} Z_j)_J$ belong to $PS(\mathbb{M}_{\mathcal{A}})$. The family $((h_j^k, d_j^k))_{k \in K_j, j \in J}$, where $(h_j^k, d_j^k))_{k \in K_j}$ is the multipushout of g_j along $c_{\mathcal{A}}(m_i)$ in $\mathcal{X}_{\mathcal{A}}$, is non-empty, since $(g_j)_J$ belongs to $PS(\mathbb{M}_{\mathcal{A}})$ and $Y_i \in Obj(\mathcal{X}_{\mathcal{A}})$. Furthermore, using 22.6.1, it is obvious that the source $(h_j^k)_{k \in K_j, j \in J}$ belongs to $PS(\mathbb{M}_{\mathcal{A}})$. Then, since Y_i belongs to $\underline{\underline{\mathcal{SC}l}}(\mathcal{A})$, all h_j^k are $c_{\mathcal{A}}$-closed $\mathcal{M}_{\mathcal{A}}$-morphisms. For each $j \in J$, let $g_j = n_j \cdot e_j$ be the $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$-factorization of g_j. Now, for each $j \in J$ and each $k \in K_j$, since $h_j^k \in \mathcal{M}_{\mathcal{A}}$ and $e_j \in \mathcal{E}_{\mathcal{A}}$, it follows, from 4.3, that there is a unique morphism t such that

$$h_j^k \cdot t = d_j^k \cdot n_j \tag{21}$$

$$\text{and} \qquad t \cdot e_j = c_{\mathcal{A}}(m_i). \tag{22}$$

From the equality (21), it follows that $h_j^k \cdot t = d_j^k \cdot c_{\mathcal{A}}(n_j) \cdot d_{\mathcal{A}}(n_j)$ and thus, since h_j^k is $c_{\mathcal{A}}$-closed, there is a unique morphism t' such that $h_j^k \cdot t' = d_j^k \cdot c_{\mathcal{A}}(n_j)$ and $t' \cdot d_{\mathcal{A}}(n_j) = t$.

$$\begin{tikzcd}
\overline{X}_i \arrow[rrr, "g_j"] \arrow[dr, "e_j"] \arrow[dd, "c_{\mathcal{A}}(m_i)"'] & & & Z_j \arrow[dd, "d_j^k"] \\
& \bullet \arrow[r, "d_{\mathcal{A}}(n_j)"] \arrow[dl, "t"'] & \bullet \arrow[ur, "c_{\mathcal{A}}(n_j)"'] \arrow[dll, "t'"] & \\
Y_i \arrow[rrr, "h_j^k"] & & & \bullet
\end{tikzcd}$$

Consequently, we have that

$$c_{\mathcal{A}}(m_i) = t \cdot e_j = t' \cdot d_{\mathcal{A}}(n_j) \cdot e_j. \tag{23}$$

Now, the fact that, in the equality (22), $e_j \in \mathcal{E}_{\mathcal{A}}$ and $c_{\mathcal{A}}(m_i) \in \mathcal{M}_{\mathcal{A}}$ implies that e_j is an isomorphism, because $(\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$ is a factorizaton system for morphisms and $\mathcal{E}_{\mathcal{A}} \subseteq$

$Epi(\mathcal{X}_\mathcal{A})$. Hence, it follows that

$$c_\mathcal{A}(m_i) \cdot e_j^{-1} = t' \cdot d_\mathcal{A}(n_j). \tag{24}$$

On the other hand, the fact that $c_\mathcal{A} : PS(\mathcal{M}_\mathcal{A}) \to PS(\mathcal{M}_\mathcal{A})$ is an idempotent, weakly hereditary closure operator implies that $d_\mathcal{A}(n_j)$ is $c_\mathcal{A}$-dense (so, it is an epimorphism) and, further, that $c_\mathcal{A}(m_i)$ is $c_\mathcal{A}$-closed.

Then, from the equality 24, there is a unique morphism s such that

$$s \cdot d_\mathcal{A}(n_j) = e_j^{-1}.$$

This equality implies that the epimorphism $d_\mathcal{A}(n_j)$ is also an isomorphism, that is, n_j is $c_\mathcal{A}$-closed. Consequently, since e_j is an isomorphism, g_j is a $c_\mathcal{A}$-closed $\mathcal{M}_\mathcal{A}$-morphism. Thus, $\overline{X}_i$ belongs to $\underline{\underline{\mathcal{SCl}}}(\mathcal{A})$.

