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PSEUDOMONADS AND DESCENT

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Abstract

This thesis consists of one introductory chapter and four single-authored papers written during my PhD studies, with minor adaptations. The original contributions of the papers are mainly within the study of pseudomonads and descent objects, including applications to descent theory, commutativity of weighted bilimits, coherence and (presentations of) categorical structures.

In Chapter 1, we give a glance of the scope of our work and briefly describe elements of the original contributions of each paper, including some connections between them. We also give a brief exposition of our main setting, which is 2-dimensional category theory. In this direction: (1) we give an exposition on the doctrinal adjunction, focusing on the Beck-Chevalley condition as used in Chapter 3, (2) we apply the results of Chapter 5 in a generalized setting of the formal theory of monads and (3) we apply the biadjoint triangle theorem of Chapter 4 to study (pseudo)exponentiable pseudocoalgebras.

Chapter 2 corresponds to the paper *Freely generated n-categories, coinserters and presentations of low dimensional categories*, *DMUC 17-20* or *arXiv:1704.04474*. We introduce and study presentations of categorical structures induced by $(n+1)$-computads and groupoidal computads. In this context, we introduce the notion of deficiency and presentations of groupoids via computads. We compare the resulting notions with those induced by monads together with a finite measure of objects. In particular, we find our notions to generalize the usual ones. One important feature of this paper is that we show that several freely generated structures are naturally given by coinserters. After recalling how the category freely generated by a graph G internal to Set is given by the coinserter of G, we introduce higher icons and present the definitions of n-computads via internal graphs of the 2-category $n\mathsf{Cat}$ of n-categories, n-functors and n-icons. Within this setting, we show that the *n-category freely generated by an n-computad is also given by a coinserter*. Analogously, we demonstrate that the *geometric realization* of a graph G consists of a left adjoint functor $\mathscr{F}_{\mathsf{Top}_1} : \mathsf{grph} \to \mathsf{Top}$ given objectwise by the *topological coinserter*. Furthermore, as a fundamental tool to study presentation of thin and locally thin categorical structures, we give a detailed construction of a 2-dimensional analogue of $\mathscr{F}_{\mathsf{Top}_1}$, denoted by $\mathscr{F}_{\mathsf{Top}_2} : \mathsf{cmp} \to \mathsf{Top}$. In the case of group presentations, $\mathscr{F}_{\mathsf{Top}_2}$ formalizes the Lyndon-van Kampen diagrams. Finally, we sketch a construction of the 3-dimensional version $\mathscr{F}_{\mathsf{Top}_3}$ which associates a 3-dimensional CW-complex to each 3-computad.

Chapter 3 corresponds to the article *Pseudo-Kan Extensions and descent theory*, *arXiv:1606.04999* under review. We develop and employ results on idempotent pseudomonads to get theorems on the general setting of descent theory, which, in our perspective, is the study of the image of pseudomonadic pseudofunctors. After giving a direct approach to prove an analogue of Fubini's Theorem for weighted bilimits and constructing pointwise pseudo-Kan extensions, we employ the results on pseudomonadic pseudofunctors to get theorems on commutativity of bilimits. In order to use these results as the

main framework to deal with classical descent theory in the context of [52], we prove that the descent category (object) of a pseudocosimplicial category (object) is its conical bilimit. We use, then, this formal approach of commutativity of bilimits to (1) recast classical theorems of descent theory, (2) prove generalizations of such theorems and (3) get new results of descent theory. In this direction, we give formal proofs of transfer theorems, embedding theorems, a pseudopullback theorem, a Galois Theorem and the Bénabou-Roubaud Theorem. We also apply the pseudopullback theorem to detect effective descent morphisms in suitable categories of enriched categories in terms of (the embedding in) internal categories.

Chapter 4 corresponds to the article *On Biadjoint Triangles*, published in *Theory and Applications of Categories, Vol 31, N. 9* (2016). The main contributions are the biadjoint triangle theorems, which have many applications in 2-dimensional category theory. Examples of which are given in this same paper: reproving the *Pseudomonadicity characterization* of [73], improving results on the 2-*monadic approach to coherence* of [9, 67, 93], improving results on *lifting of biadjoints* of [9] and introducing the suitable concept of *pointwise pseudo-Kan extension*.

Chapter 5 corresponds to the article *On lifting of biadjoints and lax algebras*, to appear in *Categories and General Algebraic Structures with Applications*. It can be seen as a complement of the precedent chapter, since it gives further theorems on lifting of biadjoints provided that we can describe the categories of morphisms of a certain domain in terms of weighted (bi)limits. This approach, together with results on lax descent objects and lax algebras, allows us to get results of lifting of biadjoints involving (full) sub-2-categories of the 2-category of lax algebras. As a consequence, we complete our treatment of the 2-monadic approach to coherence via biadjoint triangle theorems.

Resumo

Esta tese consiste em um capítulo introdutório e quatro artigos de autoria única, escritos durante os meus estudos de doutoramento. As contribuições originais dos artigos estão principalmente dentro do contexto do estudo de pseudomónadas e objetos de descida, com aplicações à teoria da descida, comutatividade de bilimites ponderados, coerência e apresentações de estruturas categoriais.

No Capítulo 1, introduzimos aspectos do escopo do trabalho e descrevemos alguns elementos das contribuições originais de cada artigo, incluindo interrelações entre elas. Damos também uma exposição básica sobre o principal assunto da tese, nomeadamente, teoria das categorias de dimensão 2. Nesse sentido, (1) introduzimos adjunção doutrinal, focando na condição de Beck-Chevalley, com a perspectiva adotada no Capítulo 3, (2) aplicamos resultados do Capítulo 5 em um contexto generalizado da teoria formal das mónadas e (3) aplicamos o teorema de triângulos biadjuntos do Capítulo 4 para estudar pseudocoalgebras (pseudo)exponenciáveis.

O Capítulo 2 corresponde ao artigo *Freely generated n-categories, coinserters and presentations of low dimensional categories*, *DMUC 17-20* ou *arXiv:1704.04474*. Neste trabalho, introduzimos e estudamos apresentações de estruturas categoriais induzidas por $(n+1)$-computadas e computadas grupoidais. Introduzimos a noção de deficiência de grupóides via computadas. Comparamos, então, as noções resultantes com as noções induzidas por mónadas junto com medidas finitas de objetos. Em particular, concluimos que nossas noções generalizam as noções clássicas. Outras contribuições do artigo consistiram em mostrar que as propriedades universais de várias estuturas livremente geradas podem ser descritas por coinserções. Começamos por relembrar que as categorias livremente geradas são dadas por coinserções de grafos e, então, introduzimos *icons* de dimensão alta e apresentamos as definições de n-computadas via grafos internos da 2-categoria $n\mathsf{Cat}$ de n-categorias, n-functores e *n-icons*. Nesse caso, mostramos que a n-categoria livremente gerada por uma n-computada é a sua coinserção. Analogamente, demonstramos que a realização geométrica de um grafo G é parte de um functor adjunto à esquerda $\mathscr{F}_{\mathsf{Top}_1} : \mathsf{grph} \to \mathsf{Top}$ definido objeto a objeto pela coinserção topológica. Além disso, como uma ferramenta fundamental para o estudo de estruturas categoriais finas e localmente finas, apresentamos uma construção detalhada de um análogo de dimensão 2 de $\mathscr{F}_{\mathsf{Top}_1} : \mathsf{grph} \to \mathsf{Top}$, denotado por $\mathscr{F}_{\mathsf{Top}_2} : \mathsf{cmp} \to \mathsf{Top}$. No caso de grupos, $\mathscr{F}_{\mathsf{Top}_2}$ formaliza e, portanto, generaliza o diagrama de Lyndon-van Kampen. Finalizamos o capítulo dando uma construção de uma versão em dimensão 3, denotada por $\mathscr{F}_{\mathsf{Top}_3}$, que associa um CW-complexo de dimensão 3 para cada 3-computada.

O Capítulo 3 corresponde ao artigo *Pseudo-Kan Extensions and descent theory*, em revisão para publicação. Desenvolvemos e aplicamos resultados sobre pseudomónadas idempotentes, obtendo teoremas no contexto geral da teoria da descida que, em nossa perspectiva, é o estudo da imagem de pseudofunctores pseudomonádicos. Depois de apresentar uma prova direta do teorema de Fubini

para bilimites ponderados e de construir pseudo-extensões de Kan, aplicamos os resultados sobre pseudomónadas para provar teoremas sobre comutatividade de bilimites ponderados. Com o objetivo de usar tais resultados como base para lidar com a teoria da descida clássica no contexto de [52], provamos que a categoria (objeto) de descida de uma categoria (objeto) pseudocosimplicial é seu bilimite cónico. Usamos, então, esse tratamento formal de comutatividade de bilimites para (1) recuperar teoremas clássicos da teoria da descida, (2) provar generalizações desses teoremas e (3) obter novos resultados de teoria da descida. Nesse sentido, apresentamos provas formais (de generalizações) de teoremas de transferência, de teoremas de mergulho, do teorema de Galois e do teorema de Bénabou-Roubaud. Provamos também um resultado sobre morfismos de descida efetiva em pseudoprodutos fibrados de categorias e o aplicamos para obter morfismos de descida efetiva em algumas categorias de categorias enriquecidas.

O Capítulo 4 corresponde ao artigo *On Biadjoint Triangles*, publicado no *Theory and Applications of Categories, Vol 31, N. 9* (2016). As contribuções principais são os teoremas de triângulos biadjuntos, os quais possuem muitas aplicações em teoria de categorias de dimensão 2. Apresentamos exemplos de aplicações no próprio artigo: provamos explicitamente o teorema de pseudomonadicidade [73], melhoramos resultados sobre o tratamento 2-monádico do problema de coerência de [9, 67, 93], generalizamos resultados de levantamentos de biadjuntos e introduzimos o conceito de *pseudo-extensões de Kan* para, então, construir as pseudo-extensões de Kan via bilimites ponderados.

O Capítulo 5 corresponde ao artigo *On lifting of biadjoints and lax algebras*, a ser publicado no *Categories and General Algebraic Structures with Applications*. O principal tema deste artigo é a demonstração de teoremas de levantamento de biadjuntos, ao assumir que conseguimos descrever a categoria de morfismos de um domínio em termos de (bi)limites ponderados. Esse tratamento, junto com resultados sobre objetos de descida lassos e álgebras lassas, nos permite obter resultados sobre levantamento de biadjuntos envolvendo sub-2-categorias (plenas) da 2-categoria de álgebras lassas. Como conseqüencia, concluímos nossos resultados de caracterização sobre o tratamento 2-monádico do problema de coerência, via teoremas de triângulos biadjuntos.

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Chapter 1

Introduction

The aim of this chapter is to introduce our main setting, which is 2-dimensional universal algebra, and to give a glimpse of the contributions of this thesis. We start by roughly explaining aspects of the interrelation between pseudomonads and descent objects in Section 1.1. Then, in Section 1.2, we introduce basic notions of 2-dimensional category theory. We take this opportunity to introduce, among other concepts, the notion of colax $\mathscr{T}$-morphisms of lax $\mathscr{T}$-algebras, which is not introduced elsewhere in this thesis. The notion of colax $\mathscr{T}$-morphisms are, then, used in Section 1.7 to relate the formal theory of monads with the problem of lifting of biadjoints studied in Chapter 5. We also use the concept of colax $\mathscr{T}$-morphisms to talk about doctrinal adjuntion in Section 1.4, which is a brief exposition of the main theorem of [57] focusing on the Beck-Chevalley condition as used in our work on the Bénabou-Roubaud Theorem in Chapter 3.

Sections 1.3, 1.5 and 1.6 are dedicated to briefly describe elements of the contributions of the Chapters 2, 3, 4 and 5, which are respectively the papers [80], [79], [77] and [78]. Finally, the last section is an application of the biadjoint triangles of Chapter 4 in the context of exponentiable objects within bicategory theory: we prove that, under suitable hypothesis, a pseudocoalgebra is (pseudo)exponentiable whenever the underlying object is (pseudo)exponentiable.

1.1 Overview: Pseudomonads and the Descent Object

In 2-dimensional category theory, by replacing strict conditions (involving commutativity of diagrams) with pseudo or lax ones (involving a 2-cell plus coherence) we get important notions and problems. We briefly describe two examples of these notions below: namely, *descent object* and *pseudomonad*.

Firstly, to give an idea of the role of the descent object in 2-dimensional universal algebra, it is useful to make an analogy with the equalizer: while the equalizer encompasses equality and commutativity of diagrams in 1-dimensional category theory, the descent object and its variations encompass 2-dimensional coherence: structure (2-cell) plus coherence.

One obvious example of the importance of the descent object is within *descent theory*, as introduced by Grothendieck, initially motivated by the problem of understanding the image of functors induced by fibrations. This theory features a 2-dimensional analogue of the sheaf condition: the (strict) gluing condition, given by an equalizer of sets, is replaced by the descent condition, given by a *descent object* of a diagram of categories.

Secondly, analogously to the case of monad theory in 1-dimensional universal algebra, *pseudomonad theory* encompasses aspects of 2-dimensional universal algebra, being useful to study many important aspects of 2-dimensional category theory. Again, in the definition of *pseudomonad*, the commutative diagrams of the definition of monad are replaced by invertible 2-cells plus coherence. In this theory, then, we have 2-dimensional versions of the features of monad theory. For instance, adjunctions are replaced by biadjunctions, and we have an Eilenberg-Moore Factorization provided that we consider the 2-category of pseudoalgebras, pseudomorphisms and algebra transformations, as it is shown in Section 4.5.

The main topic of this thesis is the study of 2-dimensional categorical structures, mostly related with descent theory and pseudomonad theory, with applications to 1-dimensional category theory. For instance:

- The contributions of Chapter 2 in this context are within the study of freely generated and finitely presented categorical structures. In particular, for instance, we deal with presentations of domain 2-categories related to the universal property of descent objects. More precisely, presentations of the inclusion of the 2-categories such that Kan extensions along such inclusion gives the (strict) descent object;
- In Chapter 3, we develop an abstract perspective of descent theory in which the fundamental problem is the existence of pseudoalgebra structures over objects: more precisely, the image of pseudomonadic pseudofunctors that induce idempotent pseudomonads. Having this goal, we develop some aspects of biadjoint triangles and lifting of pseudoalgebra structures involving pseudofunctors that induce idempotent pseudomonads and apply it to get results on commutativity of bilimits. We finish the article, then, applying our perspective to the classical context of [51, 52];
- As a (strict) morphism of algebras is given by a morphism plus the commutativity of a diagram, a pseudomorphism between pseudoalgebras is given by a morphism, an invertible 2-cell plus coherence. In Chapters 4, 5, we show that the coherence aspects of pseudomonad theory are encompassed by descent objects and their variations. More precisely, we show that the category of (lax-)(pseudo)morphisms between (lax-)(pseudo)algebras is given by (lax-)descent objects. As it is proven and explained in Chapter 5, with these results on the category of (lax-)(pseudo)morphisms, we can prove biadjoint triangle theorems and results on lifting of biadjoints.

1.2 2-Dimensional Categorical Structures

Two of the most fundamental notions of 2-dimensional categorical structures are those of *double category* and 2-*category*, both introduced by Ehresmann [34]. The former is an example of an internal category (introduced in [34]), while the latter is an example of enriched category (as introduced in [35]).

The study of the dichotomy between the theory of enriched categories and internal categories, including unification theories, is still of much interest. For instance, within the more general setting of generalized multicategories, we have the introduction of (T,V)-categories [22] (which generalizes enriched categories), T-categories [14, 45] (which generalizes internal categories) and possible unification theories [16, 24].

Since there is no definitive general framework and the approaches mentioned above are focused on the examples related to multicategories, we do not follow any of them. Instead, we follow the basic idea that internal categories and enriched categories can be seen as monads in suitable *bicategories* [5, 7].

For simplicity, we use the concept of enriched graphs. This is given in Definition 2.7.1, but we only need to recall that, given a category V, a *V-enriched graph G* is a collection of objects $G(0) = G_0$ endowed with one object $G(A,B)$ of V for each ordered pair of objects (A,B) of G_0.

Definition 1.2.1. [Bicategory [5]] A bicategory is a CAT-enriched graph $\mathfrak{B}$ endowed with:

- Identities: a functor $\mathtt{I}_A : 1 \to \mathfrak{B}(A,A)$ for each object A of $\mathfrak{B}$;
- Composition: a functor, called composition, $\circ = \circ_{ABC} : \mathfrak{B}(B,C) \times \mathfrak{B}(A,B) \to \mathfrak{B}(A,C)$ for each ordered triple (A,B,C) of objects of $\mathfrak{B}$;
- Associativity: natural isomorphisms
$$\mathfrak{a}_{ABCD} : \circ_{ABD}\left(\circ_{BCD} \times \mathrm{Id}_{\mathfrak{B}(A,B)}\right) \Rightarrow \circ_{ACD}\left(\mathrm{Id}_{\mathfrak{B}(C,D)} \times \circ_{ABC}\right);$$
for every quadruplet (A,B,C,D) of objects of $\mathfrak{B}$;
- Action of Identity: natural isomorphisms
$$\mathfrak{e}_{AB} : \circ_{ABB}\left(\mathtt{I}_B \times \mathrm{Id}_{\mathfrak{B}(A,B)}\right) \Rightarrow \mathrm{pro}^{\mathfrak{e}}_{\mathfrak{B}(A,B)},$$
$$\mathfrak{d}_{AB} : \circ_{AAB}\left(\mathrm{Id}_{\mathfrak{B}(A,B)} \times \mathtt{I}_A\right) \Rightarrow \mathrm{pro}^{\mathfrak{d}}_{\mathfrak{B}(A,B)},$$
in which $\mathrm{pro}^{\mathfrak{d}}_{\mathfrak{B}(A,B)} : \mathfrak{B}(A,B) \times 1 \to \mathfrak{B}(A,B)$ and $\mathrm{pro}^{\mathfrak{e}}_{\mathfrak{B}(A,B)} : 1 \times \mathfrak{B}(A,B) \to \mathfrak{B}(A,B)$ are the invertible projections, for each pair (A,B) of objects in $\mathfrak{B}$;

such that the diagrams

$$\begin{tikzcd}
\circ_{ABE}\left(\circ_{BCE} \times \mathrm{Id}_{\mathfrak{B}(A,B)}\right)\left(\circ_{CDE} \times \mathrm{Id}_{\mathfrak{B}(A,B,C)}\right) \arrow[r, "\mathfrak{a}_{ABCE}*\mathrm{id}"] \arrow[d, "\mathrm{id}_{\circ_{ABE}}*\left(\mathfrak{a}_{ABCE}\times\mathrm{id}_{\mathrm{Id}_{\mathfrak{B}(A,B)}}\right)"'] & \circ_{ACE}\left(\mathrm{Id}_{\mathfrak{B}(C,E)} \times \circ_{ABC}\right)\left(\circ_{CDE} \times \mathrm{Id}_{\mathfrak{B}(A,B,C)}\right) \arrow[d, "\mathfrak{a}_{ACDE}*\mathrm{id}_{\left(\mathrm{Id}_{\mathfrak{B}(C,D,E)}\times\circ_{ABC}\right)}"] \\
\circ_{ABE}\left(\circ_{BDE} \times \mathrm{Id}_{\mathfrak{B}(A,B)}\right)\left(\mathrm{Id} \times \circ_{BCD} \times \mathrm{Id}\right) \arrow[dr, "\mathfrak{a}_{ABCE}*\mathrm{id}"'] & \circ_{ADE}\left(\mathrm{Id}_{\mathfrak{B}(D,E)} \times \circ_{ACD}\right)\left(\mathrm{Id}_{\mathfrak{B}(C,D,E)} \times \circ_{ABC}\right) \\
 & \circ_{ADE}\left(\mathrm{Id}_{\mathfrak{B}(D,E)} \times \circ_{ABD}\right)\left(\mathrm{Id} \times \circ_{BCD} \times \mathrm{Id}\right) \arrow[u, "\mathrm{id}_{\circ_{ADE}}*\left(\mathrm{id}_{\mathrm{Id}_{\mathfrak{B}(D,E)}}\times\mathfrak{a}_{ABCE}\right)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\circ_{ABC}\left(\circ_{BBC}\times \mathrm{Id}_{\mathfrak{B}(A,B)}\right)(\mathrm{Id}\times \mathtt{I}_B\times \mathrm{Id}) \arrow[r, "\mathfrak{a}_{ABBC}*\mathrm{id}"] \arrow[dr, "\mathrm{id}_{\circ_{ABC}}*(\mathfrak{d}_{BC}\times \mathrm{id})"'] & \circ_{ABC}\left(\mathrm{Id}_{\mathfrak{B}(B,C)}\times \circ_{ABB}\right)(\mathrm{Id}\times \mathtt{I}_B\times \mathrm{Id}) \arrow[d, "\mathrm{id}_{\circ_{ABC}}*(\mathrm{id}\times \mathfrak{e}_{AB})"] \\
 & \circ_{ABC}\left(\mathrm{pro}_{(A,B,C)}\right)
\end{tikzcd}$$

commute for every quintuple (A,B,C,D,E) of objects in $\mathfrak{B}$, in which $\mathrm{Id}_{\mathfrak{B}(A,B,C)} := \mathrm{Id}_{\mathfrak{B}(B,C)\times\mathfrak{B}(A,B)}$, $\mathrm{pro}_{(A,B,C)} : \mathfrak{B}(B,C)\times 1\times \mathfrak{B}(A,B) \to \mathfrak{B}(B,C)\times\mathfrak{B}(A,B)$ is the invertible projection and the omitted subscripts of the identities are the obvious ones.

For simplicity, assuming that the structures are implicit, we denote such a bicategory by $(\mathfrak{B},\circ,\mathtt{I},\mathfrak{a},\mathfrak{e},\mathfrak{d})$ or just by $\mathfrak{B}$. For each pair (A,B) of objects of a bicategory $\mathfrak{B}$, if f is an object of the category $\mathfrak{B}(A,B)$, f is called an 1-cell of $\mathfrak{B}$ and it is denoted by $f : A \to B$. A morphism $\alpha : f \Rightarrow g$ of $\mathfrak{B}(A,B)$ is called a 2-cell of $\mathfrak{B}$.

Remark 1.2.2. In order to take advantage of the context of introducing monads, internal categories, double categories and enriched categories, we define 2-categories via enriched categories below. However, a brief and obvious definition of 2-category is that of a *strict bicategory*. More precisely, a 2-*category* is a bicategory such that its natural isomorphisms are identities. We assume this definition herein.

Remark 1.2.3. Since the work of this thesis is mainly within the tricategory 2-CAT of 2-categories, pseudofunctors and pseudonatural transformations (as defined in Section 4.2), the results and definitions on 2-dimensional category theory of this thesis are within the general setting of bicategories up to minor trivial adaptations. Specially in this section, since we should consider the bicategories of Definitions 1.2.9 and 1.2.10, we freely assume these adaptations.

Remark 1.2.4. In this chapter, we do not give any further comment on size issues. In this direction, when necessary, we implicitly make similar assumptions to those given in Section 2.1 or Section 5.1.

Given a bicategory $\mathfrak{B}$, there are two main duals of $\mathfrak{B}$ which give rise to four duals, including $\mathfrak{B}$ itself. The first dual, denoted by $\mathfrak{B}^{\mathrm{op}}$, comes from getting the dual of the underlying category of $\mathfrak{B}$ (that is to say, the opposite w.r.t. 1-cells), while the other dual, denoted by $\mathfrak{B}^{\mathrm{co}}$, is obtained from getting the duals of the hom-categories (that is to say, the opposite w.r.t. 2-cells). Then, we have $\mathfrak{B}$ itself and $\mathfrak{B}^{\mathrm{coop}} := (\mathfrak{B}^{\mathrm{op}})^{\mathrm{co}} \cong (\mathfrak{B}^{\mathrm{co}})^{\mathrm{op}}$.

Remark 1.2.5. We do not define tricategories [40], but we give some independent remarks. For instance, as observed in Section 4.2, 2-CAT is a tricategory and, specially in the present section, we consider the tricategory BICAT of bicategories, pseudofunctors, pseudonatural transformations and modifications as well. The dualizations mentioned above define invertible trifunctors:

$$(-)^{\mathrm{op}} : \mathsf{BICAT} \cong \mathsf{BICAT}^{\mathrm{co}}, \quad (-)^{\mathrm{co}} : \mathsf{BICAT} \cong \mathsf{BICAT}^{\mathrm{tco}}, \quad (-)^{\mathrm{coop}} : \mathsf{BICAT} \cong \mathsf{BICAT}^{\mathrm{cotco}},$$

$$(-)^{\mathrm{op}} : 2\text{-}\mathsf{CAT} \cong 2\text{-}\mathsf{CAT}^{\mathrm{co}}, \quad (-)^{\mathrm{co}} : 2\text{-}\mathsf{CAT} \cong 2\text{-}\mathsf{CAT}^{\mathrm{tco}}, \quad (-)^{\mathrm{coop}} : 2\text{-}\mathsf{CAT} \cong 2\text{-}\mathsf{CAT}^{\mathrm{cotco}},$$

in which $\mathfrak{T}^{\mathrm{tco}}$ denotes the dual of the tricategory $\mathfrak{T}$ obtained from reversing the 3-cells, and $\mathfrak{T}^{\mathrm{cotco}} := (\mathfrak{T}^{\mathrm{tco}})^{\mathrm{co}}$. These isomorphisms are 2-dimensional analogues of the invertible 2-functor $(-)^{\mathrm{op}} : \mathsf{CAT} \cong \mathsf{CAT}^{\mathrm{co}}$.

Given a bicategory $\mathfrak{B}$, we can consider the bicategory of monads of $\mathfrak{B}$. This was introduced in [100, 101] taking the point of [5] that monads in $\mathfrak{B}$ are given by lax functors between the terminal bicategory 1 and $\mathfrak{B}$. Herein, we introduce monads via lax algebras of the identity pseudomonad. This perspective takes advantage of the concepts introduced in Chapter 5 and it gives a shortcut to understand the role of lifting of biadjoints in the formal theory of monads, which is sketched in Section 1.7. It should be observed that our viewpoint also has connections with the approach of [22] to introduce (T,V)-categories.

We assume the definition of pseudomonads of Section 5.4 (or Definition 4.5.1 of pseudocomonads) and Definition 5.4.1 of the bicategory of lax algebras and lax morphisms. Within this context, it is easy to verify that:

Lemma 1.2.6. *The identity pseudofunctor* $\mathrm{Id}_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}$*, with identities* 2*-natural transformations and modifications, gives a* 2*-monad and a* 2*-comonad on* $\mathfrak{B}$*.*

Definition 1.2.7. [Bicategory of Monads] The bicategory of monads of a bicategory $\mathfrak{B}$, denoted by $\mathsf{Mnd}(\mathfrak{B})$, is the bicategory of lax $\mathrm{Id}_{\mathfrak{B}}$-algebras $\mathsf{Lax}\text{-}\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{Alg}_{\ell}$.

Following the notation of lax algebras of Definition 5.4.1, we have that a monad in a 2-category $\mathfrak{B}$ is defined by a quadruplet $\mathsf{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \overline{\underline{\mathsf{z}}}, \overline{\underline{\mathsf{z}}}_0)$ satisfying the condition of lax $\mathrm{Id}_{\mathfrak{B}}$-algebra, in which Z is an object of $\mathfrak{B}$, $\mathfrak{alg}_{\mathsf{z}} : Z \to Z$ is an endomorphism of $\mathfrak{B}$, $\overline{\underline{\mathsf{z}}} : \mathfrak{alg}_{\mathsf{z}}\mathfrak{alg}_{\mathsf{z}} \Rightarrow \mathfrak{alg}_{\mathsf{z}}$ is a 2-cell of $\mathfrak{B}$, called the multiplication of z, and $\overline{\underline{\mathsf{z}}}_0 : \mathrm{id}_Z \Rightarrow \mathfrak{alg}_{\mathsf{z}}$ is a 2-cell of $\mathfrak{B}$, called the unit of the monad z. In this case, we say that z is a monad on Z.

In order to introduce enriched categories and internal categories as monads, we define bicategories constructed from a suitable categorical structure V. These are the bicategory of matrices and the bicategory of spans, denoted respectively by V-Mat and $\mathit{Span}(V)$.

Remark 1.2.8. The bicategory of matrices V-Mat is constructed from a monoidal category V. A *monoidal category* is a bicategory $(\mathfrak{B}, \circ, \mathtt{I}, \mathfrak{a}, \mathfrak{e}, \mathfrak{d})$ which has only one object $\triangle$. The composition is called, in this case, the *monoidal product/tensor*. The *underlying category* of a monoidal category is the hom-category $\mathfrak{B}(\triangle, \triangle)$. The objects and morphisms of a monoidal category are the objects and morphisms of its underlying category.

Since we are talking about a bicategory with only one object $\triangle$, we actually have only one natural isomorphism of associativity, one identity and two natural isomorphisms of action of identity. That is to say, given a monoidal category $(\mathfrak{B}, \circ, \mathtt{I}, \mathfrak{a}, \mathfrak{e}, \mathfrak{d})$, we denote $\mathfrak{a} := \mathfrak{a}_{\triangle\triangle\triangle}$, $\mathfrak{d} := \mathfrak{d}_{\triangle\triangle}$, $\mathfrak{e} := \mathfrak{e}_{\triangle\triangle}$ and $I := \mathtt{I}_{\triangle}$. Therefore such a monoidal category is given by a sextuple $(V, \otimes, I, \mathfrak{a}, \mathfrak{e}, \mathfrak{d})$ in which $V := \mathfrak{B}(\triangle, \triangle)$ is the underlying category, $\otimes$ is the monoidal product, I is the identity and

$$\mathfrak{a} : (-\otimes -)\otimes - \longrightarrow -\otimes(-\otimes -),\ \mathfrak{e} : (I \otimes -) \longrightarrow \mathrm{id}_V,\ \mathfrak{d} : (-\otimes I) \longrightarrow \mathrm{id}_V$$

are the respective natural isomorphisms satisfying the axioms of a bicategory with the only object $\triangle$.

For simplicity, letting the natural isomorphisms implicit, a monoidal category is usually denoted by $(V, \otimes, I)$, while $(\otimes, I)$ together with the natural isomorphisms $\mathfrak{a}$, $\mathfrak{e}$, $\mathfrak{d}$ is called a monoidal structure for V. When the monoidal structure is implicit in the context, we denote the monoidal category $(V,\otimes,I)$ by V.

A *symmetric monoidal category* is a monoidal category $(V,\otimes,I)$ endowed with a natural isomorphism, called braiding, $\mathfrak{b} : -\otimes- \longrightarrow -\otimes^{\mathrm{op}}-$ in which $\otimes^{\mathrm{op}}$ is the composition of the opposite of the bicategory that corresponds to $(V,\otimes,I)$, such that

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\mathfrak{a}_{(A,B,C)}"] \arrow[d, "\mathfrak{b}_{(A,B)}\otimes\mathrm{id}_C"] & A\otimes(B\otimes C) \arrow[r, "\mathfrak{b}_{(A,B\otimes C)}"] & (B\otimes C)\otimes A \arrow[d, "\mathfrak{a}_{(B,C,A)}"] \\
(B\otimes A)\otimes C \arrow[r, "\mathfrak{a}_{(B,A,C)}"] & B\otimes(A\otimes C) \arrow[r, "\mathrm{id}_B\otimes\mathfrak{b}_{A,C}"] & B\otimes(C\otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
(A\otimes B) \arrow[r, "\mathfrak{b}_{(A,B)}"] \arrow[d, equal] & (B\otimes A) \arrow[dl, "\mathfrak{b}_{(B,A)}"] \\
A\otimes B &
\end{tikzcd}$$

commute for every triple (A,B,C) of objects of V. A *symmetric monoidal closed category* $(V,\otimes,I)$ is a symmetric monoidal category such that every object of V is exponentiable w.r.t. the monoidal product $\otimes$. In other words, this means that the representable functor $A\otimes - : V\to V$ has a right adjoint for every A of V.

If a category V has finite products, it has a natural symmetric monoidal structure called *cartesian monoidal structure*. That is to say $\otimes := \times$ and the unit is given by the terminal object 1 of V. The natural isomorphisms for associativity, actions of the identity and braiding are given by the universal property of the product. A category with finite products endowed with this monoidal structure is called a *monoidal cartesian category* or just a *cartesian category*, and it is denoted by $(V,\times,1)$.

It would be appropriate to define monoidal categories via 2-dimensional monad theory, as, for instance, it is shown in Remark 5.4.3. But our interest herein is mostly restricted to the case of cartesian closed categories. Even our result on effective descent morphisms of enriched categories, which is Theorem 3.9.11, is given within the context of cartesian closed categories. For this reason, we avoid giving further comments on general aspects of monoidal categories. We refer to [75, 76, 81] for the basics on such general aspects.

Definition 1.2.9. [Bicategory of Matrices] Let $(V,\otimes,I)$ be a symmetric monoidal closed category with finite coproducts. We define V-Mat as follows:

- The objects are the sets;
- A morphism $M : A\to B$ in V-Mat is a matrix of objects in V, that is to say, a functor $A\times B\to V$, considering A,B as discrete categories;
- The 2-cells are natural transformations. In other words, the category of morphisms for a ordered pair (A,B) of sets is the category of functors and natural transformations $\mathsf{CAT}[A\times B,\mathsf{Set}]$;

- The composition is given by the usual formula of product of matrices. More precisely, given matrices $M : A \times B \to V$ and $N : B \times C \to V$, the composition is defined by

$$\begin{array}{rll} N \circ M : & A \times C & \to V \\ & (i,j) & \mapsto \sum\limits_{k \in B} M(i,k) \otimes N(k,j) \end{array}$$

  in which $\sum$ denotes the coproduct;

- For each set A, the identity on A is the matrix

$$\begin{array}{rll} \mathrm{id}_A : & A \times A & \to V \\ & (i,j) & \mapsto \begin{cases} I, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \end{array}$$

  in which 0 is the initial object;

- The natural isomorphisms for associativity and actions of identities are given by the universal property of coproducts, the isomorphisms of the preservation of the coproducts by $\otimes$ and the isomorphisms of the monoidal structure.

In order to define the bicategory of spans $Span(V)$ of a category with pullbacks V, we denote by *span* the category with 3 objects (0, 1 and 2) whose nontrivial morphisms are given by

$$0 \xleftarrow{d^1} 2 \xrightarrow{d^0} 1.$$

Definition 1.2.10. [Bicategory of Spans] Let V be a category with pullbacks. The bicategory $Span(V)$ is defined by

- The objects are the objects of V;
- A morphism $M : A \to B$ in $Span(V)$ is a span in V between A and B, that is to say, a functor $M : span \to V$, such that $M(0) = A$ and $M(1) = B$;
- A 2-cell f between two morphisms $M, K : A \to B$ is a natural transformation $f : M \longrightarrow K$ such that $f_0 = \mathrm{id}_A$ and $f_1 = \mathrm{id}_B$. That is to say, it is a morphism $f : M(2) \to K(2)$ such that

$$\begin{tikzcd}
 & M(2) \arrow[ld, "M(d^1)"'] \arrow[rd, "M(d^0)"] \arrow[dd, "f"] & \\
A & & B \\
 & K(2) \arrow[lu, "K(d^1)"] \arrow[ru, "K(d^0)"'] &
\end{tikzcd}$$

commutes in V.

- The composition is given by the pullback. More precisely, given a span $M : span \to V$ between A and B and a span $N : span \to V$ between B and C, the composition is defined by the span

$$\begin{tikzcd}
 & & \mathrm{P}(M,N) \arrow[dl] \arrow[dr] & & \\
 & M(2) \arrow[dl, "M(d^1)"'] \arrow[dr, "M(d^0)"] & & N(2) \arrow[dl, "N(d^1)"'] \arrow[dr, "N(d^0)"] & \\
A & & B & & C
\end{tikzcd}$$

  between the objects A and C, in which $\mathrm{P}(M,N)$ is the pullback $M(2) \times_{(M(d^1),N(D^0))} N(2)$ of $M(d^1)$ along $N(D^0)$ and the unlabeled arrows are the morphisms induced by the pullback;

- For each set A, the identity on A is the span $span \to V$ constantly equal to A, taking the morphisms of the domain to the identity on A;

- The natural isomorphisms for associativity and actions of identities are given by the universal property of pullbacks.

Assuming that V is a symmetric monoidal closed category with coproducts, a (small) category enriched in V is a monad of the bicategory V-Mat, while, if V' has pullbacks, an internal category of V' is a monad of $Span(V')$. But 1-cells and 2-cells of $\mathsf{Mnd}(V\text{-Mat})$ and $\mathsf{Mnd}(Span(V'))$ do not respectively coincide with what should be 1-cells and 2-cells of the 2-category of enriched categories and the 2-category of internal categories. In order to get the appropriate notion of 1-cells, firstly we need to consider co-morphisms of monads and, secondly, we need to consider proarrow equipments.

We introduce co-morphisms between monads via colax morphisms between lax algebras. For short, we use the notation introduced in Definition 5.4.1 and we define the 2-category of lax algebras and colax morphisms w.r.t. a pseudomonad on a 2-category, assuming, then, the minor adaptations to BICAT when necessary.

Definition 1.2.11. [Colax morphisms of lax algebras] Let $\mathcal{T} = (\mathcal{T}, m, \eta, \mu, \iota, \tau)$ be a pseudomonad on a 2-category $\mathfrak{B}$. We define the 2-category of lax $\mathcal{T}$-algebras and colax $\mathcal{T}$-morphisms, denoted by $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$, as follows:

1. Objects: *lax $\mathcal{T}$-algebras* as in Definition 5.4.1.

2. Morphisms: *colax $\mathcal{T}$-morphisms* $\mathtt{f} : \mathtt{y} \to \mathtt{z}$ between lax $\mathcal{T}$-algebras

$$\mathtt{y} = (Y, \mathfrak{alg}_{\mathtt{y}}, \underline{\mathtt{y}}, \underline{\mathtt{y}}_0), \mathtt{z} = (Z, \mathfrak{alg}_{\mathtt{z}}, \underline{\mathtt{z}}, \underline{\mathtt{z}}_0)$$

   are pairs $\mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle)$ in which $f : Y \to Z$ is a morphism in $\mathfrak{B}$ and

$$\langle \overline{\mathsf{f}} \rangle : f \mathfrak{alg}_{\mathtt{y}} \Rightarrow \mathfrak{alg}_{\mathtt{z}} \mathcal{T}(f)$$

is a 2-cell of $\mathfrak{B}$ such that, defining $\widehat{\mathscr{T}(\langle\bar{\mathsf{f}}\rangle)} := \mathfrak{t}^{-1}_{(\mathfrak{alg}_{\mathtt{z}})(\mathscr{T}(f))}\mathscr{T}(\langle\bar{\mathsf{f}}\rangle)\mathfrak{t}_{(f)(\mathfrak{alg}_{\mathtt{y}})}$, the equations

$$\begin{tikzcd}
 & \mathscr{T}^2Y \arrow[r, "\mathscr{T}^2(f)"] \arrow[dl, "\mathscr{T}(\mathfrak{alg}_{\mathtt{y}})"'] & \mathscr{T}^2Z \arrow[dl, "\mathscr{T}(\mathfrak{alg}_{\mathtt{z}})"] \arrow[dr, "m_Z"] & \\
\mathscr{T}Y \arrow[r, "\mathscr{T}(f)"'] \arrow[dr, "\mathfrak{alg}_{\mathtt{y}}"'] & \mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathtt{z}}"] \arrow[ul, phantom, "\overset{\widehat{\mathscr{T}(\langle\bar{\mathsf{f}}\rangle)}}{\Longrightarrow}"] \arrow[d, phantom, "\overset{\langle\bar{\mathsf{f}}\rangle}{\Longrightarrow}"] & \overset{\bar{\mathsf{z}}}{\Rrightarrow} & \mathscr{T}Z \arrow[dl, "\mathfrak{alg}_{\mathtt{z}}"] \\
 & Y \arrow[r, "f"'] & Z &
\end{tikzcd}
=
\begin{tikzcd}
 & \mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathtt{z}}"] & Z & \\
\mathscr{T}^2Z \arrow[ur, "m_Z"] & \overset{m_f}{\Longleftarrow} & \mathscr{T}Y \arrow[ul, "\mathscr{T}(f)"'] \arrow[r, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[u, phantom, "\overset{\langle\bar{\mathsf{f}}\rangle}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\bar{\mathsf{y}}}{\Longleftarrow}"] & Y \arrow[ul, "f"'] \\
 & \mathscr{T}^2Y \arrow[ul, "\mathscr{T}^2(f)"] \arrow[ur, "m_Y"] \arrow[r, "\mathscr{T}(\mathfrak{alg}_{\mathtt{y}})"'] & \mathscr{T}Y \arrow[ur, "\mathfrak{alg}_{\mathtt{y}}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[d, "\eta_Z"'] & & Y \arrow[ll, "f"'] \arrow[dl, "\eta_Y"'] \arrow[dd, equal] \\
\mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathtt{z}}"'] \arrow[ur, phantom, "\overset{\eta_f}{\Longleftarrow}"] & \mathscr{T}Y \arrow[l, "\mathscr{T}(f)"] \arrow[dr, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[r, phantom, "\overset{\bar{\mathsf{y}}_0}{\Longleftarrow}"] & \\
Z \arrow[ur, phantom, "\overset{\langle\bar{\mathsf{f}}\rangle}{\Longleftarrow}"] & & Y \arrow[ll, "f"]
\end{tikzcd}
=
\begin{tikzcd}
 & Z \arrow[dl, "\eta_Z"'] \arrow[dd, equal] & \\
\mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathtt{z}}"'] \arrow[r, phantom, "\overset{\bar{\mathsf{z}}_0}{\Longleftarrow}"] & & \\
 & Z & Y \arrow[l, "f"']
\end{tikzcd}$$

hold. The composition of colax $\mathscr{T}$-morphisms $\mathtt{f} : \mathtt{y} \to \mathtt{z}$ and $\mathtt{g} : \mathtt{x} \to \mathtt{y}$ of lax $\mathscr{T}$-algebras is denoted by $\mathtt{fg}$ and it is defined by the pair $(fg, \langle\overline{\mathsf{fg}}\rangle)$ in which $\langle\overline{\mathsf{fg}}\rangle$ is the 2-cell defined by

$$\langle\overline{\mathsf{fg}}\rangle \quad := \quad \begin{tikzcd}
X \arrow[d, "g"'] \arrow[r, phantom, "\overset{\langle\bar{\mathsf{g}}\rangle}{\Longrightarrow}", shift right=6ex] & \mathscr{T}X \arrow[l, "\mathfrak{alg}_{\mathtt{x}}"'] \arrow[d, "\mathscr{T}(g)"] \arrow[dd, bend left=60, "\mathscr{T}(fg)"] \\
Y \arrow[d, "f"'] & \mathscr{T}Y \arrow[l, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[d, "\mathscr{T}(f)"] \arrow[r, phantom, "\overset{\mathfrak{t}_{(f)(g)}}{\Longrightarrow}"] & {} \\
Z \arrow[r, phantom, "\overset{\langle\bar{\mathsf{f}}\rangle}{\Longrightarrow}", shift left=6ex] & \mathscr{T}Z \arrow[l, "\mathfrak{alg}_{\mathtt{z}}"]
\end{tikzcd}$$

3. 2-cells: a *$\mathscr{T}$-transformation* $\mathfrak{m} : \mathtt{f} \Rightarrow \mathtt{h}$ between lax $\mathscr{T}$-morphisms $\mathtt{f} = (f, \langle\bar{\mathsf{f}}\rangle)$, $\mathtt{h} = (h, \langle\bar{\mathsf{h}}\rangle)$ is a 2-cell $\mathfrak{m} : f \Rightarrow h$ in $\mathfrak{B}$ such that the equation below holds.

$$\begin{tikzcd}
\mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[d, bend right=50, "\mathscr{T}(h)"'] \arrow[d, bend left=50, "\mathscr{T}(f)"] \arrow[d, phantom, "\overset{\mathscr{T}(\mathfrak{m})}{\Longleftarrow}"] & \overset{\langle\bar{\mathsf{f}}\rangle}{\Longleftarrow} & Y \arrow[d, "f"] \\
\mathscr{T}Z \arrow[rr, "\mathfrak{alg}_{\mathtt{z}}"'] & & Z
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[d, "\mathscr{T}(h)"'] & \overset{\langle\bar{\mathsf{h}}\rangle}{\Longleftarrow} & Y \arrow[d, bend right=50, "h"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "\overset{\mathfrak{m}}{\Leftarrow}"] \\
\mathscr{T}Z \arrow[rr, "\mathfrak{alg}_{\mathtt{z}}"'] & & Z
\end{tikzcd}$$

The compositions of $\mathscr{T}$-transformations are defined in the obvious way and these definitions make $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$ a 2-category.