Therefore, since $c_\mathcal{A}$ is weakly hereditary with respect to $PS(\mathbb{M}_\mathcal{A})$, the source

$$(X \stackrel{d_i}{\to} \overline{X}_i)_I \;=\; (X \stackrel{e_i}{\to} X_i \stackrel{d_\mathcal{A}(m_i)}{\to} \overline{X}_i)_I$$

is a $c_\mathcal{A}$-dense $PS(\mathbb{M}_\mathcal{A})$-source with codomain in $\underline{\underline{\mathcal{SCl}}}(\mathcal{A})$. We show that there is a subsource of $(d_i)_I$ which belongs to $PS(\mathbb{M}_\mathcal{A})$ and is $\mathcal{A}$-disjoint. It suffices to show that if i and i' are such that there is some commutative diagram in $\mathcal{X}_\mathcal{A}$ of the form

$$\begin{tikzcd} X \arrow[r, "d_i"] \arrow[d, "d_{i'}"] & \overline{X}_i \arrow[d, "g"] \\ \overline{X}_{i'} \arrow[r, "h"] & \bullet \end{tikzcd}$$

then $d_i \cong d_{i'}$. Let us consider such a commutative diagram. Without loss of generality, we may assume that (h, g) is a component of the multipushout in $\mathcal{X}_\mathcal{A}$ of $(d_i, d_{i'})$. Then, since d_i is part of a $PS(\mathbb{M}_\mathcal{A})$-source, it is easy to see that the morphism h is also part of a $PS(\mathbb{M}_\mathcal{A})$-source; thus, since $\overline{X}_{i'}$ is $\mathcal{A}$-strongly multiclosed, h is a $c_\mathcal{A}$-closed $\mathcal{M}_\mathcal{A}$-morphism. On the other hand, as d_i is a $c_\mathcal{A}$-dense $PC(\mathcal{M}_\mathcal{A})$-morphism, the morphism h is also $c_\mathcal{A}$-dense (from 22.6.2). Therefore, h is an isomorphism and $d_{i'}$ is factorizable through d_i. Analogously, we conclude that d_i is factorizable through $d_{i'}$. Consequenly, since d_i and $d_{i'}$ are epimorphisms, this implies the existence of an isomorphism t such that $d_{i'} = t \cdot d_i$.

Thus, let $(X \stackrel{d_j}{\longrightarrow} \overline{X}_j)_J$ be a subsource of $(d_i)_I$ which belongs to $PS(\mathbb{M}_\mathcal{A})$ and is $\mathcal{A}$-disjoint. Such a source $(d_J)_J$ belongs to $\mathcal{A}^{\underline{\perp}}$ in $\mathcal{X}_\mathcal{A}$, by 22.8. Then, since $\underline{\underline{\mathcal{SCl}}}(\mathcal{A}) \subseteq$

$\underline{\underline{\mathcal{O}}}(\mathcal{A})$, it follows that, furthermore, by 19.3.2, $\underline{\underline{\mathcal{O}}}(\mathcal{A}) = \underline{\underline{\mathcal{SC}l}}(\mathcal{A})$. Therefore, $\underline{\underline{\mathcal{O}}}(\mathcal{A})$ is the multireflective hull of $\mathcal{A}$ in $\mathcal{X}$. $\square$

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, Least and largest initial completions, *Comment. Math. Univ. Carolinae* 20, 1 (1979), 43-73.

[2] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, New York 1990.

[3] J. Adámek, H. Herrlich and J. Reiterman, Cocompleteness almost implies completeness, *Proc. Conf. "Categorical Topology, Prague 1988;· World Sci. Publ., Singapore* (1989), 246-256.

[4] J. Adámek and J. Rosický, Intersections of reflective subcategories, *Proc. Amer. Math. Soc.* 103 (1988), 710-712.

[5] J. Adámek, J. Rosický, On injectivity in locally presentable categories, *Trans. Amer. Math. Soc.* 336 (1993), 785-804.

[6] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, Cambridge Univ. Press 1994.

[7] J. Adámek, J. Rosický, V. Trnková, Are all limit-closed subcategories of locally presentable categories reflective?, *Lect. Notes in Math.*, 1348 (1988), 1-18.

[8] J. Adámek and H. Volger, On locally reflective categories of structures, MIP-9202, Fakultät für Mathematik und Informatik, Universität Passau (1992).

[9] S. Baron, Note on epi in T_0, *Canad. Math. Bull.* 11 (1968), 503-504.

[10] R.Börger, Multicoreflectivity and coprime objects, *Topol. and its Appl.* 33 (1989), 127-142.

[11] R.Börger and W. Tholen, Abschwächungen des Adjunktionsbegriffs, *Manuscripta Math.* 19, (1976) 19-45.

[12] G. Brümmer, E.Giuli, A categorical concept of completion of objects, *Commt. Math. Univ. Carolinae* 33 (1992), 131-147.

[13] G. Brümmer, E.Giuli and H.Herrlich, Epireflections which are completions, *Cahiers Topol. Géom. Differ. Cat.* 33 (1992), 71-93.