Remark 1.2.12. [Identity on a lax $\mathscr{T}$-algebra] The identities on a lax $\mathscr{T}$-algebra $\mathrm{y} = (Y, \mathfrak{alg}_{\mathrm{y}}, \underline{\overline{\mathrm{y}}}, \underline{\overline{\mathrm{y}}}_0)$ in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$ and in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell}$ are respectively given by

$$\left(\mathrm{id}_Y,\ \begin{tikzcd} \mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathrm{y}}"] \arrow[d, bend right=60, "\mathscr{T}(\mathrm{id}_Y)"'] \arrow[d, bend left=60, "\mathrm{id}_{\mathscr{T}Y}"] \arrow[d, phantom, "\overset{\mathfrak{t}_Y}{\Leftarrow}"] & = & Y \arrow[d, "\mathrm{id}_Y"] \\ \mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathrm{y}}"'] & & Y \end{tikzcd}\right) \quad \text{and} \quad \left(\mathrm{id}_Y,\ \begin{tikzcd} \mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathrm{y}}"] \arrow[d, bend right=60, "\mathscr{T}(\mathrm{id}_Y)"'] \arrow[d, bend left=60, "\mathrm{id}_{\mathscr{T}Y}"] \arrow[d, phantom, "\overset{\mathfrak{t}_Y^{-1}}{\Rightarrow}"] & = & Y \arrow[d, "\mathrm{id}_Y"] \\ \mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathrm{y}}"'] & & Y \end{tikzcd}\right).$$

Remark 1.2.13. [Colax algebras and coalgebras] By dualizing, we get below the notions of colax algebras, lax coalgebras and colax coalgebras. The dualization $(-)^{\mathrm{co}}$ preserves pseudomonads and pseudocomonads, while $(-)^{\mathrm{op}}$ takes pseudomonads to pseudocomonads and pseudocomonads to pseudomonads. In particular, given a pseudomonad $\mathscr{T}$ on a bicategory $\mathfrak{B}$ and a pseudocomonad $\mathscr{S}$ on a bicategory $\mathfrak{C}$, we have that $\mathscr{T}^{\mathrm{co}}$ is a pseudomonad on $\mathfrak{B}^{\mathrm{co}}$ and $\mathscr{S}^{\mathrm{op}}, \mathscr{S}^{\mathrm{coop}}$ are pseudomonads respectively on $\mathfrak{C}^{\mathrm{op}}$ and $\mathfrak{C}^{\mathrm{coop}}$. Therefore we can define:

- The bicategories of colax $\mathscr{T}$-algebras of the pseudomonad $\mathscr{T}$ (respectively with colax $\mathscr{T}$-morphisms and lax $\mathscr{T}$-morphisms):

$$\mathsf{Colax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell} := (\mathsf{Lax}\text{-}\mathscr{T}^{\mathrm{co}}\text{-}\mathsf{Alg}_{\ell})^{\mathrm{co}} \quad \text{and} \quad \mathsf{Colax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell} := (\mathsf{Lax}\text{-}\mathscr{T}^{\mathrm{co}}\text{-}\mathsf{Alg}_{\mathsf{c}\ell})^{\mathrm{co}};$$

- The bicategories of lax $\mathscr{S}$-coalgebras:

$$\mathsf{Lax}\text{-}\mathscr{S}\text{-}\mathsf{CoAlg}_{\ell} := (\mathsf{Lax}\text{-}\mathscr{S}^{\mathrm{op}}\text{-}\mathsf{Alg}_{\ell})^{\mathrm{op}} \quad \text{and} \quad \mathsf{Lax}\text{-}\mathscr{S}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell} := (\mathsf{Lax}\text{-}\mathscr{S}^{\mathrm{op}}\text{-}\mathsf{Alg}_{\mathsf{c}\ell})^{\mathrm{op}};$$

- The bicategory of colax $\mathscr{S}$-coalgebras:

$$\mathsf{Colax}\text{-}\mathscr{S}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell} := (\mathsf{Lax}\text{-}\mathscr{S}^{\mathrm{coop}}\text{-}\mathsf{Alg}_{\ell})^{\mathrm{coop}} \quad \text{and} \quad \mathsf{Colax}\text{-}\mathscr{S}\text{-}\mathsf{CoAlg}_{\ell} := (\mathsf{Lax}\text{-}\mathscr{S}^{\mathrm{coop}}\text{-}\mathsf{Alg}_{\mathsf{c}\ell})^{\mathrm{coop}}.$$

Remark 1.2.14. A colax $\mathscr{T}$-morphism $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$ is a $\mathscr{T}$-pseudomorphism if $\langle\overline{\mathsf{f}}\rangle$ is an invertible 2-cell. In particular, we have an inclusion 2-functor $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$, in which $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ denotes the 2-category of lax $\mathscr{T}$-algebras and $\mathscr{T}$-pseudomorphisms as introduced in Section 5.4.

Definition 1.2.15. [Co-bicategory of monads] The bicategory of monads and co-morphisms of a bicategory $\mathfrak{B}$, called herein the co-bicategory of monads and denoted by $\mathsf{Mnd}_{\mathrm{co}}(\mathfrak{B})$, is the bicategory of lax $\mathrm{Id}_{\mathfrak{B}}$-algebras and colax $\mathrm{Id}_{\mathfrak{B}}$-morphisms, that is to say: $\mathsf{Mnd}_{\mathrm{co}}(\mathfrak{B}) := \mathsf{Lax}\text{-}\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$.

The duals of the bicategories of monads are the bicategories of comonads. More precisely, the bicategories are defined by $\mathsf{CoMnd}(\mathfrak{B}) := (\mathsf{Mnd}(\mathfrak{B}^{\mathrm{co}}))^{\mathrm{co}}$ and $\mathsf{CoMnd}_{\mathrm{co}}(\mathfrak{B}) := (\mathsf{Mnd}_{\mathrm{co}}(\mathfrak{B}^{\mathrm{co}}))^{\mathrm{co}}$.

Lemma 1.2.16. *Given a bicategory $\mathfrak{B}$,*

$$\mathsf{Mnd}_{\mathrm{co}}(\mathfrak{B}) \cong \mathsf{Lax}\text{-}\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{CoAlg}_{\ell} \cong (\mathsf{Mnd}(\mathfrak{B}^{\mathrm{op}}))^{\mathrm{op}} \quad \textit{and} \quad \mathsf{Mnd}(\mathfrak{B}) \cong \mathsf{Lax}\text{-}\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}.$$

Herein, we actually do not need to give the full definition of proarrow equipment introduced in [112, 113]. Instead, we can give a much less structured version:

Definition 1.2.17. [Proarrow equipment] A proarrow equipment on a 2-category $\mathfrak{B}_0$ is a pseudofunctor $P : \mathfrak{B}_0 \to \mathfrak{B}_1$ which is the identity on objects and locally fully faithful.

Clearly, the category of proarrow equipments is a subcategory of the category of morphisms of the category of bicategories and pseudofunctors. Similarly, in the context of Remark 1.2.5, there is a tricategory of proarrow equipments which is a sub-tricategory of the tricategory of morphisms of BICAT. Thereby, it is natural to consider pseudomonads on pseudofunctors and on proarrow equipments.

Definition 1.2.18. [Pseudomonad on proarrow equipments] A pseudomonad $\mathscr{T}$ on a pseudofunctor $P : \mathfrak{B}_0 \to \mathfrak{B}_1$ is a pair $(\mathscr{T}_0, \mathscr{T}_1)$ in which $\mathscr{T}_0 = (\mathscr{T}^0, m^0, \eta^0, \mu^0, \iota^0, \tau^0)$ is a pseudomonad on $\mathfrak{B}_0$ and $\mathscr{T}_1 = (\mathscr{T}^1, m^1, \eta^1, \mu^1, \iota^1, \tau^1)$ is a pseudomonad on $\mathfrak{B}_1$ such that this pair of pseudomonads agrees with P, which means that:

$$\mathscr{T}^1 P = P\mathscr{T}^0, \quad m^1 P = Pm^0, \quad \eta^1 P = P\eta^0, \quad \mu^1 P = P\mu^0, \quad \iota^1 P = P\iota^0, \quad \tau^1 P = P\tau^0.$$

For our purposes, we could define a simpler version of pseudomonads on proarrow equipments on 2-categories. That is to say, we could say that a pseudomonad on $P : \mathfrak{B}_0 \to \mathfrak{B}_1$ is just a pseudomonad on $\mathfrak{B}_1$.

Definition 1.2.19. Given a pseudomonad $\mathscr{T} = (\mathscr{T}_0, \mathscr{T}_1)$ on a proarrow equipment $P : \mathfrak{B}_0 \to \mathfrak{B}_1$ on a 2-category $\mathfrak{B}_0$, the bicategory of lax $(\mathscr{T}, P)$-algebras, denoted herein by $\mathsf{Lax}\text{-}(\mathscr{T}, P)\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}$, is the pullback of P along the forgetful pseudofunctor $\mathsf{Lax}\text{-}\mathscr{T}_1\text{-}\mathsf{Alg}_{\mathsf{c}\ell} \to \mathfrak{B}_1$ in the category of bicategories and pseudofunctors.

The category of bicategories and pseudofunctors does not have all pullbacks (or equalizers). However, in the context of Definition 1.2.19, the pullbacks always exist. Moreover, $\mathsf{Lax}\text{-}(\mathscr{T}, P)\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}$ is always a 2-category. Explicitly, the objects of $\mathsf{Lax}\text{-}(\mathscr{T}, P)\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}$ are lax $\mathscr{T}$-algebras $\mathtt{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \overline{\underline{\mathsf{z}}}, \overline{\underline{\mathsf{z}}}_0)$ and the morphisms between two objects in $\mathsf{Lax}\text{-}(\mathscr{T}, P)\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}$ are colax $\mathscr{T}$-morphisms $\mathsf{g} = (g, \langle\overline{\mathsf{g}}\rangle)$ between lax $\mathscr{T}$-algebras such that $P(\dot{g}) = g$ for some morphism $\dot{g}$ of $\mathfrak{B}_0$. The composition of morphisms $\mathtt{f} = (P(\dot{f}), \langle\overline{\mathsf{f}}\rangle), \mathsf{g} = (P(\dot{g}), \langle\overline{\mathsf{g}}\rangle)$ is defined by $\mathtt{f} \cdot \mathsf{g} := (P(\dot{f}\dot{g}), \langle\overline{\underline{\mathsf{fg}}}\rangle)$ such that $\langle\overline{\underline{\mathsf{fg}}}\rangle$ is defined by

$$\begin{tikzcd}
 & X \arrow[dd, bend right=80, "P(\dot{f}\dot{g})"'] \arrow[d, "g=P(\dot{g})"'] & \mathscr{T}X \arrow[l, "\mathfrak{alg}_{\mathsf{x}}"'] \arrow[dd, "\mathscr{T}(fg)" description] \arrow[dd, bend left=80, "\mathscr{T}P(\dot{f}\dot{g})"] & \\
 {} \arrow[r, Rightarrow, "\mathfrak{p}^{-1}_{(\dot{f})(\dot{g})}"] & Y \arrow[d, "f=P(\dot{f})"'] \arrow[r, Rightarrow, "\langle\overline{\mathsf{fg}}\rangle"] & {} \arrow[r, Rightarrow, "\mathscr{T}\left(\mathfrak{p}_{(\dot{f})(\dot{g})}\right)"] & {} \\
 & Z & \mathscr{T}Z \arrow[l, "\mathfrak{alg}_{\mathsf{z}}"] &
\end{tikzcd}$$

in which $\langle \overline{\mathsf{fg}} \rangle$ denotes the 2-cell component of the usual composition of the colax $\mathscr{T}$-morphisms of $\mathtt{f}$ and $\mathtt{g}$.

The 2-*category of monads and co-morphisms in a proarrow equipment* $P : \mathfrak{B}_0 \to \mathfrak{B}_1$ is defined, then, by $\mathsf{Mnd}_{\mathrm{co}}(P) := \mathsf{Lax}\text{-}(\mathrm{Id}_P, P)\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$.

Definition 1.2.20. [Proarrows of Matrices] Given a symmetric closed monoidal category $(V, \otimes, I)$ with coproducts, the bicategory V-Mat gives a natural proarrow equipment $\mathsf{Set} \to V$-Mat on the locally discrete bicategory Set in which a function $f : A \to B$ is taken to $\check{f} : A \times B \to V$ defined by

$$\check{f}(i, j) = \begin{cases} I, & \text{if } f(i) = j \\ 0, & \text{otherwise} \end{cases}$$

in which 0 is the initial object.

Definition 1.2.21. [Proarrows of Spans] Given a category V with pullbacks, the bicategory $\mathit{Span}(V)$ gives a natural proarrow equipment $V \to \mathit{Span}(V)$ on the locally discrete bicategory V in which a morphism $f : A \to B$ is taken to the span

$$\begin{tikzcd} A & A \arrow[l, "\mathrm{id}_A"'] \arrow[r, "f"] & B. \end{tikzcd}$$

Definition 1.2.22. [Enriched Categories and Internal Categories] Given a symmetric closed monoidal category $(V, \otimes, I)$ with coproducts, the category of small V-enriched categories V-cat is the underlying category of $\mathsf{Mnd}_{\mathrm{co}}(\mathsf{Set} \to V\text{-Mat})$.

Given a category with pullbacks V', the category of V'-internal categories $\mathsf{Cat}(V')$ is the underlying category of $\mathsf{Mnd}_{\mathrm{co}}(V' \to \mathit{Span}(V'))$.

Remark 1.2.23. The introduction of enriched categories via monads of V-Mat does not work well for large enriched categories, unless we make tiresome considerations about enlargements of universes/completions. For our setting, however, it is enough to observe that, if V is a large symmetric monoidal closed category that has large coproducts (indexed by discrete categories), one can consider the bicategory as in Definition 1.2.9 but with discrete categories (objects of SET) instead of sets, and matrices indexed by them. By abuse of language, denoting this bicategory as V-Mat, it is clear that we can consider the category of large V-enriched categories V-Cat defined by the underlying (large) category $\mathsf{Mnd}_{\mathrm{co}}(\mathsf{SET} \to V\text{-Mat})$.

Explicitly, an internal category of a category with pullbacks V' is a span $A_1 \xleftarrow{d^1} A_2 \xrightarrow{d^0} A_1$, which we denote by $\mathtt{a}$, together with the multiplication and identity, 2-cells $\mathtt{a} \circ \mathtt{a} \Rightarrow \mathtt{a}$ and $\mathrm{id}_A \Rightarrow \mathtt{a}$ of $\mathit{Span}(V')$, satisfying the conditions of monad/lax $\mathrm{Id}_{\mathsf{Cat}}$-algebra of associativity and action of identity described in Definition 5.4.1. Recall that, by definition, the 2-cells $\mathtt{a} \circ \mathtt{a} \Rightarrow \mathtt{a}$ and $\mathrm{id}_A \Rightarrow \mathtt{a}$ are just morphisms $\underline{\mathsf{a}} : A_2 \times_{(d^0,d^1)} A_2 \to A_1$ and $\underline{\mathsf{a}}_0 : A_1 \to A_2$ of V' such that

$$\begin{tikzcd}
 & A_1 \arrow[ddr, equal] & & & A_1 & & \\
 & & & & & A_2 \arrow[ul, "d^0"'] & \\
A_2 \arrow[uur, "d^0"] \arrow[ddr, "d^1"'] & & A_1 \arrow[ll, "\underline{\mathsf{a}}_0"'] \arrow[ddl, equal] & A_2 \arrow[uur, "d^0"] \arrow[ddr, "d^1"'] & & & A_2 \times_{(d^0,d^1)} A_2 \arrow[lll, "\underline{\mathsf{a}}"'] \arrow[ul] \arrow[dl] \arrow[uull, bend right, "D^0"'] \arrow[ddll, bend left, "D^1"] \\
 & & & & & A_2 \arrow[dl, "d^1"'] & \\
 & A_1 & & & A_1 & &
\end{tikzcd}$$

commute. In this case, the object A_1 is called the *object of objects*, the object A_2 is called the *object of morphisms*, d^1 is the *domain morphism*, d^0 is the codomain morphism, the morphism $\underline{\mathsf{a}}$ is the *composition* and $\underline{\mathsf{a}}_0 : A_1 \to A_2$ is the *identity assigning*.

Definition 1.2.24. [Double Category] The category of double categories is the category of internal categories of Cat, that is to say Cat(Cat).

A double category $\mathfrak{X}$ is, then, a span $\mathfrak{X}_1 \xleftarrow{d^1} \mathfrak{X}_2 \xrightarrow{d^0} \mathfrak{X}_1$ of Cat with the composition and identity satisfying the usual conditions. Given such a double category, the objects of $\mathfrak{X}_0$ are called the objects of the double category, while the morphisms of $\mathfrak{X}_0$ are called vertical arrows. The objects f of $\mathfrak{X}_2$ are called horizontal arrows (or morphisms) of the double category $\mathfrak{X}$ and we denote it by $f : d^1(f) \to d^0(f)$ in which $d^1(f)$ is called the domain of f and $d^0(f)$ is the codomain of f. Finally, if $\alpha : f \to g$ is a morphism of $\mathfrak{X}_2$, we denote it by

$$\begin{tikzcd}
d^1(f) \arrow[r, "f"] \arrow[d, "d^1(\alpha)"'] \arrow[dr, phantom, "\alpha"] & d^0(f) \arrow[d, "d^0(\alpha)"] \\
d^1(g) \arrow[r, "g"] & d^0(g)
\end{tikzcd}$$

and we say that α is a 2-cell (or a *square*) of $\mathfrak{X}$.

A 2-*category* is just a Cat-enriched category. Clearly, if the category of objects of a double category $\mathfrak{D}$ is discrete, then it is a 2-category. More generally, the full subcategory of Cat(Cat) consisting of the double categories without nontrivial vertical arrows is isomorphic to the category of 2-categories and 2-functors Cat-Cat. For suitable categories V instead of Cat, a similar property holds. This generalization is given by Lemma 3.9.10.

1.3 Freely generated n-categories, coinserters and presentations of low dimensional categories

Chapter 2 corresponds to the paper *Freely generated n-categories, coinserters and presentations of low dimensional categories* [80], *DMUC 17-20* or *arXiv:1704.04474*. As the title suggests, the main subjects of this paper are related to development of a theory towards the study of presentations of low dimensional categories and freely generated categorical structures. Although it was the last paper to be written, it introduces some basic aspects of 2-category theory.

The chapter starts by giving basic aspects of 2-dimensional (weighted) colimits, focusing on coinserters, coequifiers and coinverters. These are the 2-dimensional colimits that have direct applications respectively in the study of adding free morphisms to a category, forcing relations between morphisms and categories of fractions [59, 61]. We are, however, interested in particular cases: freely generated categories, the left adjoint of the inclusion of thin categories and the left adjoint of the inclusion of groupoids.

The category of thin categories Prd and the category of groupoids Gr, as defined in 2.1.1, are replete reflective subcategories of Cat, with inclusions $M_1 : \mathsf{Prd} \to \mathsf{Cat}$ and $\mathscr{U}_1 : \mathsf{Gr} \to \mathsf{Cat}$. Hence there is an easy way of characterizing the images of M_1 and $\mathscr{U}_1$ via universal properties. More precisely, if we denote the reflectors by $\mathscr{M}_1 : \mathsf{Cat} \to \mathsf{Prd}$ and $\mathscr{L}_1 : \mathsf{Cat} \to \mathsf{Gr}$, we have that, given a category X of Cat, there is Y of Prd (Y of Gr) such that $M_1(Y) \cong X$ ($\mathscr{U}_1(Y) \cong X$) if and only if $M_1\mathscr{M}_1(X) \cong X$ ($\mathscr{U}_1\mathscr{L}_1(X) \cong X$). Since $\mathscr{U}_1\mathscr{L}_1(X)$ and $M_1\mathscr{M}_1(X)$ are given, respectively, by a coinverter and a coequifier, we get Proposition 2.1.4 and Theorem 2.1.7. Actually, within the context of Chapter 2, the most important fact in this direction is that we can get the category freely generated by a graph $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$, denoted by $\mathscr{F}_1(G)$, via the coinserter of G composed with the inclusion $\mathsf{Set} \to \mathsf{Cat}$. This is Lemma 2.2.3. It motivates one of the main points of the paper: to give freely generated categorical structures via coinserters.

After showing these facts, we give further background in Section 2.2. We study basic aspects of freely generated categories. We introduce basic notions of graphs (trees, forests and connectedness) relating with the groupoids and categories freely generated by them. We study reflexive graphs as well. The main importance of reflexive graphs within our context is the fact that the terminal category is freely generated by the terminal reflexive graph, while $\mathscr{F}_1$ does not preserve the terminal object. Finally, we also characterize the totally ordered sets that are free categories and show that freeness of groupoids is a property preserved by equivalences.

Then, we give the basic notions of presentations of this paper. On one hand, we show how every monad $\mathscr{T}$ induces a natural notion of presentation of $\mathscr{T}$-algebras in Section 2.3. In particular, we have that the free category monad induces a notion of presentation of categories. On the other hand, in Section 2.4 we define computads and show how it induces a notion of presentation of categories (and groupoids) with equations between morphisms. We compare both notions of presentations in Theorems 2.4.5 and 2.4.6.

Since we can see computads as free categories together with relations between morphisms, we introduce the suitable variation of the concept of computad to deal with presentation of groupoids: groupoidal computads. This has particular interest in Section 2.5 which deals with the relation between topology and computads. Moreover, the notion of presentations of groups via groupoidal computads coincides with the usual notion, as explained in Remark 2.4.19.

Section 2.5 establishes fundamental connections between topology and computads. We start by showing that the usual association of a *topological graph* to each graph, usually called *geometric realization*, consists of a left adjoint functor $\mathscr{F}_{\mathsf{Top}_1} : \mathsf{grph} \to \mathsf{Top}$ given objectwise by the *topological coinserter* introduced therein. In this context, we show that there is a distributive law between the monads induced by $\mathscr{F}_1$ and $\mathscr{F}_{\mathsf{Top}_1}$ which is constructed from the usual notion of concatenation of continuous paths. As a fundamental tool to study presentation of thin and locally thin categorical structures, using the distributive law mentioned above, we give a detailed construction of a 2-dimensional analogue of $\mathscr{F}_{\mathsf{Top}_1}$, denoted by $\mathscr{F}_{\mathsf{Top}_2} : \mathsf{cmp} \to \mathsf{Top}$, which associates each computad to a topological space.

We introduce the fundamental groupoid functor $\Pi : \mathsf{Top} \to \mathsf{gr}$ using the concept of presentation via computads. More precisely, we firstly associate a computad to each topological space and, then, Π is given by the groupoid presented by the associated computad. We prove theorems that relate the fundamental groupoid and freely generated groupoids. In particular, the last results of Section 2.5 state that the fundamental groupoid of $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g})$ is equivalent to the groupoid presented by $\mathfrak{g}$. These results show that $\mathscr{F}_{\mathsf{Top}_2}$ formalizes the usual construction of a CW-complex from each presentation of groups, the Lyndon–van Kampen diagrams [56].

In Section 2.6 we introduce our main notions of deficiency. More precisely, we introduce the notion of deficiency of a groupoid w.r.t. presentation of groupoids and the notion deficiency of a presentation of a $\mathscr{T}$-algebra w.r.t. a finite measure. Under suitable hypotheses, we find both notions to coincide in the particular case of groupoids. They also coincide with the usual notions of deficiency. In this section, mostly using the results of Section 2.5, we also develop a theory towards the study of thin categories and thin groupoids. For instance, we prove that, whenever $\mathfrak{g}$ is a computad such that $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g})$ has Euler characteristic smaller than 1, then the groupoid presented by $\mathfrak{g}$ is not thin. From this fact, we can prove that deficiency of a thin groupoid is 0, recasting and generalizing the result that says that trivial groups have deficiency 0.

Although our definition of computads is equivalent to the original one of [103], we introduce it via a graph satisfying a coincidence property, as it is shown in Remark 2.8.12. The main point of our perspective, besides giving a concise recursive definition, is that it allows us to prove that the 2-category freely generated by a computad $\mathfrak{g}$ is the coinserter of $\mathfrak{g}$, when we consider $\mathfrak{g}$ as a graph internal to an appropriate 2-category of 2-categories: the 2-category of 2-categories, 2-functors and icons [69]. In order to get freely generated n-categories via coinserters, we introduce higher dimensional analogues of icons. These concepts and results, including the general result that states that the n-category freely generated by an n-computad is its coinserter, are given in Sections 2.7 and 2.8.

We finish the paper studying aspects of presentations of 2-categories. We show simple examples of locally thin 2-categories that are not free and develop a theory to study presentations of locally thin and groupoidal 2-categories. We give efficient presentations of 2-categories related to the *strict descent object*. In the end of Chapter 2, we sketch a construction of the 3-dimensional analogue of $\mathscr{F}_{\mathsf{Top}_2}$, that is to say $\mathscr{F}_{\mathsf{Top}_3}$, which associates a 3-dimensional CW-complex to each 3-computad.

1.4 Beck-Chevalley

The *mate correspondence* [63, 98] is a fundamental framework in 2-dimensional category theory. For instance, this correspondence is in the core of the techniques of Chapter 2 to construct $\mathscr{F}_{\mathsf{Top}_2}$ and the distributive law between the monads induced by $\mathscr{F}_1$ and $\mathscr{F}_{\mathsf{Top}_1}$. Another important example is in

Chapter 3: the Beck-Chevalley condition, written in terms of a simple mate correspondence, plays an important role in the proof of the Bénabou-Robaud Theorem.

The main aim of this section is to present the Beck-Chevalley condition within the context of 2-dimensional monad theory. In order to do so, we present the most elementary version of *mate correspondence* in Theorem 1.4.6. We start by defining and giving elementary results on adjunctions in a 2-category.

Definition 1.4.1. [Adjunction] An adjunction in a 2-category $\mathfrak{A}$ is a quadruplet

$$(f : Y \to Z, g : Z \to Y, \varepsilon : fg \Rightarrow \mathrm{id}_Z, \eta : \mathrm{id}_Y \Rightarrow gf),$$

in which f, g are 1-cells and ε, η are 2-cells of $\mathfrak{A}$, such that

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}", pos=0.25] & Z \arrow[d, equal] \arrow[dl, "g" description] \\
Y \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\varepsilon}{\Longrightarrow}", pos=0.75] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[d, equal] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}", pos=0.25] & Z \arrow[l, "g"'] \arrow[d, equal] \\
Y \arrow[ur, "f" description] \arrow[ur, phantom, "\overset{\varepsilon}{\Longrightarrow}", pos=0.75] & Z \arrow[l, "g"]
\end{tikzcd}$$

are respectively the identity 2-cells $f \Rightarrow f$ and $g \Rightarrow g$. In this case, we denote the adjunction by $(f \dashv g, \varepsilon, \eta) : Y \to Z$. For short, we also denote such an adjunction by just $f \dashv g$ when the counit and unit are implicit.

Remark 1.4.2. If $(f \dashv g, \varepsilon, \eta) : Y \to Z$ is an adjunction, f is called left adjoint, g is called right adjoint, ε is called the counit and η is called the unit of the adjunction. Moreover, the equations of Definition 1.4.1 are called triangle identities.

Remark 1.4.3. It is clear that adjoints are unique up to isomorphism. More precisely, if $(\widetilde{f} \dashv g, \mu, \rho)$ and $(f \dashv g, \varepsilon, \eta) : Y \to Z$ then

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}", pos=0.25] & Z \arrow[d, equal] \arrow[dl, "g" description] \\
Y \arrow[r, "\widetilde{f}"'] \arrow[ur, phantom, "\overset{\mu}{\Longrightarrow}", pos=0.75] & Z
\end{tikzcd}
\quad \text{is the inverse of} \quad
\begin{tikzcd}
Y \arrow[r, "\widetilde{f}"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Longrightarrow}", pos=0.25] & Z \arrow[d, equal] \arrow[dl, "g" description] \\
Y \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\varepsilon}{\Longrightarrow}", pos=0.75] & Z
\end{tikzcd}$$

by the triangle identities. In particular, $f \cong \widetilde{f}$.

Let $\mathfrak{A}$ be a 2-category. We can construct a category of adjunctions $\mathfrak{A}^{\mathsf{adj}}$ of $\mathfrak{A}$. The objects of $\mathfrak{A}^{\mathsf{adj}}$ are the objects of $\mathfrak{A}$, but the morphisms are adjunctions $(f \dashv g, \varepsilon, \eta) : Y \to Z$. The identities are the adjunctions between identities with identities counit and unit. Given adjunctions $(f_2 \dashv g_2, \varepsilon_2, \eta_2) :$

$Y \to Z$ and $(f_1 \dashv g_1, \varepsilon_1, \eta_1) : X \to Y$, the composition is given by $(f_2 f_1 \dashv g_1 g_2, \varepsilon_3, \eta_3) : X \to Z$ in which ε_3 and η_3 are defined below.

$$\eta_3 := \begin{tikzcd}
X \arrow[drr, "f_1"] \arrow[dddd, equal] & & & & \\
& & Y \arrow[drr, "f_2"] \arrow[dd, "\mathrm{id}_Y"] & & \\
& \overset{\eta_1}{\Longrightarrow} & & \overset{\eta_2}{\Longrightarrow} & Z \arrow[dll, "g_2"] \\
& & Y \arrow[dll, "g_1"] & & \\
X & & & &
\end{tikzcd}
\qquad
\varepsilon_3 := \begin{tikzcd}
Z \arrow[dddd, equal] & & & & \\
& & Y \arrow[ull, "f_2"'] & & \\
& \overset{\varepsilon_2}{\Longleftarrow} & & \overset{\varepsilon_1}{\Longleftarrow} & X \arrow[ull, "f_1"'] \\
& & Y \arrow[uu, "\mathrm{id}_Y"] \arrow[urr, "g_1"'] & & \\
Z \arrow[urr, "g_2"'] & & & &
\end{tikzcd}$$

Of course, 2-functors take adjunctions to adjunctions. However, pseudofunctors do not. Instead, in this case, we can say that left (or right) adjoints are taken to left (or right) adjoints. More precisely, we have Lemma 1.4.5.

In order to prove such result, we use Lemma 1.4.4, a basic result on the image of pasting of 2-cells. Following the notation established in Definition 4.2.1, we have:

Lemma 1.4.4. *If $L : \mathfrak{A} \to \mathfrak{B}$ is a pseudofunctor, then:*

$$L\left(\begin{tikzcd}
W \arrow[rrr, "g"] \arrow[ddd, "m"] & & & X \arrow[ddd, "n"] \arrow[dddlll, "h"] \\
& \overset{\alpha}{\Longrightarrow} & & \\
& & \overset{\beta}{\Longrightarrow} & \\
Z \arrow[rrr, "f"] & & & Y
\end{tikzcd}\right) = \mathfrak{l}_{ng} \cdot \left(\begin{tikzcd}
L(W) \arrow[rrr, "L(g)"] \arrow[ddd, "L(m)"] & & & L(X) \arrow[ddd, "L(n)"] \arrow[dddlll, "L(h)"] \\
& \overset{\mathfrak{l}_{hg}^{-1} \cdot L(\alpha)}{\Longrightarrow} & & \\
& & \overset{L(\beta) \cdot \mathfrak{l}_{fh}}{\Longrightarrow} & \\
L(Z) \arrow[rrr, "L(f)"] & & & L(Y)
\end{tikzcd}\right) \cdot \mathfrak{l}_{fm}^{-1},$$

$$L\left(\begin{tikzcd}
Z & & & X \arrow[lll, "h"'] \\
& \overset{\alpha}{\Longrightarrow} & & \\
& & \overset{\gamma}{\Longrightarrow} & \\
W \arrow[uuu, "m"] \arrow[uuurrr, "g"] & & & Y \arrow[lll, "u"] \arrow[uuu, "t"']
\end{tikzcd}\right) = \mathfrak{l}_{ht} \cdot \left(\begin{tikzcd}
L(Z) & & & L(X) \arrow[lll, "L(h)"'] \\
& \overset{\mathfrak{l}_{hg}^{-1} \cdot L(\alpha)}{\Longrightarrow} & & \\
& & \overset{L(\gamma) \cdot \mathfrak{l}_{gu}}{\Longrightarrow} & \\
L(W) \arrow[uuu, "L(m)"] \arrow[uuurrr, "L(g)"] & & & L(Y) \arrow[lll, "L(u)"] \arrow[uuu, "L(t)"']
\end{tikzcd}\right) \cdot \mathfrak{l}_{mu}^{-1}.$$

Proof. Firstly, by the interchange law, it is clear that the right side of the first equation above is equal to

$$\mathfrak{l}_{ng} \cdot \big(L(\beta) * L(\mathrm{id}_g)\big) \cdot \left(\mathfrak{l}_{fh} * \mathrm{id}_{L(g)}\right) \cdot \left(\mathrm{id}_{L(f)} * \mathfrak{l}_{hg}^{-1}\right) \cdot \big(L(\mathrm{id}_f) * L(\alpha)\big) \cdot \mathfrak{l}_{fm}^{-1}.$$

Then, by the naturality of Definition 4.2.1, this is equal to

$$L(\beta * \mathrm{id}_g) \cdot \mathfrak{l}_{(fh)g} \cdot \left(\mathfrak{l}_{fh} * \mathrm{id}_{L(g)}\right) \cdot \left(\mathrm{id}_{L(f)} * \mathfrak{l}_{hg}^{-1}\right) \cdot \mathfrak{l}_{f(hg)}^{-1} \cdot L(\mathrm{id}_f * \alpha),$$

which is indeed equal to the left side of the equation above, since, by the associativity of Definition 4.2.1,

$$\begin{aligned}
\mathfrak{l}_{(fh)g} \cdot \left(\mathfrak{l}_{fh} * \mathrm{id}_{L(g)}\right) \cdot \left(\mathrm{id}_{L(f)} * \mathfrak{l}_{hg}^{-1}\right) \cdot \mathfrak{l}_{f(hg)}^{-1} &= \mathfrak{l}_{f(hg)} \cdot \left(\mathrm{id}_{L(f)} * \mathfrak{l}_{hg}\right) \cdot \left(\mathrm{id}_{L(f)} * \mathfrak{l}_{hg}\right)^{-1} \cdot \mathfrak{l}_{f(hg)}^{-1} \\
&= \mathrm{id}_{L(fhg)}
\end{aligned}$$

and $L(\beta * \mathrm{id}_g) \cdot L(\mathrm{id}_f * \alpha) = L\big((\beta * \mathrm{id}_g) \cdot (\mathrm{id}_f * \alpha)\big)$. This proves the first equation. The proof of the second one is analogous. ☐

Lemma 1.4.5. *If $L : \mathfrak{A} \to \mathfrak{B}$ is a pseudofunctor and $(f \dashv g, \varepsilon, \eta) : Y \to Z$ is an adjunction in $\mathfrak{A}$, then*

$$(L(f) \dashv L(g), \mathfrak{l}_Z^{-1} \cdot L(\varepsilon) \cdot \mathfrak{l}_{fg}, \mathfrak{l}_{gf}^{-1} \cdot L(\eta) \cdot \mathfrak{l}_Y)$$

is an adjunction in $\mathfrak{B}$. Whenever $(f \dashv g, \varepsilon, \eta)$ is implicit, we usually denote the induced adjunction above by $L(f) \dashv L(g)$.

Proof. By Lemma 1.4.4, we get that:

$$L\left(\begin{tikzcd}
Y \arrow[d, equal] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}" near start] & Z \arrow[l, "g"'] \arrow[d, equal] \\
Y \arrow[ur, "f" description] \arrow[ur, phantom, "\overset{\varepsilon}{\Longrightarrow}" near end, shift right=4] & Z \arrow[l, "g"]
\end{tikzcd}\right) = \mathfrak{l}_{g\,\mathrm{id}_Z} \cdot \left(\begin{tikzcd}
L(Y) \arrow[d, equal] \arrow[dr, phantom, "\overset{\mathfrak{l}_{gf}^{-1}\cdot L(\eta)}{\Longrightarrow}" near start] & L(Z) \arrow[l, "L(g)"'] \arrow[d, equal] \\
L(Y) \arrow[ur, "L(f)" description] \arrow[ur, phantom, "\overset{L(\varepsilon)\cdot\mathfrak{l}_{fg}}{\Longrightarrow}" near end, shift right=4] & L(Z) \arrow[l, "L(g)"]
\end{tikzcd}\right) \cdot \mathfrak{l}^{-1}_{\mathrm{id}_Y g},$$

$$L\left(\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}" near start] & Z \arrow[d, equal] \arrow[dl, "g" description] \\
Y \arrow[r, "f"'] \arrow[ur, phantom, "\overset{\varepsilon}{\Longrightarrow}" near end, shift right=4] & Z
\end{tikzcd}\right) = \mathfrak{l}_{\mathrm{id}_Z f} \cdot \left(\begin{tikzcd}
L(Y) \arrow[r, "L(f)"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\mathfrak{l}_{gf}^{-1}\cdot L(\eta)}{\Longrightarrow}" near start] & L(Z) \arrow[d, equal] \arrow[dl, "L(g)" description] \\
L(Y) \arrow[r, "L(f)"'] \arrow[ur, phantom, "\overset{L(\varepsilon)\cdot\mathfrak{l}_{fg}}{\Longrightarrow}" near end, shift right=4] & L(Z)
\end{tikzcd}\right) \cdot \mathfrak{l}^{-1}_{f\,\mathrm{id}_Y}.$$

Since, by Equation 2 of Definition 4.2.1,

$$\mathfrak{l}^{-1}_{f\,\mathrm{id}_Y} = \mathrm{id}_{L(f)} * \mathfrak{l}_Y, \quad \mathfrak{l}^{-1}_{\mathrm{id}_Y g} = \mathfrak{l}_Y * \mathrm{id}_{L(g)}, \quad \mathfrak{l}_{\mathrm{id}_Z f} = \mathfrak{l}_Z^{-1} * \mathrm{id}_{L(f)}, \quad \mathfrak{l}_{g\,\mathrm{id}_Z} = \mathrm{id}_{L(g)} * \mathfrak{l}_Z^{-1},$$

the result follows. ☐

Theorem 1.4.6 (Mate Correspondence)**.** *Let $(f \dashv g) := (f \dashv g, \varepsilon, \eta) : Z \to Y$ and $(l \dashv u) := (l \dashv u, \mu, \rho) : W \to X$ be adjunctions in a 2-category $\mathfrak{A}$. Given 1-cells $m : X \to Y$ and $n : W \to Z$ of $\mathfrak{A}$, there is a bijection $\mathfrak{A}(X,Z)(nu, gm) \cong \mathfrak{A}(W,Y)(fn, ml)$, given by $\alpha \mapsto \overline{\alpha}^{f\dashv g}_{l\dashv u}$ in which $\overline{\alpha}^{f\dashv g}_{l\dashv u}$ is defined by:*

$$\overline{\alpha}^{(f\dashv g)}_{(l\dashv u)} \quad := \quad \begin{tikzcd}
W \arrow[r, "l"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\rho}{\Rightarrow}" near start] & X \arrow[d, "m"] \arrow[dl, "u" description] \\
W \arrow[d, "n"'] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & Y \arrow[d, equal] \arrow[dl, "g" description] \arrow[dl, phantom, "\overset{\varepsilon}{\Rightarrow}", shift right=4] \\
Z \arrow[r, "f"'] & Y
\end{tikzcd}$$

We call $\overline{\alpha}_{l\dashv u}^{f\dashv g}$ *the mate of* α *under the adjunction* $(l\dashv u,\mu,\rho)$ *and the adjunction* $(f\dashv g,\varepsilon,\eta)$*.*

Proof. The map $\beta\mapsto\underline{\beta}_{l\dashv u}^{f\dashv g}$ defined by

$$\underline{\beta}_{(l\dashv u)}^{(f\dashv g)} \quad := \quad \begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & W \arrow[d, "n"] \arrow[dl, "l"'] \arrow[dl, phantom, "\overset{\mu}{\Leftarrow}", shift right=3ex] \\ X \arrow[d, "m"'] \arrow[r, phantom, "\overset{\beta}{\Leftarrow}"] & Z \arrow[dl, "f"] \arrow[d, equal] \arrow[dl, phantom, "\overset{\eta}{\Leftarrow}", shift left=3ex] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is clearly the inverse of $\alpha\mapsto\overline{\alpha}_{(l\dashv u)}^{(f\dashv g)}$. ☐

Actually, we can say much more about this correspondence. For instance, this is part of an isomorphism of double categories. More precisely, given a 2-category $\mathfrak{A}$, we define two double categories $\mathsf{RAdj}(\mathfrak{A})$ and $\mathsf{LAdj}(\mathfrak{A})$. The objects and the horizontal arrows of both double categories are the objects and 1-cells of $\mathfrak{A}$, while the vertical arrows are adjunctions $(f\dashv g,\varepsilon,\eta)$ of $\mathfrak{A}$. Given vertical arrows $(f\dashv g,\varepsilon,\eta):Z\to Y$, $(l\dashv u,\mu,\rho):W\to X$ and horizontal arrows $m:X\to Y$, $n:W\to Z$, the squares of $\mathsf{RAdj}(\mathfrak{A})$ are 2-cells $\alpha:nu\Rightarrow gm$ of $\mathfrak{A}$, while the squares of $\mathsf{LAdj}(\mathfrak{A})$ are 2-cells $\beta:fn\Rightarrow ml$. The composition of squares are given by pasting of 2-cells, the composition of horizontal arrows is the composition of 1-cells and the composition of vertical arrows is the composition of adjunctions as in $\mathfrak{A}^{\mathsf{adj}}$. It is clear, then, that the mate correspondence induces an isomorphism between $\mathsf{RAdj}(\mathfrak{A})$ and $\mathsf{LAdj}(\mathfrak{A})$. In particular, the mate correspondence respects vertical and horizontal compositions.

As a first application of these observations on the mate correspondence, we give another proof of the statement of Remark 1.4.3. Indeed, in the context of Remark 1.4.3, we take the 2-cells that are mates of the identity

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "g"'] & Z \arrow[d, "g"] \\ Y \arrow[r, equal] \arrow[ur, phantom, "="] & Y \end{tikzcd}$$

under the adjunctions $(\widetilde{f}\dashv g,\mu,\rho)$ and $(f\dashv g,\varepsilon,\eta)$, and under the adjunctions $(f\dashv g,\varepsilon,\eta)$ and $(\widetilde{f}\dashv g,\mu,\rho)$. They are respectively denoted by $\psi:f\mathrm{id}_Y\Rightarrow\mathrm{id}_Z\widetilde{f}$ and $\psi':\widetilde{f}\mathrm{id}_Y\Rightarrow\mathrm{id}_Z f$ (which actually are the 2-cells of Remark 1.4.3). Since the mate correspondence preserves horizontal composition, the compositions $\psi'\psi$ and $\psi\psi'$ are respectively the mates of the 2-cell $\mathrm{id}_Y g=g\mathrm{id}_Z$ above under $f\dashv g$ and itself, and under $\widetilde{f}\dashv\widetilde{g}$ and itself; that is to say, the identity on f and the identity on $\widetilde{f}$. In particular, $\psi:f\Rightarrow\widetilde{f}$ is an isomophism.

Remark 1.4.7. If $(f\dashv g,\varepsilon,\eta):Y\to Z$ is an adjunction in the 2-category Cat, we know that $Z(f-,-)\cong Y(-,g-)$. The mate correspondence generalizes this fact since, assuming now that $(l\dashv u,\mu,\rho):Y'\to Z'$ is an adjunction in a 2-category $\mathfrak{A}$, as a particular case of Theorem 1.4.6, we conclude that

$$\mathfrak{A}(X,Y')(-,u-)\cong\mathfrak{A}(X,Z')(l-,-)$$

for any object X of $\mathfrak{A}$. Still, up to size considerations, the Yoneda structure [110] of CAT implies that: given a functor $\widetilde{f}:Y\to Z$, $\widetilde{f}\dashv\widetilde{g}$ in Cat if and only if there is a natural isomorphism

$$\begin{tikzcd}
(\mathsf{CAT}\,[Y,\mathsf{SET}])^{\mathrm{op}} \arrow[r, "(\mathsf{CAT}[\widetilde{g},\mathsf{SET}])^{\mathrm{op}}"] \arrow[dr, phantom, "\cong"] & (\mathsf{CAT}\,[Z,\mathsf{SET}])^{\mathrm{op}} \\
Y \arrow[u, "\mathscr{Y}^{\mathrm{op}}_{Y^{\mathrm{op}}}"] \arrow[r, "\widetilde{f}"'] & Z \arrow[u, "\mathscr{Y}^{\mathrm{op}}_{Z^{\mathrm{op}}}"']
\end{tikzcd} \qquad (\varphi)$$

in which $\mathscr{Y}_{Y^{\mathrm{op}}}$, $\mathscr{Y}_{Z^{\mathrm{op}}}$ denote the Yoneda embeddings. Moreover, it should be noted that, if $\widetilde{f}\dashv\widetilde{g}$, then $(\mathsf{CAT}\,[\widetilde{g},\mathsf{SET}])^{\mathrm{op}}\dashv\left(\mathsf{CAT}\left[\widetilde{f},\mathsf{SET}\right]\right)^{\mathrm{op}}$ by the 2-functoriality of $(\mathsf{CAT}\,[-,\mathsf{SET}])^{\mathrm{op}}:\mathsf{CAT}^{\mathrm{coop}}\to\mathsf{CAT}$.

It is clear that the images of the mates by 2-functors are the mates of the images. In the case of pseudofunctors, it follows from Lemma 1.4.4 that:

Lemma 1.4.8. *Let $E:\mathfrak{A}\to\mathfrak{B}$ be a pseudofunctor and $(f\dashv g,\varepsilon,\eta)$, $(l\dashv u,\mu,\rho)$ adjunctions in $\mathfrak{A}$. Given a 2-cell $\alpha:nu\Rightarrow gm$,*

$$\widehat{E\left(\overline{\alpha}^{f\dashv g}_{l\dashv u}\right)}=\overline{\widehat{E(\alpha)}}^{E(f)\dashv E(g)}_{E(l)\dashv E(u)},$$

in which $\widehat{E\left(\overline{\alpha}^{f\dashv g}_{l\dashv u}\right)}:=\mathfrak{e}_{ml}^{-1}\cdot E\left(\overline{\alpha}^{f\dashv g}_{l\dashv u}\right)\cdot\mathfrak{e}_{fn}$ and $\widehat{E(\alpha)}:=\mathfrak{e}_{gm}^{-1}\cdot E(\alpha)\cdot\mathfrak{e}_{nu}$. In other words,

$$\mathfrak{e}_{ml}^{-1}\cdot E\left(\overline{\alpha}^{f\dashv g}_{l\dashv u}\right)\cdot\mathfrak{e}_{fn}$$

is the mate of $\mathfrak{e}_{gm}^{-1}\cdot E(\alpha)\cdot\mathfrak{e}_{nu}$ under $E(l)\dashv E(u)$ and $E(f)\dashv E(g)$.

We also have an important result relating mates and pseudonatural transformations:

Lemma 1.4.9. *Let $\lambda:E\longrightarrow L$ be a pseudonatural transformation between pseudofunctors. If $(f\dashv g,\varepsilon,\eta):Y\to Z$ is an adjunction of $\mathfrak{A}$, then the mate of*

$$\begin{tikzcd}
 & E(Y) \arrow[dl, "E(f)"'] \arrow[d, "\lambda_Y"] \\
E(Z) \arrow[d, "\lambda_Z"'] & L(Y) \arrow[l, Rightarrow, "\lambda_f"'] \arrow[dl, "L(f)"] \\
L(Z) &
\end{tikzcd}$$

under the adjunction $(E(f)\dashv E(g),\mathfrak{e}_Z^{-1}\cdot E(\varepsilon)\cdot\mathfrak{e}_{fg},\mathfrak{e}_{gf}^{-1}\cdot E(\eta)\cdot\mathfrak{e}_Y)$ and $(L(f)\dashv L(g),\mathfrak{l}_Z^{-1}\cdot L(\varepsilon)\cdot\mathfrak{l}_{fg},\mathfrak{l}_{gf}^{-1}\cdot L(\eta)\cdot\mathfrak{l}_Y)$ is equal to λ_g^{-1}.

Proof. In order to simplify the terminology, we prove it for a pseudonatural transformation $\lambda:E\longrightarrow L$ between 2-functors. The proof in the case of pseudofunctors is analogous.

The proof consists in verifying that the mate $\lambda_{f\;E(f)\dashv E(g)}^{\;L(f)\dashv L(g)}$ composed with λ_g is equal to the identity $L(g)\lambda_Z \Rightarrow L(g)\lambda_Z$. Firstly, observe that this composition is equal to

$$\begin{tikzcd}
 & E(Z) \arrow[dl, equal] \arrow[d, "E(g)"] \arrow[dr, "\lambda_Z"] \arrow[dl, phantom, "\overset{E(\varepsilon)}{\Leftarrow}", shift left=3] & \\
E(Z) \arrow[dd, "\lambda_Z"'] & E(Y) \arrow[l, "E(f)"'] \arrow[d, "\lambda_Y"] \arrow[r, phantom, "\overset{\lambda_g}{\Leftarrow}"] & L(Z) \arrow[dl, "L(g)"] \\
 & L(Y) \arrow[dl, "L(f)"'] \arrow[d, equal] \arrow[l, phantom, "\overset{\lambda_f}{\Leftarrow}"] & \\
L(Z) \arrow[r, "L(g)"'] \arrow[ur, phantom, "\overset{L(\eta)}{\Leftarrow}", shift right=3] & L(Y) &
\end{tikzcd}$$

Since $\lambda_{\mathrm{id}_Z} = \mathrm{id}_{\lambda_Z}$, by Equations 1 and 3 of Definition 4.2.2 we get that this composition is equal to

$$\begin{tikzcd}
E(Z) \arrow[r, "\lambda_Z"] \arrow[dr, "E(g)"] \arrow[dd, equal] & L(Z) \arrow[dr, "L(g)"] & & & E(Z) \arrow[r, "\lambda_Z"] \arrow[dd, equal] & L(Z) \arrow[dr, "L(g)"] \arrow[dd, equal] & \\
 & E(Y) \arrow[dl, "E(f)"] \arrow[l, phantom, "\overset{E(\varepsilon)}{\Leftarrow}"] \arrow[r, phantom, "\overset{\lambda_{fg}}{\Leftarrow}"] & L(Y) \arrow[dl, "L(f)"'] \arrow[d, equal] & = & {} \arrow[r, phantom, "\overset{\mathrm{id}_{\lambda_Z}}{\Leftrightarrow}"] & {} & L(Y) \arrow[dl, "L(f)"'] \arrow[d, equal] \arrow[l, phantom, "\overset{L(\varepsilon)}{\Leftarrow}"] \\
E(Z) \arrow[r, "\lambda_Z"'] & L(Z) \arrow[r, "L(g)"'] \arrow[ur, phantom, "\overset{L(\eta)}{\Leftarrow}", shift right=3] & L(Y) & & E(Z) \arrow[r, "\lambda_Z"'] & L(Z) \arrow[r, "L(g)"'] \arrow[ur, phantom, "\overset{L(\eta)}{\Leftarrow}", shift right=3] & L(Y)
\end{tikzcd}$$

which is clearly equal to the identity $L(g)\lambda_Z \Rightarrow L(g)\lambda_Z$, since $(L(f) \dashv L(g), L(\varepsilon), L(\eta))$ is an adjunction. □

Definition 1.4.10. [Beck-Chevalley condition] Let $(f \dashv g) := (f \dashv g, \varepsilon, \eta) : Z \to Y$ and $(l \dashv u) := (l \dashv u, \mu, \rho) : W \to X$ be adjunctions in a 2-category $\mathfrak{A}$. Assume that $\alpha : nu \Rightarrow gm$ is a 2-cell of $\mathfrak{A}$. We say that

$$\begin{tikzcd}
 & X \arrow[dl, "u"'] \arrow[d, "m"] \\
W \arrow[d, "n"'] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & Y \arrow[dl, "g"] \\
Z &
\end{tikzcd}$$

satisfies the Beck-Chevalley condition if the mate of α under $l \dashv u$ and $f \dashv g$ is an invertible 2-cell.

Outside any context, the meaning of the Beck-Chevalley condition might seem vacuous. Even when some context is provided, it is many times considered as an isolated technical condition. In this thesis, however, this condition is always applied in the context of *doctrinal adjunction*. More precisely, our informal perspective is that, whenever the Beck-Chevalley condition plays an important role, we can frame our problem in terms of 2-dimensional monad theory, getting a problem of lifting of adjunctions of the base 2-category to the 2-category of pseudoalgebras. The Beck-Chevalley condition is precisely the obstruction condition to this lifting. The most important example of this approach in this thesis is Chapter 3 or, more specifically, the proof of the Bénabou-Robaud Theorem presented therein.

Below, we briefly explain the Beck-Chevalley condition within the context of 2-dimensional monad theory. This section can be considered, then, as prerequisite to the understanding of Section 3.7, since herein we do not assume familiarity with the doctrinal adjunction. We also show in 1.4.17 how, within our context, Kock-Zöberlein pseudomonads encompass the situation of “the Beck-Chevalley condition always holding”.

We start by showing the most elementary version of an important bijection between colax and lax $\mathcal{T}$-structures in adjoint morphisms. Again, the mate correspondence is the basic technique to introduce this bijection.

Let $\mathcal{T}$ be a pseudomonad on a 2-category $\mathfrak{B}$ and $g : Z \to Y$ a morphism of $\mathfrak{B}$. Given lax $\mathcal{T}$-algebras $\mathsf{y} = (Y, \mathfrak{alg}_{\mathsf{y}}, \overline{\mathsf{y}}, \underline{\overline{\mathsf{y}}}_0)$ and $\mathsf{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \underline{\overline{\mathsf{z}}}, \underline{\overline{\mathsf{z}}}_0)$, the *collection of lax $\mathcal{T}$-structures for $g : Z \to Y$ w.r.t.* z *and* y, denoted by $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\ell}(\mathsf{z},\mathsf{y})_g$, is the pullback of the inclusion of g in the category of morphisms $\mathfrak{B}(Z,Y)$, $1 \to \mathfrak{B}(Z,Y)$, along the functor $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\ell}(\mathsf{z},\mathsf{y}) \to \mathfrak{B}(Z,Y)$ induced by the forgetful 2-functor. Analogously, given a morphism $f : Y \to Z$ of $\mathfrak{B}$, $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}(\mathsf{y},\mathsf{z})_f$ is the pullback of the inclusion of f into $\mathfrak{B}(Y,Z)$ along the forgetful functor $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}(\mathsf{y},\mathsf{z}) \to \mathfrak{B}(Y,Z)$.

It is clear, then, that a lax $\mathcal{T}$-morphism in $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\ell}(\mathsf{z},\mathsf{y})_g$ corresponds to a 2-cell $\langle \overline{\mathsf{g}} \rangle : \mathfrak{alg}_{\mathsf{y}} \mathcal{T}(g) \Rightarrow g\mathfrak{alg}_{\mathsf{z}}$ of $\mathfrak{B}$ satisfying the axioms of Definition 5.4.1, while a colax $\mathcal{T}$-morphism in $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}(\mathsf{y},\mathsf{z})_f$ corresponds to a 2-cell $\langle \overline{\mathsf{f}} \rangle : f\mathfrak{alg}_{\mathsf{y}} \Rightarrow \mathfrak{alg}_{\mathsf{z}} \mathcal{T}(f)$ of $\mathfrak{B}$ satisfying the axioms of Definition 1.2.11.

Moreover, we can consider the *category of lax $\mathcal{T}$-structures for f w.r.t. lax $\mathcal{T}$-algebras* which is the pullback of the inclusion of the morphism f into $\mathfrak{B}$, $2 \to \mathfrak{B}$, along the forgetful 2-functor $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\ell} \to \mathfrak{B}$. Finally, the *category of colax $\mathcal{T}$-structures for g w.r.t. lax $\mathcal{T}$-algebras* is the pullback of the inclusion of the morphism g into $\mathfrak{B}$ along $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell} \to \mathfrak{B}$.

Theorem 1.4.11 (Colax and lax structures in adjoints)**.** *Let $\mathcal{T}$ be a pseudomonad on $\mathfrak{B}$ and*

$$(f \dashv g, \varepsilon, \eta) : Y \to Z$$

an adjunction in $\mathfrak{B}$. Given lax $\mathcal{T}$-algebras $\mathsf{y} = (Y, \mathfrak{alg}_{\mathsf{y}}, \underline{\overline{\mathsf{y}}}, \underline{\overline{\mathsf{y}}}_0)$ *and* $\mathsf{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \underline{\overline{\mathsf{z}}}, \underline{\overline{\mathsf{z}}}_0)$*, the mate correspondence under the adjunction* $(\mathcal{T}(f) \dashv \mathcal{T}(g), \mathfrak{t}_{Z}^{-1} \cdot \mathcal{T}(\varepsilon) \cdot \mathfrak{t}_{fg}, \mathfrak{t}_{gf}^{-1} \cdot \mathcal{T}(\eta) \cdot \mathfrak{t}_{Y})$ *and the adjunction* $(f \dashv g, \varepsilon, \eta)$ *induces a bijection*

$$\diamond : \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\ell}(\mathsf{z},\mathsf{y})_g \cong \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}(\mathsf{y},\mathsf{z})_f.$$

These bijections induce an isomorphism between the category of lax $\mathcal{T}$-structures for $g : Z \to Y$ and category of colax $\mathcal{T}$-structures for $f : Y \to Z$ w.r.t. lax $\mathcal{T}$-algebras.

Proof. Assume that $(f \dashv g, \varepsilon, \eta) : Y \to Z$ is an adjunction in $\mathfrak{B}$ and $\mathcal{T}$ is a pseudomonad on $\mathfrak{B}$ as in the hypothesis. Given 2-cells $\langle \overline{\mathsf{g}} \rangle : \mathfrak{alg}_{\mathsf{z}} \mathcal{T}(g) \Rightarrow g\mathfrak{alg}_{\mathsf{y}}$ and $\langle \overline{\mathsf{f}} \rangle : f\mathfrak{alg}_{\mathsf{y}} \Rightarrow \mathfrak{alg}_{\mathsf{z}} \mathcal{T}(f)$ that are mates under the adjunctions $(\mathcal{T}(f) \dashv \mathcal{T}(g), \mathfrak{t}_{Z}^{-1} \cdot \mathcal{T}(\varepsilon) \cdot \mathfrak{t}_{fg}, \mathfrak{t}_{gf}^{-1} \cdot \mathcal{T}(\eta) \cdot \mathfrak{t}_{Y})$ and $(f \dashv g, \varepsilon, \eta)$, we have that:

1. By Lemmas 1.4.8 and 1.4.9, the 2-cells

$$\begin{tikzcd}
 & \mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"] & Z & \\
\mathscr{T}^2Z \arrow[ur, "m_Z"] \arrow[rr, phantom, "\overset{m_f}{\Longleftarrow}"] & & \mathscr{T}Y \arrow[ul, "\mathscr{T}(f)"] \arrow[r, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[d, phantom, "\overset{\overline{\mathsf{y}}}{\Longleftarrow}"] & Y \arrow[ul, "f"'] \\
 & \mathscr{T}^2Y \arrow[ul, "\mathscr{T}^2(f)"] \arrow[ur, "m_Y"] \arrow[r, "\mathscr{T}(\mathfrak{alg}_{\mathsf{y}})"'] & \mathscr{T}Y \arrow[ur, "\mathfrak{alg}_{\mathsf{y}}"'] &
\arrow[from=1-2, to=2-4, phantom, "\overset{\langle\bar{\mathsf{f}}\rangle}{\Longleftarrow}"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathscr{T}^2Y \arrow[r, "\mathscr{T}^2(f)"] \arrow[dl, "\mathscr{T}(\mathfrak{alg}_{\mathsf{y}})"'] \arrow[d, phantom, "\overset{\widehat{\mathscr{T}(\langle\bar{\mathsf{f}}\rangle)}}{\Longrightarrow}"] & \mathscr{T}^2Z \arrow[dl, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"] \arrow[dr, "m_Z"] & \\
\mathscr{T}Y \arrow[r, "\mathscr{T}(f)"'] \arrow[dr, "\mathfrak{alg}_{\mathsf{y}}"'] & \mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[d, phantom, "\overset{\langle\bar{\mathsf{f}}\rangle}{\Longrightarrow}"] \arrow[rr, phantom, "\overset{\overline{\mathsf{z}}}{\Longrightarrow}"] & & \mathscr{T}Z \arrow[dl, "\mathfrak{alg}_{\mathsf{z}}"] \\
 & Y \arrow[r, "f"'] & Z &
\end{tikzcd}$$

are respectively the mates of

$$\begin{tikzcd}
 & \mathscr{T}^2Z \arrow[r, "\mathscr{T}^2(g)"] \arrow[dl, "m_Z"'] \arrow[d, phantom, "\overset{m_g^{-1}}{\Longleftarrow}"] & \mathscr{T}^2Y \arrow[dl, "m_Y"] \arrow[dr, "\mathscr{T}(\mathfrak{alg}_{\mathsf{y}})"] & \\
\mathscr{T}Z \arrow[r, "\mathscr{T}(g)"'] \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"'] & \mathscr{T}Y \arrow[dr, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[d, phantom, "\overset{\langle\bar{\mathsf{g}}\rangle}{\Longleftarrow}"] \arrow[rr, phantom, "\overset{\overline{\mathsf{y}}}{\Longleftarrow}"] & & \mathscr{T}Y \arrow[dl, "\mathfrak{alg}_{\mathsf{y}}"] \\
 & Z \arrow[r, "g"'] & Y &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathsf{y}}"] & Y & \\
\mathscr{T}^2Y \arrow[ur, "\mathscr{T}(\mathfrak{alg}_{\mathsf{y}})"] \arrow[rr, phantom, "\overset{\widehat{\mathscr{T}(\langle\bar{\mathsf{g}}\rangle)}}{\Longrightarrow}"] & & \mathscr{T}Z \arrow[ul, "\mathscr{T}(g)"] \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[d, phantom, "\overset{\overline{\mathsf{z}}}{\Longrightarrow}"] & Z \arrow[ul, "g"'] \\
 & \mathscr{T}^2Z \arrow[ul, "\mathscr{T}^2(g)"] \arrow[ur, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"] \arrow[r, "m_Z"'] & \mathscr{T}Z \arrow[ur, "\mathfrak{alg}_{\mathsf{z}}"'] &
\arrow[from=1-2, to=2-4, phantom, "\overset{\langle\bar{\mathsf{g}}\rangle}{\Longrightarrow}"]
\end{tikzcd}$$

under the adjunctions $\mathscr{T}^2(f) \dashv \mathscr{T}^2(g)$ and $f \dashv g$.

2. By Lemma 1.4.9, the 2-cells

$$\begin{tikzcd}
Z \arrow[d, "\eta_Z"'] \arrow[dr, phantom, "\overset{\eta_f}{\Longleftarrow}"] & & Y \arrow[ll, "f"'] \arrow[dl, "\eta_Y"] \arrow[dd, equal] \\
\mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"'] \arrow[dr, phantom, "\overset{\langle\bar{\mathsf{f}}\rangle}{\Longleftarrow}"] & \mathscr{T}Y \arrow[l, "\mathscr{T}(f)"] \arrow[dr, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[r, phantom, "\overset{\overline{\mathsf{y}}_0}{\Longleftarrow}"] & \\
Z & & Y \arrow[ll, "f"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & Z \arrow[dl, "\eta_Z"'] \arrow[dd, equal] & \\
\mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"'] \arrow[r, phantom, "\overset{\overline{\mathsf{z}}_0}{\Longleftarrow}"] & & \\
 & Z & Y \arrow[l, "f"']
\end{tikzcd}$$

are respectively the mates of

$$\begin{tikzcd}
Z \arrow[rr, "g"] \arrow[d, "\eta_Z"'] \arrow[dr, phantom, "\overset{\eta_g^{-1}}{\Longleftarrow}"] & & Y \arrow[dl, "\eta_Y"'] \arrow[dd, equal] \\
\mathscr{T}Z \arrow[r, "\mathscr{T}(g)"'] \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"'] \arrow[dr, phantom, "\overset{\langle\bar{\mathsf{g}}\rangle}{\Longleftarrow}"] & \mathscr{T}Y \arrow[dr, "\mathfrak{alg}_{\mathsf{y}}"'] \arrow[r, phantom, "\overset{\overline{\mathsf{y}}_0}{\Longleftarrow}"] & \\
Z \arrow[rr, "g"'] & & Y
\end{tikzcd}$$

$$\begin{tikzcd}
 & Z \arrow[dl, "\eta_Z"'] \arrow[dd, equal] & \\
\mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"'] \arrow[r, phantom, "\overset{\overline{\mathsf{z}}_0}{\Longleftarrow}"] & & \\
 & Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

under $f \dashv g$ and itself.

Therefore $\langle\overline{\mathtt{g}}\rangle : \mathfrak{alg}_{\mathtt{z}}\mathscr{T}(g) \Rightarrow g\mathfrak{alg}_{\mathtt{y}}$ corresponds to a lax $\mathscr{T}$-morphism in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell}(\mathtt{z},\mathtt{y})_g$ if and only if $\langle\overline{\mathtt{f}}\rangle : f\mathfrak{alg}_{\mathtt{y}} \Rightarrow \mathfrak{alg}_{\mathtt{z}}\mathscr{T}(f)$ corresponds to a colax $\mathscr{T}$-morphism in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}(\mathtt{y},\mathtt{z})_f$. □

Remark 1.4.12. Clearly, we have dual results. For instance, given colax $\mathscr{T}$-algebras $\mathtt{z},\mathtt{y}$ and an adjunction $(f \dashv g, \varepsilon, \eta) : Y \to Z$ in the base 2-category $\mathfrak{B}$, we can analogously define the collections $\mathsf{Colax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}(\mathtt{y},\mathtt{z})_f$ and $\mathsf{Colax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell}(\mathtt{z},\mathtt{y})_g$ of colax $\mathscr{T}$-structures for f and lax $\mathscr{T}$-structures for g. The mate correspondence induces a bijection between such collections.

Now, we can introduce the Beck-Chevalley condition in the context of lax $\mathscr{T}$-morphisms, being a particular case of Definition 1.4.10. The relevance of this case is demonstrated in Theorem 1.4.14.

Definition 1.4.13. [Beck-Chevalley within 2-dimensional monad theory] Let $\mathscr{T}$ be a pseudomonad on $\mathfrak{B}$ and $(f \dashv g, \varepsilon, \eta) : Y \to Z$ an adjunction in $\mathfrak{B}$. Assume that $\mathtt{g} = (g, \langle\overline{\mathtt{g}}\rangle) : \mathtt{z} \to \mathtt{y}$ is a lax $\mathscr{T}$-morphism between the lax $\mathscr{T}$-algebras $\mathtt{z} = (Z, \mathfrak{alg}_{\mathtt{z}}, \underline{\mathtt{z}}, \underline{\mathtt{z}}_0)$ and $\mathtt{y} = (Y, \mathfrak{alg}_{\mathtt{y}}, \underline{\mathtt{y}}, \underline{\mathtt{y}}_0)$.

We say that $\mathtt{g}$ *satisfies the Beck-Chevalley condition* if the corresponding colax $\mathscr{T}$-morphism $(f, \diamond\langle\overline{\mathtt{g}}\rangle) : \mathtt{z} \to \mathtt{y}$ via the bijection of Theorem 1.4.11 is a $\mathscr{T}$-pseudomorphism. That is to say, $\mathtt{g}$ satisfies the Beck-Chevalley condition if

$$\diamond\langle\overline{\mathtt{g}}\rangle = \begin{tikzcd}
\mathscr{T}Y \arrow[rr, "\mathscr{T}(f)"] \arrow[dd, equal] & & \mathscr{T}Z \arrow[ddll, "\mathscr{T}(g)" description] \arrow[dd, "\mathfrak{alg}_{\mathtt{z}}"] \\
 & \overset{\mathfrak{t}^{-1}_{gf}\cdot\mathscr{T}(\eta)\cdot\mathfrak{t}_Y}{\Longrightarrow} & \\
\mathscr{T}Y \arrow[dd, "\mathfrak{alg}_{\mathtt{y}}"'] & \overset{\langle\overline{\mathtt{g}}\rangle}{\Longrightarrow} & Z \arrow[ddll, "g" description] \arrow[dd, equal] \\
 & \overset{\varepsilon}{\Longrightarrow} & \\
Z \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

is an invertible 2-cell.

Given a 2-monad $\mathscr{T}$ on a 2-category $\mathfrak{B}$, while the forgetful 2-functors $\mathscr{T}\text{-}\mathsf{Alg}_{\mathrm{s}} \to \mathfrak{B}$ and $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathfrak{B}$ respectively reflect isomorphisms and equivalences, the forgetful 2-functor

$$\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell} \to \mathfrak{B}$$

reflects right adjoints that satisfy the Beck-Chevalley condition. More generally, the *Doctrinal Adjunction* characterizes when the unit and the counit of an adjunction satisfy the condition of being a $\mathscr{T}$-transformation (given in in Eq. 3 of Definition 5.4.1).

Theorem 1.4.14 (Doctrinal Adjunction)**.** *Let $\mathscr{T}$ be a pseudomonad on $\mathfrak{B}$ and*

$$\mathtt{g} = (g, \langle\overline{\mathtt{g}}\rangle) : \mathtt{z} \to \mathtt{y}, \mathtt{f} = (f, \langle\overline{\mathtt{f}}\rangle) : \mathtt{y} \to \mathtt{z}$$

lax $\mathscr{T}$-morphisms. Assume that $(f \dashv g, \varepsilon, \eta) : Y \to Z$ is an adjunction in $\mathfrak{B}$. The 2-cells ε, η give $\mathscr{T}$-transformations $\widetilde{\varepsilon} : \mathtt{fg} \Rightarrow \mathrm{id}_{\mathsf{z}}$ and $\widetilde{\eta} : \mathrm{id}_{\mathsf{y}} \Rightarrow \mathtt{gf}$ if and only if $\langle \overline{\mathsf{f}} \rangle$ is invertible and $\diamond \langle \overline{\mathsf{g}} \rangle = \langle \overline{\mathsf{f}} \rangle^{-1}$. In this case, $(\mathtt{f} \dashv \mathtt{g}, \widetilde{\varepsilon}, \widetilde{\eta})$ is an adjunction in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$.

Proof. Assume that ε, η give $\mathscr{T}$-transformations $\widetilde{\varepsilon} : \mathtt{fg} \Rightarrow \mathrm{id}_{\mathsf{z}}$ and $\widetilde{\eta} : \mathrm{id}_{\mathsf{y}} \Rightarrow \mathtt{gf}$ in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$. That is to say, by hypothesis, ε, η satisfy Eq.3 of Definition 5.4.1. Denoting $\mathfrak{t}_{gf}^{-1} \cdot \mathscr{T}(\eta) \cdot \mathfrak{t}_Y$ by $\widehat{\mathscr{T}(\eta)}$, we have that $\langle \overline{\mathsf{f}} \rangle \cdot (\diamond \langle \overline{\mathsf{g}} \rangle)$ and $(\diamond \langle \overline{\mathsf{g}} \rangle) \cdot \langle \overline{\mathsf{f}} \rangle$ are respectively equal to

$$\begin{tikzcd}
\mathscr{T}Y \arrow[d, "\mathfrak{alg}_{\mathsf{y}}"'] \arrow[dr, "\mathscr{T}(f)"] & & \mathscr{T}Y \arrow[ll, "\mathrm{id}_{\mathscr{T}Y}"'] \arrow[dd, equal] \\
Y \arrow[dr, "f"'] \arrow[r, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Leftarrow}"] & \mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[dr, "\mathscr{T}(g)"] \arrow[r, phantom, "\overset{\widehat{\mathscr{T}(\eta)}}{\Longleftarrow}"] & \\
& Z \arrow[d, equal] \arrow[dr, "g"] \arrow[r, phantom, "\overset{\langle\overline{\mathsf{g}}\rangle}{\Leftarrow}"] & \mathscr{T}Y \arrow[d, "\mathfrak{alg}_{\mathsf{y}}"] \\
& Z \arrow[ur, phantom, "\overset{\varepsilon}{\Leftarrow}"] & Y \arrow[l, "f"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"'] \arrow[dr, "\mathscr{T}(g)"] & \mathscr{T}Y \arrow[l, "\mathscr{T}(f)"'] \arrow[d, equal] \arrow[dl, phantom, "\overset{\widehat{\mathscr{T}(\eta)}}{\Longleftarrow}"] & \\
Z \arrow[dd, equal] \arrow[dr, "g"'] \arrow[r, phantom, "\overset{\langle\overline{\mathsf{g}}\rangle}{\Leftarrow}"] & \mathscr{T}Y \arrow[d, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[dr, "\mathscr{T}(f)"] & \\
& Y \arrow[dr, "f"'] \arrow[r, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longleftarrow}"] \arrow[dl, phantom, "\overset{\varepsilon}{\Longleftarrow}"] & \mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"] \\
Z & & Z \arrow[ll, "\mathrm{id}_Z"]
\end{tikzcd}$$

which, by Eq.3 of Definition 5.4.1 and the triangle identities (of Definition 1.4.1), are respectively equal to the identities $f\mathfrak{alg}_{\mathsf{y}} \Rightarrow f\mathfrak{alg}_{\mathsf{y}}$ and $\mathfrak{alg}_{\mathsf{z}}\mathscr{T}(f) \Rightarrow \mathfrak{alg}_{\mathsf{z}}\mathscr{T}(f)$. Therefore the proof of the first part is complete.

Reciprocally, assume now that $\diamond \langle \overline{\mathsf{g}} \rangle = \langle \overline{\mathsf{f}} \rangle^{-1}$. We prove below that η gives a $\mathscr{T}$-transformation $\widetilde{\eta} : \mathrm{id}_{\mathsf{y}} \Rightarrow \mathtt{gf}$. On one hand, the mate of the 2-cell

$$\left(\begin{tikzcd}
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[d, bend right=50, "\mathscr{T}(\mathrm{id}_Y)"'] \arrow[d, bend left=50, "\mathscr{T}(gf)"] \arrow[d, phantom, "\overset{\widehat{\mathscr{T}(\eta)}}{\Longrightarrow}"] & Y \arrow[d, "gf"] \arrow[dl, phantom, "\overset{\langle\overline{gf}\rangle}{\Longrightarrow}"] \\
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathsf{y}}"] & Y
\end{tikzcd}\right) \cdot \left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{y}}} * \mathfrak{t}_Y\right) \quad = \quad \begin{tikzcd}
\mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[dr, "\mathscr{T}(f)"] \arrow[dd, equal] & \arrow[d, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longrightarrow}"] & Y \arrow[d, "f"] \\
\arrow[r, phantom, "\overset{\widehat{\mathscr{T}(\eta)}}{\Longrightarrow}"] & \mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[dl, "\mathscr{T}(g)"] \arrow[d, phantom, "\overset{\langle\overline{\mathsf{g}}\rangle}{\Longrightarrow}"] & Z \arrow[d, "g"] \\
\mathscr{T}Y \arrow[rr, "\mathfrak{alg}_{\mathsf{y}}"] & \, & Y
\end{tikzcd}$$

under the adjunction of identities and the adjunction $(f \dashv g, \varepsilon, \eta)$ is equal to $\langle \overline{\mathsf{f}} \rangle \cdot (\diamond \langle \overline{\mathsf{g}} \rangle)$ which by hypothesis is equal to $\mathrm{id}_{f\mathfrak{alg}_{\mathsf{y}}}$. On the other hand, the mate of $\eta * \mathrm{id}_{\mathfrak{alg}_{\mathsf{y}}}$ under the adjunction of identities and the adjunction $(f \dashv g, \varepsilon, \eta)$ is also equal to $\mathrm{id}_{f\mathfrak{alg}_{\mathsf{y}}}$ by the triangle identity. Therefore, by the mate correspondence (Theorem 1.4.6), we conclude that the left side of the equation above is equal to

$\eta * \mathrm{id}_{\mathfrak{alg}_{\mathtt{y}}}$. Therefore

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[d, bend right=50, "\mathscr{T}(\mathrm{id}_Y)"'{name=A}] \arrow[d, bend left=50, "\mathscr{T}(gf)"{name=B}] & Y \arrow[d, "gf"] \\
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathtt{y}}"'] \arrow[ur, phantom, "\overset{\langle\overline{\mathtt{gf}}\rangle}{\Longrightarrow}"] & Y
\arrow[Rightarrow, from=A, to=B, "\mathscr{T}(\eta)"]
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathtt{y}}"] \arrow[d, bend right=30, "\mathscr{T}(\mathrm{id}_Y)"'] & Y \arrow[d, bend right=50, "\mathrm{id}_Y"'{name=C}] \arrow[d, bend left=50, "gf"{name=D}] \\
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathtt{y}}"'] \arrow[ur, phantom, "\overset{\mathrm{id}_{\mathfrak{alg}_{\mathtt{y}}} * \mathfrak{t}_Y^{-1}}{\Longrightarrow}"] & Y
\arrow[Rightarrow, from=C, to=D, "\eta"]
\end{tikzcd}$$

which, by Remark 1.2.12, shows that η satisfies Eq.3 of Definition 5.4.1. This proves that indeed η gives a $\mathscr{T}$-transformation $\widetilde{\eta} : \mathrm{id}_{\mathtt{y}} \Rightarrow \mathtt{gf}$. The proof for ε is analogous. □

Corollary 1.4.15. *Let $U : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ be the forgetful 2-functor. Given a lax $\mathscr{T}$-morphism $\mathtt{f} : \mathtt{y} \to \mathtt{z}$:*

- *$\mathtt{f}$ is left adjoint in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$ if and only if $U(\mathtt{f})$ is left adjoint and $\mathtt{f}$ is a $\mathscr{T}$-pseudomorphism;*
- *$\mathtt{f}$ is right adjoint if and only if $U(\mathtt{f})$ is right adjoint and $\mathtt{f}$ satisfies the Beck-Chevalley condition.*

In the case of pseudomorphisms, the second condition remains equally, but, for the case of lifting of left adjoints, we still need to assure that the right adjoint is going to be a pseudomorphism. More precisely:

Corollary 1.4.16. *Let $U : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ be the forgetful 2-functor. Given a $\mathscr{T}$-pseudomorphism $\mathtt{f} = (f, \langle\overline{\mathtt{f}}\rangle) : \mathtt{y} \to \mathtt{z}$:*

- *$\mathtt{f}$ is left adjoint in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$ if and only if $U(\mathtt{f})$ is left adjoint and $\diamond^{-1}\langle\overline{\mathtt{f}}\rangle$ is an invertible 2-cell;*
- *$\mathtt{f}$ is right adjoint if and only if $U(\mathtt{f})$ is right adjoint and $\mathtt{f}$ satisfies the Beck-Chevalley condition.*

1.4.17 Kock-Zöberlein pseudomonads

The concept of Kock-Zöberlein doctrine was originally introduced by Kock [65] and Zöberlein [114]. We adopt the natural extended notion of Kock-Zöberlein pseudomonad [83], called herein lax idempotent pseudomonad.

Furthermore, since, in our context, the most important property of a lax idempotent pseudomonad $\mathscr{T}$ is the fact that the forgetful 2-functor $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ is fully faithful, we get a shortcut, defining lax idempotent pseudomonads via this property:

Definition 1.4.18. A pseudomonad $(\mathscr{T}, m, \eta, \mu, \iota, \tau)$ on a 2-category $\mathfrak{B}$ is called a *lax idempotent* if the forgetful 2-functor $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ is fully faithful (meaning that it is locally an isomorphism).

It should be noted that our definition is actually equivalent to the usual Kock-Zöberlein adjoint property as stated below: the proof of this fact for the strict case is originally given in [60]. Since the Kock-Zöberlein adjoint property has no important role in our observation, we avoid the proof.

Proposition 1.4.19. *A pseudomonad* $(\mathscr{T}, m, \eta, \mu, \iota, \tau)$ *is lax idempotent if and only if it satisfies the Kock-Zöberlein adjoint structure property: that is to say, there is a modification* $\gamma : \mathrm{Id}_{\mathscr{T}^2} \Longrightarrow (\eta\mathscr{T})(m)$ *such that* $(m \dashv \eta\mathscr{T}, \iota, \gamma)$ *is an adjunction.*

In some situations, it can be easier to verify whether a pseudomonad $\mathscr{T}$ satisfies the Kock-Zöberlein adjoint property than to verify whether the forgetful 2-functor $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ is fully faithful. However our main observation on lax idempotent pseudomonads relies on the last property. More precisely:

Remark 1.4.20. Given a lax idempotent pseudomonad $(\mathscr{T}, m, \eta, \mu, \iota, \tau)$, with a forgetful 2-functor $U : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$, it is clear that:

- The forgetful 2-functor $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ is fully faithful as well;
- Given an object Z of $\mathfrak{B}$, if there is a lax $\mathscr{T}$-algebra z such that $U(\mathsf{z}) = Z$, it is unique up to isomorphism;
- For every adjunction $(f \dashv U(\mathsf{g}), \varepsilon, \eta) : Y \to Z$ in $\mathfrak{B}$, there is an adjunction $(\mathtt{f} \dashv \mathsf{g}, \widetilde{\varepsilon}, \widetilde{\eta})$ in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ such that $U(\mathtt{f}) = f$;
- For every adjunction $(U(\mathtt{f}) \dashv g, \varepsilon, \eta) : Y \to Z$ in $\mathfrak{B}$, there is an adjunction $(\mathtt{f} \dashv \mathsf{g}, \widetilde{\varepsilon}, \widetilde{\eta})$ in $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ such that $U(\mathsf{g}) = g$.

By Corollaries 1.4.15,1.4.16 and, by Remark 1.4.20, we get that, for every lax $\mathscr{T}$-morphism $\mathsf{g} : \mathsf{z} \to \mathsf{y}$ such that $U(\mathsf{g})$ is right adjoint, g satisfies the Beck-Chevalley condition. More precisely:

Corollary 1.4.21. *Assume that* $\mathscr{T}$ *is a lax idempotent pseudomonad,* $U : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \to \mathfrak{B}$ *is the forgetful* 2*-functor and* $(f \dashv g, \varepsilon, \eta) : U(\mathsf{y}) \to U(\mathsf{z})$ *is an adjunction in* $\mathfrak{B}$.

- *There is only one lax* $\mathscr{T}$*-morphism* $\mathtt{f} : \mathsf{y} \to \mathsf{z}$ *such that* $U(\mathtt{f}) = f$*. Furthermore,* $\mathtt{f}$ *is a* $\mathscr{T}$*-pseudomorphism which is left adjoint in* $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$*;*
- *There is only one lax* $\mathscr{T}$*-morphism* $\mathsf{g} : \mathsf{z} \to \mathsf{y}$ *such that* $U(\mathsf{g}) = g$*. Furthermore,* g *is right adjoint to* $\mathtt{f}$ *in* $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$ *and* g *satisfies the Beck-Chevalley condition.*

This shows how Kock-Zöberlein pseudomonads encompass situations when "the Beck-Chevalley conditions always hold". In other words, given such a lax idempotent pseudomonad, whenever g is a right adjoint between objects in the base 2-category that can be endowed with lax $\mathscr{T}$-algebra structure, the unique lax $\mathscr{T}$-structure for g always satisfies the Beck-Chevalley condition.

We can now work on examples of lax idempotent pseudomonads. Besides the idempotent pseudomonads of Chapter 3, the examples we should mention are the cocompletion pseudomonads [62, 95], which motivated the definition of Kock-Zöberlein doctrines. Our aim is to show how the elementary result that says that left adjoints preserve colimits can be stated in our context.

The definition of cocompletion 2-monads for enriched categories is given in [95] and, by direct verification, via Definition 1.4.18 or via the Kock-Zöberlein adjoint property, one can see that cocompletion 2-monads are lax idempotent pseudomonads. We are more interested on the particular case of the Kock-Zöberlein pseudomonad of cocompletion on CAT. In order to work out our example, we need the list of well known properties of the cocompletion pseudomonad below. Although we do not present proofs, they can be found in any of the main references [58].

1. There exists a lax idempotent pseudomonad $(\mathscr{P}, m, \eta, \mu, \iota, \tau)$ on CAT such that $\mathscr{P}X$ is the free cocompletion of X for every category X;
2. The $\mathscr{P}$-pseudoalgebras (as the lax $\mathscr{P}$-algebras) are the cocomplete categories;
3. Clearly, since $\mathscr{P}$ is lax idempotent, the lax $\mathscr{P}$-morphisms are just functors between cocomplete categories;
4. If $\mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle) : \mathtt{y} \to \mathtt{z}$ is a lax $\mathscr{P}$-morphism, $\langle \overline{\mathsf{f}} \rangle$ is given by the natural comparisons of the colimit of the image and the image of the colimit of diagrams. In particular, the $\mathscr{P}$-pseudomorphisms are exactly the cocontinuous functors (which means that they preserve all the colimits).

By Corollary 1.4.21, we get that right adjoint functors between cocomplete categories always satisfy the Beck-Chevalley condition w.r.t. the pseudomonad $\mathscr{P}$. Or, in other words, left adjoints between cocomplete categories always induce $\mathscr{P}$-pseudomorphisms.

Corollary 1.4.22. *Left adjoint functors between cocomplete categories are cocontinuous.*

This shows how our result on Beck-Chevalley condition for Kock-Zöberlein pseudomonads can be seen as a generalization of this elementary result. Of course, left adjoints in general preserve colimits. But we can see this fact as a consequence of Corollary 1.4.22. More precisely, given an adjunction $(\widetilde{f} \dashv \widetilde{g}) : Y \to Z$ in CAT, $(\mathsf{CAT}[\widetilde{g}, \mathsf{SET}])^{\mathrm{op}} \dashv \left(\mathsf{CAT}\left[\widetilde{f}, \mathsf{SET}\right]\right)^{\mathrm{op}}$ by Remark 1.4.7. Hence, by Corollary 1.4.22, $(\mathsf{CAT}[\widetilde{g}, \mathsf{SET}])^{\mathrm{op}}$ is cocontinuous since it is a left adjoint between cocomplete categories. By the natural isomorphism φ of Remark 1.4.7, since $\mathscr{Y}^{\mathrm{op}}_{Y^{\mathrm{op}}} : Y \to (\mathsf{CAT}[Y, \mathsf{Set}])^{\mathrm{op}}$ preserves and reflects colimits, we conclude that $\widetilde{f}$ preserves colimits as well.

1.5 Pseudo-Kan Extensions and Descent Theory

Chapter 3 is the article *Pseudo-Kan Extensions and descent theory* [79], under review. We give a formal approach to descent theory, framing classical descent theory in the context of idempotent pseudomonads. Within this perspective, we recast and generalize most of the classical results of the context of [51, 52], including transfer results, embedding results and the Bénabou-Roubaud Theorem.

The chapter starts by giving an outline of the setting, presenting basic problems and results of the classical context of descent theory. We give an outline of the classical results that are proved and generalized in that paper, including the results mentioned above.

In Section 3.2, we prove theorems on pseudoalgebra structures and biadjoint triangles, always focusing in the case of idempotent pseudomonads. The main advantages on focusing our study on idempotent pseudomonads are the following: the pseudoalgebra structures w.r.t. an idempotent pseudomonad are easier to study. In this case, if a pseudoalgebra structure over an object X exists, it is unique (up to isomorphism) and, moreover, the pseudoalgebra structure over an object X exists if and only if the unit of the pseudomonad on X is an equivalence. This fact allows us to study situations when we "almost have" a pseudoalgebra structure over an object X, which correspond to the situations when the component of the unit on X is faithful or fully faithful. This is important, later, to study descent and almost descent morphisms.

The results on pseudoalgebra structures and biadjoint triangles give the formal account to study descent theory. In order to study classical descent theory in the context of [52], the first step was to give results on commutativity of bilimits. More precisely, we firstly give a direct approach to prove an analogue of Fubini's Theorem for weighted bilimits. This allows us to construct pointwise pseudo-Kan extensions and prove the basic results about them. Secondly, since we prove that

$$[\mathrm{t},\mathfrak{B}]_{PS} : \left[\dot{\mathfrak{A}},\mathfrak{B}\right]_{PS} \to [\mathfrak{A},\mathfrak{B}]$$

is pseudomonadic and induces an idempotent pseudomonad whenever t is locally fully faithful and $\dot{\mathfrak{A}}$ is a small 2-category, we are able to get results on commutativity of bilimits as direct consequences of our results on pseudomonadic pseudofunctors.

Section 3.4 introduces the descent objects, giving key results to finally frame the classical context of descent theory. The main result of this section is that the conical bilimit of a pseudocosimplicial object is its descent object. In other words, it shows that our definition of descent object coincides with the usual definition (as, for instance, given in [104]). More concisely, within the language of pseudo-Kan extensions, we prove that

$$\mathrm{Ps}\mathscr{R}an_{\mathrm{j}}\mathscr{A}(0) \simeq \mathrm{Ps}\mathscr{R}an_{\mathrm{j}_3}\mathscr{A}\mathrm{t}_3$$

in which $\mathscr{A} : \Delta \to \mathfrak{H}$ is a pseudofunctor, $\mathfrak{H}$ is a bicategorically complete 2-category, $\mathrm{j} : \Delta \to \dot{\Delta}$ is the full inclusion of the category of finite nonempty ordinals into the category of finite ordinals and order preserving functions, $\mathrm{t}_3 : \Delta_3 \to \Delta$ is the inclusion of the 2-category given by the faces and degeneracies

$$\begin{tikzcd}
1 \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & 2 \arrow[l, "s^0" description] \arrow[r, shift left=3, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=3, "\partial^2"'] & 3
\end{tikzcd}$$

into Δ, and j_3 is the inclusion of Δ_3 into the 2-category

$$\begin{tikzcd}
0 \arrow[r, "d"] & 1 \arrow[r, shift left=3, "d^0"] \arrow[r, shift right=3, "d^1"'] & 2 \arrow[l, "s^0" description] \arrow[r, shift left=3, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=3, "\partial^2"'] & 3
\end{tikzcd}$$

which is the 2-category obtained from the addition of an initial object to Δ_3. This proves in particular that the definition of descent category via biased descent data on objects, which corresponds to $\mathrm{Ps}\mathscr{R}an_{\mathrm{j}_3}\mathscr{A}\mathrm{t}_3$, is equivalent to the definition of the descent category via unbiased descent data on objects, which corresponds to the case $\mathrm{Ps}\mathscr{R}an_{\mathrm{j}}\mathscr{A}(0)$. But the main points of this result are (1) this gives a very simple universal property of the descent category/object and (2) this gives a way of easily fitting the descent object in our language.

After this detailed work on descent objects, we turn to elementary and known examples. The Eilenberg-Moore objects and the monadicity of functors also fit easily in our context of weighted bilimits/pseudoalgebra structures, once we follow the ideas of [97]. This is explained in Section 3.6.

Finally, in Section 3.7, we show how our perspective on the Beck-Chevalley condition (as explained in Section 1.4) allows us to get results on pseudoalgebra structures/commmutativity of bilimits and monadicity. This leads to our first result of the type of Bénabou-Roubaud Theorem. After that, we finally show how our results work in the context of classical descent theory. We recast and generalize classical results as direct consequences of our previous work.

We then give refinements of our results on commutativity of bilimits. In the context of descent theory, this allows us to give better results on effective descent morphisms of weighted bilimits of 2-categories. It also gives the Galois result of [49] as a direct consequence.