[14] C. Cassidy, M. Hébert, G. M.Kelly, Reflective subcategories, localizations and factorizations systems, *J.Austral. Math. Soc.* 38 (1985), 287-329

[15] G. Castellini, Closure operators, monomorphisms and epimorphisms in categories og groups, *Cahiers Topol. Géom. Differ. Cat.* 27 (1986), 151-167.

[16] B. J. Day and G. M. Kelly, On topological quotient maps preserved by pullbacks and products, *Proc. Cambridge Phil. Soc.* 67 (1970), 553-558.

[17] Y. Diers, *Catègories localisables*, These de doctorat d'état, Université Pierre et Marie Curie - Paris 6, 1977.

[18] D. Dikranjan and E. Giuli, Closure operators I, *Topol. and its Appl.* 27 (1987), 129-143.

[19] D. Dikranjan and E. Giuli, Urysohn-closed spaces, old and new, preprint.

[20] D. Dikranjan, E. Giuli and W. Tholen, Closure operators II, in: *Categorical Topology and its relations to Analysis, Algebra and Combinatorics*(Conference Proceedings, Prague 1988), pp.297-335.

[21] D. Dikranjan, W. Tholen, *Categorical Structure of Closure Operators*, Kluwer Academic Publishers 1995.

[22] P. J. Freyd and G. M. Kelly, Categories of continuous functors I, *J. Pure Appl. Algebra* 2 (1972), 169-191.

[23] J. R. Harvey, Reflective subcategories, *Illinois J. Math.* 29 (1985), 365-369.

[24] H. Herrlich, Categorical Topology 1971-1981, in: *General Topology and its Relations to Modern Analysis and Algebra* V, Proceedings Fifth Prague Topological Symposium 1981 (Heldermann Verlag, Berlin 1982), 279-383.

[25] H. Herrlich, Categorical Topology, *Gen. Topol. and its Appl.* 1 (1971), 1-15.

[26] H. Herrlich, Epirefective subcategories need not be cowellpowered, *Comment. Math. Univ. Carolinae*, 16(4) (1975), 713-715.

[27] H. Herrlich, Initial completions, *Math. Z.* 150 (1976), 101-110.

[28] H. Herrlich, Initial and final completions, *Lect. Notes in Math.* 719 (1978), 137-149.

[29] H. Herrlich and M. Hušek, Some open categorical problems in Top, *Appl. Cat. Structures* 1 (1993), 1-19.

[30] H. Herrlich, G. E. Strecker, Coreflective subcategories, *Trans. Amer. Math. Soc.* 157 (1971), 205-226.

[31] R.-E. Hoffmann, Charakterisierung nüchterner Räume, *Manuscripta Math.* 15 (1975), 185-191.

[32] R.-E. Hoffmann, Reflective hulls of finite topological spaces, *Arch. Math.* 33 (1979), 258-262.

[33] R.-E. Hoffmann, Co-well-powered reflective subcategories, *Proc. Amer. Math. Soc.* 90 (1984), 45-46.

[34] R.-E. Hoffmann, Topological functors admitting generalized Cauchy-completions, in: Proc. Conf. "Categorical Topology, Manheim 1975", *Lect. Notes in Math.* 540 (1976) pp. 286-344.

[35] R.-E. Hoffmann, Semi-identifying lifts and a generalization of the duality theorem for topological functors, *Math. Nachr.* 74 (1976), 295-307.

[36] R.-E. Hoffmann, Full reflective restrictions of topological functors, *Math. Colloq. Univ. Cape Town* 11 (1977), 65-88.

[37] R.-E. Hoffmann, Note on semi-topological functors, *Math. Z.* 160 (1978), 69-74.

[38] R.-E. Hoffmann, Minimal topological completion of $kBan1 \to kVec$, *Lect. Notes in Math.* 962 (1982), 123-132.

[39] S. S. Hong, Extensive subcategories of the category of T_0-spaces, *Canad. J. Math.* 27, 311-318.

[40] J. R. Isbell, Epimorphisms and dominions, in:Proc. Conf. Categorical Algebra (La Jolla 1965), Springer-Verlag, Berlin-Heidelberg-New York 1966, pp. 232-246.

[41] G. Janelidze and W. Tholen, How algebraic is the change-of-base functor?, Proc. Conf. Category Theory, Como 1990, *Lect. Notes in Math.* 1488, (1991) pp. 174-186.

[42] T. Jech, *Set Theory*, Academic Press, New York 1978.

[43] V. Kannan and M. Rajagopalan, Constructions and applications of rigid spaces I, *Advances Math.* 29 (1978), 89-130.

[44] J. Kaput, Locally adjuncyable functors, *Illinois J. Math.* 16 (1972) 86-94.

[45] E. W. Kiss, L. Márki, P. Pröhle and W. Tholen, Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity, *Studia Scientiarum Mathematicarum Hungarica* 18 (1983), 79-141.

[46] V. Koubek, Each concrete category has a representation by T_2 paracompact topological spaces, *Comment. Math. Univ. Carolinae*, 15 (1974), 655-664.

[47] S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, Berlin-Heidelberg-New York 1971.

[48] S. Mantovani, Epireflective hulls of spaces of ordinals in T_0, *Quaestiones Math.* 7 (1984), 203-211.

[49] J. D. Monk, *Introduction to Set Theory*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York 1969.

[50] L. D. Nel and R. G. Wilson, Epireflections in the category of T_0-spaces, *Fund. Math.* LXXV (1972), 69-74.

[51] G. Preuß, *Allgemeine Topologie*, Berlin-Heidelberg-New York, Springer 1972.

[52] D. Pumplün, Universelle und spezielle Probleme, *Math. Ann.* 198 (1972), 131-146.

[53] J. Reiterman and W. Tholen, Effective descent maps of topological spaces, preprint (York University, 1991).

[54] C. M. Ringel, Monofunctors as reflectors, *Trans. Amer. Math. Soc.* 161 (1971), 293-306.

[55] J. Rosický, Concrete categories and infinitary language, *J. Pure Appl. Algebra* 22 (1981), 309-339.

[56] J. Rosický, Semi-initial completions, *J. Pure Appl. Algebra* 40 (1986), 177-183.

[57] J. Rosický, Correction to "Semi-initial Competions", *J. Pure Appl. Algebra* 46 (1987), 109.

[58] J. Rosický and W. Tholen, Orthogonal and prereflective subcategories, *Cahiers Topol. Géom. Differ. Cat.* 29 (1988), 203-215.

[59] J. Rosický, V.Trnková, J.Adámek, Unnexpected properties of locally presentable categories, *Alg. Univ.* 27 (1990), 153-170

[60] S. Salbany, Reflective subcategories and closure operators, in: Proc. Conf. "Categorical Topology, Manheim 1975", *Lect. Notes in Math.* 540 (1976), pp. 548-565.

[61] G. Salicrup, Multiepireflective subcategories, *Topology and its Appl.* 13 (1982), 191-200.

[62] F. Schwarz and S. Weck-Schwarz, Internal description of hulls: a unifying approach, *Category Theory at Work*, H. Herrlich, H.-E. Porst (eds.), Heldermann Verlag Berlin 1991, 21-34.

[63] L. Skula, On a reflective subcategory of the category of all topological spaces, *TRans Amer. Math. Soc.* 142 (1969), 37-41.

[64] M. Sobral, Absolutely closed spaces and categories of algebras, *Portugaliae Math.* 47 (1990), 341-351.

[65] M. Sobral and W. Tholen, Effective descent morphisms and effective equivalence relations, *Canad. Math. Soc.*, Conf. Proc., Vol. 13 (1992), 421-432.

[66] L. Sousa, Solid hulls of concrete categories, *Appl. Categ. Structures* 3 (1995), 105-118.

[67] L. Sousa, Orthogonality and closure operators, *Cahiers Topol. Géom. Differ. Cat.* 36 (1995), 323-343.

[68] L. Sousa, α-sober spaces via the orthogonal closure operator, *Appl. Categ. Structures* 4 (1996), 87-95.

[69] L. Sousa, Note on multisolid categories (to appear in *J. Pure App. Algebra*).

[70] W. Tholen, M-functors, Mathematik Arbeitspapiere Nr. 7, Universitat Bremen (1976), 178-185.

[71] W. Tholen, Semi-topological functors I, *J. Pure App. Algebra* 15 (1979), 53-73.

[72] W. Tholen, Factorizations, localizations, and the orthogonal subcategory problem, *Math. Nachr.* 114 (1983), 63-85.

[73] W. Tholen, Reflective subcategories, *Topol. and its Appl.* 27 (1987), 201-212.

[74] W. Tholen, MacNeille completions of concrete categories with local properties, *Comment. Math., Univ. St. Pauli*, 28 (1979), 179-202.

[75] V. Trnková, Automata and categories, *Lect. Notes in Computer Science* 32 (1975), 138-152.

[76] V. Trnková, Non-constant continuous mappings of metric or compact Hausdorff spaces, *Comment. Math. Univ. Carolinae* 13 (1972), 283-295.

[77] V. Trnková, J. Adámek and J. Rosický, Topological reflections revisited, *Proc. Amer. Math. Soc.* 108 (1990), 605-612.

[78] S. Willard, *General Topology*, Addison-Wesley Publishing Company (1970)

[79] H. Wolff, Free monads and the orthogonal subcategory problem, *J. Pure Appl. Algebra* 13 (1978), 233-242.

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