One particular result obtained from our setting of commutativity of bilimits is the pseudopullback theorem. It gives conditions to get effective descent morphisms (w.r.t. basic fibration) of well behaved pseudopullbacks of categories. We finish this chapter applying this result to detect effective descent morphisms in categories of enriched categories. Firstly, we prove that, for suitable cartesian categories V, we have an embedding $V\text{-}\mathsf{Cat} \to \mathsf{Cat}(V)$ that is actually induced by a pseudopullback of categories. Then, using the pseudopullback theorem, we prove that such embedding reflects effective descent morphisms.

1.6 Biadjoint Triangles and Lifting of Biadjoints

Chapter 4 corresponds to the article *On Biadjoint Triangles* [77], published in *Theory and Applications of Categories, Vol 31, N. 9* (2016). The main contributions are the biadjoint triangle theorems, which can be seen as 2-dimensional analogues of the adjoint triangle theorem of [30]. As mentioned in Section 1.1, in order to prove the main results, we use the fact that the category of pseudomorphisms between two pseudoalgebras has the universal property of the descent object. More precisely, assuming that

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "R"'] & \simeq & \mathfrak{B} \arrow[dl, "U"] \\
& \mathfrak{C} &
\end{tikzcd}$$

is a pseudonatural equivalence, in which R, J, U are pseudofunctors, U is pseudopremonadic and R has a right biadjoint. We prove that J has a left biadjoint G, provided that $\mathfrak{A}$ has some needed codescent objects. We also study the unit and the counit of the obtained biadjunction: we give sufficient conditions under which the unit and counit are pseudonatural equivalences. Finally, we show that, under suitable conditions, it is possible to construct a (strict) left 2-adjoint.

Similarly to the case of adjoint triangles in 1-dimensional category theory, the biadjoint triangles have many applications in 2-dimensional category theory. Examples of which are given in this same paper:

– *Pseudomonadicity characterization*: without avoiding pseudofunctors, using the biadjoint triangle theorem and the results on counit and unit, we give a explicit proof of the pseudomonadicity characterization due to Le Creurer, Marmolejo and Vitale in [73];

– 2-*monadic approach to coherence*: as immediate consequences of our main theorems, we recast and improve results on the 2-monadic approach to coherence developed in [9, 67, 73, 93]. More precisely, we characterize when there is a left 2-adjoint (and biadjoint) to the inclusion of the

2-category of strict algebras into the 2-category of pseudoalgebras,

$$\mathscr{T}\text{-}\mathsf{Alg}_{\mathrm{s}} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$$

of a given 2-monad $\mathscr{T}$. We also characterize when the unit of such biadjunction is a pseudonatural equivalence;

- *Lifting of biadjoints*: the biadjoint triangle theorem gives biadjoints to algebraic pseudofunctors, that is to say, lifting of biadjoints. These results recover and generalize, for instance, results of Blackwell, Kelly and Power [9].
- *Pointwise pseudo-Kan extension*: We originally introduce the notion of pseudo-Kan extension and, using the results on lifting of biadjoints, in the presence of weighted bilimits, we construct pseudo-Kan extensions with them. This result, hence, gives the notion of pointwise pseudo-Kan extension. It also gives a way of recovering the construction of weighted bilimits via descent objects, cotensor (bi)products and (bi)products given originally in [104], if we assume the construction of the pointwise pseudo-Kan extension via Fubini's Theorem for weighted bilimits given in Chapter 3.

  Similarly to the pointwise Kan extension in 1-dimensional category theory, the concept of pointwise pseudo-Kan extension plays a relevant role in 2-dimensional category theory. As mentioned in Section 1.5, one instance of application of this concept given in this thesis is within the study of commutativity of weighted bilimits of Chapter 3. Other examples are within the study of 2-dimensional flat pseudofunctors of [27] and within the study of formal aspects of 2-dimensional category theory via Gray-categories [28].

Chapter 5 corresponds to the article *On lifting of biadjoints and lax algebras* [78], to appear in *Categories and General Algebraic Structures with Applications*. It gives further theorems on lifting of biadjoints provided that we can describe the categories of morphisms of a certain domain 2-categories in terms of weighted (bi)limits. This gives an abstract account of the main idea of some proofs of Chapter 4. Still, we show that this setting allows us to get results outside of the context of Chapter 4.

In particular, this approach, together with results on lax descent objects and lax algebras, allows us to give results on lifting of biadjoints involving (full) sub-2-categories of the 2-category of lax algebras. This gives biadjoint triangle theorems involving the 2-category of lax algebras. As a immediate consequence, we complete our treatment of the 2-monadic approach to coherence via biadjoint triangle theorems.

Remark 1.6.1. Unlike in the case of Chapter 5, our results on biadjoint triangles involving the 2-category of lax algebras, Theorems 5.5.2 and 5.5.3, lack the study of the counit and the unit of the obtained biadjunctions. The study of counit and the unit in this setting could lead to new applications. One example is given in Remark 1.7.8.

1.7 Lifting of Biadjoints and Formal Theory of Monads

In this section, we talk about applications of the results of Chapter 5 in the context of the formal theory of monads. In order to do so, we assume most of the prerequisites of that chapter, including the concept of weighted limits and colimits in a 2-category w.r.t. the Cat-enrichment, usually called 2-limits and 2-colimits. In this direction, we adopt the terminology and definitions of Section 2.1.

Every adjunction induces a monad. This was originally shown in [48] for the 2-category Cat. However it works for any 2-category. Indeed, given an adjunction $(f \dashv g, \varepsilon, \eta) : Y \to Z$ in a 2-category $\mathfrak{B}$, $(Y, gf, \mathrm{id}_g * \varepsilon * \mathrm{id}_f, \eta)$ is a monad on Y, that is to say, a lax $\mathrm{Id}_{\mathfrak{B}}$-algebra structure on Y.

Remark 1.7.1. Given a monad $\mathrm{y} = (Y, \mathfrak{alg}_{\mathrm{y}}, \underline{\overline{\mathrm{y}}}, \underline{\overline{\mathrm{y}}}_0)$ in a 2-category $\mathfrak{B}$, we define the category of y-adjunctions $\mathrm{y}\text{-}\mathsf{adj}(\mathfrak{B})$ as follows:

- The objects of $\mathrm{y}\text{-}\mathsf{adj}(\mathfrak{B})$ are adjunctions $(f \dashv g, \varepsilon, \eta) : Y \to Z$ that induce y;
- A morphism of $\mathrm{y}\text{-}\mathsf{adj}(\mathfrak{B})$ between two adjunctions $(f \dashv g, \varepsilon, \eta) : Y \to Z$ and $(\widetilde{f} \dashv \widetilde{g}, \widetilde{\varepsilon}, \eta) : Y \to \widetilde{Z}$ is a morphism $j : Z \to \widetilde{Z}$ such that $jf = \widetilde{f}$ and $g = \widetilde{g}j$.

If there exists, the terminal object of $\mathrm{y}\text{-}\mathsf{adj}(\mathfrak{B})$ is called an *Eilenberg-Moore adjunction* for the monad y. In this case, the domain of the right adjoint of such adjunction is called the *Eilenberg-Moore object of* y and denoted by Y^{y}. Dually, the initial object (if it exists) is called the Kleisli adjunction for the monad y. In this case, the domain of the right adjoint of such adjunction is called the *Kleisli object of* y and denoted by Y_{y}.

There is an Eilenberg-Moore adjunction and a Kleisli adjunction for each monad in Cat. These results were shown respectively in [36] and [64]. However, it is easy to construct counterexamples of 2-categories not having all the Eilenberg-Moore (or Kleisli) adjunctions. In order to give a non-artificial easy to check example, we consider bicategories. More precisely, we can consider the suspension of the monoidal cartesian category $(\mathsf{Set}, \times, 1)$, that is to say, we see such a monoidal category as a bicategory with only one object $\triangle$ as in Remark 1.2.8. A monad in such bicategory is the same as a (classical) monoid. There are plenty nontrivial monoids, while the suspension of $(\mathsf{Set}, \times, 1)$ has only the trivial adjunction.

Remark 1.7.2. Clearly there is a bijection between monads in $\mathfrak{B}$ and monads in $\mathfrak{B}^{\mathrm{op}}$. More precisely, the contraviariant 2-functor $\mathfrak{B} \to \mathfrak{B}^{\mathrm{op}}$ takes monads in $\mathfrak{B}$ to monads in $\mathfrak{B}^{\mathrm{op}}$. So, by abuse of language, if y is a monad in $\mathfrak{B}$, we denote by y the corresponding monad in $\mathfrak{B}^{\mathrm{op}}$.

We can, then, give precise meaning to the fact that the notion of Eilenberg-Moore objects is dual to the notion of Kleisli objects. Indeed, the Kleisli object for a monad y in a 2-category $\mathfrak{B}$ is, if it exists, the Eilenberg-Moore object of y in $\mathfrak{B}^{\mathrm{op}}$.

In [101], it is observed that the Eilenberg-Moore object has a concise universal property. Namely, given a 2-category $\mathfrak{B}$, there is an inclusion 2-functor $\mathfrak{B} \to \mathsf{Mnd}(\mathfrak{B})$ which takes each object Z to the monad $(Z, \mathrm{id}_Z, \mathrm{id}_{\mathrm{id}_Z}, \mathrm{id}_{\mathrm{id}_Z})$. The Eilenberg-Moore object of a monad $\mathrm{y} = (Y, \mathfrak{alg}_{\mathrm{y}}, \underline{\overline{\mathrm{y}}}, \underline{\overline{\mathrm{y}}}_0)$ is given by the right 2-reflection of y along $\mathfrak{B} \to \mathsf{Mnd}(\mathfrak{B})$, if it exists. In particular, a morphism $f : X \to Y^{\mathrm{y}}$ corresponds to a pair $\mathtt{f} = (\widetilde{f}, \langle \overline{\mathsf{f}} \rangle)$ in which $\widetilde{f} : X \to Y$ is a morphism and $\langle \overline{\mathsf{f}} \rangle : \mathfrak{alg}_{\mathrm{y}} \widetilde{f} \Rightarrow \widetilde{f}$ is a 2-cell, such that the equations

$$\begin{tikzcd}
Y \arrow[r, "\mathfrak{alg}_{\mathrm{y}}"] \arrow[dr, phantom, "\overset{\overline{\mathrm{y}}}{\Rightarrow}", near start] & Y \\
Y \arrow[u, "\mathfrak{alg}_{\mathrm{y}}"] \arrow[ur, "\mathfrak{alg}_{\mathrm{y}}"'] & X \arrow[l, "\widetilde{f}"] \arrow[u, "\widetilde{f}"'] \arrow[ul, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longrightarrow}", near start]
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "\mathfrak{alg}_{\mathrm{y}}"] & Y \\
Y \arrow[u, "\mathfrak{alg}_{\mathrm{y}}"] \arrow[ur, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longrightarrow}", near start] & X \arrow[l, "\widetilde{f}"] \arrow[u, "\widetilde{f}"'] \arrow[ul, "\widetilde{f}"] \arrow[ul, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longrightarrow}", near end, shift left=6]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Y \arrow[r, equal] \arrow[dr, phantom, "\overset{\overline{\mathrm{y}}_{\mathrm{Q}}}{\Rightarrow}", near start] & Y \\
Y \arrow[u, equal] \arrow[ur, "\mathfrak{alg}_{\mathrm{y}}"'] & X \arrow[l, "\widetilde{f}"] \arrow[u, "\widetilde{f}"'] \arrow[ul, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longrightarrow}", near start]
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[d, bend right=50, "\widetilde{f}"'] \arrow[d, bend left=50, "\widetilde{f}"] \arrow[d, phantom, "\overset{\mathrm{id}_{\widetilde{f}}}{\Longrightarrow}"] \\
X
\end{tikzcd}$$

hold. Since the Kleisli object of y is the Eilenberg-Moore object of y in $\mathfrak{B}^{\mathrm{op}}$, we have that the Kleisli object is given by the right 2-reflection of y along $\mathfrak{B}^{\mathrm{op}} \to \mathsf{Mnd}(\mathfrak{B}^{\mathrm{op}})$ which is the same as the left 2-reflection of $\mathfrak{B} \to (\mathsf{Mnd}(\mathfrak{B}^{\mathrm{op}}))^{\mathrm{op}} \cong \mathsf{Mnd}_{\mathrm{co}}(\mathfrak{B})$.

Moreover, [101] generalizes the Eilenberg-Moore and the Kleisli constructions. More precisely, if X is a category, [101] constructs the right 2-adjoint to the inclusion $[X,\mathsf{Cat}] \to [X,\mathsf{Cat}]_{Lax}$ of the 2-category of lax functors $X \to \mathsf{Cat}$, lax natural transformations and modifications into the 2-category of 2-functors, 2-natural transformations and modifications. In [101], Street also constructs the left 2-adjoint to the inclusion $[X,\mathsf{Cat}] \to [X,\mathsf{Cat}]_{Lax_{\mathsf{c}}}$, in which $[X,\mathsf{Cat}]_{Lax_{\mathsf{c}}}$ denotes the 2-category of lax functors, colax natural transformations and modifications.

In order to verify that these 2-adjoints actually are generalizations of the Eilenberg-Moore and Kleisli objects, we should observe that, considering the inclusions $\mathsf{Cat} \to \mathsf{Mnd}_{\mathrm{co}}(\mathsf{Cat})$ and $\mathsf{Cat} \to \mathsf{Mnd}(\mathsf{Cat})$, we actually have isomorphisms $\mathsf{Mnd}(\mathsf{Cat}) \cong [1,\mathsf{Cat}]_{Lax}$ and $\mathsf{Mnd}_{\mathrm{co}}(\mathsf{Cat}) \cong [1,\mathsf{Cat}]_{Lax_{\mathsf{c}}}$ such that the diagrams

$$\begin{tikzcd}
\mathsf{Cat} \arrow[r, "\cong"] \arrow[d] & {[1,\mathsf{Cat}]} \arrow[d] \\
\mathsf{Mnd}(\mathsf{Cat}) \arrow[r, "\cong"] & {[1,\mathsf{Cat}]_{Lax}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Cat} \arrow[r, "\cong"] \arrow[d] & {[1,\mathsf{Cat}]} \arrow[d] \\
\mathsf{Mnd}_{\mathrm{co}}(\mathsf{Cat}) \arrow[r, "\cong"] & {[1,\mathsf{Cat}]_{Lax_{\mathsf{c}}}}
\end{tikzcd}$$

commute. More generally, in our context, this is given by the fact that, given a 2-category $\mathfrak{B}$, the diagrams

$$\begin{tikzcd}
\mathsf{Lax}\text{-}\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell} \arrow[dr] & \mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{CoAlg}_{\mathrm{s}} \cong \mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{Alg}_{\mathrm{s}} \arrow[l] \arrow[r] \arrow[d] & \mathsf{Lax}\text{-}\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{Alg}_{\mathsf{c}\ell} \arrow[dl] \\
& \mathfrak{B} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[1,\mathfrak{B}]_{Lax}} \arrow[dr] & {[1,\mathfrak{B}]} \arrow[l] \arrow[r] \arrow[d] & {[1,\mathfrak{B}]_{Lax_{\mathsf{c}}}} \arrow[dl] \\
& \mathfrak{B} &
\end{tikzcd}$$

are naturally isomorphic, in which the horizontal arrows are the obvious inclusions while the non-horizontal arrows are the forgetful 2-functors. In particular, the inclusion $[1,\mathfrak{B}] \to [1,\mathfrak{B}]_{Lax}$ is actually the inclusion $\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \cong \mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{Alg}_{\mathsf{s}} \to \mathsf{Lax}\text{-}\mathrm{Id}_{\mathfrak{B}}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}$.

More generally, if $\mathfrak{A}$ is any 2-category, denoting by $\mathfrak{A}_0$ the discrete 2-category of objects, the inclusion $\mathfrak{A}_0 \to \mathfrak{A}$ induces a restriction 2-functor $[\mathfrak{A},\mathfrak{B}] \to [\mathfrak{A}_0,\mathfrak{B}]$. If $\mathfrak{B}$ has suitable weighted limits and colimits and $\mathfrak{A}$ is small, this restriction has right and left 2-adjoints given by the (global) pointwise right and left Kan extensions. Assuming that this restriction has right and left 2-adjoints, we have a 2-monad $\overline{\mathscr{L}an}$ and a 2-comonad $\overline{\mathscr{R}an}$ on the 2-category $[\mathfrak{A}_0,\mathfrak{B}]$. In this case, the diagrams

$$\begin{tikzcd}
\mathsf{Lax}\text{-}\overline{\mathscr{R}an}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell} \arrow[rd] & \overline{\mathscr{R}an}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \cong \overline{\mathscr{L}an}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[l] \arrow[r] \arrow[d] & \mathsf{Lax}\text{-}\overline{\mathscr{L}an}\text{-}\mathsf{Alg}_{\mathsf{c}\ell} \arrow[ld] \\
 & {[\mathfrak{A}_0,\mathfrak{B}]} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[\mathfrak{A},\mathfrak{B}]_{Lax}} \arrow[rd] & {[\mathfrak{A},\mathfrak{B}]} \arrow[l] \arrow[r] \arrow[d] & {[\mathfrak{A},\mathfrak{B}]_{Lax_{\mathsf{c}}}} \arrow[ld] \\
 & {[\mathfrak{A}_0,\mathfrak{B}]} &
\end{tikzcd}$$

are naturally isomorphic. These observations immediately show how the results of Chapter 5 generalize the construction: they actually characterize when it is possible to get such constructions. More precisely, using the techniques of that chapter, we are able to study the existence of the right 2-adjoint to $\mathscr{S}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \to \mathsf{Lax}\text{-}\mathscr{S}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}$ for any given 2-comonad $\mathscr{S}$, or, equivalently, a left 2-adjoint to $\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$ for any given 2-monad $\mathscr{T}$. So it is clear that this generalizes the constructions of [101]. Since we do not explicitly deal with colax morphisms in Chapter 5, we briefly describe below how we get the results for our context. We omit most of the proofs, since some of them are slight variations of the proofs on lax morphisms of Chapter 5, while the rest of the proofs follow directly from results of that chapter.

Definition 1.7.3. $[\mathsf{t}^{\mathsf{c}} : \Delta_\ell^{\mathsf{c}} \to \dot{\Delta}_\ell^{\mathsf{c}}]$ We denote by $\dot{\Delta}_\ell^{\mathsf{c}}$ the 2-category generated by the diagram

$$\begin{tikzcd}
\mathsf{0} \arrow[r, "d"] & \mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=2, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=2, "\partial^2"'] & \mathsf{3}
\end{tikzcd}$$

with the 2-cells:

$$\begin{aligned}
\sigma_{00} &: \partial^0 d^0 \Rightarrow \partial^1 d^0, & n_0 &: \mathrm{id}_{\mathsf{1}} \Rightarrow s^0 d^0, \\
\sigma_{01} &: \partial^0 d^1 \Rightarrow \partial^2 d^0, & n_1 &: \mathrm{id}_{\mathsf{1}} \Rightarrow s^0 d^1, \\
\sigma_{21} &: \partial^2 d^1 \Rightarrow \partial^1 d^1, & \vartheta &: d^0 d \Rightarrow d^1 d,
\end{aligned}$$

satisfying

- Associativity:

$$\begin{tikzcd}
\mathsf{0} \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Rightarrow}"] & \mathsf{1} \arrow[d, "d^0"] \\
\mathsf{1} \arrow[r, "d^0"] \arrow[d, "d^0"'] \arrow[dr, phantom, "\overset{\sigma_{00}}{\Rightarrow}"] & \mathsf{2} \arrow[d, "\partial^1"] \\
\mathsf{2} \arrow[r, "\partial^0"'] & \mathsf{3}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{3} & & \mathsf{2} \arrow[ll, "\partial^1"'] \\
\arrow[r, phantom, "\overset{\sigma_{01}}{\Rightarrow}"] & \mathsf{2} \arrow[ul, "\partial^2"] \arrow[r, phantom, "\overset{\sigma_{21}}{\Rightarrow}", shift left=4ex] & \mathsf{1} \arrow[l, "d^1"] \arrow[u, "d^0"'] \\
\mathsf{2} \arrow[uu, "\partial^0"] & \mathsf{1} \arrow[l, "d^1"'] \arrow[u, "d^0"'] \arrow[r, phantom, "\overset{\vartheta}{\Rightarrow}"] & \\
\mathsf{1} \arrow[u, "d^0"] \arrow[ur, phantom, "\overset{\vartheta}{\Rightarrow}"] & \mathsf{0} \arrow[l, "d"] \arrow[u, "d"'] \arrow[uur, bend right, "d"'] &
\end{tikzcd}$$

- Identity:

$$\begin{tikzcd}
\mathsf{0} \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Rightarrow}"] & \mathsf{1} \arrow[d, "d^1"] \\
\mathsf{1} \arrow[r, "d^0"'] \arrow[dr, bend right, equal] \arrow[dr, phantom, "\overset{n_0}{\Rightarrow}"] & \mathsf{2} \arrow[d, "s^0"] \\
& \mathsf{1}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{0} \arrow[d, "d"'] & \\
\mathsf{1} \arrow[r, "d^1"'] \arrow[dr, bend right, equal] \arrow[dr, phantom, "\overset{n_1}{\Rightarrow}"] & \mathsf{2} \arrow[d, "s^0"] \\
& \mathsf{1}
\end{tikzcd}$$

The 2-category Δ_ℓ^{c} is, herein, the full sub-2-category of $\dot{\Delta}_\ell^{\mathsf{c}}$ with objects 1, 2 and 3. We denote the inclusion by $\mathsf{t}^{\mathsf{c}} : \Delta_\ell^{\mathsf{c}} \to \dot{\Delta}_\ell^{\mathsf{c}}$.

Remark 1.7.4. [Colax descent object and category] If $\mathscr{A} : \Delta_\ell^{\mathsf{c}} \to \mathfrak{B}$ and $\mathscr{B} : (\Delta_\ell^{\mathsf{c}})^{\mathrm{op}} \to \mathfrak{B}$ are 2-functors, if it exists, the weighted limit $\left\{\dot{\Delta}_\ell^{\mathsf{c}}(\mathsf{0}, \mathsf{t}^{\mathsf{c}}-), \mathscr{A}\right\}$ is called the *strict colax descent object* of $\mathscr{A}$, while the weighted colimit $\dot{\Delta}_\ell^{\mathsf{c}}(\mathsf{0}, \mathsf{t}^{\mathsf{c}}-) * \mathscr{B}$ is called the *strict colax codescent object* of $\mathscr{B}$ (if it exists).

In the case of strict colax descent categories, we have a result similar to that described by Remark 5.3.4. More precisely, if $\mathscr{D} : \Delta_\ell \to \mathsf{Cat}$ is a 2-functor, then

$$\left\{\dot{\Delta}_\ell^{\mathsf{c}}(\mathsf{0}, \mathsf{t}^{\mathsf{c}}-), \mathscr{D}\right\} \cong [\Delta_\ell^{\mathsf{c}}, \mathsf{Cat}]\left(\dot{\Delta}_\ell^{\mathsf{c}}(\mathsf{0}, \mathsf{t}^{\mathsf{c}}-), \mathscr{D}\right).$$

Thereby, we can describe the strict colax descent object of $\mathscr{D} : \Delta_\ell^{\mathsf{c}} \to \mathsf{Cat}$ explicitly as follows:

1. Objects are 2-natural transformations $\mathtt{f} : \dot{\Delta}_\ell^{\mathsf{c}}(\mathsf{0}, \mathsf{t}^{\mathsf{c}}-) \longrightarrow \mathscr{D}$. We have a bijective correspondence between such 2-natural transformations and pairs $(f, \langle\bar{\mathsf{f}}\rangle)$ in which f is an object of $\mathscr{D}1$ and $\langle\bar{\mathsf{f}}\rangle : \mathscr{D}(d^0)f \to \mathscr{D}(d^1)f$ is a morphism in $\mathscr{D}2$ satisfying the following equations:
   - Associativity:

$$\left(\mathscr{D}(\sigma_{01})_f\right)\left(\mathscr{D}(\partial^0)(\langle\bar{\mathsf{f}}\rangle)\right)\left(\mathscr{D}(\sigma_{21})_f\right)\left(\mathscr{D}(\partial^2)(\langle\bar{\mathsf{f}}\rangle)\right) = \left(\mathscr{D}(\partial^1)(\langle\bar{\mathsf{f}}\rangle)\right)\left(\mathscr{D}(\sigma_{00})_f\right)$$

   - Identity:

$$\left(\mathscr{D}(s^0)(\langle\bar{\mathsf{f}}\rangle)\right)\left(\mathscr{D}(n_0)_f\right) = \left(\mathscr{D}(n_1)_f\right)$$

If $\mathtt{f} : \dot{\Delta}(0,-) \longrightarrow \mathscr{D}$ is a 2-natural transformation, we get such pair by the correspondence $\mathtt{f} \mapsto (\mathtt{f}_1(d), \mathtt{f}_2(\vartheta))$.

2. The morphisms are modifications. In other words, a morphism $\mathfrak{m} : \mathtt{f} \to \mathtt{h}$ is determined by a morphism $\mathfrak{m} : f \to g$ in $\mathscr{D}1$ such that $\mathscr{D}(d^1)(\mathfrak{m})\left\langle \overline{\mathsf{f}} \right\rangle = \left\langle \overline{\mathsf{h}} \right\rangle \mathscr{D}(d^0)(\mathfrak{m})$.

Similarly to Proposition 4.5.5 and 5.4.5, we clearly have:

Proposition 1.7.5. *Let $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ be a pseudomonad on a 2-category $\mathfrak{B}$. Given lax $\mathscr{T}$-algebras $\mathsf{y} = (Y, \mathfrak{alg}_{\mathsf{y}}, \underline{\overline{\mathsf{y}}}, \underline{\overline{\mathsf{y}}}_0)$, $\mathsf{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \underline{\overline{\mathsf{z}}}, \underline{\overline{\mathsf{z}}}_0)$ the category $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}(\mathsf{y},\mathsf{z})$ is the strict colax descent object of the diagram $\mathbb{T}^{\mathsf{y}}_{\mathsf{z}} : \Delta_\ell \to \mathsf{Cat}$*

$$\begin{tikzcd}[column sep=huge]
\mathfrak{B}(U\mathsf{y},U\mathsf{z}) \arrow[r, shift left=8, "\mathfrak{B}(\mathscr{T}U\mathsf{y},\mathfrak{alg}_{\mathsf{z}})\circ \mathscr{T}_{(U\mathsf{y},U\mathsf{z})}"] \arrow[r, shift right=8, "\mathfrak{B}(\mathfrak{alg}_{\mathsf{y}},U\mathsf{z})"'] & \mathfrak{B}(\mathscr{T}U\mathsf{y},U\mathsf{z}) \arrow[l, "\mathfrak{B}(\eta_{U\mathsf{y}},U\mathsf{z})" description] \arrow[r, "\mathfrak{B}(m_{U\mathsf{y}},U\mathsf{z})" description] \arrow[r, shift left=8, "\mathfrak{B}(\mathscr{T}^2U\mathsf{y},\mathfrak{alg}_{\mathsf{z}})\circ \mathscr{T}_{(\mathscr{T}U\mathsf{y},U\mathsf{z})}"] \arrow[r, shift right=8, "\mathfrak{B}(\mathscr{T}(\mathfrak{alg}_{\mathsf{y}}),U\mathsf{z})"'] & \mathfrak{B}(\mathscr{T}^2U\mathsf{y},U\mathsf{z})
\end{tikzcd}$$

in which $U : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell} \to \mathfrak{B}$ denotes the forgetful 2-functor and

$$\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(\sigma_{01})_f := \left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * \mathfrak{t}^{-1}_{(f)(\mathfrak{alg}_{\mathsf{y}})}\right)$$

$$\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(\sigma_{21})_f := \left(\mathrm{id}_f * \underline{\overline{\mathsf{y}}}\right) \qquad \mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(\sigma_{00})_f := \left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * m_f^{-1}\right)\cdot\left(\underline{\overline{\mathsf{z}}} * \mathrm{id}_{\mathscr{T}^2(f)}\right)\cdot\left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * \mathfrak{t}^{-1}_{(\mathfrak{alg}_{\mathsf{z}})(\mathscr{T}(f))}\right)$$

$$\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(n_1)_f := \left(\mathrm{id}_f * \underline{\overline{\mathsf{y}}}_0\right) \qquad \mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(n_0)_f := \left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * \eta_f^{-1}\right)\cdot\left(\underline{\overline{\mathsf{z}}}_0 * \mathrm{id}_f\right)$$

Furthermore, the strict descent object of $\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}$ is $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}(\mathsf{y},\mathsf{z})$.

Remark 1.7.6. Similarly to the case of Remark 5.4.6, we can actually define a pseudofunctor $\mathbb{T}^{\mathsf{y}} : \Delta^{\mathsf{c}}_\ell \times \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Cat}$ in which $\mathbb{T}^{\mathsf{y}}(-,\mathsf{z}) := \mathbb{T}^{\mathsf{y}}_{\mathsf{z}}$, since the morphisms defined above are actually pseudonatural in z w.r.t. $\mathscr{T}$-pseudomorphisms and $\mathscr{T}$-transformations. More importantly to our context, if $\mathscr{T}$ is a 2-monad, the restriction of $\mathbb{T}^{\mathsf{y}}$ to $\Delta^{\mathsf{c}}_\ell \times \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$, in which $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$ denotes the locally full sub-2-category of lax algebras and (strict) $\mathscr{T}$-morphisms and lax algebras, is actually a 2-functor.

By Remark 1.7.6 and Proposition 1.7.5, as a consequence of the results of Chapter 5, we conclude as a particular case that:

Theorem 1.7.7. *Let $\mathscr{T} = (\mathscr{T}, m, \eta)$ be a 2-monad on $\mathfrak{B}$ and $(E \dashv R, \varepsilon, \eta) : \mathfrak{B} \to \mathscr{T}\text{-}\mathsf{Alg}_s$ the Eilenberg-Moore 2-adjunction induced by $\mathscr{T}$. The inclusion $J : \mathscr{T}\text{-}\mathsf{Alg}_s \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$ has a left 2-adjoint if and only if $\mathscr{T}\text{-}\mathsf{Alg}_s$ has the strict colax codescent object of*

$$\begin{tikzcd}[column sep=huge]
EU\mathsf{y} \arrow[r, "E(\eta_{U\mathsf{y}})" description] & E\mathscr{T}U\mathsf{y} \arrow[l, shift right=8, "\varepsilon_{EU\mathsf{y}}E(\mathfrak{alg}_{JEU\mathsf{y}}\mathscr{T}(\eta_{U\mathsf{y}}))"'] \arrow[l, shift left=8, "E(\mathfrak{alg}_{\mathsf{y}})"] & E\mathscr{T}^2U\mathsf{y} \arrow[l, "E(m_{U\mathsf{y}})" description] \arrow[l, shift right=8, "\varepsilon_{E\mathscr{T}U\mathsf{y}}E(\mathfrak{alg}_{JE\mathscr{T}U\mathsf{y}}\mathscr{T}(\eta_{\mathscr{T}U\mathsf{y}}))"'] \arrow[l, shift left=8, "E\mathscr{T}(\mathfrak{alg}_{\mathsf{y}})"]
\end{tikzcd} \qquad (\mathscr{B}_{\mathsf{y}})$$

(with omitted 2-cells) in which $U : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell} \to \mathfrak{B}$ is the forgetful 2-functor. In this case, the left 2-adjoint is given by $G\mathsf{y} = \dot{\Delta}^{\mathsf{c}}_\ell(0, \mathsf{t}^{\mathsf{c}}-) * \mathscr{B}_{\mathsf{y}}$.

As a corollary, if $\mathcal{T}$ is a 2-monad on $\mathfrak{B}$, the inclusion $\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \to \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{c}\ell}$ has a left 2-adjoint if $\mathfrak{B}$ has and $\mathcal{T}$ preserves strict colax codescent objects. Dually, if $\mathcal{S}$ is a 2-comonad on $\mathfrak{B}$, $\mathcal{S}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \to \mathsf{Lax}\text{-}\mathcal{S}\text{-}\mathsf{CoAlg}_{\mathsf{c}\ell}$ has a right 2-adjoint if $\mathfrak{B}$ has and $\mathcal{S}$ preserves strict colax descent objects.

Since the 2-monad $\overline{\mathcal{L}an}$ on $[\mathfrak{A}_0, \mathfrak{B}]$ preserves strict colax codescent objects whenever $\mathfrak{A}$ is small and $\mathfrak{B}$ has 2-colimits, our result gives a left 2-adjoint to the inclusion $[\mathfrak{A}, \mathfrak{B}] \to [\mathfrak{A}, \mathfrak{B}]_{Lax_{\mathsf{c}}}$. Dually, the 2-comonad $\overline{\mathcal{R}an}$ on $[\mathfrak{A}_0, \mathfrak{B}]$ preserves strict colax descent objects whenever $\mathfrak{B}$ has 2-limits, hence our result gives the right 2-adjoint of $[\mathfrak{A}, \mathfrak{B}] \to [\mathfrak{A}, \mathfrak{B}]_{Lax}$. These facts explain how our results generalize greatly the constructions of [101]. In particular, unlike [101], our setting includes the case of lax actions of monoidal categories (called graded monads): or, more precisely, the case in which $\mathfrak{A}$ is a bicategory with only one object (see Remark 5.4.3 and [37]).

Applying to the very special case of the classical theory of monads, we get that the Eilenberg-Moore category of a monad (seen as a lax coalgebra) $\mathrm{y} = (Y, \mathfrak{coalg}_{\mathrm{y}}, \underline{\overline{\mathrm{y}}}, \underline{\overline{\mathrm{y}}}_0)$ is the colax descent category of

$$\begin{tikzcd}[column sep=huge]
Y \arrow[r, shift left=3, "\mathfrak{coalg}_{\mathrm{y}}"] \arrow[r, shift right=3, "\mathrm{id}_Y"'] & Y \arrow[l, "\mathrm{id}_Y" description] \arrow[r, shift left=3, "\mathfrak{coalg}_{\mathrm{y}}"] \arrow[r, "\mathrm{id}_Y" description] \arrow[r, shift right=3, "\mathrm{id}_Y"'] & Y
\end{tikzcd} \qquad (\mathcal{D})$$

in which $\mathcal{D}(\sigma_{00}) = \underline{\overline{\mathrm{y}}}$, $\mathcal{D}(n_0) = \underline{\overline{\mathrm{y}}}_0$ and the images of σ_{01}, σ_{21} and n_1 are the identities.

Remark 1.7.8. The article [101] also recasts the original universal properties w.r.t. adjunctions. That is to say, it generalizes the (universal property) of the (generalized) Eilenberg-Moore and Kleisli adjunctions to its setting. In order to do so, it relies on the study of the counit and unit of the 2-adjunctions $[X, \mathfrak{B}] \to [X, \mathfrak{B}]_{Lax_{\mathsf{c}}}$ and $[X, \mathfrak{B}] \to [X, \mathfrak{B}]_{Lax}$ constructed therein. This fact shows that the study of counit and unit of the obtained biadjunctions (and 2-adjunctions) in the context of Chapter 5 could be interesting to recast the Eilenberg-Moore and Kleisli adjunctions in our context, in order to generalize the setting of [101].

1.8 Pseudoexponentiability

There is a vast literature on exponentiability of objects and morphisms within 1-dimensional category theory [90, 91]. As mentioned in Section 1.2.8, we could consider exponentiability of objects w.r.t. other monoidal structures in V, but we are particularly interested in the case of exponentiation w.r.t. the cartesian structure.

An object A of a cartesian category V is exponentiable if the functor $A \times - : V \to V$ is left adjoint. In this case, the right adjoint is usually denoted by $[A, -]$. A morphism $f : A \to B$ is exponentiable if it is an exponentiable object of the comma category V/B defined by:

- The objects are morphisms with B as codomain;

- A morphism $f \to g$ is a morphism h of V between the domains of f and g such that

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"'] & & A' \arrow[dl, "g"] \\
& B &
\end{tikzcd}$$

  commutes in V;

- The composition and identities are given by the composition and identities of V.

The main problem is to characterize objects and morphisms that are exponentiable in a given category V of interest. For instance, the characterization of the exponentiable morphisms (functors) of Cat is given in [23, 39], while in [19, 20] the exponentiable morphisms (enriched functors) between V-enriched categories are characterized, for suitable monoidal categories V.

As many concepts of 1-dimensional category theory, exponentiability is very strict to most of the cases within bicategory theory. Hence we should consider a weaker version: pseudoexponentiability. Firstly, we consider bicategorical products instead of products. Secondly, we consider a biadjunction instead of an adjunction. That is to say:

Definition 1.8.1. An object Y of a 2-category $\mathfrak{B}$ with bicategorical products is *pseudoexponentiable* if the pseudofunctor $Y \times - : \mathfrak{B} \to \mathfrak{B}$ has a right biadjoint, while Y is exponentiable if $Y \times -$ is a 2-functor and it has a right 2-adjoint.

As briefly mentioned in Section 1.6, the adjoint triangle theorem of [30] has many applications in 1-dimensional category theory. In particular, it is very useful within the study of exponentiable objects and morphisms. For instance, some of the results of the theory developed in [33] can be seen as applications of the adjoint triangle theorem.

This fact suggests the possibility of applying the biadjoint triangle theorem proved in Chapter 4 to develop an analogue theory for pseudoexponentiable objects and morphisms. We however do not do this here. Instead, we finish the chapter giving what seems to be a folklore result on exponentiability of coalgebras. Then, employing the biadjoint triangle theorem, we give the bicategorical analogue that studies the (pseudo)exponentiability of pseudocoalgebras.

Theorem 1.8.2 (Exponentiability of Coalgebras)**.** *Let $\mathscr{S}$ be a comonad on a finitely complete category $\mathfrak{B}$. If $\mathscr{S}$ preserves finite limits, then the forgetful functor $L : \mathscr{S}\text{-}\mathsf{CoAlg} \to \mathfrak{B}$ reflects exponentiable objects.*

Proof. In this setting, L creates finite limits. In particular, given an $\mathscr{S}$-coalgebra $\mathrm{y} = (Y, \mathfrak{coalg}_{\mathrm{y}})$, we get that

$$\begin{tikzcd}
\mathscr{S}\text{-}\mathsf{CoAlg}_{\mathrm{s}} \arrow[r, "\mathrm{y}\times -"] \arrow[d, "L"'] & \mathscr{S}\text{-}\mathsf{CoAlg}_{\mathrm{s}} \arrow[d, "L"] \\
\mathfrak{B} \arrow[r, "(L\mathrm{y})\times -"'] & \mathfrak{B}
\end{tikzcd}$$

commutes. If $(L\mathrm{y}) \times - \dashv [(L\mathrm{y}), -]$, we have that $(L\mathrm{y}) \times L(-) \dashv U\,[(L\mathrm{y}), -]$. Since L is comonadic, by the adjoint triangle theorem of Dubuc, we conclude that $\mathrm{y} \times -$ has a right adjoint. □

Within the context of the theorem above, given an $\mathcal{S}$-coalgebra y, by the Beck theorem, it is clear that L induces a functor $L/\mathrm{y} : \mathcal{S}\text{-}\mathsf{CoAlg}_{\mathrm{s}}/\mathrm{y} \to \mathfrak{B}/L(\mathrm{y})$ which creates limits and is comonadic as well. Therefore:

Corollary 1.8.3. *Let $\mathcal{S}$ be a comonad on a finitely complete category $\mathfrak{B}$. If $\mathcal{S}$ preserves finite limits, then the forgetful functor $L : \mathcal{S}\text{-}\mathsf{CoAlg}_s \to \mathfrak{B}$ reflects exponentiable morphisms.*

Remark 1.8.4. An elementary application is given by the category of functors. For instance, if X, Y are categories such that X is small and Y is complete, there exists a (global) pointwise right Kan extension $\mathsf{Cat}[X_0, Y] \to \mathsf{Cat}[X, Y]$. Since the forgetful functor $\mathsf{Cat}[X, Y] \to \mathsf{Cat}[X_0, Y]$ creates equalizers, we conclude that $\mathsf{Cat}[X, Y] \to \mathsf{Cat}[X_0, Y]$ is comonadic.

By Theorem 1.8.2, we conclude that $\mathsf{Cat}[X, Y] \to \mathsf{Cat}[X_0, Y]$ reflects exponentiable objects. More generally, a natural transformation is exponentiable in $\mathsf{Cat}[X, Y]$ whenever it is objectwise exponentiable.

From an argument entirely analogous to that given in the proof of Theorem 1.8.2, using the enriched version of the adjoint triangle theorem as presented in Section 4.1, we get that:

Theorem 1.8.5 (Exponentiability of Strict Coalgebras)**.** *Let $\mathcal{S}$ be a 2-comonad on a 2-category $\mathfrak{B}$. If $\mathfrak{B}$ has and $\mathcal{S}$ preserves products and equalizers, then the forgetful 2-functor $L : \mathcal{S}\text{-}\mathsf{CoAlg}_s \to \mathfrak{B}$ reflects exponentiable objects.*

Recall the definition of (strict) descent objects given in Section 4.3. Employing the strict version of the biadjoint triangle theorem given in Theorem 4.5.10, we can study the exponentiability of pseudocoalgebras.

Theorem 1.8.6 (Exponentiability of Pseudocoalgebras)**.** *Let $\mathcal{S}$ be a 2-comonad on a 2-category $\mathfrak{B}$. If $\mathfrak{B}$ has and $\mathcal{S}$ preserves products and strict descent objects, then the forgetful 2-functor $L : \mathsf{Ps}\text{-}\mathcal{S}\text{-}\mathsf{CoAlg} \to \mathfrak{B}$ reflects exponentiable objects.*

Finally, by the biadjoint triangle theorem given in Theorem 4.5.10, we conclude that:

Theorem 1.8.7 (Pseudoexponentiability of Pseudocoalgebras)**.** *Let $\mathcal{S}$ be a pseudocomonad on a 2-category $\mathfrak{B}$. If $\mathfrak{B}$ has and $\mathcal{S}$ preserves biproducts and descent objects, then the forgetful 2-functor $L : \mathsf{Ps}\text{-}\mathcal{S}\text{-}\mathsf{CoAlg} \to \mathfrak{B}$ reflects pseudoexponentiable objects.*

Remark 1.8.8. Similarly to the case of Remark 1.8.4, we can use the results above to study exponentiability and pseudoexponentiability of the 2-category of 2-functors $[\mathfrak{A}, \mathfrak{B}]$ (with 2-natural transformations and modifications) and in the 2-category of pseudofunctors $[\mathfrak{A}, \mathfrak{B}]_{PS}$ (with pseudonatural transformations and modifications) as defined in Section 4.2.

For instance, if $\mathfrak{A}$ is small and $\mathfrak{B}$ is 2-complete, we conclude that a 2-functor is exponentiable in $[\mathfrak{A}, \mathfrak{B}]$ if it is objectwise exponentiable. Moreover, using the pointwise pseudo-Kan extension constructed in 4.9.2 (or in Section 3.3.5), we get an analogous result for pseudofunctors. More precisely, assuming that $\mathfrak{B}$ is bicategorically complete and $\mathfrak{A}$ is small, we get that a pseudofunctor in $[\mathfrak{A}, \mathfrak{B}]_{PS}$ is pseudoexponentiable whenever it is objectwise pseudoexponentiable.

Chapter 2

Freely generated n-categories, coinserters and presentations of low dimensional categories

Composing with the inclusion $\mathsf{Set} \to \mathsf{Cat}$, a graph G internal to Set becomes a graph of discrete categories, the coinserter of which is the category freely generated by G. Introducing a suitable definition of n-computad, we show that a similar approach gives the n-category freely generated by an n-computad. Suitable n-categories with relations on n-cells are presented by these $(n+1)$-computads, which allows us to prove results on presentations of thin groupoids and thin categories. So motivated, we introduce a notion of deficiency of (a presentation of) a groupoid via computads and prove that every small connected thin groupoid has deficiency 0. We compare the resulting notions of deficiency and presentation with those induced by monads. In particular, we find our notion of group deficiency to coincide with the classical one. Finally, we study presentations of 2-categories via 3-computads, focusing on locally thin groupoidal 2-categories. Under suitable hypotheses, we give efficient presentations of some locally thin and groupoidal 2-categories. A fundamental tool is a 2-dimensional analogue of the association of a "topological graph" to every graph internal to Set. Concretely, we construct a left adjoint $\mathscr{F}_{\mathsf{Top}_2} : 2\text{-}\mathsf{cmp} \to \mathsf{Top}$ associating a 2-dimensional CW-complex to each small 2-computad. Given a 2-computad $\mathfrak{g}$, the groupoid it presents is equivalent to the fundamental groupoid of $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g})$. Finally, we sketch the 3-dimensional version $\mathscr{F}_{\mathsf{Top}_3}$.

Introduction

The category of small categories cat is monadic over the category of small graphs grph. The left adjoint $\mathscr{F}_1 : \mathsf{grph} \to \mathsf{cat}$ is defined as follows: $\mathscr{F}_1(G)$ has the same objects of G and the morphisms between two objects are lists of composable arrows in G between them. The composition of such morphisms is defined by juxtaposition of composable lists and the identities are the empty lists.

Recall that a small graph is a functor $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ in which $\mathfrak{G}$ is the category with two objects and two parallel morphisms between them. If we compose G with the inclusion $\mathsf{Set} \to \mathsf{cat}$, we get a diagram $G' : \mathfrak{G}^{\mathrm{op}} \to \mathsf{cat}$. The benefit of this perspective is that the category freely generated by G is the *coinserter* of G', which is a type of (weighted) 2-colimit introduced in [59].

In the higher dimensional context, we have as primary structures the so called *n-computads*, firstly introduced for dimension 2 in [103]. There are further developments of the theory of computads [15, 44, 82, 88, 94], including generalizations such as in [3] and the proof of the monadicity of the category of the strict ω-categories over the category of ω-computads in [89].

In this paper, we give a concise definition of the classical (strict) n-computad such that the (strict) n-category freely generated by a computad is the coinserter of this computad. More precisely, we define an n-computad as a graph of $(n-1)$-categories satisfying some properties (given in Remark 2.8.12) and, then, we demonstrate that the n-category freely generated by it is just the coinserter of this graph composed with the inclusion $(n-1)\text{-}\mathsf{Cat} \to n\mathsf{Cat}$, getting in this way the free n-category functor whose induced monad is denoted by $\overline{\mathscr{F}_n}$. More generally, we show that this approach works for an n-dimensional analogue of the notion of *derivation scheme*, introduced for dimension 2 in [106].

Since we are talking about coinserters, we of course consider a 2-category of n-categories. Instead of n-natural transformations, we have to consider the n-dimensional analogues of icons, introduced for dimension 2 in [69]. We get, then, 2-categories $n\mathsf{Cat}$ of n-categories, n-functors and n-icons.

Every monad $\mathscr{T}$ on a category $\mathfrak{X}$ induces a notion of presentation of algebras given in Definition 2.3.1, which we refer as $\mathscr{T}$-presentation. If, furthermore, $\mathfrak{X}$ has a strong notion of measure μ of objects, we also get a (possibly naive) notion of deficiency (of a presentation) of a $\mathscr{T}$-algebra induced by μ (given in Remark 2.6.19). In the case of algebras over Set (together with *cardinality of sets*) given in 2.6.1, we get the classical notions of deficiency of a (presentation of a) finitely presented group, deficiency of a (presentation of a) finitely presented monoid and dimension of a finitely presented vector space.

Higher computads also give notions of presentations of higher categories. More precisely, using the description of n-computads of this paper, the coequalizer of an n-computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$, denoted by $\mathscr{P}_{(n-1)}(\mathfrak{g})$, is what we call the $(n-1)$-category presented by this n-computad in which the n-cells of the computad correspond to "relations of the presentation". In this context, an n-computad gives a presentation of an $(n-1)$-category with only equations between $(n-1)$-cells.

We show that every presentation of $(n-1)$-categories via n-computads are indeed particular cases of $\overline{\mathscr{F}_n}$-presentations. Moreover, on one hand, the notion of $\overline{\mathscr{F}_1}$-presentation of a monoid does not coincide with the (classical) notion of $\overline{\mathscr{F}_0}$-presentation, since there are $\overline{\mathscr{F}_1}$-presentations that are not $\overline{\mathscr{F}_0}$-presentations of a monoid. On the other hand, the notion of presentation of a monoid via computads does coincide with the classical one.

We present, then, the topological aspects of this theory. In order to do so, we construct two particular adjunctions. Firstly, we give the construction in Remark 2.5.1 of the left adjoint functor $\mathscr{F}_{\mathsf{Top}_1} : \mathsf{Grph} \to \mathsf{Top}$ which gives the "topological graph" associated to each graph $\mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ via a topological enriched version of the coinserter. Secondly, we show how the usual concatenation of continuous paths in a topological space gives rise to a monad functor/morphism $\overline{\mathscr{F}_1} \longrightarrow \overline{\mathscr{F}_{\mathsf{Top}_1}}$ between the free category monad and the monad induced by the left adjoint $\mathscr{F}_{\mathsf{Top}_1}$. Finally, using this monad morphism, we construct a left adjoint functor $\mathscr{F}_{\mathsf{Top}_2} : 2\text{-}\mathsf{cmp} \to \mathsf{Top}$.

The adjunction $\mathscr{F}_{\mathsf{Top}_2} \dashv \mathscr{C}_{\mathsf{Top}_2}$ gives a way of describing the fundamental groupoid of a topological space: $\mathscr{C}_{\mathsf{Top}_2}(X)$ presents the fundamental groupoid of X. More precisely, it is clear that $\mathscr{P}_1\mathscr{C}_{\mathsf{Top}_2}(X) \cong \Pi(X)$ which we adopt as the definition of the fundamental groupoid of a topological space X in Section 2.5.

Denoting by $\mathscr{L}_1 : \mathsf{cat} \to \mathsf{gr}$ the functor left adjoint to the inclusion of the category of small groupoids into the category of small categories, we show that the fundamental groupoid of a graph is equivalent to the groupoid freely generated by this graph, proving that there is a natural transformation which is objectwise an equivalence between $\mathscr{L}_1\mathscr{F}_1$ and $\Pi.\mathscr{F}_{\mathsf{Top}_1} \cong \mathscr{P}_1\mathscr{C}_{\mathsf{Top}_2}\mathscr{F}_{\mathsf{Top}_1}$. We also show that, given a small 2-computad $\mathfrak{g}$, there is an equivalence $\mathscr{P}_1\overline{\mathscr{F}_{\mathsf{Top}_2}}(\mathfrak{g}) \simeq \mathscr{L}_1\mathscr{P}_1(\mathfrak{g})$, which means that there is an equivalence between the fundamental groupoid of the CW-complex/topological space associated to a small 2-computad $\mathfrak{g}$ and the groupoid presented by $\mathfrak{g}$.

In the context of presentation of groups via computads, the left adjoint functor $\mathscr{F}_{\mathsf{Top}_2}$ formalizes the usual association of each classical ($\overline{\mathscr{L}_0\mathscr{F}_0}$-)presentation of a group G with a 2-dimensional CW-complex X such that $\pi_1(X) \cong G$.

We study freely generated categories and presentations of categories via computads, focusing on the study of thin groupoids and thin categories. By elementary results on Euler characteristic of CW-complexes, the results on $\mathscr{F}_{\mathsf{Top}_2}$ described above imply in Theorem 2.6.10 which, together with Theorem 2.6.16, motivate the definition of deficiency of a groupoid (w.r.t. presentations via groupoidal computads). We compare this notion of deficiency with the previously presented ones: for instance, in Remark 2.6.19, we compare with the notion of deficiency induced by the free groupoid monad $\overline{\mathscr{L}_1\mathscr{F}_1}$ together with the "measure" Euler characteristic, while in Propostion 2.6.14 we show that the classical concept of deficiency of groups coincides with the concept of deficiency of the suspension of a group w.r.t. presentations via computads.

By Theorem 2.6.16 and Theorem 2.6.10, the deficiency of thin "finitely generated" groupoids are 0, what generalizes the elementary fact that the trivial group has deficiency 0. Moreover, this implies that Theorem 2.6.16 gives efficient presentations, meaning that it has the least number of 2-cells (equations) of the finitely presented thin groupoids.

We lift some of these results to presentations of thin categories and give some further aspects of presentations of thin categories as well. Finally, supported by these results and the characterization of thin categories that are free $\overline{\mathscr{F}_1}$-algebras, we give comments towards the deficiency of a thin finitely presented category, considering a naive generalization of the concept of deficiency of groupoid introduced previously.

The final topic of this paper is the study of presentations of locally thin 2-categories via 3-computads. Similarly to the 1-dimensional case, we firstly describe aspects of freely generated 2-categories, including straightforward sufficient conditions to conclude that a given (locally thin) 2-category is not a free $\overline{\mathscr{F}_2}$-algebra. We conclude that there are interesting locally thin 2-categories that are not free, what gives a motivation to study presentations of locally thin 2-categories. In order to study such locally thin 2-categories, we study presentations of some special locally thin $(2,0)$-categories which are, herein, 2-categories with only invertible cells. With suitable conditions, we can lift such presentations to presentations of locally thin and groupoidal 2-categories.

We also give a sketch of the construction of a left adjoint functor $\mathscr{F}_{\mathsf{Top}_3} : 3\text{-}\mathsf{cmp} \to \mathsf{Top}$ which allows us to give a result (Corollary 2.10.18) towards a 3-dimensional version of Theorem 2.6.10.

This result shows that the presentations of $(2,0)$-categories given previously have the least number of 3-cells (equations): they are efficient presentations.

In [77–79], we introduce 2-dimensional versions of bicategorical replacements of the category $\dot{\Delta}'_3$ of the ordinals $\mathsf{0}$, $\mathsf{1}$, $\mathsf{3}$ and order-preserving functions between them without nontrivial morphisms $\mathsf{3} \to \mathsf{n}$. We apply our theory to give an efficient presentation of the bicategorical replacement of the category $\dot{\Delta}_2$ and study the presentation of the locally groupoidal 2-category $\dot{\Delta}_{\mathrm{Str}}$ introduced in [79] (which corresponds to Chapter 3).

This work was realized in the course of my PhD studies at University of Coimbra. I wish to thank my supervisor Maria Manuel Clementino for giving me useful pieces of advice, feedback and insightful lessons.

2.1 Preliminaries

The most important hypothesis is that $\mathsf{Cat}, \mathsf{CAT}$ are cartesian closed categories of categories such that Cat is an internal category of the subcategory of discrete categories of CAT. So, herein, a *category* $\mathfrak{X}$ means an object of CAT. Moreover, if X, Y are objects of Cat, we denote by $\mathsf{Cat}[X,Y]$ its internal hom and by $\mathsf{Cat}(X,Y)$ the *discrete category* of functors between X and Y. We also assume that the category of sets Set is an object of Cat. The category of *small categories* is $\mathsf{cat} := \mathsf{int}(\mathsf{Set})$, that is to say, cat is the category of internal categories of the category of sets.

If V is a symmetric monoidal closed category, we denote by V-Cat the category of V-enriched categories. We refer the reader to [7, 31, 58] for *enriched categories* and *weighted limits*. It is important to ratify that herein the collection of objects of a V-category X of V-Cat is a discrete category in Cat, while V-cat denotes the category of small V-categories.

Inductively, we define the category n-Cat by $(n+1)$-$\mathsf{Cat} := (n\text{-}\mathsf{Cat})$-$\mathsf{Cat}$ and 1-$\mathsf{Cat} := \mathsf{Cat}$. Therefore there are full inclusions $(n+1)$-$\mathsf{Cat} \to \mathsf{int}(n\text{-}\mathsf{Cat})$ and n-Cat is cartesian closed, being an $(n+1)$-category which is not an object of $(n+1)$-Cat. In particular, Cat is a 2-category which is not an object of 2-Cat.

We deal mainly with weighted limits in the Cat-enriched context, the so called 2-categorical limits. The basic references are [59, 103]. Let $\mathscr{W} : \mathfrak{S} \to \mathsf{Cat}, \mathscr{W}' : \mathfrak{S}^{\mathrm{op}} \to \mathsf{Cat}$ and $\mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ be 2-functors with a small domain. If it exists, we denote the *weighted limit* of $\mathscr{D}$ with weight $\mathscr{W}$ by $\{\mathscr{W}, \mathscr{D}\}$. Dually, we denote by $\mathscr{W}' * \mathscr{D}$ the *weighted colimit* provided that it exists. Recall that, by definition, there is a 2-natural isomorphism (in X)

$$\mathfrak{A}(\mathscr{W}' * \mathscr{D}, X) \cong [\mathfrak{S}^{\mathrm{op}}, \mathsf{Cat}]\,(\mathscr{W}', \mathfrak{A}(\mathscr{D}-, X)) \cong \left\{\mathscr{W}', \mathfrak{A}(\mathscr{D}-, X)\right\}$$

in which $[\mathfrak{S}^{\mathrm{op}}, \mathsf{Cat}]$ denotes the 2-category of 2-functors $\mathfrak{S}^{\mathrm{op}} \to \mathsf{Cat}$, 2-natural transformations and modifications.

In the last section, we apply 2-monad theory to construct 2-categories $n\mathsf{Cat}$ for each natural number n. We refer the reader to [9] for the basics of 2-monad theory. The category $n\mathsf{Cat}$ is one of the possible higher dimensional analogues of the 2-category of 2-categories, 2-functors and icons introduced in [69].

The category $\dot{\Delta}$ is the *category of finite ordinals*, denoted by $\mathsf{0},\mathsf{1},\mathsf{2},\ldots,\mathsf{n},\ldots$, and order-preserving functions between them. We denote by Δ the full *subcategory of nonempty ordinals*. There are full inclusions $\Delta \to \dot{\Delta} \to \mathsf{cat} \to \mathsf{Cat}$. Often, we use n also to denote its image by these inclusions. Thereby the category n is the category

$$\mathsf{0} \to \mathsf{1} \to \cdots \to \mathsf{n}-\mathsf{1}.$$

For each n of Δ, the n-*truncated category of* $\dot{\Delta}$, denoted by $\dot{\Delta}_{\mathsf{n}}$, is the full subcategory of $\dot{\Delta}$ with only $\mathsf{0},\mathsf{1},\ldots,\mathsf{n}$ as objects. The truncated category Δ_{n} is analogously defined. For instance, the category Δ_2 is generated by the faces d^0, d^1 and by the degeneracy s^0 as follows:

$$\begin{tikzcd}
\mathsf{1} \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d^0"'] & \mathsf{2} \arrow[l, "s^0" description]
\end{tikzcd}$$

in which, after composing with the inclusions $\Delta_2 \to \Delta \to \mathsf{Cat}$, d^0 and d^1 are respectively the inclusions of the codomain and the domain of morphism $\mathsf{0} \to \mathsf{1}$ of $\mathsf{2}$.

Moreover, the *category* $\mathfrak{G}$ is, herein, the subcategory of Δ_2 without the degeneracy $\mathsf{2} \to \mathsf{1}$ and with all the faces $\mathsf{1} \to \mathsf{2}$ of Δ_2 as it is shown below. Again, considering $\mathfrak{G}$ as a subcategory of Cat, d^1 is the inclusion of the domain and d^0 is the inclusion of the codomain.

$$\begin{tikzcd}
\mathsf{1} \arrow[r, shift left, "d^1"] \arrow[r, shift right, "d^0"'] & \mathsf{2}
\end{tikzcd}$$

We denote by $\mathscr{I} : \mathfrak{G} \to \mathsf{Cat}$ the inclusion given by the composition of the inclusions $\mathfrak{G} \to \Delta_2 \to \mathsf{Cat}$. The 2-functor $\mathscr{I}$ defines the weight of the limits called *inserters*, while $\mathscr{I}$-weighted colimits are called *coinserters*. Also, we have the weight $\mathscr{U}_1\mathscr{L}_1\mathscr{I} : \mathfrak{G} \to \mathsf{Cat}$ which gives the notions of *isoinserter* and *isocoinserter*, defined as follows:

$$\begin{tikzcd}
\mathsf{1} \arrow[r, shift left] \arrow[r, shift right] & \nabla\mathsf{2}
\end{tikzcd}$$

in which $\nabla\mathsf{2}$ is the category with two objects and one isomorphism between them and $\mathscr{U}_1\mathscr{L}_1\mathscr{I}(d^0)$, $\mathscr{U}_1\mathscr{L}_1\mathscr{I}(d^1)$ are the inclusions of the two different objects.

Let $\mathsf{2}_2$ be the 2-category below with two parallel nontrivial 1-cells and only one nontrivial 2-cell between them. We define the weight $\mathrm{J}_{\mathsf{2}_2}$ by

$$\mathrm{J}_{\mathsf{2}_2} : \mathsf{2}_2 \to \mathsf{Cat}$$

$$\begin{tikzcd}
* \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] \arrow[Rightarrow, from=U, to=D] & * \arrow[r, mapsto] & \mathsf{1} \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2}] \arrow[Rightarrow, from=U2, to=D2] & \nabla\mathsf{2},
\end{tikzcd}$$

in which the image of the 2-cell is the only possible natural isomorphism between the inclusion of the domain and the inclusion of the codomain. The $\mathrm{J}_{\mathsf{2}_2}$-weighted colimits are called *coinverters*.

Finally, let $\mathfrak{G}_2$ be the 2-category with two parallel nontrivial 1-cells and only two parallel nontrivial 2-cells between them. We define the weight $\mathrm{J}_{\mathfrak{G}_2}$ by

$$\mathrm{J}_{\mathfrak{G}_2} : \mathfrak{G}_2 \to \mathsf{Cat}$$

$$\begin{tikzcd}
* \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] \arrow[Rightarrow, from=U, to=D, shift right] \arrow[Rightarrow, from=U, to=D, shift left] & * \arrow[r, mapsto] & \mathsf{1} \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2}] \arrow[Rightarrow, from=U2, to=D2] & \mathsf{2},
\end{tikzcd}$$

in which the images of the 2-cells are the only possible natural transformation between the inclusion of the domain and the inclusion of the codomain. The $\mathrm{J}_{\mathfrak{G}_2}$-weighted colimits are called *coequifiers*.

2.1.1 Thin Categories and Groupoids

A category X is a groupoid if every morphism of X is invertible. The 2-category of groupoids of Cat is denoted by Gr. The inclusion $\mathscr{U}_1 : \mathsf{Gr} \to \mathsf{Cat}$ has a left 2-adjoint $\mathscr{L}_1$. Also, the category of *locally groupoidal* 2*-categories* is, by definition, Gr-Cat and the previous adjunction induces a left adjoint $\mathscr{L}_2$ to the inclusion $\mathscr{U}_2 : \mathsf{Gr}\text{-}\mathsf{Cat} \to 2\text{-}\mathsf{Cat}$.

Definition 2.1.2. [Connected Category] A category X of Cat is *connected* if every object of $\mathscr{L}_1(X)$ is weakly terminal. In particular, a groupoid Y is connected if and only if every object of Y is weakly terminal.

A category X of Cat is *thin* if between any two objects of X there is at most one morphism. Again, we can consider *locally thin* 2*-categories*, which are categories enriched over the category of thin categories of Cat. We denote by Prd the category of thin categories. The inclusion $M_1 : \mathsf{Prd} \to \mathsf{Cat}$ has a left 2-adjoint $\mathscr{M}_1$. Again, it induces a left adjoint $\mathscr{M}_2$ to the inclusion $\mathsf{Prd}\text{-}\mathsf{Cat} \to 2\text{-}\mathsf{Cat}$.

Remark 2.1.3. The 2-functors $\mathscr{U}_1 : \mathsf{Gr} \to \mathsf{Cat}, M_1 : \mathsf{Prd} \to \mathsf{Cat}$ are 2-monadic and the 2-monads induced by them are idempotent, since $\mathscr{U}_1, M_1$ are fully faithful. Therefore $\mathscr{U}_1, M_1$ create 2-limits.

The functor $\mathscr{U}_1$ is left adjoint: hence, as $\mathscr{U}_1$ is monadic, $\mathscr{U}_1$ creates coequalizers and coproducts. But it does not preserve tensor with 2. Finally, Prd is isomorphic to the 2-category of categories enriched over 2 and, hence, it is 2-cocomplete.

Proposition 2.1.4. *Let X be an object of* Gr *or* Cat. *We have that X is a thin category if and only if X is (isomorphic to) the coequifier of*

$$\begin{tikzcd}(\mathfrak{G}^{\mathrm{op}} \pitchfork X)_0 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\alpha\Downarrow\ \beta\Downarrow"] & X\end{tikzcd}$$

in which $\mathsf{Cat}(\mathfrak{G}^{\mathrm{op}}, X) \cong (\mathfrak{G}^{\mathrm{op}} \pitchfork X)_0$ *is the discrete category of internal graphs of X, $\alpha_G = G(d^0)$ and $\beta_G = G(d^1)$.*

Theorem 2.1.5. *There are categories X, Y in* Cat *such that $\mathscr{L}_1(X)$ and Y are thin, but X and $\mathscr{L}_1(Y)$ are not thin. In particular, $\mathscr{L}_1$ is not faithful.*

Proof. For instance, we define Y to be the category generated by the graph

$$\begin{tikzcd}\ast \arrow[r] \arrow[rrr, bend left=15] & \ast & \ast \arrow[l] \arrow[r] & \ast\end{tikzcd} \tag*{(\textit{example of weak tree})}$$

in which there is no nontrivial composition and X can be defined as

$$\begin{tikzcd}\ast \arrow[r, "h"] & \ast \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \ast\end{tikzcd}$$

satisfying the equation $fh = gh$. □

A category X satisfies the *cancellation law* if every morphism of X is a monomorphism and an epimorphism.

Theorem 2.1.6. *If X satisfies the cancellation law and $\mathscr{L}_1(X)$ is a thin groupoid, then X is a thin category.*

Proof. The components of the unit on the categories that satisfy the cancellation law of the adjunction $\mathscr{L}_1 \dashv \mathscr{U}_1$ are faithful. Thereby, if X satifies the cancellation law and $\mathscr{L}_1(X)$ is thin, X is thin. $\square$

Theorem 2.1.7. *Let X be an object of* Prd *or* Cat. *X is a groupoid if and only if X is the coinverter of*

$$\begin{tikzcd}(2 \pitchfork X)_0 \arrow[r, shift left, ""{name=U}] \arrow[r, shift right, ""'{name=D}] \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] & X\end{tikzcd}$$

in which $(2 \pitchfork X)_0$ is the discrete category of morphisms in X and $\alpha_f = f$.

Remark 2.1.8. As a consequence, since $\mathscr{M}_1$ preserves 2-colimits and, for each Y in Prd, the induced functor $\mathsf{Prd}((2 \pitchfork \mathscr{M}_1(X))_0, Y) \to \mathsf{Prd}(\mathscr{M}_1(2 \pitchfork X)_0, Y)$ is fully faithful, $\mathscr{M}_1$ preserves groupoids.

2.2 Graphs

We start studying aspects of graphs and freely generated categories. An *internal graph* of a category $\mathfrak{X}$ is a functor $G : \mathfrak{G}^{\mathrm{op}} \to \mathfrak{X}$, while the *category of graphs internal to* $\mathfrak{X}$, denoted by $\mathsf{Grph}(\mathfrak{X})$, is the category of functors and natural transformations $\mathsf{CAT}[\mathfrak{G}^{\mathrm{op}}, \mathfrak{X}]$.

Herein, a *graph* is an internal graph of discrete categories in Cat. That is to say, a graph is a functor $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ that factors through the inclusion of the discrete categories $\mathsf{SET} \to \mathsf{Cat}$. This defines the category of graphs $\mathsf{Grph} := \mathsf{Grph}(\mathsf{SET})$. Although the basic theory works for larger graphs and computads in the setting of Section 2.1, the combinatorial part is of course just suited for small graphs and computads. We define the *category of small graphs* by $\mathsf{grph} := \mathsf{Cat}[\mathfrak{G}^{\mathrm{op}}, \mathsf{Set}]$, while the category of *finite/countable graphs* is the full subcategory of small graphs G such that $G(1)$ is finite/countable.

If $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is a graph, $G(1)$ is the *discrete category/collection of objects of G*, while $G(2)$ is *the discrete category/collection of arrows (or edges) of G*. An arrow a of G is denoted by $a : x \to z$, if $G(d^0)(a) = z$ and $G(d^1)(a) = x$. As usual, in this case, z is called the *codomain* and x is called the *domain of the edge a*.

We also consider *the category of reflexive graphs* $\mathsf{RGrph} := \mathsf{Cat}\left[\Delta_2^{\mathrm{op}}, \mathsf{SET}\right]$ and *the category of small reflexive graphs* $\mathsf{Rgrph} := \mathsf{Cat}\left[\Delta_2^{\mathrm{op}}, \mathsf{Set}\right]$. If G is a reflexive graph, the collection in the image of $G(s^0)$ is called the collection/discrete category of *trivial arrows/identity arrows/identities* of G.

The inclusion $\mathfrak{G}^{\mathrm{op}} \to \Delta_2^{\mathrm{op}}$ induces a forgetful functor $\mathscr{R} : \mathsf{RGrph} \to \mathsf{Grph}$ and the left Kan extensions along this inclusion provide a left adjoint to this forgetful functor, denoted by $\mathscr{E}$.

Lemma 2.2.1. *The forgetful functor $\mathscr{R}$* : $\mathsf{RGrph} \to \mathsf{Grph}$ *has a left adjoint $\mathscr{E}$.*

Remark 2.2.2. The terminal object of RGrph is denoted by $\bullet$. It has only one object and its trivial arrow. It should be noted that RGrph is not equivalent to Grph, since $\bullet$ is also weakly initial in RGrph while the terminal graph $\mathscr{R}(\bullet) \cong \bigcirc$ is not.

The inclusion $\mathsf{SET} \to \mathsf{Cat}$ has a right adjoint $(-)_0 : \mathsf{Cat} \to \mathsf{SET}$, the forgetful functor. The comonad induced by this adjunction is also denoted by $(-)_0$. On one hand, we define $\mathscr{C}_1 : \mathsf{Cat} \to \mathsf{Grph}$

by $\mathscr{C}_1(X) := \mathsf{Cat}(\mathscr{I}-,X) = (\mathsf{Cat}[\mathscr{I}-,X])_0$. On the other hand, if $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is any 2-functor, we have:

$$\mathsf{Cat}[\mathscr{I} * G, X] \cong [\mathfrak{G}^{\mathrm{op}}, \mathsf{Cat}]\,(G, \mathsf{Cat}[\mathscr{I}-,X]),$$

since $\mathscr{I} * G \cong G * \mathscr{I}$. This induces an adjunction between the category of categories and the category of internal graphs of Cat. If $G(2)$ is a set, this shows how the coinserter encompasses the notion of freely adding morphisms to a category $G(1)$. In particular, if G is a graph, this induces a (natural) bijection between natural transformations $G \longrightarrow \mathsf{Cat}(\mathscr{I}-,X)$ and functors $\mathscr{I} * G \longrightarrow X$. Therefore:

Lemma 2.2.3. $\mathscr{F}_1 : \mathsf{Grph} \to \mathsf{Cat}, \mathscr{F}_1(G) = \mathscr{I} * G$ *gives the left adjoint to* $\mathscr{C}_1$.

Informally, we get the result above once we realize that if X is a category in Cat then a functor $f : \mathscr{F}_1(G) \to X$ needs to correspond to a pair (f_0, α^f), in which $f_0 : G(1) \to (X)_0$ is a morphism of SET and $\alpha^f : f_0 G(d^1) \longrightarrow f_0 G(d^0)$ is a natural transformation. This is precisely an object of the inserter of $\mathsf{Cat}(G-,X)$.

Remark 2.2.4. [Categories freely generated by reflexive graphs] We can also consider the inclusion $\mathscr{I}^{\mathscr{R}} : \Delta_2 \to \mathsf{Cat}$ and this inclusion induces the functor

$$\mathscr{C}_1^{\mathscr{R}} : \mathsf{Cat} \to \mathsf{RGrph}, \mathscr{C}_1^{\mathscr{R}}(X) = \mathsf{Cat}(\mathscr{I}^{\mathscr{R}}-,X).$$

Analogously, this functor has a left adjoint defined by $\mathscr{F}_1^{\mathscr{R}}(G) = \mathscr{I}^{\mathscr{R}} * G$. It is easy to verify that there is a natural isomorphism $\mathscr{F}_1 \cong \mathscr{F}_1^{\mathscr{R}} \mathscr{E}$.

If G is a reflexive graph and x is an object of G, we say that $G(s^0)(x)$ is the trivial arrow/identity arrow of x. In particular, the image of $G(s^0)$ is called the discrete category/collection of the trivial arrows of G.

Remark 2.2.5. Since $(1 \amalg 1) \cong (2)_0$ in Cat and Cat is lextensive, recall that $X \times (2)_0 \cong (X \times 1) \amalg (X \times 1)$ for any object X of Cat.

If G is an object of Grph, we can construct $\mathscr{F}_1(G)$ via the pushout of the morphism $G(2) \times (2)_0 \to G(1)$ induced by $(G(d^0), G(d^1))$ along the functor $G(2) \times (2)_0 \to G(2) \times 2$ given by the product of the identity with the inclusion $(2)_0 \to 2$ induced by the counit of the comonad $(-)_0 : \mathsf{Cat} \to \mathsf{Cat}$.

Remark 2.2.6. The functor $\mathscr{C}_1$ is monadic since it is right adjoint, reflects isomorphisms, preserves coequalizers and Cat is cocomplete. Hence, each component of the counit of $\mathscr{F}_1 \dashv \mathscr{C}_1$ gives a functor $\mathrm{comp}_X : \mathscr{F}_1 \mathscr{C}_1(X) \to X$ which is a regular epimorphism.

The forgetful functor $\mathscr{C}_1 \mathscr{U}_1 : \mathsf{Gr} \to \mathsf{Grph}$ has an obvious left adjoint given by $\mathscr{L}_1 \mathscr{F}_1 : \mathsf{Grph} \to \mathsf{Gr}$. If $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is a graph, $\mathscr{L}_1 \mathscr{F}_1(G)$ is called the *groupoid freely generated by G*.

We denote respectively by $\overline{\mathscr{F}_1}$ and $\overline{\mathscr{L}_1 \mathscr{F}_1}$ the monads induced by the adjunctions $\mathscr{F}_1 \dashv \mathscr{C}_1$ and $\mathscr{L}_1 \mathscr{F}_1 \dashv \mathscr{C}_1 \mathscr{U}_1$.The free $\overline{\mathscr{F}_1}$-algebras are called *free categories*, while we call *free groupoids* the free algebras of the monad $\overline{\mathscr{L}_1 \mathscr{F}_1}$.

Lemma 2.2.7. $\mathscr{L}_1 \mathscr{F}_1(G) \cong \mathscr{I} * (\mathscr{L}_1 G) \cong (\mathscr{U}_1 \mathscr{L}_1 \mathscr{I}) * (\mathscr{L}_1 G)$ *gives the left adjoint to* $\mathscr{C}_1$.

Observe that $\mathscr{L}_1 G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Gr}$ is nothing but G itself as an internal graph of discrete groupoids since $\mathscr{L}_1$ takes discrete categories to discrete groupoids. Also, $\mathscr{U}_1 \mathscr{L}_1(\mathscr{F}_1(G)) \cong (\mathscr{U}_1 \mathscr{L}_1 \mathscr{I}) * G$ in Cat. That is to say, the groupoid freely generated by G is its isocoinseter in Cat.

Remark 2.2.8. [Characterization of Free Categories [106]] The category Grph has terminal object $\bigcirc$, namely the graph with only one object and only one arrow. If we denote by $\Sigma(\mathbb{N})$ the resulting category from the suspension of the monoid of non-negative integers $\mathbb{N}$, we have that $\mathscr{F}_1(\bigcirc) \cong \Sigma(\mathbb{N})$. Therefore, every graph G comes with a functor

$$\ell^G : \mathscr{F}_1(G) \to \Sigma(\mathbb{N})$$

which is by definition the morphism $\mathscr{F}_1(G \to \bigcirc)$. The functor ℓ^G is called *length functor*. It satisfies a property called *unique lifting of factorizations*, usually refereed as *ulf*. In this case, this means in particular that, if $\ell^G(f) = m$, then there are unique morphisms $f_m, \dots, f_1, f_0$ such that

- $f_m \cdots f_1 f_0 = f$;
- $\ell^G(f_t) = 1 \ \forall t \in \{1, \dots, m\}$ and f_0 is the identity.

This property characterizes free categories. More precisely, $X \cong \mathscr{F}_1(G)$ for some graph G if and only if there is a functor $\ell_X : X \to \Sigma(\mathbb{N})$ satisfying the unique lifting of factorizations property.

A morphism f has length m if $\ell^G(f) = m$. It is easy to see that the morphisms of $\mathscr{F}_1(G)$ with length 1 correspond to the edges of G. Roughly, the unique lifting property of ℓ^G says that every morphism $f : x \to z$ is a composition $f = a_1 \dots a_m$ of arrows with length 1 which corresponds to a list of arrows in G satisfying $G(d^1)(a_t) = G(d^0)(a_{t+1})$ for all $t \in \{1, \dots, m-1\}$, while the identities of $\mathscr{F}_1(G)$ correspond to empty lists. Following this viewpoint, the composition is given by juxtaposition of these lists. A morphism of $f : x \to z$ of $\mathscr{F}_1(G)$ is often called a *path (of length $\ell^G(f)$) between x an z in the graph G.*

It is clear that the length functors reflect isomorphisms. More precisely, if ℓ^G is a length functor, then $\ell^G(f) = 0$ implies that $f = \mathrm{id}$.

As a particular consequence of the characterization given in Remark 2.2.8, we get that:

Theorem 2.2.9. *For any graph G, $\mathscr{F}_1(G)$ satisfies the cancellation law.*

Remark 2.2.10. Let X be a category. By the natural isomorphism of Remark 2.2.4, we have that $X \cong \mathscr{F}_1(G)$ for some graph G if and only if $X \cong \mathscr{F}_1^{\mathscr{R}} \mathscr{E}(G)$. Also, $X \cong \mathscr{F}_1^{\mathscr{R}}(G)$ for some reflexive graph G if and only if $X \cong \mathscr{F}_1(G^{\mathscr{E}})$, in which $G^{\mathscr{E}} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ has the same objects of G and the nontrivial arrows of G. More precisely, $G^{\mathscr{E}}(2) = G(2) - G(s^0)(G(1))$, $G^{\mathscr{E}}(1) = G(1)$. Therefore the characterization of categories freely generated by reflexive graphs is equivalent to the characterization given in Remark 2.2.8.

It should be noted that $(-)^{\mathscr{E}}$ is a functor between the subcategories of monomorphisms of RGrph and Grph.

Remark 2.2.11. [Characterization of Free Groupoids] A natural extension of the Remark 2.2.8 gives a characterization of free groupoids. More precisely, for each graph G, there is functor

$$\mathscr{L}_1(\ell^G) = \mathscr{L}_1 \mathscr{F}_1(G \to \bigcirc) : \mathscr{L}_1 \mathscr{F}_1(G) \to \Sigma(\mathbb{Z})$$

in which $\Sigma(\mathbb{Z})$ is the suspension of the group of integers. This functor has the ulf property. In this case, this means that, if $\mathscr{L}_1(\ell^G)(f) = m$, then there are unique morphisms $f_n, \dots, f_1, f_0$ such that

- $f_n \cdots f_1 f_0 = f$;
- $\mathscr{L}_1(\ell^G)(f_t) \in \{-1,1\}$, for all $t \in \{1,\dots,n\}$ and f_0 is identity;
- $\sum \mathscr{L}_1(\ell^G)(f_t) = m$.

This property characterizes free groupoids. That is to say, $X \cong \mathscr{L}_1\mathscr{F}_1(G)$ for some graph G if and only if X is a groupoid and there is a functor $\ell_X : X \to \Sigma(\mathbb{Z})$ satisfying the unique lifting of factorizations property.

It is easy to see that the morphisms of $\mathscr{L}_1\mathscr{F}_1(G)$ with length 1 correspond to the arrows of G, while the morphisms with length -1 correspond to formal inversions of arrows of G.

Definition 2.2.12. A graph G is called:

- *connected* if $\mathscr{F}_1(G)$ is connected;
- a *weak forest* if $\mathscr{F}_1(G)$ is thin;
- a *forest* if $\mathscr{L}_1\mathscr{F}_1(G)$ is thin;
- a *tree/weak tree* if G is a connected forest/weak forest.

Theorem 2.2.13. *If G is a forest, then it is a weak forest as well.*

Proof. By Theorem 2.1.6 and Theorem 2.2.9, if $\mathscr{L}_1\mathscr{F}_1(G)$ is thin, then $\mathscr{F}_1(G)$ is thin as well. □

The converse of Theorem 2.2.13 is not true. For instance, a counterexample is given in Remark 2.2.16.

Remark 2.2.14. [Maximal Tree] By Zorn's Lemma, every small connected graph G has maximal trees and maximal weak trees. This means that, given a small connected graph G, the preordered set of trees and the preordered set of weak trees of G have maximal objects. Of course, these results do not depend on Zorn's Lemma if G is countable.

Lemma 2.2.15. *G_{mtree} is a maximal tree of a connected graph G if and only if the following properties are satisfied:*

- *G_{mtree} is a subgraph of G;*
- *G_{mtree} is a tree;*
- *G_{mtree} has every object of G.*

Remark 2.2.16. By the last result, a tree in a small connected graph G is maximal if and only if it has all the objects of G. Such a characterization does not hold for maximal weak trees. For instance, the graph `T` given by the *example of weak tree* is a weak tree which is not a tree. Hence, the maximal tree of this graph is an example of a weak tree that has all the objects of the graph `T` without being a maximal weak tree. However, one of the directions holds. Namely, every maximal weak tree of a small connected graph G has every object of G.

Remark 2.2.17. All definitions and results related to trees and forests have analogues for reflexive graphs. In fact, for instance, a reflexive graph G is a *reflexive tree* if $\mathscr{F}_1^{\mathscr{R}}(G)$ is a connected thin category. Then, we get that G is a *reflexive tree* if and only if the graph $G^{\mathscr{E}}$ (defined in Remark 2.2.10) is a tree.

In particular, G_{mtree} is a maximal reflexive tree of a connected reflexive graph G if and only if $G^{\mathscr{E}}_{\mathrm{mtree}}$ is a maximal tree of the graph $G^{\mathscr{E}}$.

Definition 2.2.18. [Fair Graph] An object G of Grph is a *fair graph* if it has a maximal weak tree which is a tree.

Remark 2.2.19. From Zorn's Lemma, we also get that every small graph G has a maximal fair subgraph which contains a maximal tree of G. Again, we can avoid Zorn's Lemma if we restrict our attention to countable graphs.

There are thin categories which are not free $\overline{\mathscr{F}_1}$-algebras. For instance, as a particular case of Lemma 2.2.20, the category $\nabla 2$ is thin and is not a free category. Furthermore, by Theorem 2.2.22, $\mathbb{R}$ and $\mathbb{Q}$ are examples of small thin categories without nontrivial isomorphisms that are not free categories.

Lemma 2.2.20. *If X is a category and it has a nontrivial isomorphism, then X is not a free category.*

Proof. There is only one isomorphism in $\Sigma(\mathbb{N})$, namely the identity 0. If f is an isomorphism of $\mathscr{F}_1(G)$, then $\ell^G(f) = 0$. Since ℓ^G reflects identities, we conclude that f is an identity. □

We can also consider the *thin category freely generated by a graph G*, since $\mathscr{M}_1\mathscr{F}_1 \dashv \mathscr{C}_1 M_1$. It is clear that $\mathscr{C}_1 M_1$ is fully faithful and, hence, it induces an idempotent monad $\overline{\mathscr{M}_1\mathscr{F}_1}$. In particular, every $\overline{\mathscr{M}_1\mathscr{F}_1}$-algebra is a free $\overline{\mathscr{M}_1\mathscr{F}_1}$-algebra. That is to say, every thin category is a thin category freely generated by a graph.

Proposition 2.2.21. *If $\mathscr{F}_1(G)$ is a totally ordered set then, for each object x of $\mathscr{F}_1(G)$ and each length m, there is at most one morphism of length m with x as domain in $\mathscr{F}_1(G)$. Moreover, if x is not the terminal object, then there is a unique morphism of length* 1 *with x as domain in $\mathscr{F}_1(G)$.*

Proof. In fact, suppose there are morphisms $b : x \to z', a : x \to z$ of length m. Since $\mathscr{F}_1(G)$ is totally ordered, we can assume without losing generality that there is a morphism $c : z \leq z'$ of some length n.

As $\mathscr{F}_1(G)$ is thin, $ca = b$. In particular, $n + m = \ell^G(ca) = \ell^G(b) = m$. Hence $n = 0$. This means that c is the empty path (identity) and $z = z'$. Again, since $\mathscr{F}_1(G)$ is thin, $a = b$.

It remains to prove the existence of a morphism of length 1 with x as domain whenever x is not the top element. In this case, there is a morphism $x \to z''$ of length $m > 0$ By Remark 2.2.8, we conclude that there is a unique list $x < z_1 < \ldots < z_{m-1} < z''$ such that $z_t < z_{t+1}$ corresponds to a morphism of length 1. In particular, $x < z_1$ has length 1. □

Theorem 2.2.22. *If $\mathscr{F}_1(G)$ is a totally ordered set, then it is isomorphic to one of the following ordered sets:*

- *The finite ordinals* $\mathsf{0}, \mathsf{1}, \ldots, \mathsf{n}, \ldots$*;*

– *The totally ordered sets* $\mathbb{N}, \mathbb{N}^{\mathrm{op}}$ *and* $\mathbb{Z}$.

Proof. – Assuming that $\mathscr{F}_1(G)$ has bottom $\bot$ and top $\top$ elements:
If $1 \ncong \mathscr{F}_1(G) \ncong 2$, $\bot \to \top$ has a length, say $m-1 \geq 2$. This means that

$$\mathscr{F}_1(G) \cong \{\bot < 1 < \ldots < m-2 < \top\} \cong \mathsf{m}.$$

– Assuming that $\mathscr{F}_1(G)$ has a bottom element $\bot$ but it does not have a top element:

We can define $s : \mathbb{N} \to \mathscr{F}_1(G)$ in which $s(0) := \bot$ and $s(n+1)$ is the codomain of the unique morphism of length 1 with $s(n)$ as domain. Of course, s is order preserving.

It is easy to see by induction that $\bot < s(n)$ has length n. Hence it is obvious that s is injective. Also, given an object x of $\mathscr{F}_1(G)$, there is m' such that $\bot \to x$ has length m'. By Proposition 2.2.21, it follows that $s(m') = x$. This proves that s is actually a bijection.

– Assuming that $\mathscr{F}_1(G)$ has a top element $\top$ but it does not have a bottom element:

By duality, we get that $\mathbb{N}^{\mathrm{op}} \cong \mathscr{F}_1(G)$.

– Assuming that $\mathscr{F}_1(G)$ does not have top nor bottom elements:

If $\mathscr{F}_1(G) \ncong 0$, given an object y of $\mathscr{F}_1(G)$, take the subcategories

$$\{x \in \mathscr{F}_1(G) : x \leq y\} \text{ and } \{x \in \mathscr{F}_1(G) : y \leq x\}.$$

By what we proved, these subcategories are isomorphic respectively to $\mathbb{N}^{\mathrm{op}}$ and $\mathbb{N}$. By the uniqueness of pushouts, we get $\mathscr{F}_1(G) \cong \mathbb{Z}$. ☐

Corollary 2.2.23. *If $\mathscr{F}_1(G)$ is a small thin category, then it is isomorphic to a colimit of ordinals* $\mathsf{0}, \mathsf{1}, \ldots, \mathsf{n}, \ldots$ *or/and* $\mathbb{N}, \mathbb{N}^{\mathrm{op}}, \mathbb{Z}$.

Remark 2.2.24. There are non-free categories which are subcategories of free categories. But subgroupoids of freely generated small groupoids are freely generated. In fact, this follows from:

Theorem 2.2.25. *A small groupoid is free if and only if its skeleton is free. In particular, freeness is a property preserved by equivalences of groupoids. As a consequence, subgroupoids of free groupoids are free.*

Proof. Since $\mathscr{L}_1\mathscr{F}_1$ creates coproducts and every groupoid is a coproduct of connected groupoids, it is enough to prove the statement for connected groupoids.

If a connected groupoid is free, this means that it is isomorphic to $\mathscr{L}_1\mathscr{F}_1(G)$ for a connected graph G. It is easy to see that the skeleton of $\mathscr{L}_1\mathscr{F}_1(G)$ is isomorphic to $\mathscr{L}_1\mathscr{F}_1(G)/\mathscr{L}_1\mathscr{F}_1(G_{\mathrm{mtree}})$ for any maximal tree G_{mtree}. Therefore $\mathscr{L}_1\mathscr{F}_1(G/G_{\mathrm{mtree}})$ is isomorphic to the skeleton and, hence, the skeleton is free.

Reciprocally, if the skeleton of a connected groupoid X is free, it follows that for any object y of X, the full subgroupoid with only x as object, often denoted by $\pi(X,y)$, is free. We take $\pi(X,y) \cong \mathscr{F}_1(H)$

and define the graph $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{cat}$ by:

- $G(2) := H(2) \amalg (\mathsf{cat}(1,X) - \{y\})$;
- $G(1) := \mathsf{cat}(1,X)$;
- $G(d^1)$ is constant equal to y;
- $G(d^0)(a) := y$ if $a \in H(2)$;
- $G(d^0)(z) := z$ if $z \in \mathsf{cat}(1,X) - \{y\}$.

Of course, $\mathscr{L}_1\mathscr{F}_1(G) \cong X$. The consequence follows from Nielsen-Schreier theorem for groups, since every small groupoid is equivalent to a coproduct of groups. $\square$

2.3 Presentations

If $\mathscr{T} = (\mathscr{T}, m, \eta)$ is a monad on a category $\mathfrak{X}$, we denote respectively by $\mathfrak{X}^{\mathscr{T}}$ and $\mathfrak{X}_{\mathscr{T}}$ the category of Eilenberg-Moore $\mathscr{T}$-algebras and the Kleisli category. Every such monad comes with a notion of presentation of a $\mathscr{T}$-algebra. More precisely, a diagram in $\mathfrak{X}$

$$G_2 \rightrightarrows \mathscr{T}(G_1) \tag{$\mathscr{T}$-presentation diagram}$$

can be seen as a graph in $\mathfrak{X}_{\mathscr{T}}$ and, hence, it can be seen as a graph $\mathfrak{G}^{\mathrm{op}} \to \mathfrak{X}^{\mathscr{T}}$ of free $\mathscr{T}$-algebras in $\mathfrak{X}^{\mathscr{T}}$. We say that the graph above is a *presentation* of the $\mathscr{T}$-algebra $(G', \mathscr{T}(G') \to G')$ if this algebra is (isomorphic to) the coequalizer of the corresponding diagram $\mathfrak{G}^{\mathrm{op}} \to \mathfrak{X}^{\mathscr{T}}$ of free $\mathscr{T}$-algebras in $\mathfrak{X}^{\mathscr{T}}$. Every $\mathscr{T}$-algebra admits a presentation, since every $\mathscr{T}$-algebra is a coequalizer of free $\mathscr{T}$-algebras. If $\mathfrak{X}^{\mathscr{T}}$ has all coequalizers of free algebras, denoting by $\mathsf{Grph}(\mathfrak{X}_{\mathscr{T}}) = \mathsf{Cat}\,[\mathfrak{G}^{\mathrm{op}}, \mathfrak{X}_{\mathscr{T}}]$ the category of graphs internal to the Kleisli category, there is a functor $\mathsf{Grph}(\mathfrak{X}_{\mathscr{T}}) \to \mathfrak{X}^{\mathscr{T}}$ which takes each graph to the category presented by it.

Definition 2.3.1. [$\mathscr{T}$-presentation] Let $\mathscr{T} = (\mathscr{T}, m, \eta)$ be a monad on a category $\mathfrak{X}$. Consider the comma category $(\mathrm{Id}_{\mathfrak{X}}/\mathscr{T})$. We have a functor $K_{\mathscr{T}} : (\mathrm{Id}_{\mathfrak{X}}/\mathscr{T}) \to \mathsf{Grph}(\mathfrak{X}^{\mathscr{T}})$ given by the composition of the comparisons $(\mathrm{Id}_{\mathfrak{X}}/\mathscr{T}) \to \mathsf{Grph}(\mathfrak{X}_{\mathscr{T}}) \to \mathsf{Grph}(\mathfrak{X}^{\mathscr{T}})$.

Consider also the full subcategory $\mathsf{Grph}'(\mathfrak{X}^{\mathscr{T}})$ of $\mathsf{Grph}(\mathfrak{X}^{\mathscr{T}})$ whose objects are graphs G such that the coequalizer of G exists in $\mathfrak{X}^{\mathscr{T}}$. The *category of $\mathscr{T}$-presentations*, denoted by $\mathscr{P}\mathrm{re}(\mathscr{T})$, is the pullback of $K_{\mathscr{T}}$ along the inclusion $\mathsf{Grph}'(\mathfrak{X}^{\mathscr{T}}) \to \mathsf{Grph}(\mathfrak{X}^{\mathscr{T}})$.

We get then a natural functor $K'_{\mathscr{T}} : \mathscr{P}\mathrm{re}(\mathscr{T}) \to \mathsf{Grph}'(\mathfrak{X}^{\mathscr{T}})$. The *functor presentation*, denoted by $\mathscr{P}_{\mathscr{T}} : \mathscr{P}\mathrm{re}(\mathscr{T}) \to \mathfrak{X}^{\mathscr{T}}$, is the composition of the coequalizer $\mathsf{Grph}'(\mathfrak{X}^{\mathscr{T}}) \to \mathfrak{X}^{\mathscr{T}}$ with $K'_{\mathscr{T}}$.

Lemma 2.3.2. *$\mathscr{P}_{\mathscr{T}}$ is essentially surjective. This means that every $\mathscr{T}$-algebra has at least one presentation.*

Remark 2.3.3. We ratify that if $\mathscr{T}$ is a monad such that $\mathfrak{X}^{\mathscr{T}}$ has coequalizers of free algebras, then the definition of $\mathscr{P}\mathrm{re}(\mathscr{T})$ is easier. More precisely, $\mathscr{P}\mathrm{re}(\mathscr{T}) := (\mathrm{Id}_{\mathfrak{X}}/\mathscr{T})$.

We denote by $\overline{\mathscr{L}_0\mathscr{F}_0}$ the *free group monad* on Set whose category of algebras is the category of groups Group. A *$\overline{\mathscr{L}_0\mathscr{F}_0}$-presentation of a group* is a pair $\langle S, R\rangle$ in which S is a set and $R : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ is a small graph such that $R(1) = \overline{\mathscr{L}_0\mathscr{F}_0}(S)$. This induces a graph $\overline{R} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Group}$ of free groups. The coequalizer of this graph is precisely the *group presented by* $\langle S, R\rangle$. Analogously, we get the notion of *$\overline{\mathscr{F}_0}$-presentation of monoids* induced by the *free monoid monad* $\overline{\mathscr{F}_0}$ on Set.

Remark 2.3.4. Recall, for instance, the basics of presentations of groups [54]. The classical definition of a presentation of a group is not usually given explicitly by a graph as it is described above. Instead, the usual definition of a presentation of a group is given by a pair $\langle S,R\rangle$ in which S is a set and R is a "set of relations or equations". However, this is of course the same as an $\overline{\mathscr{L}_0\mathscr{F}_0}$-presentation. That is to say, it is a graph

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & \overline{\mathscr{L}_0\mathscr{F}_0}(S) \end{tikzcd}$$

in Set such that the first arrow gives one side of the equations and the second arrow gives the other side of the equations. For instance, in computing the fundamental group of a torus via the van Kampen Theorem and the quotient of the square [74], one usually gets it via the presentation $\langle \{a,b\}, ab = ba\rangle$. This is the same as the graph

$$\begin{tikzcd} * \arrow[r, shift left] \arrow[r, shift right] & \overline{\mathscr{L}_0\mathscr{F}_0}(\{a,b\}) \end{tikzcd}$$

in which the image of $*$ by the first arrow is the word ab and the image by the second arrow is ba. Of course, this is the presentation of $\mathbb{Z}\times\mathbb{Z}$, as $\mathbb{Z}\times\mathbb{Z}$ is the coequalizer of the corresponding diagram of free groups in the category of groups.

The free category monad $\overline{\mathscr{F}_1}$ on Grph induces a notion of presentation of categories. More precisely, an $\overline{\mathscr{F}_1}$-presentation of a category X is a graph $\mathfrak{g} : \mathfrak{G} \to \mathsf{Grph}_{\overline{\mathscr{F}_1}}$ such that, after composing $\mathfrak{g}$ with $\mathsf{Grph}_{\overline{\mathscr{F}_1}} \to \mathsf{Grph}^{\overline{\mathscr{F}_1}} \simeq \mathsf{Cat}$, its coequalizer in Cat is isomorphic to X. Analogously, the free groupoid monad $\overline{\mathscr{L}_1\mathscr{F}_1}$ gives rise to the notion of $\overline{\mathscr{L}_1\mathscr{F}_1}$-presentation of groupoids.

Remark 2.3.5. [Suspension] The forgetful functor $\mathfrak{u}_1 : \mathsf{Grph} \to \mathsf{SET}$ has left and right adjoints. The left adjoint $\mathrm{i}_1 : \mathsf{SET} \to \mathsf{Grph}$ is defined by $\mathrm{i}_1(X)(2) = \emptyset$ and $\mathrm{i}_1(X)(1) = X$. The right adjoint $\sigma_1 = \Sigma' : \mathsf{SET} \to \mathsf{Grph}$ is defined by $\Sigma'(X)(2) = X$ and $\Sigma'(X)(1) = *$ is the terminal set.

Indeed, σ_1 is part of monad (mono)morphisms $\overline{\mathscr{F}_0} \to \overline{\mathscr{F}_1}$ and $\overline{\mathscr{L}_0\mathscr{F}_0} \to \overline{\mathscr{L}_1\mathscr{F}_1}$. We conclude that presentation of monoids are particular cases of presentations of categories and presentations of groups are particular cases of presentations of groupoids. More precisely, there are inclusions

$$\begin{tikzcd} \mathscr{P}\mathrm{re}(\overline{\mathscr{L}_0\mathscr{F}_0}) \arrow[r] \arrow[d] & \mathscr{P}\mathrm{re}(\overline{\mathscr{F}_0}) \arrow[d] \\ \mathscr{P}\mathrm{re}(\overline{\mathscr{L}_1\mathscr{F}_1}) \arrow[r] & \mathscr{P}\mathrm{re}(\overline{\mathscr{F}_1}) \end{tikzcd}$$

but it is important to note that they are not essentially surjective.

Roughly, $\overline{\mathscr{F}_1}$-presentations and $\overline{\mathscr{L}_1\mathscr{F}_1}$-presentations can be seen as freely generated graphs with equations between 1-cells and equations between 0-cells. More precisely, we have:

Definition 2.3.6. If $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}_{\overline{\mathscr{F}_1}}$ is a presentation of a category, we denote by $\mathfrak{g}(d^0)_1$ the component of the graph morphism $\mathfrak{g}(d^0)$ in 1. If $\mathfrak{g}(d^0)_1 = \mathfrak{g}(d^1)_1$ and they are inclusions, $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}_{\overline{\mathscr{F}_1}}$ is called an 1-cell presentation.

Theorem 2.3.7. *If* $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}_{\overline{\mathscr{F}_1}}$

$$\begin{tikzcd} \mathfrak{g}(2) \arrow[r, shift left, "\mathfrak{g}(d^0)"] \arrow[r, shift right, "\mathfrak{g}(d^1)"'] & \overline{\mathscr{F}_1}(\mathfrak{g}_1) \end{tikzcd}$$

is a presentation of a category X, then there is an induced 1*-cell presentation* $\underline{\mathfrak{g}}$ *of* X

$$\begin{tikzcd}\underline{\mathfrak{g}}(2) \arrow[r, shift left, "\underline{\mathfrak{g}}(d^0)"] \arrow[r, shift right, "\underline{\mathfrak{g}}(d^1)"'] & \overline{\mathscr{F}_1}(\underline{\mathfrak{g}}_1)\end{tikzcd}$$

in which $\underline{\mathfrak{g}}(2)(1)$ *is the coequalizer of the graph of objects induced by* $\underline{\mathfrak{g}}$.

Example 2.3.8. We denote by $\widehat{2}$ the graph such that $\mathscr{F}_1(\widehat{2}) = 2$. It is clear that $\mathscr{I}$ can be lifted through $\mathscr{C}_1$. That is to say, there is a functor $\widehat{\mathscr{I}} : \mathfrak{G} \to \mathsf{Grph}$ such that $\mathscr{C}_1\widehat{\mathscr{I}} = \mathscr{I}$. Then $\mathscr{F}_1\widehat{\mathscr{I}}$ composed with the isomorphism $\mathfrak{G}^{\mathrm{op}} \cong \mathfrak{G}$ gives a graph of free $\overline{\mathscr{F}_1}$-algebras. Therefore it gives an $\overline{\mathscr{F}_1}$-presentation

$$\begin{tikzcd}\bullet \arrow[r, shift left] \arrow[r, shift right] & \overline{\mathscr{F}_1}(\widehat{2})\end{tikzcd}$$

of the category (suspension of the monoid) $\Sigma(\mathbb{N})$. Actually, the corresponding 1-cell presentation is just

$$\begin{tikzcd}\emptyset \arrow[r, shift left] \arrow[r, shift right] & \overline{\mathscr{F}_1}(\bigcirc) \cong \Sigma(\mathbb{N}).\end{tikzcd}$$

Remark 2.3.9. Of course, we also have the notion of $\overline{\mathscr{F}_1^{\mathscr{R}}}$-presentations of categories. Although the category of $\overline{\mathscr{F}_1}$-presentations is not isomorphic to the category of $\overline{\mathscr{F}_1^{\mathscr{R}}}$-presentations, we have an obvious inclusion between these categories which is essentially surjective.

2.4 Definition of Computads

In Section 2.8, we give the definition of the n-category freely generated by an n-computad by induction. The starting point of the induction is the definition of a category freely generated by a graph. Thereby graphs are called 1-computads and we define respectively the *category of* 1*-computads* and the category of *small* 1*-computads* by $1\text{-}\mathsf{Cmp} := \mathsf{Grph}$ and $1\text{-}\mathsf{cmp} := \mathsf{grph}$.

In the present section, we give a concise definition of 2-computads and of the category $2\text{-}\mathsf{Cmp}$. This concise definition is precisely what allows us to get its freely generated 2-category via a coinserter. We also introduce the notion of a category presented by a computad, which is going to be our canonical notion of presentation of categories.

Definition 2.4.1. [Derivation Schemes and Computads] Consider the functor $(- \times \mathfrak{G}) : \mathsf{SET} \to \mathsf{Cat}, Y \mapsto Y \times \mathfrak{G}$ and the functor $\mathscr{F}_1 : \mathsf{Grph} \to \mathsf{Cat}$. The *category of derivation schemes* is the comma category $\mathsf{Der} := (- \times \mathfrak{G}/\mathrm{Id}_{\mathsf{Cat}})$. The *category of* 2*-computads* is the comma category $2\text{-}\mathsf{Cmp} := (- \times \mathfrak{G}/\mathscr{F}_1)$.

Considering the restrictions $(- \times \mathfrak{G}) : \mathsf{Set} \to \mathsf{cat}$ and $\mathscr{F}_1 : \mathsf{grph} \to \mathsf{cat}$, we define the *category of small* 2*-computads* as $2\text{-}\mathsf{cmp} := (- \times \mathfrak{G}/\mathscr{F}_1)$. We also define the *category of small computads over reflexive graphs* (or just *category of reflexive computads*) as $\mathsf{Rcmp} := (- \times \mathfrak{G}/\mathscr{F}_1^{\mathscr{R}})$. There is an obvious left adjoint inclusion $\mathsf{cmp} \to \mathsf{Rcmp}$ induced by $\mathscr{E}$. We denote the induced adjunction by $\mathscr{E}_{\mathsf{cmp}} \dashv \mathscr{R}_{\mathsf{cmp}}$.

Derivations schemes were first defined in [106]. Respecting the original terminology of [103], the word *computad* without any index means 2-computad. Also, we set the notation: $\mathsf{Cmp} := 2\text{-}\mathsf{Cmp}$ and $\mathsf{cmp} := 2\text{-}\mathsf{cmp}$.

The pushout of the inclusion $(2)_0 \to 2$ of Remark 2.2.5 along itself is (isomorphic to) $\mathfrak{G}$. Hence, by definition, a *derivation scheme* is pair $(\mathfrak{d}, \mathfrak{d}_2)$ in which $\mathfrak{d}_2$ is a discrete category and $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is an internal graph

$$\begin{tikzcd} \mathfrak{d}_2 \times 2 \arrow[r, shift left] \arrow[r, shift right] & \mathfrak{d}(1) \end{tikzcd} \qquad (\mathfrak{d}\text{-diagram})$$

such that, for every α of $\mathfrak{d}_2$:

$$\mathfrak{d}(d^0)(\alpha,0) = \mathfrak{d}(d^1)(\alpha,0) \qquad\qquad \mathfrak{d}(d^0)(\alpha,1) = \mathfrak{d}(d^1)(\alpha,1).$$

In this direction, by definition, a *computad* is a triple $(\mathfrak{g}, \mathfrak{g}_2, G)$ in which $(\mathfrak{g}, \mathfrak{g}_2)$ is a derivation scheme and $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is a graph such that $\mathfrak{g}(1) = \mathscr{F}_1(G)$. We usually adopt this viewpoint.

Definition 2.4.2. [Groupoidal Computad] Consider the functor $(- \times \mathscr{L}_1(\mathfrak{G})) : \mathsf{SET} \to \mathsf{Gr}, X \mapsto X \times \mathscr{L}_1(\mathfrak{G})$ and the functor $\mathscr{L}_1\mathscr{F}_1 : \mathsf{Grph} \to \mathsf{Gr}$. The category of *groupoidal computads* is the comma category $\mathsf{Cmp}_{\mathsf{Gr}} := (- \times \mathscr{L}_1(\mathfrak{G})/\mathscr{L}_1\mathscr{F}_1)$. Analogously, the *category of groupoidal computads over reflexive graphs* is defined by $\mathsf{Rcmp}_{\mathsf{gr}} := (- \times \mathscr{L}_1(\mathfrak{G})/\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}})$.

We denote by $\widehat{\mathfrak{G}}$ the graph below with two objects and two arrows between them. It is clear that $\mathscr{F}_1(\widehat{\mathfrak{G}}) \cong \mathfrak{G}$. Hence, there is a natural morphism $\widehat{\mathfrak{G}} \to \mathscr{C}_1(\mathfrak{G})$ induced by the unit of $\mathscr{F}_1 \dashv \mathscr{C}_1$. Moreover, it is important to observe that $\widehat{\mathfrak{G}}$ is not isomorphic to $\mathscr{C}_1(\mathfrak{G})$.

$$\begin{tikzcd} * \arrow[r, shift left] \arrow[r, shift right] & * \end{tikzcd}$$

Theorem 2.4.3. *Consider the functor* $(\mathrm{i}_1(-) \times \widehat{\mathfrak{G}}) : \mathsf{Set} \to \mathsf{grph}, X \mapsto \mathrm{i}_1(X) \times \widehat{\mathfrak{G}}$. *There are isomorphisms of categories* $\mathsf{Cmp} \cong (\mathrm{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathscr{F}_1})$ *and* $\mathsf{Cmp}_{\mathsf{Gr}} \cong (\mathrm{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathscr{L}_1\mathscr{F}_1})$.

Moreover, considering suitable restrictions of $(\mathrm{i}_1(-) \times \widehat{\mathfrak{G}})$ *and* $\overline{\mathscr{F}_1}$ *(to* Set *and* grph *respectively), we have that* $\mathsf{cmp} \cong (\mathrm{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathscr{F}_1})$. *Analogously,* $\mathsf{cmp}_{\mathsf{Gr}} \cong (\mathrm{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathscr{L}_1\mathscr{F}_1})$.

Definition 2.4.4. [Presentation of a category via a computad] We say that a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ *presents a category* X if the coequalizer of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is isomorphic to X. We have, then, a functor $\mathscr{P}_1 : \mathsf{Cmp} \to \mathsf{Cat}$ which gives the category presented by each computad. Of course, there is also a presentation functor $\mathscr{P}_1^{\mathscr{R}} : \mathsf{Rcmp} \to \mathsf{cat}$.

Analogously, we say that a groupoidal computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ *presents a groupoid* X if the coequalizer of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Gr}$ is isomorphic to X. Again, we have presentation functors $\mathscr{P}_{(1,0)} : \mathsf{Cmp}_{\mathsf{Gr}} \to \mathsf{Gr}$ and $\mathscr{P}_{(1,0)}^{\mathscr{R}} : \mathsf{Rcmp}_{\mathsf{gr}} \to \mathsf{gr}$.

Theorem 2.4.5. *Every presentation via computads is an* $\overline{\mathscr{F}_1}$*-presentation. That is to say, there is a natural inclusion* $\mathsf{Cmp} \to \mathscr{P}\mathrm{re}(\overline{\mathscr{F}_1})$. *Analogously, every groupoidal computad is an* $\overline{\mathscr{L}_1\mathscr{F}_1}$*-presentation.*

Proof. By Theorem 2.4.3, $\mathsf{Cmp} \cong (\mathrm{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathscr{F}_1})$. So, it is enough to consider the natural inclusion between comma categories

$$(\mathrm{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathscr{F}_1}) \to (\mathrm{Id}_{\mathsf{Grph}}/\overline{\mathscr{F}_1}).$$

$\square$

Every category admits a presentation via a computad and, analogously, every groupoid admits a presentation via a groupoidal computad. These results follow from Theorem 2.3.7 and:

Theorem 2.4.6. *There is a functor* $\mathsf{Cmp} \to \mathscr{P}\mathrm{re}(\overline{\mathscr{F}_1}), \mathfrak{g} \mapsto \underline{\mathscr{C}_1\mathfrak{g}}$ *which is essentially surjective in the subcategory of* 1*-cell presentations* $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}$ *of categories such that the graph* $\mathfrak{g}(2)$ *has no isolated objects (that is to say, every object is the domain or codomain of some arrow). Moreover, there is a natural isomorphism*

$$\begin{tikzcd}
\mathsf{Cmp} \arrow[rr] \arrow[rd, "\mathscr{P}_1"'] & \arrow[d, phantom, "\cong"] & \mathscr{P}\mathrm{re}(\overline{\mathscr{F}_1}) \arrow[ld, "\mathscr{P}_{\overline{\mathscr{F}_1}}"] \\
 & \mathsf{Cat} &
\end{tikzcd}$$

Example 2.4.7. The computad $(\mathfrak{g}^{\dot{\Delta}_2}, \mathfrak{g}_2^{\dot{\Delta}_2}, G_{\dot{\Delta}_2})$ defined below presents the truncated category $\dot{\Delta}_2$.

$$\begin{aligned}
G_{\dot{\Delta}_2}(1) &:= \{0,1,2\} & G_{\dot{\Delta}_2}(2) &:= \left\{d, s^0, d^0, d^1\right\}\\
G_{\dot{\Delta}_2}(d^1)(d^i) &:= 1, \forall i & G_{\dot{\Delta}_2}(d^1)(s^0) &:= 2\\
G_{\dot{\Delta}_2}(d^0)(d^i) &:= 2, \forall i & G_{\dot{\Delta}_2}(d^0)(s^0) &:= 1\\
G_{\dot{\Delta}_2}(d^1)(d) &:= 0, \forall i & G_{\dot{\Delta}_2}(d^0)(d) &:= 1, \forall i
\end{aligned}$$

$$\mathfrak{g}_2^{\dot{\Delta}} := \{n_0, n_1, \vartheta\}$$

$$\begin{aligned}
\mathfrak{g}^{\dot{\Delta}_2}(d^1)(n_0, 0 \to 1) &:= s^0 \cdot d^0 & \mathfrak{g}^{\dot{\Delta}_2}(d^0)(n_1, 0 \to 1) &:= s^0 \cdot d^1\\
\mathfrak{g}^{\dot{\Delta}_2}(d^0)(n_0, 0 \to 1) &:= \mathrm{id}_1 & \mathfrak{g}^{\dot{\Delta}_2}(d^1)(n_1, 0 \to 1) &:= \mathrm{id}_1\\
\mathfrak{g}^{\dot{\Delta}_2}(d^0)(\vartheta, 0 \to 1) &:= d^0 \cdot d & \mathfrak{g}^{\dot{\Delta}_2}(d^1)(\vartheta, 0 \to 1) &:= d^1 \cdot d.
\end{aligned}$$

This computad can also be described by the graph

$$\begin{tikzcd}
0 \arrow[r, "d"] & 1 \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d^0"'] & 2 \arrow[l, "s^0" description]
\end{tikzcd}$$

with the following 2-cells:

$$n_0 : s^0 \cdot d^0 \Rightarrow \mathrm{id}_1, \qquad n_1 : \mathrm{id}_1 \Rightarrow s^0 \cdot d^1, \qquad \vartheta : d^1 \cdot d \Rightarrow d^0 \cdot d.$$

Lemma 2.4.8. *The category* $\dot{\Delta}_2$ *is the coequalizer of the computad* $\mathfrak{g}^{\dot{\Delta}_2}$.

Example 2.4.9. The usual presentation of the category $\dot{\Delta}$ via faces and degeneracies is given by the computad $(\mathfrak{g}^{\dot{\Delta}}, \mathfrak{g}_2^{\dot{\Delta}}, G_{\dot{\Delta}})$ which is defined by

$$\begin{tikzcd}
\mathfrak{g}_2^{\dot{\Delta}} \times 2 \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}_1(G_{\dot{\Delta}})
\end{tikzcd}$$

in which $G_{\dot{\Delta}}(1) := (\mathbb{N})_0$ is the discrete category of the non-negative integers and

$$G_{\dot{\Delta}}(2) := \left\{(d^i, m) : (i,m) \in \mathbb{N}^2, i \leq m\right\} \cup \left\{(s^k, m) : (k,m) \in \mathbb{N}^2, k \leq m-1 \geq 0\right\}$$

$$\begin{aligned}
G_{\dot{\Delta}}(d^1)(d^i, m) &:= m & G_{\dot{\Delta}}(d^1)(s^k, m) &:= m+1\\
G_{\dot{\Delta}}(d^0)(d^i, m) &:= m+1 & G_{\dot{\Delta}}(d^0)(s^k, m) &:= m
\end{aligned}$$

$$\begin{aligned}\mathfrak{g}_2^{\dot{\Delta}} := &\left\{(d^k,d^i,m) : (i,k,m)\in\mathbb{N}^3, m\geq i<k\right\}\\ &\cup\left\{(s^k,s^i,m) : (i,k,m)\in\mathbb{N}^3, 0\leq m-1\geq k\geq i\right\}\\ &\cup\left\{(s^k,d^i,m) : (i,k,m)\in\mathbb{N}^3, k\leq m-1\geq 0\right\}\end{aligned}$$

$$\begin{aligned}\mathfrak{g}^{\dot{\Delta}}(d^1)((d^k,d^i,m),0\to 1) &:= (d^k,m+1)\cdot(d^i,m)\\ \mathfrak{g}^{\dot{\Delta}}(d^1)((s^k,s^i,m),0\to 1) &:= (s^k,m)\cdot(s^i,m+1)\\ \mathfrak{g}^{\dot{\Delta}}(d^1)((s^k,d^i,m),0\to 1) &:= (s^k,m+1)\cdot(d^i,m)\\ \mathfrak{g}^{\dot{\Delta}}(d^0)((d^k,d^i,m),0\to 1) &:= (d^i,m+1)\cdot(d^{k-1},m)\\ \mathfrak{g}^{\dot{\Delta}}(d^0)((s^k,s^i,m),0\to 1) &:= (s^i,m)\cdot(s^{k+1},m+1)\\ \mathfrak{g}^{\dot{\Delta}}(d^0)((s^k,d^i,m),0\to 1) &:= (d^i,m-1)\cdot(s^{k-1},m),\text{ if } k>i\\ \mathfrak{g}^{\dot{\Delta}}(d^0)((s^k,d^i,m),0\to 1) &:= \mathrm{id}_m,\text{ if } i=k \text{ or } i=k+1\\ \mathfrak{g}^{\dot{\Delta}}(d^0)((s^k,d^i,m),0\to 1) &:= (d^{i-1},m-1)\cdot(s^k,m-1),\text{ if } i>k+1.\end{aligned}$$

Lemma 2.4.10. *The category $\dot{\Delta}$ is the coequalizer of the computad $\mathfrak{g}^{\dot{\Delta}}$.*

Every computad induces a presentation of groupoids via a groupoidal computad, since we have an obvious functor $\mathsf{Cmp}\to\mathsf{Cmp}_{\mathsf{Gr}}$ induced by $\mathscr{L}_1$. More precisely, the functor $\mathscr{L}_1^{\mathsf{Cmp}} : \mathsf{Cmp}\to\mathsf{Cmp}_{\mathsf{Gr}}$ is defined by $\mathfrak{g}\mapsto\mathscr{L}_1\mathfrak{g}$. Observe that the groupoidal computad $\mathscr{L}_1\mathfrak{g}$ gives a presentation of the coequalizer of $\mathscr{L}_1\mathfrak{g}$ in Gr which is (isomorphic to) $\mathscr{L}_1\mathscr{P}_1(\mathfrak{g})$. In this case, we say that the computad $\mathfrak{g}$ presents the groupoid $\mathscr{L}_1\mathscr{P}_1(\mathfrak{g})$.

Proposition 2.4.11. *There is a natural isomorphism $\mathscr{P}_{(1,0)}\mathscr{L}_1^{\mathsf{Cmp}}\cong\mathscr{L}_1\mathscr{P}_1$.*

Remark 2.4.12. If $\mathscr{P}_1(\mathfrak{g})$ is a groupoid, there is no confusion between the groupoid presented by $\mathfrak{g}$ and the category presented by $\mathfrak{g}$, since, in this case, they are actually isomorphic. More precisely, in this case, $\mathscr{L}_1\mathscr{P}_1(\mathfrak{g})\cong\mathscr{P}_1(\mathfrak{g})$.

Theorem 2.4.13. *If the groupoid presented by a computad $(\mathfrak{g},\mathfrak{g}_2,G)$ is thin, then $(\mathfrak{g},\mathfrak{g}_2,G)$ presents a thin category as well provided that $\mathscr{P}_1(\mathfrak{g},\mathfrak{g}_2,G)$ satisfies the cancellation law.*

Proof. By Theorem 2.1.6, if $\mathscr{L}_1\mathscr{P}_1(\mathfrak{g})$ is thin, then $\mathscr{P}_1(\mathfrak{g})$ is thin. □

Definition 2.4.14. [2-cells of computads] Let $(\mathfrak{g},\mathfrak{g}_2,G)$ be a computad. The discrete category $\mathfrak{g}_2$ is called *the discrete category of the* 2*-cells of the computad* $\mathfrak{g}$. Moreover, we say that α is a 2-cell between f and g, denoted by $\alpha : f\Rightarrow g$, if $\mathfrak{g}(d^1)(\alpha,0\to 1)=f$ and $\mathfrak{g}(d^0)(\alpha,0\to 1)=g$. In this case, the *domain* of α is f while the *codomain* is g.

Sometimes, we need to be even more explicit and denote the 2-cell α by $\alpha : f\Rightarrow g : x\to y$ whenever $\mathfrak{g}(d^1)(\alpha,0\to 1)=f$, $\mathfrak{g}(d^0)(\alpha,0\to 1)=g$, $\mathfrak{g}(d^0)(\alpha,0)=x$ and $\mathfrak{g}(d^0)(\alpha,1)=y$.

In the context of presentation of categories, the 2-cells of a computad $(\mathfrak{g},\mathfrak{g}_2,G)$ correspond to the equations of the presentation induced by this computad. If $\mathfrak{g}$ has more than one 2-cell between two arrows of $\mathfrak{g}(1)$, then it is a redundant presentation of the coequalizer of $\mathfrak{g}$. Yet, we also have interesting examples of redundant presentations. For instance, in the next section, we give the definition of the fundamental groupoid via a redundant presentation.

Remark 2.4.15. [Sigma] There is an obvious forgetful functor $\mathfrak{u}_2 : \mathsf{cmp} \to \mathsf{grph}$. This forgetful functor has left and right adjoints. The left adjoint $\mathfrak{i}_2 : \mathsf{grph} \to \mathsf{cmp}$ is defined by $\mathfrak{i}_2(G) = (G^{\mathfrak{i}_2}, \emptyset, G)$. Sometimes, we denote $G^{\mathfrak{i}_2}$ by $\mathfrak{i}_2(G)$ and, of course, it is defined as follows:

$$\mathfrak{i}_2(G) : \emptyset \rightrightarrows \mathscr{F}_1(G).$$

The right adjoint $\sigma_2 : \mathsf{grph} \to \mathsf{cmp}$ is defined by $\sigma_2(G) = (G^{\sigma_2}, G_2^{\sigma_2}, G)$ in which $\sigma_2(G)(2) = G_2^{\sigma_2} \times 2$ and the set of 2-cells $G_2^{\sigma_2}$ is the pullback of $(\mathscr{F}_1(G)(d^1), \mathscr{F}_1(G)(d^0)) : \mathscr{F}_1(G)(2) \to \mathscr{F}_1(G)(1) \times \mathscr{F}_1(G)(1)$ along itself. Finally, the images of $G^{\sigma_2}(G)(d^1), G^{\sigma_2}(d^0)$ are induced by the obvious projections. Sometimes we write $\sigma_2(G) = (\sigma_2(G), \sigma_2(G)_2, G)$ as follows

$$\sigma_2(G) : G_2^{\sigma_2} \times 2 \rightrightarrows \mathscr{F}_1(G).$$

Remark 2.4.16. [Sigma$^{\mathsf{Gr}}$] Of course, we also have a forgetful functor $\mathfrak{u}_2^{\mathsf{Gr}} : \mathsf{cmp}_{\mathsf{Gr}} \to \mathsf{grph}$. The left adjoint of this functor is defined by $\mathfrak{i}_2^{\mathsf{Gr}} := \mathscr{L}_1^{\mathsf{Cmp}} \mathfrak{i}_2$, while the right adjoint is defined by $\sigma_2^{\mathsf{Gr}} := \mathscr{L}_1^{\mathsf{Cmp}} \sigma_2$.

Proposition 2.4.17. *There is a natural isomorphism* $\mathscr{P}_1 \mathfrak{i}_2 \cong \mathscr{F}_1$.

Definition 2.4.18. [Connected Computad] A computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ is *connected* if $\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G) = G$ is connected.

Remark 2.4.19. Let X be a group. We consider the full subcategory $\mathscr{P}\mathrm{re}(\overline{\mathscr{L}_0\mathscr{F}_0}, X)$ of $\mathscr{P}\mathrm{re}(\overline{\mathscr{L}_0\mathscr{F}_0})$ consisting of the presentations of X. This subcategory is isomorphic to the full subcategory of $\mathsf{cmp}^{\mathsf{Gr}}$ consisting of the groupoidal computads which presents $\Sigma(X)$. This fact shows that presentations of groupoids by groupoidal computads generalizes the notion of $\overline{\mathscr{L}_0\mathscr{F}_0}$-presentations of groups. Moreover, unlike the case of $\overline{\mathscr{L}_1\mathscr{F}_1}$-presentations, the notion of presentations of (suspensions of) groups by groupoidal computads is precisely the same of $\overline{\mathscr{L}_0\mathscr{F}_0}$-presentations.

Analogously, given a monoid Y the category of $\overline{\mathscr{F}_0}$-presentations $\mathscr{P}\mathrm{re}(\overline{\mathscr{F}_0}, Y)$ is isomorphic to the category of computads which presents $\Sigma(Y)$.

2.5 Topology and Computads

We introduce topological aspects of our theory. We refer the reader to [87] for basic notions and results of algebraic topology, including the van Kampen theorem for fundamental groupoids.

We start with the relation between the fundamental groupoids and groupoids freely generated by small graphs. By the classical van Kampen theorem, the fundamental group of a (topological) graph with only one object is the group freely generated by the set of edges/arrows. We show that it also holds for fundamental groupoids: roughly, the groupoid freely generated by a small graph G is equivalent to its fundamental groupoid. Although this is a straightforward result, this motivates the relation between topology and small computads: that is to say, the association of each small computad with a CW-complex presented in 2.5.12.

We always consider small computads, small graphs and small categories throughout this section. Moreover, we use the appropriate restrictions of the functors $\mathscr{F}_1, \mathscr{L}_1, \mathscr{U}_1, \mathscr{C}_1$. Finally, Top denotes any suitable cartesian closed category of topological spaces: for instance, compactly generated spaces. Then we can consider weighted colimits in Top w.r.t. the Top-enrichment.

Remark 2.5.1. [Topological Graph] There is an obvious left adjoint inclusion $D_2 : \mathsf{cat} \to \mathsf{Top\text{-}Cat}$ induced by the fully faithful (discrete topology) functor $D : \mathsf{Set} \to \mathsf{Top}$ left adjoint to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$. We denote by $\mathfrak{G}$ and $\mathfrak{G}^{\mathrm{op}}$ the images $D_2(\mathfrak{G})$ and $D_2(\mathfrak{G}^{\mathrm{op}})$ respectively, whenever there is no confusion. If $I = [0,1]$ is the unit interval with the usual topology and $*$ is the terminal topological space, then the Top-weight $\mathscr{I}_{\mathsf{Top}_1} : \mathfrak{G} \to \mathsf{Top}$ defined by

$$\begin{tikzcd} \ast \arrow[r, shift left, "1"] \arrow[r, shift right, "0"'] & I \end{tikzcd}$$

gives the definition of Top-*isoinserters* and Top-*isocoinserters*.

If $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ is a small graph, $DG : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Top}$ is actually compatible with the Top-enrichment. More precisely, since D_2 is left adjoint, there is a Top-functor $D_2(\mathfrak{G}^{\mathrm{op}}) \to \mathsf{Top}$ which is the mate of $DG : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Top}$. Again, by abuse of notation, the mate $D_2(\mathfrak{G}^{\mathrm{op}}) \to \mathsf{Top}$ is also denoted by $DG : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Top}$.

Any small graph $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ has an associated topological (undirected) graph given by the Top-isocoinserter of the Top-functor DG. This gives a functor $\mathscr{F}_{\mathsf{Top}_1} : \mathsf{grph} \to \mathsf{Top}$ which is left adjoint to the functor $\mathscr{C}_{\mathsf{Top}_1} : \mathsf{Top} \to \mathsf{grph}$, $E \mapsto \mathsf{Top}(\mathscr{I}_{\mathsf{Top}_1} -, E)$. We denote the monad induced by this adjunction by $\overline{\mathscr{F}_{\mathsf{Top}_1}}$.

A *path in a topological space E* is an edge of $\mathscr{C}_{\mathsf{Top}_1}(E)$, that is to say, a path in E is a continuous map $a : I \to E$.

Lemma 2.5.2. *A small graph G is connected if and only if* $\mathscr{F}_{\mathsf{Top}_1}(G)$ *is a path connected topological space.*

Remark 2.5.3. We also have an adjunction $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_1} \dashv \mathscr{C}^{\mathscr{R}}_{\mathsf{Top}_1}$ in which $\mathscr{C}^{\mathscr{R}}_{\mathsf{Top}_1} = \mathscr{C}_{\mathsf{Top}_1}\mathscr{R}$. This adjunction is induced by a weight analogue of $\mathscr{I}_{\mathsf{Top}_1}$. Namely, if we denote by Δ_2 the image of itself by $\mathsf{cat} \to \mathsf{Top\text{-}Cat}$, the Top-functor $\mathscr{I}^{\mathscr{R}}_{\mathsf{Top}_1} : \Delta_2 \to \mathsf{Top}$ defined by

$$\begin{tikzcd} \ast \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "0"'] & I \arrow[l] \end{tikzcd}$$

in which $\mathscr{I}^{\mathscr{R}}_{\mathsf{Top}_1}$ composed with the inclusion $\mathfrak{G} \to \Delta_2$ is equal to $\mathscr{I}_{\mathsf{Top}_1}$. This weight gives rise to the notion of *reflexive* Top-*isoinserters* and *reflexive* Top-*isocoinserters*. Finally, $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_1}(G) = \mathscr{I}^{\mathscr{R}}_{\mathsf{Top}_1} * DG$ and $\mathscr{C}_{\mathsf{Top}_1} : \mathsf{Top} \to \mathsf{Rgrph}$, $E \mapsto \mathsf{Top}(\mathscr{I}^{\mathscr{R}}_{\mathsf{Top}_1} -, E)$.

Given an arrow f of $\overline{\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1}(E)$, we have that there is a unique finite list of arrows $a^f_0, \ldots, a^f_{m-1}$ of $\mathscr{C}_{\mathsf{Top}_1}(E)$ such that $f = a^f_{m-1} \cdots a^f_0$ by the ulf property of the length functor. Since, by definition, $a^f_0, \ldots, a^f_{m-1}$ are continuous maps $I \to E$, we can define a continuous map $\lceil \underline{f} \rceil_E : I \to E$ by $\lceil \underline{f} \rceil_E(t) = a^f_n(mt - n)$ whenever $t \in [n/m, (n+1)/m]$. This gives a morphism of graphs

$$\lceil \_ \rceil_E : \mathscr{C}_1\mathscr{F}_1\mathscr{C}_{\mathsf{Top}_1}(E) \to \mathscr{C}_{\mathsf{Top}_1}(E)$$

which is identity on objects and takes each arrow $f = a^f_{m-1} \cdots a^f_0$ of length m to the arrow $\lceil \underline{f} \rceil_E$ of $\mathscr{C}_{\mathsf{Top}_1}(E)$. These graph morphisms define a natural transformation

$$\lceil \_ \rceil : \overline{\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1} \longrightarrow \mathscr{C}_{\mathsf{Top}_1}.$$

Remark 2.5.4. It is very important to observe that, if f is an arrow of $\mathscr{C}_1\mathscr{F}_1\mathscr{C}_{\mathsf{Top}_1}(E)$ of length $m > 1$, then $\lceil \underline{f} \rceil_E : x \to z$ is not the same as the morphism $f : x \to z$ itself. The former is an edge of $\mathscr{C}_{\mathsf{Top}_1}(E)$, which means that, as morphism of $\overline{\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1}(E)$, its length is 1.

Remark 2.5.5. We have also a natural transformation $\lceil \_ \rceil^{\mathsf{Gr}} : \overline{\mathscr{L}_1\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1} \longrightarrow \mathscr{C}_{\mathsf{Top}_1}$. Observe that, by the ulf property of the length functor and by the definition of $\mathscr{C}_{\mathsf{Top}_1}$, if f is an arrow of $\overline{\mathscr{L}_1\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1}(E)$ of lenght k, then $f = a^f_{m-1} \cdots a^f_0$ for a unique list $(a^f_{m-1}, \ldots, a^f_0)$ of paths or formal inverses of paths in E and we can define $\lceil \underline{f} \rceil^{\mathsf{Gr}}_E : I \to E$ by:

$$\lceil \underline{f} \rceil^{\mathsf{Gr}}_E (t) = \begin{cases} a^f_n(mt - n), & \text{if } t \in [n/m, (n+1)/m] \text{ and } a^f_n \text{ is a path in } E, \\ \\ b^f_n(-mt + n + 1), & \text{if } t \in [n/m, (n+1)/m] \text{ and } a^f_n \\ & \text{is a formal inverse of an arrow } b^f_n \text{ of } \mathscr{C}_{\mathsf{Top}_1}(E). \end{cases}$$

On one hand, this defines morphisms of graphs $\overline{\mathscr{L}_1\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1}(E) \longrightarrow \mathscr{C}_{\mathsf{Top}_1}(E)$ for each topological space E. On the other hand, these morphisms define the natural transformation $\lceil \_ \rceil^{\mathsf{Gr}} : \overline{\mathscr{L}_1\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1} \longrightarrow \mathscr{C}_{\mathsf{Top}_1}$.

Theorem 2.5.6. *The mate of* $\lceil \_ \rceil : \overline{\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1} \longrightarrow \mathscr{C}_{\mathsf{Top}_1}$ *under the adjunction* $\mathscr{F}_{\mathsf{Top}_1} \dashv \mathscr{C}_{\mathsf{Top}_1}$ *and the identity adjunction is a natural transformation*

$$\lceil\ \rceil : \overline{\mathscr{F}_1} \longrightarrow \overline{\mathscr{F}_{\mathsf{Top}_1}}$$

which is a part of a monad functor/morphism $(\mathrm{Id}_{\mathsf{Grph}}, \lceil\ \rceil) : \overline{\mathscr{F}_{\mathsf{Top}_1}} \to \overline{\mathscr{F}_1}$. *Analogously, the mate* $\lceil \_ \rceil^{\mathsf{Gr}}$ *under the same adjunctions is a natural transformation* $\lceil\ \rceil^{\mathsf{Gr}}$ *which is part of a monad functor* $(\mathrm{Id}_{\mathsf{Grph}}, \lceil\ \rceil^{\mathsf{Gr}}) : \overline{\mathscr{F}_{\mathsf{Top}_1}} \to \overline{\mathscr{L}_1\mathscr{F}_1}$.

Remark 2.5.7. It is also important to consider the mate $\lceil \cdot \rceil : \mathscr{F}_{\mathsf{Top}_1}\overline{\mathscr{F}_1} \longrightarrow \mathscr{F}_{\mathsf{Top}_1}$ of the natural transformation $\lceil \_ \rceil : \overline{\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1} \longrightarrow \mathscr{C}_{\mathsf{Top}_1}$ under the adjunction $\mathscr{F}_{\mathsf{Top}_1} \dashv \mathscr{C}_{\mathsf{Top}_1}$ and itself. Again, we can consider the case of groupoids: the mate of $\lceil \_ \rceil^{\mathsf{Gr}}$ under $\mathscr{F}_{\mathsf{Top}_1} \dashv \mathscr{C}_{\mathsf{Top}_1}$ and itself is denoted by $\lceil \cdot \rceil^{\mathsf{Gr}} : \mathscr{F}_{\mathsf{Top}_1}\overline{\mathscr{L}_1\mathscr{F}_1} \longrightarrow \mathscr{F}_{\mathsf{Top}_1}$.

Let S^1 be the circle (complex numbers with norm 1) and B^2 the closed ball (complex numbers whose norm is smaller than or equal to 1). We denote the usual inclusion by $\mathrm{h} : S^1 \to B^2$. We consider also the embeddings:

$$\mathrm{h}_0 : I \to B^2,\ t \mapsto e^{\pi i t} \qquad\qquad \mathrm{h}_1 : I \to B^2,\ t \mapsto e^{\pi i(-t)}.$$

Recall that, if E is a topological space and $a, b : I \to E$ are continuous maps, a *homotopy of paths* $H : a \simeq b$ is a continuous map $H : B^2 \to E$ such that $H\mathrm{h}_0 = a$ and $H\mathrm{h}_1 = b$. If there is such a homotopy, we say that a and b are *homotopic*.

There is a functor $\mathscr{C}_{\mathsf{Top}_2} : \mathsf{Top} \to \mathsf{cmp}$ given by $\mathscr{C}_{\mathsf{Top}_2}(E) = (\mathfrak{g}^E, \mathfrak{g}_2^E, G^E)$ in which $G^E := \mathscr{C}_{\mathsf{Top}_1}(E)$ and

$$\begin{aligned} \mathfrak{g}_2^E \quad := \quad & \Big\{ (f, g, H : \lceil \underline{f} \rceil_E \simeq \lceil \underline{g} \rceil_E) : \\ & H \text{ is a homotopy of paths and } f, g \in \overline{\mathscr{F}_1} \mathscr{C}_{\mathsf{Top}_1}(E)(2) \Big\} . \end{aligned}$$

Also, $\mathfrak{g}^E(d^1)(f, g, H : \lceil \underline{f} \rceil_E \simeq \lceil \underline{g} \rceil_E, 0 \to 1) := f$ and $\mathfrak{g}^E(d^0)(f, g, H : \lceil \underline{f} \rceil_E \simeq \lceil \underline{g} \rceil_E, 0 \to 1) := g$. By an elementary result of algebraic topology, the image of $\mathscr{P}_1 \mathscr{C}_{\mathsf{Top}_2} : \mathsf{Top} \to \mathsf{Cat}$ is inside *the category of small groupoids* gr. More precisely, there is a functor $\Pi : \mathsf{Top} \to \mathsf{gr}$ such that $\mathscr{U}_1 \Pi \cong \mathscr{P} \mathscr{C}_{\mathsf{Top}_2}$. If E is a topological space, $\Pi(E)$ is called the *fundamental groupoid of* E. Given a point $e \in E$, recall that the fundamental group $\pi_1(E, e)$ is by definition the full subcategory of $\Pi(E)$ with only e as object.

Example 2.5.8. The van Kampen theorem [29] for groupoids (see, for instance, [12, 29]) gives the fundamental groupoid $\Pi(S^1)$ by the pushout of the inclusion $\{0, 1\} \to \Pi(I)$ along itself. This is equivalent to the pushout of the inclusion $(2)_0 \to 2$ of Remark 2.2.5 along $(2)_0 \to 1$, which is given by the $\overline{\mathscr{L}_1 \mathscr{F}_1}$-presentation

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \overline{\mathscr{L}_1 \mathscr{F}_1}(\hat{2}) \end{tikzcd}$$

induced by the $\overline{\mathscr{F}_1}$-presentation of Example 2.3.8. We conclude that this is isomorphic to $\mathscr{L}_1(\Sigma(\mathbb{N})) \cong \Sigma(\mathbb{Z})$.

Proposition 2.5.9. *There is a natural isomorphism* $\mathscr{C}_{\mathsf{Top}_1} \cong \mathfrak{u}_2 \mathscr{C}_{\mathsf{Top}_2}$.

Remark 2.5.10. The groupoid freely generated by a given small graph is equivalent to the fundamental groupoid of the respective topological graph. To see that, since $\mathscr{F}_1^{\mathscr{R}} \mathscr{E} \cong \mathscr{F}_1$ and $\mathscr{F}_{\mathsf{Top}_1}^{\mathscr{R}} \mathscr{E} \cong \mathscr{F}_{\mathsf{Top}_1}$, it is enough to prove that, for each small reflexive graph G,

$$\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G) \simeq \Pi \mathscr{F}_{\mathsf{Top}_1}^{\mathscr{R}}(G).$$

On one hand, if G is a reflexive tree, then both $\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G), \Pi \mathscr{F}_{\mathsf{Top}_1}^{\mathscr{R}}(G)$ are thin (and connected): therefore, they are equivalent. On the other hand, if G is a reflexive graph with only one object, then $\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G)$ and $\Pi \mathscr{F}_{\mathsf{Top}_1}^{\mathscr{R}}(G)$ are equivalent to the group freely generated by the set of nontrivial edges/arrows of G.

If a reflexive graph G is not a reflexive tree and it has more than one object, we can choose a maximal reflexive tree G_{mtree} of G. Then, if we denote by $\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G) / \mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G_{\mathrm{mtree}})$ the pushout of the inclusion $\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G_{\mathrm{mtree}}) \to \mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G)$ along the unique functor between $\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G_{\mathrm{mtree}})$ and the terminal groupoid, we get:

$$\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G) \simeq \mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G) / \mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G_{\mathrm{mtree}}) \cong \mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G / G_{\mathrm{mtree}}),$$

in which, analogously, G / G_{mtree} denotes the pushout of the morphism induced by the inclusion $G_{\mathrm{mtree}} \to G$ along the unique morphism $G_{\mathrm{mtree}} \to \bullet$ in the category of reflexive graphs Rgrph.

Since the reflexive graph G / G_{mtree} has only one object, we have that

$$\mathscr{L}_1 \mathscr{F}_1^{\mathscr{R}}(G / G_{\mathrm{mtree}}) \simeq \Pi \mathscr{F}_{\mathsf{Top}_1}^{\mathscr{R}}(G / G_{\mathrm{mtree}}) \cong \Pi \left(\mathscr{F}_{\mathsf{Top}_1}^{\mathscr{R}}(G) / \mathscr{F}_{\mathsf{Top}_1}^{\mathscr{R}}(G_{\mathrm{mtree}}) \right)$$

in which the last isomorphism follows from the fact that $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_1}$ is left adjoint. Since

$$\Pi\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_1}(G)/\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_1}(G_{\mathrm{mtree}})\right) \simeq \Pi\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_1}(G),$$

the proof is complete. This actually can be done in a pseudonatural equivalence as we show below.

Theorem 2.5.11. *There is a natural transformation $\mathscr{L}_1\mathscr{F}_1 \longrightarrow \Pi\mathscr{F}_{\mathsf{Top}_1}$ which is an objectwise equivalence.*

Proof. Consider the unit of the adjunction $\mathscr{F}_{\mathsf{Top}_1} \dashv \mathscr{C}_{\mathsf{Top}_1}$, denoted in this proof by η. We have that the horizontal composition $\mathrm{Id}_{\mathscr{P}_1\mathrm{i}_2} * \eta$ gives a natural transformation $\mathscr{P}_1\mathrm{i}_2 \longrightarrow \mathscr{P}_1\mathrm{i}_2\overline{\mathscr{F}_{\mathsf{Top}_1}}$. We, then, compose this natural transformation with the obvious isomorphism $\mathscr{P}_1\mathrm{i}_2\overline{\mathscr{F}_{\mathsf{Top}_1}} \longrightarrow \mathscr{P}_1\mathrm{i}_2\mathfrak{u}_2\mathscr{C}_{\mathsf{Top}_2}\mathscr{F}_{\mathsf{Top}_1}$ obtained from the isomorphism of Proposition 2.5.9

Now, we suitably past this natural transformation with the counit of $\mathrm{i}_2 \dashv \mathfrak{u}_2$ and get a natural transformation $\mathscr{P}_1\mathrm{i}_2 \longrightarrow \mathscr{P}_1\mathscr{C}_{\mathsf{Top}_2}\mathscr{F}_{\mathsf{Top}_1}$, which, after composing with the isomorphism of Proposition 2.4.17, gives $\mathscr{F}_1 \longrightarrow \mathscr{P}_1\mathscr{C}_{\mathsf{Top}_2}\mathscr{F}_{\mathsf{Top}_1}$.

The horizontal composition of this natural isomorphism with $\mathrm{Id}_{\mathscr{L}_1}$ gives our natural transformation $\mathscr{L}_1\mathscr{F}_1 \longrightarrow \mathscr{L}_1\mathscr{P}_1\mathscr{C}_{\mathsf{Top}_2}\mathscr{F}_{\mathsf{Top}_1} \cong \Pi\mathscr{F}_{\mathsf{Top}_1}$. It is an exercise of basic algebraic topology to show that, as a consequence of the considerations of Remark 2.5.10, this natural transformation is an objectwise equivalence. □

2.5.12 Further on Topology

To get the relation between small computads and topological spaces, we use the isomorphism of Theorem 2.4.3. In particular, an object $(\mathfrak{g},\mathfrak{g}_2,G)$ of cmp is a diagram $\mathfrak{g}: \mathsf{2} \to \mathsf{grph}$

$$\widehat{\mathfrak{G}} \times \mathrm{i}_1(\mathfrak{g}_2) \to \overline{\mathscr{F}_1}(G),$$

in which $\mathfrak{g}_2$ is a set and G is a small graph. We also fix the homeomorphism $\mathrm{cir}^{-1}: \mathscr{F}_{\mathsf{Top}_1}(\widehat{\mathfrak{G}}) \to S^1$ which is the mate of the morphism of graphs $\mathrm{cir}': \widehat{\mathfrak{G}} \to \mathscr{C}_{\mathsf{Top}_1}(S^1)$ which takes the edges of $\widehat{\mathfrak{G}}$ to the continuous maps $\mathrm{h}'_1, \mathrm{h}'_0 : I \to S^1$, $\mathrm{h}'_1(t) := \mathrm{h}_1(t)$, $\mathrm{h}'_0(t) := \mathrm{h}_0(t)$ (which are edges between 0 and 1 in $\mathscr{C}_{\mathsf{Top}_1}(S^1)$). More generally, for each set $\mathfrak{g}_2$, we fix the homeomorphism

$$\mathrm{cir} \times \mathfrak{g}_2 : S^1 \times D(\mathfrak{g}_2) \to \mathscr{F}_{\mathsf{Top}_1}(\widehat{\mathfrak{G}} \times \mathrm{i}_1(\mathfrak{g}_2)).$$

Analogously to the case of graphs, we can associate each computad with a “topological computad”, which is a CW-complex of dimension 2. The functor $\mathscr{C}_{\mathsf{Top}_2}: \mathsf{Top} \to \mathsf{cmp}$ is actually right adjoint to the functor $\mathscr{F}_{\mathsf{Top}_2}: \mathsf{cmp} \to \mathsf{Top}$ defined as follows: if $(\mathfrak{g},\mathfrak{g}_2,G)$ is a small computad $\mathfrak{g}: \widehat{\mathfrak{G}} \times \mathrm{i}_1(\mathfrak{g}_2) \to \overline{\mathscr{F}_1}(G)$, then $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$ is the pushout of $\mathrm{h} \times D(\mathfrak{g}_2) : S^1 \times D(\mathfrak{g}_2) \to B^2 \times D(\mathfrak{g}_2)$ along the composition of the morphisms

$$S^1 \times D(\mathfrak{g}_2) \xrightarrow{(\mathrm{cir}\times\mathfrak{g}_2)} \mathscr{F}_{\mathsf{Top}_1}(\widehat{\mathfrak{G}} \times \mathrm{i}_1(\mathfrak{g}_2)) \xrightarrow{(\mathscr{F}_{\mathsf{Top}_1}\mathfrak{g})} \mathscr{F}_{\mathsf{Top}_1}\overline{\mathscr{F}_1}(G) \xrightarrow{\lceil\cdot\rceil_G} \mathscr{F}_{\mathsf{Top}_1}(G)$$

in which $\lceil\cdot\rceil : \mathscr{F}_{\mathsf{Top}_1}\overline{\mathscr{F}_1} \longrightarrow \mathscr{F}_{\mathsf{Top}_1}$ is the natural transformation of Remark 2.5.7.

Lemma 2.5.13. *A small computad* $(\mathfrak{g},\mathfrak{g}_2,G)$ *is connected if and only if* $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$ *is a path connected topological space.*

Let $\mathfrak{g}=(\mathfrak{g},\mathfrak{g}_2,G)$ be a small connected computad. We denote by $\mathtt{T}$ the maximal tree of $\mathfrak{u}_2(\mathfrak{g},\mathfrak{g}_2,G)$. Consider the pushout of $\mathscr{F}_{\mathsf{Top}_2}\mathfrak{i}_2(\mathtt{T})\to *$ along the composition

$$\mathscr{F}_{\mathsf{Top}_2}\mathfrak{i}_2(\mathtt{T})\to\mathscr{F}_{\mathsf{Top}_2}\mathfrak{i}_2\mathfrak{u}_2(\mathfrak{g},\mathfrak{g}_2,G)\to\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$$

in which $\mathscr{F}_{\mathsf{Top}_2}\mathfrak{i}_2(\mathtt{T})\to\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{i}_2\mathfrak{u}_2(\mathfrak{g},\mathfrak{g}_2,G))$ is induced by the inclusion of the maximal tree of the graph $\mathfrak{u}_2(\mathfrak{g},\mathfrak{g}_2,G)$ and $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{i}_2\mathfrak{u}_2(\mathfrak{g},\mathfrak{g}_2,G))\to\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$ is induced by the counit of $\mathfrak{i}_2\dashv\mathfrak{u}_2$. Since this is actually a pushout of a homotopy equivalence along a cofibration (that is to say, this is a homotopy pushout along a homotopy equivalence), we get that $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$ has the same homotopy type of the obtained pushout which is a wedge of spheres, balls and circumferences.

Theorem 2.5.14. *For each small computad* $(\mathfrak{g},\mathfrak{g}_2,G)$*, there is an equivalence*

$$\Pi\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\simeq\mathscr{L}_1\mathscr{P}_1(\mathfrak{g},\mathfrak{g}_2,G).$$

Remark 2.5.15. It is clear that the adjunction $\mathscr{F}_{\mathsf{Top}_2}\dashv\mathscr{C}_{\mathsf{Top}_2}$ can be lifted to an adjunction $\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}\dashv\mathscr{C}^{\mathsf{Gr}}_{\mathsf{Top}_2}$ in which $\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}:\mathsf{cmp}_{\mathsf{Gr}}\to\mathsf{Top}$ is defined as follows: if $(\mathfrak{g},\mathfrak{g}_2,G)$ is a small groupoidal computad,

$$\mathfrak{g}:\widehat{\mathfrak{G}}\times\mathfrak{i}_1(\mathfrak{g}_2)\to\overline{\mathscr{L}_1\mathscr{F}_1}(G),$$

then $\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$ is the pushout of $\mathrm{h}\times D(\mathfrak{g}_2):S^1\times D(\mathfrak{g}_2)\to B^2\times D(\mathfrak{g}_2)$ along $\lceil\cdot\rceil^{\mathsf{Gr}}_G\cdot(\mathscr{F}_{\mathsf{Top}_1}\mathfrak{g})\cdot(\mathrm{cir}\times\mathfrak{g}_2)$. We have an isomorphism $\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}\mathscr{L}^{\mathsf{cmp}}_1\cong\mathscr{F}_{\mathsf{Top}_2}$.

Theorem 2.5.16. *For each small groupoidal computad* $(\mathfrak{g},\mathfrak{g}_2,G)$*, there is an equivalence*

$$\Pi\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\simeq\mathscr{P}_{(1,0)}(\mathfrak{g},\mathfrak{g}_2,G).$$

2.6 Deficiency

In this section, we study presentations of small categories/groupoids, focusing on thin groupoids and categories. Roughly, the main result of this section computes the minimum of equations/2-cells necessary to get a presentation of a groupoid generated by a given graph G with finite Euler characteristic. This result motivates our definition of deficiency of a (finitely presented) groupoid/category. We start by giving the basic definitions of deficiency of algebras over Set.

2.6.1 Algebras over the category of sets

Let $\mathscr{T}=(\mathscr{T},m,\eta)$ be a monad on Set. We denote a $\mathscr{T}$-presentation $R:\mathfrak{G}^{\mathrm{op}}\to\mathrm{Set}$,

$$R(2)\rightrightarrows\mathscr{T}(S),$$

by $\langle S,R\rangle$. If S and $R(2)$ are finite, the presentation $\langle S,R\rangle$ is called *finite*. If a $\mathscr{T}$-algebra $(A,\mathscr{T}(A)\to A)$ has a finite presentation $\langle S,R\rangle$, it is called *finitely ($\mathscr{T}$-)presented.*

In this context, the *($\mathscr{T}$-)deficiency of a $\mathscr{T}$-presentation* $\langle S, R\rangle$ is defined by

$$\operatorname{def}_{\mathscr{T}}(\langle S, R\rangle) := |S| - |R(2)|$$

in which $|-|$ gives the cardinality of the set. The *($\mathscr{T}$-)deficiency of a finitely presented $\mathscr{T}$-algebra* $(A, \mathscr{T}(A) \to A)$, denoted by $\operatorname{def}_{\mathscr{T}}(A, \mathscr{T}(A) \to A)$, is the maximum of the set

$$\{\operatorname{def}_{\mathscr{T}}(\langle S, R\rangle) : \langle S, R\rangle \text{ presents } (A, \mathscr{T}(A) \to A)\}.$$

Example 2.6.2. Consider the *free real vector space monad* and the notion of presentation of vector spaces induced by it. In this context, the notion of finitely presented vector space coincides with the notion of finite dimensional vector space and it is a consequence of the rank-nulity theorem that the deficiency of a finite dimensional vector space is its dimension.

The notion of deficiency and finite presentations induced by the free group monad $\overline{\mathscr{L}_0\mathscr{F}_0}$ coincide with the usual notions (see [54]). Analogously, the respective usual notions of deficiency and finite presentations are induced by the free monoid monad and free abelian group monad.

It is well known that, if a (finitely presented) group has positive deficiency, then this group is nontrivial (actually, it is not finite). Indeed, if H is a group which has a presentation with positive deficiency, then $\mathsf{Group}(H, \mathbb{R})$ is a vector space with a presentation with positive deficiency. This implies that $\mathsf{Group}(H, \mathbb{R})$ has positive dimension and, then, H is not trivial. In particular, we conclude that the trivial group has deficiency 0.

We present a suitable definition of deficiency of groupoids and, then, we prove that thin groupoids have deficiency 0. Before doing so, we recall elementary aspects of Euler characteristics and define what we mean by finitely presented category.

2.6.3 Euler characteristic

If X is a topological space, we denote by $\mathrm{H}^i(X)$ its ordinary i-th cohomology group with coefficients in $\mathbb{R}$. Assuming that the dimensions of the cohomology groups of a topological space X are finite, recall that the *Euler characteristic of a topological space X* is given by

$$\chi(X) := \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{H}^i(X)$$

whenever all but a finite number of terms of this sum are 0.

If G is a small graph, it is known that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G)) = |G(1)| - |G(2)|$ whenever the cardinality of the sets $G(1), G(2)$ are finite. Also, a connected small graph G is a tree if and only if $\chi(\mathscr{F}_{\mathsf{Top}_1}(G)) = 1$. As a corollary of Theorem 2.2.13, we get:

Corollary 2.6.4. *Let G be a connected small graph. If $\chi(\mathscr{F}_{\mathsf{Top}_1}(G)) = 1$, $\mathscr{L}_1\mathscr{F}_1(G)$ and $\mathscr{F}_1(G)$ are thin.*

If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a connected small computad, since $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ has the same homotopy type of a wedge of spheres, closed balls and circumferences, $\mathrm{H}^0\left(\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g})\right) = \mathbb{R}$ and $\mathrm{H}^i\left(\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g})\right) = 0$

for all $i > 2$. Furthermore, assuming that $\chi(\mathscr{F}_{\mathsf{Top}_1}\mathfrak{u}_2(\mathfrak{g},\mathfrak{g}_2,G)) = \chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ and $\mathfrak{g}_2$ are finite, we have that:

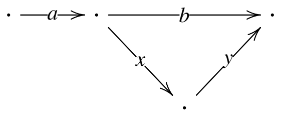
$$\chi\left(\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right) = \chi(\mathscr{F}_{\mathsf{Top}_1}(G)) + |\mathfrak{g}_2|.$$

Remark 2.6.5. All considerations about $\mathscr{F}_{\mathsf{Top}_2}$ have analogues for $\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}$. In particular, if $(\mathfrak{g},\mathfrak{g}_2,G)$ is a connected small groupoidal computad $\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$ is a CW-complex and has the same homotopy type of a wedge of spheres, closed balls and circumferences. Moreover, $\chi\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right) = \chi(\mathscr{F}_{\mathsf{Top}_1}(G)) + |\mathfrak{g}_2|$ provided that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ and $\mathfrak{g}_2$ are finite.

2.6.6 Deficiency of a Groupoid

Observe that $\sigma_2(G)$ gives a (natural) presentation of the thin category freely generated by G. More precisely, $\mathscr{P}_1\sigma_2 \cong \overline{\mathscr{M}_1}\mathscr{F}_1$. Yet, $\sigma_2(G)$ gives always a presentation of $\overline{\mathscr{M}_1}\mathscr{F}_1(G)$ with more equations than necessary.

Example 2.6.7. Let G be the graph below. In this case, the set of 2-cells $\sigma_2(G)_2$ is given by $\{(w,w) : w \in \mathscr{F}_1(G)(2)\} \cup \{(yx,b),(yxa,ba),(ba,yxa),(b,yx)\}$ with obvious projections.



On one hand, the computad $\sigma_2(G)$ induces the presentation of $\overline{\mathscr{M}_1}\mathscr{F}_1(G)$ with the equations:

$$\begin{cases} w = w \text{ if } w \in \mathscr{F}_1(G)(2) \\ yx = b \\ yxa = ba \\ b = yx \\ ba = yxa. \end{cases}$$

On the other hand, the computad

$$\begin{tikzcd} 2 \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}_1(G), \end{tikzcd}$$

in which the image of one functor is the arrow yx while the image of the other functor is b, gives a presentation of $\overline{\mathscr{M}_1}\mathscr{F}_1(G)$ with less equations than $\sigma_2(G)$.

The main theorem about presentations of thin groupoids in low dimension is Theorem 2.6.10. This result gives a lower bound to the number of equations we need to present a thin groupoid. We start with our first result, which is a direct corollary of Theorem 2.5.16:

Corollary 2.6.8. *Let $(\mathfrak{g},\mathfrak{g}_2,G)$ be a small connected groupoidal computad. $\mathscr{P}_{(1,0)}(\mathfrak{g},\mathfrak{g}_2,G)$ is thin if and only if $\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$ is 1-connected which means that the fundamental group $\pi_1(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G))$ is trivial.*

Proof. The fundamental group $\pi_1(\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G))$ is trivial if and only if $\Pi(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G))$ is thin. By Theorem 2.5.16, we conclude that $\pi_1(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G))$ is trivial if and only if $\mathscr{P}_{(1,0)}(\mathfrak{g},\mathfrak{g}_2,G)$ is thin. □

Remark 2.6.9. Of course, last corollary applies also to the case of presentation of groupoids via computads. More precisely, if $(\mathfrak{g},\mathfrak{g}_2,G)$ is a small connected computad,

$$\mathscr{L}_1\mathscr{P}_1(\mathfrak{g},\mathfrak{g}_2,G)\cong\mathscr{P}_{(1,0)}\mathscr{L}_1^{\mathsf{Cmp}}(\mathfrak{g},\mathfrak{g}_2,G)$$

is thin if and only if the fundamental group of

$$\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}\mathscr{L}_1^{\mathsf{Cmp}}(\mathfrak{g},\mathfrak{g}_2,G)\cong\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)$$

is trivial.

Theorem 2.6.10. *If $(\mathfrak{g},\mathfrak{g}_2,G)$ is a small connected groupoidal computad and*

$$\mathbb{Z}\ni\chi\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)<1,$$

then $\mathscr{P}_{(1,0)}(\mathfrak{g},\mathfrak{g}_2,G)$ is not thin.

Proof. Recall that

$$\chi\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)=1-\dim\mathrm{H}^1\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)+\dim\mathrm{H}^2\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right).$$

Therefore, by hypothesis,

$$\dim\mathrm{H}^1\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)>\dim\mathrm{H}^2\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right).$$

In particular, we conclude that $\dim\mathrm{H}^1\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)>0$. By the Hurewicz isomorphism theorem and by the universal coefficient theorem, this fact implies that the fundamental group $\pi_1\left(\mathscr{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)$ is not trivial. By Corollary 2.6.8, we get that $\mathscr{P}_{(1,0)}(\mathfrak{g},\mathfrak{g}_2,G)$ is not thin. □

Corollary 2.6.11. *If $(\mathfrak{g},\mathfrak{g}_2,G)$ is a small connected groupoidal computad which presents a thin groupoid and $\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ is finite, then*

$$\chi(\mathscr{F}_{\mathsf{Top}_1}(G))+|\mathfrak{g}_2|-1\geq 0.$$

In particular, Corollary 2.6.11 implies that, if G is such that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ is finite, we need at least $1-\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ equations to get a presentation of $\overline{\mathscr{M}_1}\,\overline{\mathscr{L}_1\mathscr{F}_1}(G)$.

Definition 2.6.12. [Finitely Presented Groupoids and Categories] A groupoid/category X is *finitely presented* if there is a small connected groupoidal computad/small connected computad $(\mathfrak{g},\mathfrak{g}_2,G)$ which presents X, such that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ and $|\mathfrak{g}_2|$ are finite.

Recall the definition of deficiency of groups w.r.t. the free group monad $\overline{\mathscr{L}_0\mathscr{F}_0}$ given in 2.6.1. Definition 2.6.12 agrees with the definition of finitely $\overline{\mathscr{L}_0\mathscr{F}_0}$-presented groups. Moreover, as explained in Proposition 2.6.14, Definition 2.6.13 also agrees with the definition of $\overline{\mathscr{L}_0\mathscr{F}_0}$-deficiency of groups.

Definition 2.6.13. [Deficiency of a Groupoid] Let X be a finitely presented groupoid. The *deficiency of a presentation* of X by a small connected groupoidal computad $(\mathfrak{g},\mathfrak{g}_2,G)$ is defined by $\operatorname{def}(\mathfrak{g},\mathfrak{g}_2,G) := 1-|\mathfrak{g}_2|-\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$, provided that $|\mathfrak{g}_2|$ and $\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ are finite.

Moreover, the *deficiency of the groupoid* X is the maximum of the set

$$\left\{\left(1-\chi\left(\mathscr{F}^{\mathsf{gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)\right)\in\mathbb{Z} : \mathscr{P}_{(1,0)}(\mathfrak{g},\mathfrak{g}_2,G)\cong X \text{ and } \chi(\mathscr{F}_{\mathsf{Top}_1}(G))\in\mathbb{Z}\right\}.$$

Proposition 2.6.14. *If X is a finitely presented group, the deficiency of $\Sigma(X)$ w.r.t. presentations by groupoidal computads is equal to* $\operatorname{def}_{\overline{\mathscr{L}_0\mathscr{F}_0}}(X)$.

Proof. This result follows from Remark 2.4.19. ☐

Theorem 2.6.11 is the first part of Corollary 2.6.18. The second part is Theorem 2.6.16 which is easy to prove: but we need to give some explicit constructions to give further consequences in 2.6.20. To do that, we need the terminology introduced in:

Remark 2.6.15. Given a small reflexive graph G, a morphism f of $\mathscr{F}_1^{\mathscr{R}}(G)$ determines a subgraph of G, namely, the smallest (reflexive) subgraph G' of G, called the image of f, such that f is a morphism of $\mathscr{F}_1^{\mathscr{R}}(G')$. More generally, given a small computad $\mathfrak{g}=(\mathfrak{g},\mathfrak{g}_2,G)$ of Rcmp, it determines a subgraph G' of G, called the *image of the computad* $\mathfrak{g}$ in G, which is the smallest graph G' satisfying the following: there is a computad $\mathfrak{g}' : \mathfrak{g}_2\times\mathfrak{G}\to\mathscr{F}_1^{\mathscr{R}}(G')$ such that

$$\begin{tikzcd}
\mathfrak{g}_2\times\mathfrak{G} \arrow[r, "\mathfrak{g}'"] \arrow[dr, "\mathfrak{g}"'] & \mathscr{F}_1^{\mathscr{R}}(G') \arrow[d] \\
 & \mathscr{F}_1^{\mathscr{R}}(G)
\end{tikzcd}$$

commutes. We also can consider the *graph domain* and the *graph codomain* of a small computad $\mathfrak{g}=(\mathfrak{g},\mathfrak{g}_2,G)$, $\mathfrak{g}:\mathfrak{G}^{\mathrm{op}}\to\mathsf{Cat}$, which are respectively the smallest subgraphs $\mathfrak{g}^{d^1}$ and $\mathfrak{g}^{d^0}$ of G such that $\mathfrak{g}(d^1)(\mathfrak{g}_2\times 2)$ and $\mathfrak{g}(d^0)(\mathfrak{g}_2\times 2)$ are respectively in $\mathscr{F}_1^{\mathscr{R}}(\mathfrak{g}^{d^1})$ and $\mathscr{F}_1^{\mathscr{R}}(\mathfrak{g}^{d^0})$.

Of course, we can consider the notions introduced above in the category of computads or groupoidal computads as well.

Theorem 2.6.16. *Let G be a small connected graph such that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G))\in\mathbb{Z}$ (equivalently, $\pi_1(\mathscr{F}_{\mathsf{Top}_1}(G))$ is finitely generated). There is a groupoidal computad $(\widehat{\mathfrak{g}},\mathfrak{g}_2,G)$ which presents $\overline{\mathscr{M}_1}\,\overline{\mathscr{L}_1}\mathscr{F}_1(G)$ such that $|\mathfrak{g}_2|=1-\chi(\mathscr{F}_{\mathsf{Top}_1}(G))$.*

Proof. Without losing generality, in this proof we consider reflexive graphs, and computads over reflexive graphs. Let G be a small reflexive connected graph such that its fundamental group is finitely generated. If G_{mtree} is the maximal (reflexive) tree of G, we know that the image of the (natural) morphism of reflexive graphs $G\to G/G_{\mathrm{mtree}}$ by the functor $\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}$ is an equivalence. That is to say, we have a natural equivalence $\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G)\to\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G/G_{\mathrm{mtree}})$ which is in the image of $\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}$.

In particular, each arrow f of G/G_{mtree} corresponds to a unique arrow $\widehat{f}$ of G such that $\widehat{f}$ is not an arrow of G_{mtree} and the image of $\widehat{f}$ by $G\to G/G_{\mathrm{mtree}}$ is f.

Recall that, since G/G_{mtree} has only one object, $\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G/G_{\mathrm{mtree}})$ is the suspension of the group freely generated by the set $G/G_{\mathrm{mtree}}(2)$ of arrows. By hypothesis, this set is finite and has $1-\chi(\mathscr{F}_{\mathsf{Top}_1}(G))\in\mathbb{N}$ elements. Thereby we have a computad $\mathfrak{g}:\mathfrak{G}^{\mathrm{op}}\to\mathsf{cat}$,

$$\begin{tikzcd}\mathfrak{g}_2\times 2 \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}_1^{\mathscr{R}}(G/G_{\mathrm{mtree}}),\end{tikzcd}$$

in which $\mathfrak{g}_2:=G/G_{\mathrm{mtree}}(2)$, $\mathfrak{g}(d^0)(f,0\to 1)=f$ and $\mathfrak{g}(d^0)(f,0\to 1)=\mathrm{id}$. The computad $\mathscr{L}_1^{\mathsf{Rcmp}}(\mathfrak{g}):\mathfrak{G}^{\mathrm{op}}\to\mathsf{gr}$ gives a presentation of the trivial group.

The computad $\mathfrak{g}$ lifts through $G\to G/G_{\mathrm{mtree}}$ to a (small) groupoidal computad $\widehat{\mathfrak{g}}:\mathfrak{G}^{\mathrm{op}}\to\mathsf{cat}$ over G. More precisely, we define $\widehat{\mathfrak{g}}=(\mathfrak{g},\mathfrak{g}_2,G)$,

$$\widehat{\mathfrak{g}}(1)=\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G),\quad \widehat{\mathfrak{g}}(2)=\mathfrak{g}_2\times 2,\quad \widehat{\mathfrak{g}}(d^0)(f,0\to 1)=\widehat{f}\quad\text{and}\quad \widehat{\mathfrak{g}}(d^1)(f,0\to 1)=\dot{f}$$

in which $\dot{f}$ is the unique morphism of the (thin) subgroupoid $\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G_{\mathrm{mtree}})$ of $\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G)$ such that the domain and codomain of $\dot{f}$ coincide respectively with the domain and codomain of $\widehat{f}$.

Of course, this construction provides a 2-natural transformation which is pointwise an equivalence $\widehat{\mathfrak{g}}\longrightarrow\mathscr{L}_1^{\mathsf{Rcmp}}(\mathfrak{g})$,

$$\begin{tikzcd}\mathfrak{g}_2\times 2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G) \arrow[d] \\ \mathfrak{g}_2\times 2 \arrow[r, shift left] \arrow[r, shift right] & \mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G/G_{\mathrm{mtree}}).\end{tikzcd}$$

It is easy to see that, in this case, it induces an equivalence between the coequalizers. Thereby $\widehat{\mathfrak{g}}$ presents a thin groupoid, which is $\mathscr{L}_1\mathscr{P}_1(\sigma_2^{\mathsf{Gr}}(G))$. This completes the proof. □

Remark 2.6.17. The graph domain and the graph codomain of the computad $\widehat{\mathfrak{g}}$ constructed in the proof above are, respectively, inside and outside the maximal tree G_{mtree}. More precisely, for every $\alpha\in\widehat{\mathfrak{g}}_2=\mathfrak{g}_2$, the $\widehat{\mathfrak{g}}(d^1)(\alpha,0\to 1)$ is a morphism of $\mathscr{L}_1\mathscr{F}_1^{\mathscr{R}}(G_{\mathrm{mtree}})$ and $\widehat{\mathfrak{g}}(d^0)(\alpha,0\to 1)$ is a morphism of length one which is not an arrow of G_{mtree}.

By Theorem 2.6.16 and Theorem 2.6.10 we get:

Corollary 2.6.18. *The deficiency of a finitely presented thin groupoid is* 0. *In particular, this result generalizes the fact that the deficiency of the trivial group is* 0.

Remark 2.6.19. [Finite measure and deficiency] Let $\mathbb{R}_\infty^+$ be the category whose structure comes from the totally ordered set of the non-negative real numbers with a top element ∞ with the usual order. The initial object is of course 0, while the terminal object is ∞.

Let $\mathfrak{X}'$ be the subcategory of monomorphisms of a category $\mathfrak{X}$. A *finite (strong/naive) measure* on $\mathfrak{X}$ is a functor $\mu:\mathfrak{X}'\to\mathbb{R}_\infty^+$ that preserves finite coproducts (including the empty coproduct, which is the initial object).

A pair $(\mathfrak{X},\mu)$ together with a monad $\mathscr{T}$ on $\mathfrak{X}$ give rise to a notion of *finite $\mathscr{T}$-presentation*: a presentation as in the $\mathscr{T}$-presentation diagram is μ-finite if $\mu(G_1)$ and $\mu(G_2)$ are finite. In this case, we define the *$(\mathscr{T},\mu)$-deficiency* of such a μ-finite $\mathscr{T}$-presentation by $\mathrm{def}_{(\mathscr{T},\mu)}:=\mu(G_1)-\mu(G_2)$. If X is a $\mathscr{T}$-algebra which admits a finite presentation, X is called finitely $\mathscr{T}$- presented.

For instance, *cardinality* is a measure on the category of sets Set which induces the notions of finite $\mathscr{T}$-presentation and $\mathscr{T}$-deficiency of algebras over sets given in 2.6.1.

Finally, consider the category of graphs $\mathsf{Grph}_{\mathsf{finEu}}$ with finite Euler characteristic: the measure *Euler characteristic* χ and the monad $\overline{\mathscr{L}_1\mathscr{F}_1}$ induce the notion of $(\overline{\mathscr{L}_1\mathscr{F}_1}, \chi)$-deficiency of an $\overline{\mathscr{L}_1\mathscr{F}_1}$-presentation. If we consider the inclusion of Theorem 2.4.5, this notion of deficiency coincides with the notion of deficiency of a presentation via groupoidal computad given in 2.6.6.

2.6.20 Presentation of Thin Categories

The results on presentations of thin groupoids can be used to study presentations of thin categories. For instance, if a presentation of a thin groupoid can be lifted to a presentation of a category, then this category is thin provided that the lifting presents a category that satisfies the cancellation law. To make this statement precise (which is Proposition 2.6.22), we need:

Definition 2.6.21. [Lifting Groupoidal Computads] We denote by $\mathsf{cmp}_{\mathsf{lift}}$ the pseudopullback (iso-comma category) of $\mathscr{P}_{(1,0)} : \mathsf{cmp}_{\mathsf{gr}} \to \mathsf{gr}$ along $\mathscr{L}_1\mathscr{P}_1 : \mathsf{cmp} \to \mathsf{gr}$. A small computad $\mathfrak{g}$ is called a *lifting of the small groupoidal computad* $\mathfrak{g}'$ if there is an object $\zeta^{\mathfrak{g}}_{\mathfrak{g}'}$ of $\mathsf{cmp}_{\mathsf{lift}}$ such that the images of this object by the functors

$$\mathsf{cmp}_{\mathsf{lift}} \to \mathsf{cmp}_{\mathsf{gr}}, \qquad \mathsf{cmp}_{\mathsf{lift}} \to \mathsf{cmp}$$

are respectively $\mathfrak{g}'$ and $\mathfrak{g}$.

Proposition 2.6.22. *If* $\mathfrak{g}'$ *is a groupoidal computad that presents a thin groupoid and* $\mathscr{P}_1(\mathfrak{g})$ *satisfies the cancellation law, then* $\mathfrak{g}$ *presents a thin category.*

Proof. By hypothesis, $\mathscr{P}_{(1,0)}(\mathfrak{g}') \cong \mathscr{L}_1\mathscr{P}_1(\mathfrak{g})$ is a thin groupoid and $\mathscr{P}_1(\mathfrak{g})$ satisfies the cancellation law. Hence $\mathscr{P}_1(\mathfrak{g})$ is a thin category. □

Theorem 2.6.23. *If G is small connected fair graph such that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}$, then there is a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ of* cmp *such that $|\mathfrak{g}_2| = 1 - \chi(\mathscr{F}_{\mathsf{Top}_1}(G))$ which presents the groupoid $\overline{\mathscr{M}_1}\mathscr{L}_1\mathscr{F}_1(G)$.*

Proof. Let G_{mtree} be a maximal weak tree of G which is also the maximal tree. Let $(\widehat{\mathfrak{g}}, \mathfrak{g}_2, G)$ be the groupoidal computad constructed in the proof of Theorem 2.6.16 using the maximal tree G_{mtree}.

We will prove that the groupoidal computad $(\widehat{\mathfrak{g}}, \mathfrak{g}_2, G)$ can be lifted to a computad. In order to do so, we need to prove that, for each $\alpha \in \mathfrak{g}_2$, the restriction $\widehat{\mathfrak{g}}|_\alpha : \mathfrak{G}^{\text{op}} \to \mathsf{gr}$,

$$\begin{tikzcd} \{\alpha\} \times 2 \arrow[r, shift left] \arrow[r, shift right] & \mathscr{L}_1\mathscr{F}_1(G), \end{tikzcd}$$

can be lifted to a small computad. By Remark 2.6.17, we know that $\widehat{\mathfrak{g}}(d^1)(\alpha, 0 \to 1)$ is a morphism of $\mathscr{L}_1\mathscr{F}_1(G_{\text{mtree}})$ and $\widehat{\mathfrak{g}}(d^0)(\alpha, 0 \to 1)$ is a morphism of length one which is not an arrow of G_{mtree}. Since G_{mtree} is a maximal weak tree, we conclude that the image of $\widehat{\mathfrak{g}}|_\alpha$ is not a weak tree. Hence there are parallel morphisms f_0, f_1 of $\mathscr{F}_1(G)$ that determine the same graph determined by the image of $\widehat{\mathfrak{g}}|_\alpha$ (see Remark 2.6.15) such that f_0 is a morphism of $\mathscr{F}_1(G_{\text{mtree}})$. Therefore, $\widehat{\mathfrak{g}}$ can be lifted to $(\mathfrak{g}|_\alpha, \{\alpha\}, G)$ given by $\mathfrak{g}|_\alpha : \mathfrak{G}^{\text{op}} \to \mathsf{cat}$, $\mathfrak{g}|_\alpha(d^0)(\alpha, 0 \to 1) = f_0$ and $\mathfrak{g}|_\alpha(d^1)(\alpha, 0 \to 1) = f_1$. □

As a corollary of the proof, we get:

Corollary 2.6.24. *If G is small connected fair graph such that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}$ and $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small computad which presents $\overline{\mathscr{M}_1}\mathscr{F}_1(G)$, then $|\mathfrak{g}_2| \geq 1 - \chi(\mathscr{F}_{\mathsf{Top}_1}(G))$.*

Proof. As consequence of the constructions involved in the last proof, for each 2-cell of the computad $\mathfrak{g}$ of Theorem 2.6.23, there are parallel morphisms in $\mathscr{F}_1(G)$ such that they can be represented by (completely) different lists of arrows of G. □

However, in the conditions of the result above, often we need more than $1 - \chi(\mathscr{F}_{\mathsf{Top}_1}(G)))$ equations. The point is that the lifting given in Theorem 2.6.23 often does not present a category that satisfies the cancellation law. As consequence of the proof of Theorem 2.6.23, we get a generalization. More precisely:

Corollary 2.6.25. *Let G be a small connected graph such that $\chi(\mathscr{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}$. Consider the groupoidal computad $(\widehat{\mathfrak{g}}, \mathfrak{g}_2, G)$ constructed in Theorem 2.6.16.*

There is a largest groupoidal computad of the type $(\widehat{\mathfrak{h}}, \mathfrak{h}_2, G)$ which is a subcomputad of $\widehat{\mathfrak{g}}$ and can be lifted to a computad $(\mathfrak{h}, \mathfrak{h}_2, G)$ in the sense of Definition 2.6.21. We have that

$$\min\left\{|\mathfrak{x}_2| : \mathscr{P}_1(\mathfrak{x}, \mathfrak{x}_2, G) \cong \overline{\mathscr{M}_1}\mathscr{F}_1(G)\right\} \geq |\mathfrak{h}_2|.$$

Definition 2.6.26. A pair (G, G_{mtree}) is called a *monotone graph* if G is a small connected graph, $\chi(\mathscr{F}_{\mathsf{Top}_1}(G))) \in \mathbb{Z}$, G_{mtree} is a maximal weak tree of G and, whenever there exists an arrow $f : x \to y$ in G, either $x \leq y$ or $y \leq x$ in which $\leq$ is the partial order of the poset $\mathscr{F}_1(G_{\mathrm{mtree}})$.

If (G, G_{mtree}) is a *monotone graph* and $f : x \to y$ is an arrow such that $y \leq x$, f is called a *nonincreasing arrow* of the monotone graph. Finally, if (G, G_{mtree}) does not have nonincreasing arrows, (G, G_{mtree}) is called a *strictly increasing graph.*

Theorem 2.6.27. *Let (G, G_{mtree}) be a strictly increasing graph. There is a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ such that $|\mathfrak{g}_2| = 1 - \chi(\mathscr{F}_{\mathsf{Top}_1}(G)))$ which presents $\overline{\mathscr{M}_1}\mathscr{F}_1(G, G_{\mathrm{mtree}})$.*

Proof. For each arrow $f : x \to y$ outside the maximal weak tree G_{mtree}, there is a unique morphism $\dot{f} : x \to y$ in $\mathscr{F}_1(G_{\mathrm{mtree}})$. It is enough, hence, to define

$$\mathfrak{g}_2 := \left\{\alpha_f : f \in G(2) - G_{\mathrm{mtree}}(2)\right\}, \quad \mathfrak{g}(d^0)(\alpha_f, 0 \to 1) := \dot{f}, \quad \mathfrak{g}(d^1)(\alpha_f, 0 \to 1) := f.$$

It is clear that this is a lifting of the groupoidal computad $\widehat{\mathfrak{g}}$ of Theorem 2.6.16. Actually, $\mathfrak{g}$ is precisely the lifting given by Theorem 2.6.23. Moreover, it is also easy to see that $\mathscr{P}_1(\mathfrak{g})$ satisfies the cancellation law. Therefore the category presented by $\mathfrak{g}$ is thin. □

As a consequence of Corollary 2.6.24 and Theorem 2.6.27, we get:

Corollary 2.6.28. *Let (G, G_{mtree}) be a strictly increasing graph. The minimum of the set*

$$\left\{|\mathfrak{g}_2| : \mathscr{P}_1(\mathfrak{g}, \mathfrak{g}_2, G) \cong \overline{\mathscr{M}_1}\mathscr{F}_1(G)\right\}$$

is equal to $1 - \chi(\mathscr{F}_{\mathsf{Top}_1}(G))$.

Theorem 2.6.29. *Let (G, G_{mtree}) be a monotone graph with precisely n nonincreasing arrows. There is a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ such that $|\mathfrak{g}_2| = 1 - \chi(\mathscr{F}_{\mathsf{Top}_1}(G))) + \mathsf{n}$ which presents $\overline{\mathscr{M}_1}\mathscr{F}_1(G, G_{\mathrm{mtree}})$.*

Proof. For each nonincreasing arrow $f : x \to y$ outside the maximal weak tree G_{mtree}, either there is a unique morphism $\dot{f} : y \to x$ in $\mathscr{F}_1(G_{\text{mtree}})$ or there is a unique $\dot{f} : x \to y$ in $\mathscr{F}_1(G_{\text{mtree}})$. We define A^* the set of the nonincreasing arrows of G outside G_{mtree} and $\mathsf{A} := G(2) - G_{\text{mtree}}(2) - \mathsf{A}^*$. We define

$$\mathfrak{g}_2 := \left\{\alpha_f : f \in \mathsf{A}\right\} \cup \left\{\beta_{(f,j)} : f \in \mathsf{A}^* \text{ and } j \in \{-1,1\}\right\},$$

$$\mathfrak{g}(d^0)(\alpha_f, 0 \to 1) := \dot{f}, \quad \mathfrak{g}(d^1)(\alpha_f, 0 \to 1) := f,$$

$$\mathfrak{g}(d^0)(\beta_{(f,1)}, 0 \to 1) := \dot{f}f, \quad \mathfrak{g}(d^1)(\beta_{(f,1)}, 0 \to 1) := \text{id},$$

$$\mathfrak{g}(d^0)(\beta_{(f,-1)}, 0 \to 1) := f\dot{f}, \quad \mathfrak{g}(d^1)(\beta_{(f,-1)}, 0 \to 1) := \text{id}.$$

It is clear that is a lifting of the groupoidal computad $\widehat{\mathfrak{g}}$ of Theorem 2.6.16. Actually, the lifting given by Theorem 2.6.23 is a subcomputad of $\mathfrak{g}$. Moreover, it is also easy to see that $\mathscr{P}_1(\mathfrak{g})$ satisfies the cancellation law. Therefore the category presented by $\mathfrak{g}$ is thin. ☐

Remark 2.6.30. If we generalize the notion of deficiency of a groupoid and define: the *deficiency of a finitely presented category X* (by presentations via computads) is, if it exists, the maximum of the set

$$\left\{\left(1 - \chi\left(\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right)\right) \in \mathbb{Z} : \mathscr{P}_1(\mathfrak{g}, \mathfrak{g}_2, G) \cong X \text{ and } \chi(\mathscr{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}\right\},$$

then, given a strictly increasing graph (G, G_{mtree}), the deficiency of $\overline{\mathscr{M}_1}\mathscr{F}_1(G, G_{\text{mtree}})$ is 0. However, for instance, the deficiency of the thin category (by presentation of computads) $\nabla 2$ is not 0: it is -1. More generally, by Corollary 2.6.24 the deficiency of category freely generated by a tree (characterized in Theorem 2.2.22 and Corollary 2.2.23) is 0, while the deficiency of category freely generated by a weak tree G is $\chi(G) - 1$. Furthermore, if (G, G_{mtree}) is a monotone graph, the deficiency (by presentations via computads) of $\overline{\mathscr{M}_1}\mathscr{F}_1(G, G_{\text{mtree}})$ is $-\mathsf{n}$ in which n is the number of nontrivial isomorphisms of X (see Theorem 2.6.29).

2.7 Higher Dimensional Icons

Icons were originally defined in [69]. They were introduced as a way of organizing bicategories in a 2-category, recovering information of the tricategory of bicategories, pseudofunctors, pseudonatural/oplax natural transformations and modifications. Thereby, icons allow us to study aspects of these 2-categories of 2-categories/bicategories via 2-dimensional universal algebra.

There are examples of applications of this concept in [69, 70]. In this setting, on one hand, we get a 2-category $2\mathsf{Cat}$ which is the 2-category of 2-categories, 2-functors and icons. On the other hand, we have the 2-category Bicat of bicategories, pseudofunctors and icons.

The inclusion $2\mathsf{Cat} \to \mathsf{Bicat}$ can be seen as an inclusion of the 2-category of strict algebras into the 2-category of pseudoalgebras of a 2-monad. Therefore, we can apply 2-monad theory to get results about these categories of algebras. The 2-monadic coherence theorem [9, 67, 77] can be applied to this case and we get, in particular, the celebrated result that states that "every bicategory is biequivalent to a 2-category".

In Section 2.8, we show that the 2-categories $2\mathsf{Cat}$ and Bicat provide a concise way of constructing freely generated 2-categories as coinserters. We also show analogous descriptions for n-categories.

In order to do so, we give a definition of higher dimensional icon and construct 2-categories $n\mathsf{Cat}$ of n-categories in this section. It is important to note that there are many higher dimensional versions of icons and, of course, the best choice depends on the context. The definition of 3-dimensional icon presented herein is similar to that of "ico-icon" introduced in [38], but the scope herein is limited to strict n-categories.

Definition 2.7.1. [V-graphs] Let V be a 2-category. An object G of the 2-category $V\mathsf{Grph}$ is a discrete category $G(1) = G_0$ of Cat with a hom-object $G(A,B)$ of V for each ordered pair (A,B) of objects of $G(1)$.

A 1-cell $F : G \to H$ of $V\mathsf{Grph}$ is a functor $F_0 : G(1) \to H(1)$ with a collection of 1-cells $\left\{F_{(A,B)} : G(A,B) \to H(F_0(A),F_0(B))\right\}_{(A,B)\in G_0\times G_0}$ of V. The composition of 1-cells in $V\mathsf{Grph}$ is defined in the obvious way.

A 2-cell $\alpha : F \Rightarrow G$ is a collection of 2-cells $\left\{\alpha_{(A,B)} : F_{(A,B)} \Rightarrow G_{(A,B)}\right\}_{(A,B)\in G_0^2}$ in V. It should be noted that the existence of such a 2-cell implies, in particular, that $F_0 = G_0$. The horizontal and vertical compositions of 2-cells in $V\mathsf{Grph}$ come naturally from the horizontal and vertical compositions in V.

Let V be a 2-category with finite products and large coproducts (indexed in discrete categories of Cat). Assume that V is distributive w.r.t. these large coproducts. We can define a 2-monad $\mathscr{T}_V$ on $V\mathsf{Grph}$ such that $\mathscr{T}_V(G)_0 = G_0$ and

$$\mathscr{T}_V(G)(A,B) = \sum_{j\in\mathbb{N}} \sum_{(C_1,\dots,C_j)\in G_0^j} G(C_j,B)\times\cdots\times G(C_1,C_2)\times G(A,C_1),$$

in which $\sum$ denotes coproduct and this coproduct includes the term for $j = 0$ which is $G(A,B)$. The actions of $\mathscr{T}_V$ on the 1-cells and 2-cells are defined in the natural way. The component $m_G : \mathscr{T}_V^2(G) \longrightarrow \mathscr{T}_V(G)$ of the multiplication is identity on objects, while the 1-cells between the hom-objects are induced by the isomorphisms given by the distributivity and identities $G(C_j,B)\times\cdots\times G(A,C_1) = G(C_j,B)\times\cdots\times G(A,C_1)$. The component $\eta_G : G \longrightarrow \mathscr{T}_V(G)$ of the unit is identity on objects and the 1-cells between the hom-objects are given by the natural morphisms $G(A,B) \to \sum_{j\in\mathbb{N}}\sum_{(C_1,\dots,C_j)\in G_0^j} G(C_j,B)\times\cdots\times G(A,C_1)$ which correspond to the "natural inclusions" for $j = 0$.

In this context, we denote by V-Cat the category of V-enriched categories (described in Section 2.1) w.r.t. the underlying cartesian category of V.

Lemma 2.7.2. *Let V be a 2-category satisfying the properties above. The underlying category of the 2-category of strict 2-algebras $\mathscr{T}_V$-$\mathsf{Alg}_{\mathsf{s}}$ is equivalent to V-Cat.*

Proof. This follows from a classical result that states that the enriched categories are the Eilenberg-Moore algebras of the underlying monad of $\mathscr{T}_V$. See, for instance, [7]. □

Remark 2.7.3. We could consider the general setting of a 2-category V with a monoidal structure which preserves (large) coproducts (see, for instance, [99]).

Corollary 2.7.4. *The underlying category of the 2-category of strict 2-algebras $\mathscr{T}_{\mathsf{Cat}}$-$\mathsf{Alg}_{\mathsf{s}}$ is equivalent to 2-Cat.*

Definition 2.7.5. [nCat] We define $2\mathsf{Cat} := \mathscr{T}_{\mathsf{Cat}}\text{-}\mathsf{Alg}_{\mathsf{s}}$ and $\mathsf{Bicat} := \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$. An icon is just a 2-cell of Bicat. More generally, we define

$$n\mathsf{Cat} := \mathscr{T}_{(n-1)\mathsf{Cat}}\text{-}\mathsf{Alg}_{\mathsf{s}}.$$

The 2-cells of nCat are called *n-icons*. Following this definition, *icons* are also called 2-*icons* and 1-*icons* are just natural transformations between functors.

Proposition 2.7.6. *The underlying category of* $n\mathsf{Cat}$ *is the category of n-categories and n-functors* n-Cat.

Remark 2.7.7. We say that an internal graph $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to m\text{-}\mathsf{Cat}$ satisfies the *n-coincidence property* if, whenever $\mathbb{N} \ni r \leq n$, $\mathfrak{d}(d^1)(\kappa) = \mathfrak{d}(d^0)(\kappa)$ for every r-cell κ of X.

If $F, G : X \to Y$ are m-functors, $n > 1$ and there is an m-icon $\alpha : F \Rightarrow G$, then, in particular, the pair (F, G) defines an internal graph

$$\begin{tikzcd} X \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & Y \end{tikzcd}$$

in mCat (or m-Cat) that satisfies the $(m-2)$-coincidence property. For instance, if there is an icon $\alpha : F \Rightarrow G$ between 2-functors (or pseudofunctors), then the internal graph defined by (F, G) satisfies the 0-coincidence property: this means that $F(\kappa) = G(\kappa)$ for any 0-cell (object) κ of X.

Definition 2.7.8. [Universal n-cell] For each $n \in \mathbb{N}$, we denote by 2_n the n-category with a nontrivial n-cell $\widehat{\kappa}$ with the following universal property: *if κ is an n-cell of an n-category X, then there is a unique n-functor $F : 2_n \to X$ such that $F(\widehat{\kappa}) = \kappa$.*

Remark 2.7.9. We have isomorphisms $2_1 \cong 2$ and $2_0 \cong 1$. Moreover, in general, 2_n is an n-category but we also denote by 2_n the image of this n-category by the inclusion $n\text{-}\mathsf{Cat} \to (n+m)\text{-}\mathsf{Cat}$ for $m \geq 1$. Therefore, for instance, we can consider inclusions $2_n \to 2_{n+m}$ which are $(n+m)$-functors, *i.e.* morphisms of $(n+m)$-Cat.

Of course, 2_n has a unique nontrivial n-cell. This n-cell is *denoted herein by* $\widehat{\kappa}_n$, or just $\widehat{\kappa}$ whenever it does not cause confusion.

Theorem 2.7.10. *Let $F, G : 2_n \to Y$ be $(n+1)$-functors such that $F(\kappa) = G(\kappa)$ for all m-cell κ, provided that $m < n$. There is a one-to-one correspondence between the $(n+1)$-cells $F(\widehat{\kappa}) \Longrightarrow G(\widehat{\kappa})$ of Y and $(n+1)$-icons $F \Rightarrow G$.*

2.8 Higher Computads

Recall that a derivation scheme is a pair $(\mathfrak{d}, \mathfrak{d}_2)$ in which $\mathfrak{d}_2$ is a discrete category and $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is an internal graph with the same format of $\mathfrak{d}$-diagram (described in Section 2.4) satisfying the 0-coincidence property. Roughly, the 2-category freely generated by a derivation scheme is the category $\mathfrak{d}(1)$ freely added with the 2-cells of $\mathfrak{d}_2$ in the following way, for each $\alpha \in \mathfrak{d}_2$, we freely add a 2-cell

$$\alpha : \mathfrak{d}(d^1)(\alpha, 0 \to 1) \Rightarrow \mathfrak{d}(d^0)(\alpha, 0 \to 1).$$

This construction is described in [106]. More precisely, it is constructed a 2-category $\mathscr{F}_{2\text{-}\mathsf{Der}}(\mathfrak{d})$ with the following universal property: a 2-functor $G : \mathscr{F}_{2\text{-}\mathsf{Der}}(\mathfrak{d}) \to X$ is uniquely determined by a pair (G_1, G_2) in which $G_1 : \mathfrak{d}(1) \to X$ is a 2-functor (between categories) and $G_2 : \mathfrak{d}_2 \to 2\text{-}\mathsf{Cat}(2_2, X)$ is a 2-functor (between discrete categories) satisfying the codomain and domain conditions, which means that, given $\alpha \in \mathfrak{d}_2$, the 1-cell domain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^1)(\alpha, 0 \to 1)$ and the codomain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^0)(\alpha, 0 \to 1)$.

Theorem 2.8.1. *There is a functor* $\mathscr{F}_{2\text{-}\mathsf{Der}} : \mathsf{Der} \to 2\text{-}\mathsf{Cat}$ *which gives the* 2*-category freely generated by each derivation scheme. Furthermore, for each derivation scheme* $(\mathfrak{d}, \mathfrak{d}_2)$*,*

$$\mathscr{F}_{2\text{-}\mathsf{Der}}(\mathfrak{d}) \cong \mathscr{I} * \mathfrak{d},$$

in which, by abuse of language, $\mathscr{I} * \mathfrak{d}$ *denotes the coinserter in* $2\mathsf{Cat}$ *of the internal graph* $\mathfrak{d}$ *composed with the inclusion* $\mathsf{Cat} \to 2\mathsf{Cat}$*.*

Proof. An object of the inserter

$$2\mathsf{Cat}(\mathfrak{d}(1), X) \rightrightarrows 2\mathsf{Cat}(\mathfrak{d}_2 \times 2, X).$$

is a 2-functor $G_1 : \mathfrak{d}(1) \to X$ and an icon $G_1\left(\mathfrak{d}(d^1)\right) \Rightarrow G_1\left(\mathfrak{d}(d^0)\right)$ which means a 2-cell $G_2(\alpha)$ for each $\alpha \in \mathfrak{d}_2$ by Theorem 2.7.10 such that the 1-cell domain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^1)(\alpha, 0 \to 1)$ and the codomain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^0)(\alpha, 0 \to 1)$. This proves that the coinserter is determined by the universal properties of the 2-category freely generated by the derivation scheme of $\mathfrak{d}$. □

We already can construct the 2-category freely generated by a computad. This is precisely the 2-category freely generated by its underlying derivation scheme. More precisely, there is an obvious forgetful functor $\mathsf{Cmp} \to \mathsf{Der}$ and the functor $\mathscr{F}_2 : \mathsf{Cmp} \to 2\text{-}\mathsf{Cat}$ is obtained from the composition of such forgetful functor with $\mathscr{F}_{2\text{-}\mathsf{Der}}$.

Definition 2.8.2. $[2_n]$ For each $n \in \mathbb{N}$, of course, there are precisely two inclusions $2_{(n-1)} \to 2_n$. This gives an n-functor

$$\mathscr{I}_n : \mathfrak{G} \to n\text{-}\mathsf{Cat}, \qquad 2_{(n-1)} \rightrightarrows 2_n.$$

Definition 2.8.3. $[\mathfrak{G}_n]$ Consider the usual forgetful functor $(n+1)\text{-}\mathsf{Cat} \to n\text{-}\mathsf{Cat}$. The image of $2_{(n+1)}$ by this forgetful functor is denoted by $\mathfrak{G}_n$.

Lemma 2.8.4. *The internal graph of Definition 2.8.2 induces an n-functor* $2_{(n-1)} \amalg 2_{(n-1)} \to 2_n$*. The pushout in* $n\text{-}\mathsf{Cat}$ *of this n-functor along itself is isomorphic to* $\mathfrak{G}_n$*.*

Furthermore, there is an inclusion n-functor $\mathfrak{G}_{(n-1)} \to 2_n$ *induced by the counit of the adjunction with right adjoint being* $n\text{-}\mathsf{Cat} \to (n-1)\text{-}\mathsf{Cat}$*. The pushout in* $n\text{-}\mathsf{Cat}$ *of this inclusion along itself is isomorphic to* $\mathfrak{G}_n$*.*

Definition 2.8.5. [Higher Derivation Schemes] Consider the functor $(- \times \mathfrak{G}_{n-1}) : \mathsf{SET} \to (n-1)\text{-}\mathsf{Cat}, Y \mapsto Y \times \mathfrak{G}$. The *category of derivation n-schemes* is the comma category $n\text{-}\mathsf{Der} := (- \times \mathfrak{G}_{n-1}/\mathrm{Id}_{\mathsf{Cat}})$.

Remark 2.8.6. Of course, $\mathsf{Der} = 2\text{-}\mathsf{Der}$. Also, it is clear that the derivation n-scheme is just a pair $(\mathfrak{d}, \mathfrak{d}_2)$ in which $\mathfrak{d}_2$ is a discrete category and $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$ is an internal graph

$$\mathfrak{d}_2 \times 2_{(n-1)} \rightrightarrows \mathfrak{d}(1)$$

satisfying the $(n-2)$-coincidence property.

We can define a forgetful functor $\mathscr{C}_{n\text{-}\mathsf{Der}} : n\text{-}\mathsf{Cat} \to n\text{-}\mathsf{Der}$ where $\mathscr{C}_{n\text{-}\mathsf{Der}}(X) : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$ is an internal graph (derivation scheme)

$$n\text{-}\mathsf{Cat}(2_n, X) \times 2_{(n-1)} \rightrightarrows X$$

in which $n\text{-}\mathsf{Cat}(2_n, X)$ denotes the set of n-functors $2_n \to X$ and, by abuse of language, X is the underlying $(n-1)$-category of X. This graph is obtained from the graph $n\text{-}\mathsf{Cat}[\mathscr{I}_n -, X] : \mathfrak{G}^{\mathrm{op}} \to n\text{-}\mathsf{Cat}$: firstly, we compose each nontrivial morphism in the image of this graph with the inclusion $n\text{-}\mathsf{Cat}(2_n, X) \to n\text{-}\mathsf{Cat}[2_n, X]$ (induced by the counit of the adjunction given by the inclusion and underlying set) as follows:

$$\mathsf{Cat}(2_n, X) \longrightarrow n\text{-}\mathsf{Cat}[2_n, X] \rightrightarrows n\text{-}\mathsf{Cat}[2_{(n-1)}, X]$$

and, then, we take the mates:

$$n\text{-}\mathsf{Cat}(2_n, X) \times 2_{(n-1)} \rightrightarrows X. \qquad (\mathscr{C}_{n\text{-}\mathsf{Der}}(X)\text{-diagram})$$

Finally, we compose this internal graph $\mathfrak{G}^{\mathrm{op}} \to n\text{-}\mathsf{Cat}$ with the underlying functor $n\text{-}\mathsf{Cat} \to (n-1)\text{-}\mathsf{Cat}$.

The universal property that defines $\mathscr{F}_{2\text{-}\mathsf{Der}}$ is precisely the universal property of being left adjoint to $\mathscr{C}_{2\text{-}\mathsf{Der}}$, namely a morphism of derivation schemes $G : \mathfrak{d} \to \mathscr{C}_{2\text{-}\mathsf{Der}}(X)$ corresponds to a pair of 2-functors (G_1, G_2) with the universal property described in the proof of Theorem 2.8.1.

Theorem 2.8.7. *There is an adjunction* $\mathscr{F}_{2\text{-}\mathsf{Der}} \dashv \mathscr{C}_{2\text{-}\mathsf{Der}}$. *More generally, there is an adjunction* $\mathscr{F}_{n\text{-}\mathsf{Der}} \dashv \mathscr{C}_{n\text{-}\mathsf{Der}}$ *in which* $\mathscr{F}_{n\text{-}\mathsf{Der}}(\mathfrak{d}) := \mathscr{I} * \mathfrak{d}$ *where, by abuse of language,* $\mathscr{I} * \mathfrak{d}$ *denotes the coinserter in* $n\mathsf{Cat}$ *of the derivation n-scheme* $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$ *with the inclusion* $(n-1)\text{-}\mathsf{Cat} \to n\mathsf{Cat}$.

Proof. Similarly to the proof of Theorem 2.8.1, this result follows from the universal property of the coinserter and from Theorem 2.7.10. ☐

Proposition 2.8.8. *In this proposition, we denote by* $\mathscr{I}_n$ *the functor* $\mathscr{I}_n : \mathfrak{G} \to n\text{-}\mathsf{Cat}$ *composed with the isomorphism* $\mathfrak{G}^{\mathrm{op}} \to \mathfrak{G}$. *In this case,* $\mathscr{I}_n$ *is itself a higher derivation scheme. Then* $\mathscr{F}_{(n+1)\text{-}\mathsf{Der}}(\mathscr{I}_n)$ *is isomorphic to* $2_{(n+1)}$.

Remark 2.8.9. The inclusion $\mathsf{Cmp} \to \mathsf{Der}$ has a right adjoint $(-)_{\mathsf{Cmp}} : \mathsf{Der} \to \mathsf{Cmp}$ such that, given a derivation scheme $\mathfrak{d} : \mathfrak{d}_2 \times \mathfrak{G} \to X$, $(\mathfrak{d})_{\mathsf{Cmp}}$ is the pullback $\mathrm{comp}_X^*(\mathfrak{d})$ in Cat of the morphism $\mathfrak{d}$ along comp_X. It is clear that this adjunction is induced by the adjunction $\mathscr{F}_1 \dashv \mathscr{C}_1$.

Theorem 2.8.10. *There is an adjunction* $\mathscr{F}_2 \dashv \mathscr{C}_2$ *such that* $\mathscr{F}_2 : \mathsf{Cmp}\ 2\text{-}\mathsf{Cat}$ *gives the* 2*-category freely generated by each computad. More precisely, given a computad* $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ *in the format of the* $\mathfrak{d}$*-diagram,* $\mathscr{F}_2(\mathfrak{g})$ *is the coinserter in* $2\mathsf{Cat}$ *of* $\mathfrak{g}$ *composed with the inclusion* $\mathsf{Cat} \to 2\mathsf{Cat}$.

Proof. It is enough to define the adjunction $\mathscr{F}_2 \dashv \mathscr{C}_2$ as the composition of the adjunctions $- \dashv (-)_{\mathsf{Cmp}}$ and $\mathscr{F}_{2\text{-}\mathsf{Der}} \dashv \mathscr{C}_{2\text{-}\mathsf{Der}}$. $\square$

Definition 2.8.11. [n-computads] For each $n \in \mathbb{N}$, consider the functor $(- \times \mathfrak{G}_n) : \mathsf{SET} \to n\text{-}\mathsf{Cat}, Y \mapsto Y \times \mathfrak{G}_n$. The *category of* $(n+1)$*-computads* is defined by the comma category

$$(n+1)\text{-}\mathsf{Cmp} := (- \times \mathfrak{G}_n / \mathscr{F}_n)$$

in which $\mathscr{F}_n$ is the composition of the inclusion n-$\mathsf{Cmp} \to n$-Der with $\mathscr{F}_{n\text{-}\mathsf{Der}}$.

Remark 2.8.12. By Lemma 2.8.4, it is easy to see that an n-computad is just a triple $(\mathfrak{g}, \mathfrak{g}_2, G)$ in which $\mathfrak{g}_2$ is a discrete category, G is a $(n-1)$-computad and $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)$-$\mathsf{Cat}$ is an internal graph

$$\mathfrak{g}_2 \times \mathsf{2}_{(n-1)} \rightrightarrows \mathscr{F}_{(n-1)}(G) \tag{n-computad diagram}$$

satisfying the $(n-2)$-coincidence property. Or, more concisely, by Remark 2.8.6, an n-computad is just a derivation n-scheme $(\mathfrak{g}, \mathfrak{g}_2)$ with a $(n-1)$-computad G such that $\mathfrak{g}(1) = \mathscr{F}_{(n-1)}(G)$.

Theorem 2.8.13 (Freely Generated n-Categories)**.** *For each $n \in \mathbb{N}$, there is a functor $\mathscr{F}_n : n$-$\mathsf{Cmp} \to n$-Cat such that, given an n-computad as in the n-computad diagram, $\mathscr{F}_n(\mathfrak{g})$ is given by the coinserter in $n\mathsf{Cat}$ of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)$-$\mathsf{Cat}$ composed with the inclusion $(n-1)$-$\mathsf{Cat} \to n\mathsf{Cat}$. This functor is left adjoint to a functor $\mathscr{C}_n : n$-$\mathsf{Cat} \to n$-Cmp which gives the underlying n-computad of each n-category.*

Proof. Of course, $\mathscr{F}_n$ coincides with the functor $\mathscr{F}_n$ of Definition 2.8.11. We prove by induction that $\mathscr{F}_n$ is left adjoint. It is clear that $\mathscr{F}_1 \dashv \mathscr{C}_1$. We assume by induction that we have an adjunction $\mathscr{F}_m \dashv \mathscr{C}_m$.

We have that $\mathscr{F}_m \dashv \mathscr{C}_m$ induces an adjunction $(-) \dashv (-)_{(m+1)\text{-}\mathsf{Cmp}}$ in which the left adjoint is the inclusion m-$\mathsf{Cmp} \to m$-Der similarly to what is described in Remark 2.8.9. That is to say, $(\mathfrak{d})_{(m+1)\text{-}\mathsf{Cmp}}$ is the pullback of $\mathfrak{d}$ along the component of the counit of $\mathscr{F}_m \dashv \mathscr{C}_m$ on $\mathfrak{d}(1)$.

Finally, we compose the adjunction $\mathscr{F}_{(m+1)\text{-}\mathsf{Der}} \dashv \mathscr{C}_{(m+1)\text{-}\mathsf{Der}}$ with the adjunction $(-) \dashv (-)_{(m+1)\text{-}\mathsf{Cmp}}$ to get the desired adjunction $\mathscr{F}_{(m+1)} \dashv \mathscr{C}_{(m+1)}$. $\square$

An n-category X is a *free n-category* if there is an n-computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)$-$\mathsf{Cat}$ such that $\mathscr{F}_n(\mathfrak{g}) \cong X$.

Definition 2.8.14. Let $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, G)$ be an n-computad. The objects of $\mathfrak{g}_2$ are called n-cells of $\mathfrak{g}$, while, whenever $n \geq m > 0$, an $(n-m)$-cell of $\mathfrak{g}$ is an $(n-m)$-cell of the $(n-1)$-computad G. In this context, we use the following terminology for graphs in Grph: the 0-cells of a graph are the objects and its 1-cells are the arrows.

Similarly to the 2-dimensional case, we denote an n-cell by $\iota : \alpha \Longrightarrow \alpha'$ if $\mathfrak{g}(d^0)(\alpha, \widehat{\kappa}) = \alpha'$ and $\mathfrak{g}(d^1)(\alpha, \widehat{\kappa}) = \alpha$.

Remark 2.8.15. For each $n \in \mathbb{N}$ such that $n > 1$, there is a forgetful functor $\mathfrak{u}_n : n$-$\mathsf{Cmp} \to (n-1)$-Cmp, $(\mathfrak{g}, \mathfrak{g}_2, G) \mapsto G$. This forgetful functor has a left adjoint $\mathfrak{i}_n : (n-1)$-$\mathsf{Cmp} \to n$-Cmp such that

$$\mathfrak{i}_n(\mathfrak{g}) : \quad \emptyset \times \mathsf{2}_{(n-1)} \rightrightarrows \mathscr{F}_{(n-1)}(\mathfrak{g})$$

and a right adjoint $\sigma_n : (n-1)\text{-}\mathsf{Cmp} \to n\text{-}\mathsf{Cmp}$, defined by $\sigma_n(G) = (G^{\sigma_n}, G_2^{\sigma_n}, G)$ in which there is precisely one n-cell $\iota_{(\alpha,\alpha')} : \alpha \Longrightarrow \alpha'$ for each ordered pair (α, α') with same domain and codomain of $\mathscr{F}_{(n-1)}(G)$. Actually, it should be observed that the description of these functors are similar to those given in Remark 2.4.15.

2.9 Freely Generated 2-Categories

Recall the adjunction $\mathscr{E}_{\mathsf{Cmp}} \dashv \mathscr{R}_{\mathsf{Cmp}}$ in which $\mathscr{E}_{\mathsf{Cmp}} : \mathsf{Cmp} \to \mathsf{RCmp}$ is the inclusion (see Definition 2.4.1). We also can consider the 2-category freely generated by computad over a reflexive graph. More precisely, given a computad $\mathfrak{g}$ of RCmp, $\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ is the coinserter of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to 2\mathsf{Cat}$. It is clear that $\mathscr{F}_2^{\mathscr{R}}$ is left adjoint to a forgetful functor $\mathscr{C}_2^{\mathscr{R}}$. Moreover, $\mathscr{R}_{\mathsf{Cmp}}\mathscr{C}_2^{\mathscr{R}} \cong \mathscr{C}_2$ and $\mathscr{F}_2^{\mathscr{R}}\mathscr{E}_{\mathsf{Cmp}} \cong \mathscr{F}_2$.

In this section, following our approach of the 1-dimensional case, we give some results relating free 2-categories with locally thin categories and locally groupoidal categories. In order to do so, we also consider the (strict) concept of $(2,0)$-category given in Definition 2.9.6 and the $(2,0)$-category freely generated by a computad which provides a way of studying some elementary aspects of free 2-categories. We start by giving some sufficient conditions to conclude that a 2-category is not free.

Remark 2.9.1. [Length [106]] Recall that $\sigma_2 : \mathsf{Grph} \to \mathsf{Cmp}$ is right adjoint and $\bigcirc$ is the terminal graph in Grph. Therefore $\sigma_2(\bigcirc) : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is the terminal computad. If $\mathfrak{g}$ is a computad, the length 2-functor is defined by $\ell^{\mathfrak{g}} := \mathscr{F}_2(\mathfrak{g} \to \sigma_2(\bigcirc))$. It should be noted that $\ell^{\mathfrak{g}}$ reflects identity 2-cells.

The 2-category $\mathscr{F}_2\sigma_2(\bigcirc)$ is described in [106]. The unit of the adjunction $\mathscr{F}_2 \dashv \mathscr{C}_2$ induces a morphism of computads $\sigma_2(\bigcirc) \to \mathscr{C}_2\mathscr{F}_2\sigma_2(\bigcirc)$. The image of the 2-cells of $\sigma_2(\bigcirc)$ are called herein *simple* 2*-cells*. If α is a composition in $\mathscr{F}_2\sigma_2(\bigcirc)$ of a simple 2-cell with (only) 1-cells (identity 2-cells), α is called a *whiskering of a simple* 2*-cell*. It is clear that every 2-cell of $\sigma_2(\bigcirc)$ is given by successive vertical compositions of whiskering of simple 2-cells. It is also easy to see that $\sigma_2(\bigcirc)$ does not have nontrivial invertible 2-cells.

The counit of the adjunction $\mathscr{F}_2 \dashv \mathscr{C}_2$ induces a 2-functor $\mathrm{past}_X : \mathscr{F}_2\mathscr{C}_2(X) \to X$ for each 2-category X, called *pasting*.

Remark 2.9.2. Similarly to the 1-dimensional case, the terminal reflexive computad of Rcmp is the computad with only one 0-cell, the trivial 1-cell and only one 2-cell. That is to say, the computad $\mathfrak{G} \to \mathscr{F}_1^{\mathscr{R}}(\bullet)$ which is the unique functor between $\mathfrak{G}$ and the terminal category $\mathscr{F}_1^{\mathscr{R}}(\bullet)$. If $\mathfrak{h}$ is a subcomputad of $\mathfrak{g}$ in Rcmp, we denote by $\mathfrak{g}/\mathfrak{h}$ the pushout of the inclusion $\mathfrak{h} \to \mathfrak{g}$ along the unique morphism of reflexive computads between $\mathfrak{h}$ and the terminal reflexive computad in Rcmp.

As a particular case of Proposition 2.9.3, if a 2-category X has a nontrivial invertible 2-cell, then X is not a free 2-category. Consequently, any locally thin 2-category that has a nontrivial invertible 2-cell is not a free 2-category.

Proposition 2.9.3. *Let α be an invertible* 2*-cell of a* 2*-category X in* 2-Cat. *If we can write α as pasting of* 2*-cells in which at least one of the* 2*-cells is nontrivial, then X is not free.*

Proof. Let α be a pasting of 2-cells in $\mathscr{F}_2(\mathfrak{g})$. We have that $\ell^{\mathfrak{g}}(\alpha)$ is a pasting of 2-cells of $\mathscr{F}_2\sigma_2(\bigcirc)$ with at least one nontrivial 2-cell. Therefore $\ell^{\mathfrak{g}}(\alpha)$ is not identity and, hence, α is not invertible. □

Recall that there is an adjunction $\mathscr{M}_2 \dashv M_2$ which induces a monad $\overline{\mathscr{M}_2}$, in which $M_2 : \mathsf{Prd\text{-}Cat} \to 2\text{-}\mathsf{Cat}$ is the inclusion.

Corollary 2.9.4. *Let X be a 2-category in $2\text{-}\mathsf{Cat}$. Assume that $\beta : f \Rightarrow g$ is a 2-cell of X such that $f \neq g$. If the pasting of β with another 2-cell is a 2-cell $\alpha : h \Rightarrow h$, then $\overline{\mathscr{M}_2}(X)$ is not a free 2-category.*

Proof. The unit of the monad $\overline{\mathscr{M}_2}$ gives, in particular, a 2-functor $X \to \overline{\mathscr{M}_2}(X)$. Therefore, the image of $\alpha : h \Rightarrow h$ by this 2-functor is also the pasting of a nontrivial 2-cell with other 2-cells, but, since $\overline{\mathscr{M}_2}(X)$ is locally thin, α is the identity. Therefore $\overline{\mathscr{M}_2}(X)$ is not free by Proposition 2.9.3. □

Proposition 2.9.5. *Consider the computad $\mathfrak{g}^{\dot{\Delta}_2} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ defined in Example 2.4.7. The locally thin 2-category $\overline{\mathscr{M}_2\mathscr{F}_2}(\mathfrak{g}^{\dot{\Delta}_2})$ is not a free 2-category. In particular, $\mathscr{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ and $\overline{\mathscr{L}_2\mathscr{F}_2}(\mathfrak{g}^{\dot{\Delta}_2})$ are not locally thin.*

Proof. Since $\mathscr{M}_2\mathscr{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ is locally thin, we conclude that:

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[rd, phantom, "\overset{\vartheta}{\Longleftarrow}"] & 1 \arrow[d, "d^1"] \arrow[rdd, equal, bend left] \arrow[rdd, phantom, "\overset{n_1}{\Longleftarrow}"] & & & 0 \arrow[dd, bend right, "d"'] \arrow[dd, bend left, "d"] \arrow[dd, phantom, "="] \\
1 \arrow[r, "d^0"'] \arrow[rrd, equal, bend right] \arrow[rrd, phantom, "\overset{n_0}{\Longleftarrow}"] & 2 \arrow[rd, "s^0"] & & = & \\
& & 1 & & 1
\end{tikzcd}$$

(identity descent diagram)

Therefore, by Corollary 2.9.4, the proof is complete. □

If a 2-category X is locally groupoidal and free, then every 2-cell of X is identity. Hence, in this case, X is locally discrete, that is to say, it is a free 1-category.

We call $\mathscr{M}_2\mathscr{F}_2(\mathfrak{g})$ the *locally thin 2-category freely generated by* $\mathfrak{g}$. But we often consider such a 2-category as an object of 2-Cat, that is to say, we often consider $\overline{\mathscr{M}_2\mathscr{F}_2}(\mathfrak{g})$.

Definition 2.9.6. [(n,m)-Categories] If $m < n$, an *(n,m)-category* X is an n-category of n-Cat such that, whenever $n \geq r > m$, all r-cells of X are invertible. The full subcategory of n-Cat consisting of the (n,m)-categories is denoted by (n,m)-Cat.

For instance, groupoids are $(1,0)$-categories and locally groupoidal categories are $(2,1)$-categories. The adjunction $\mathscr{L}_1 \dashv \mathscr{U}_1$ also induces an adjunction $\mathscr{L}_{(2,0)} \dashv \mathscr{U}_{(2,0)}$ in which $\mathscr{U}_{(2,0)} : (2,0)\text{-}\mathsf{Cat} \to 2\text{-}\mathsf{Cat}$ is the inclusion. Thereby, given a computad $\mathfrak{g}$ of Cmp, we can consider the *locally groupoidal 2-category $\mathscr{L}_2\mathscr{F}_2(\mathfrak{g})$ freely generated by the computad* $\mathfrak{g}$, as well as the *$(2,0)$-category $\mathscr{L}_{(2,0)}\mathscr{F}_2(\mathfrak{g})$ freely generated by* $\mathfrak{g}$.

Sometimes, we denote $\mathscr{L}_{(2,1)} := \mathscr{L}_2$ and $\mathscr{U}_{(2,1)} := \mathscr{U}_{(2,1)}$.

Remark 2.9.7. Let X be a $(2,0)$-category of $(2,0)$-Cat and assume that Y is a sub-2-category of X. We denote by X/Y the pushout of the inclusion $Y \to X$ along the unique 2-functor between Y and the terminal 2-category. If Y is locally discrete and thin (that is to say, a thin category), then X/Y is isomorphic to X.

Definition 2.9.8. A 2-category X satisfies the $(2,1)$-*cancellation law* if it satisfies the cancellation law w.r.t. the vertical composition of 2-cells (that is to say, it satisfies the cancellation law locally).

A 2-category X satisfies the $(2,0)$-*cancellation law* if it satisfies the $(2,1)$-*cancellation law* and, whenever X has 1-cells f,g and 2-cells α,β such that $\mathrm{id}_f * \alpha * \mathrm{id}_g = \mathrm{id}_f * \beta * \mathrm{id}_g$, $\alpha = \beta$.

It is clear that, if a 2-category X satisfies the $(2,0)$-cancellation law, in particular, the underlying category of X satisfies the cancellation law. Moreover, every $(2,1)$-category satisfies the $(2,1)$-cancellation law and every $(2,0)$-category satisfies the $(2,0)$-cancellation law.

Finally, the components of the units of the adjunctions $\mathscr{L}_2 \dashv \mathscr{U}_2$ and $\mathscr{L}_{(2,0)} \dashv \mathscr{U}_{(2,0)}$ are locally faithful on 2-categories satisfying respectively the $(2,1)$-cancellation law and the $(2,0)$-cancellation law. Thereby:

Theorem 2.9.9. *Let X be a 2-category. If X satisfies the $(2,1)$-cancellation law and $\mathscr{L}_2(X)$ is locally thin, then X is locally thin as well. Analogously, if X satisfies the $(2,0)$-cancellation law and $\mathscr{L}_{(2,0)}(X)$ is locally thin, then X is locally thin as well*

Corollary 2.9.10. *Let $\mathfrak{g}$ be an object of* Cmp. *Consider the following statements:*

(a) $\mathscr{L}_{(2,0)}\mathscr{F}_2(\mathfrak{g})$ *is locally thin;*

(b) $\mathscr{L}_2\mathscr{F}_2(\mathfrak{g})$ *is locally thin;*

(c) $\mathscr{F}_2(\mathfrak{g})$ *is locally thin.*

We have that (a) implies (b) implies (c).

Proof. It is clear that $\mathscr{F}_2(\mathfrak{g})$ and $\mathscr{L}_2\mathscr{F}_2(\mathfrak{g})$ satisfies the $(2,0)$-cancellation law. Therefore we get the result by Theorem 2.9.9. □

Definition 2.9.11. A 2-category X satisfies the *underlying terminal property* or *u.t.p.* if the underlying category of X is the terminal category.

On one hand, by the Eckman-Hilton argument, given any small 2-category X with only one object $*$, the vertical composition of 2-cells $\mathrm{id} \Rightarrow \mathrm{id}$ coincides with the horizontal one and they are commutative. Therefore, in this context, the set of 2-cells $\mathrm{id} \Rightarrow \mathrm{id}$ endowed with the vertical composition is a commutative monoid, denoted by $\Omega^2(X) := X(*,*)(\mathrm{id},\mathrm{id})$.

On the other hand, given a commutative monoid Y, the suspension $\Sigma(Y)$ is naturally a monoidal category (in which the monoidal structure coincides with the composition). This allows us to consider the *double suspension* $\Sigma^2(Y)$ which is a 2-category satisfying *u.t.p.* and the set of 2-cells $\mathrm{id} \Rightarrow \mathrm{id}$ is the underlying set of Y, while the vertical and horizontal compositions of $\Sigma^2(Y)$ are given by the operation of Y. More precisely, there is a fully faithful functor

$$\Sigma^2 : \mathsf{AbGroup} \to (2,1)\text{-}\mathsf{cat}$$

between the category of abelian groups and the category of small locally groupoidal 2-categories which is essentially surjective on the full subcategory of 2-categories satisfying *u.t.p.* such that $\Omega^2\Sigma^2 \cong \mathrm{Id}_{\mathsf{AbGroup}}$.

If $(\mathfrak{g},\mathfrak{g}_2,G)$ is a small 2-computad in which $G = \bullet^{\mathscr{E}}$ is the connected graph without arrows, then $\mathscr{L}_{(2,0)}\mathscr{F}_2(\mathfrak{g},\mathfrak{g}_2,G)$ is isomorphic to the double suspension of a free abelian group.

Theorem 2.9.12. *If* $(\mathfrak{g}, \mathfrak{g}_2, \bullet)$ *is a small reflexive computad, then* $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}) \cong \mathscr{L}_2\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}) \cong \Sigma^2\pi_2(\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g}))$.

Proof. Since $\mathscr{F}_1^{\mathscr{R}}(\bullet)$ is the terminal category, $\Omega^2\left(\mathscr{L}_2\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})\right)$ is the abelian group freely generated by the set $\mathfrak{g}_2$ that is also isomorhic to $\pi_2(\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g}))$.

To complete the proof, it is enough to observe that $\mathscr{L}_{(2,0)}(X) \cong \mathscr{L}_2(X)$ whenever X does not have nontrivial 1-cells. □

We say that a computad $\mathfrak{g}$ is 1-*connected* if $\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g})$ is simply connected. By Corollary 2.6.8, a computad $\mathfrak{g}$ is 1-connected if and only if $\mathscr{L}_1\mathscr{P}_1(\mathfrak{g})$ is connected and thin.

Definition 2.9.13. [$f.c.s.$] Let $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, G)$ be a computad of Rcmp with only one 0-cell and let $\mathfrak{h}$ be a subcomputad of $\mathfrak{g}$.

We call $\mathfrak{g}^{\mathfrak{b}} := \mathfrak{h}$ a *full contractible subcomputad of* $\mathfrak{g}$ or, for short, $f.c.s.$ of $\mathfrak{g}$, if $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\mathfrak{b}})$ has a unique 2-cell $f \Rightarrow \mathrm{id}$ or a 2-cell $\mathrm{id} \Rightarrow f$ for each 1-cell f of $\mathfrak{g}$. In particular, if $\mathfrak{g}^{\mathfrak{b}}$ is an $f.c.s.$ of $\mathfrak{g}$, $\mathfrak{g}^{\mathfrak{b}}$ has every 1-cell of $\mathfrak{g}$.

It should be noted that, if $\mathfrak{g}^{\mathfrak{b}}$ is an $f.c.s.$ of $\mathfrak{g}$, we are already assuming that $\mathfrak{g}$ is an object of Rcmp.

There are small (reflexive) computads with only one 0-cell and no full contractible subcomputad. For instance, consider the computad $\mathfrak{x}$ with two 1-cells f, g and with 2-cells $\alpha : gf \Rightarrow \mathrm{id}$ and $\beta : \mathrm{id} \Rightarrow g$. The number of 2-cells of any subcomputad belongs to $\{0, 1, 2\}$. It is clear that the subcomputads with only one 2-cell are not full contractible subcomputads. It remains to prove that the whole computad is not an $f.c.s.$ of itself. Indeed, the 2-cells $\mathrm{id}_{g^{-1}} * (\beta \cdot \alpha)$ and $\alpha \cdot \left(\beta * \mathrm{id}_f\right)$ below are both 2-cells $f \Rightarrow \mathrm{id}$ of $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{x})$.

$$\begin{tikzcd}
\ast \arrow[r, "f"] \arrow[d, bend right=70, equal, ""{name=L}] & \ast \\
\ast \arrow[u, bend right=40, "g"', ""{name=G}] \arrow[r, "\mathrm{id}"'] & \ast \arrow[u, "\mathrm{id}"'] \arrow[from=L, to=G, phantom, "\overset{\beta}{\Longrightarrow}"] \arrow[from=G, to=2-2, phantom, "\overset{\alpha}{\Longrightarrow}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\ast \arrow[d, bend right=80, "gf"', ""{name=A}] \arrow[d, bend left=80, "g", ""{name=B}] \arrow[d, equal, ""{name=E}] \\
\ast \arrow[d, "g^{-1}"'] \\
\ast
\arrow[from=A, to=E, phantom, "\overset{\alpha}{\Longrightarrow}"] \arrow[from=E, to=B, phantom, "\overset{\beta}{\Longrightarrow}"]
\end{tikzcd}$$

Theorem 2.9.14. *If* $\mathfrak{g}^{\mathfrak{b}}$ *is an* $f.c.s.$, *then the* 2-*categories* $\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\mathfrak{b}})$, $\mathscr{L}_2\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\mathfrak{b}})$ *and* $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\mathfrak{b}})$ *are locally thin.*

Proposition 2.9.15. *If* $\mathfrak{g}^{\mathfrak{b}} = (\mathfrak{g}^{\mathfrak{b}}, \mathfrak{g}_2^{\mathfrak{b}}, G)$ *is an* $f.c.s.$, *then* $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g}^{\mathfrak{b}})$ *is contractible. In particular, it is simply connected and, hence,* $\mathfrak{g}^{\mathfrak{b}}$ *is* 1-*connected.*

Proof. It is enough to see that $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g}^{\mathfrak{b}})$ is a wedge of (closed) balls. □

Theorem 2.9.16. *Assume that* $\mathfrak{g}^{\mathfrak{b}}$ *is an* $f.c.s.$ *of* $(\mathfrak{g}, \mathfrak{g}_2, G)$. *The following statements are equivalent:*

- $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}\left(\mathfrak{g}/\mathfrak{g}^{\mathfrak{b}}\right)$ *is locally thin;*
- $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ *is locally thin;*
- $\mathscr{L}_2\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ *is locally thin;*

– $\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ *is locally thin.*

Proof. $\mathfrak{g}/\mathfrak{g}^{\mathfrak{b}}$ is the computad $(\mathfrak{h}, \mathfrak{h}_2, \bullet)$ in which $\mathfrak{h}_2 = \mathfrak{g}_2 - \mathfrak{g}_2^{\mathfrak{b}}$. Therefore $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}\left(\mathfrak{g}/\mathfrak{g}^{\mathfrak{b}}\right)$ is locally thin if and only if $\mathfrak{g}_2 = \mathfrak{g}_2^{\mathfrak{b}}$, which means that $\mathfrak{g} = \mathfrak{g}^{\mathfrak{b}}$. Since $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\mathfrak{b}})$ is locally thin, the proof is complete. □

Let $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, G)$ be a small connected computad of Rcmp. Assume that G_{mtree} is a maximal tree of G. We have that the computad

$$\begin{tikzcd} \mathfrak{g}_2 \times 2 \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}_1^{\mathscr{R}}(G) \arrow[r] & \mathscr{F}_1^{\mathscr{R}}(G/G_{\text{mtree}}) \end{tikzcd}$$

obtained from the composition of the morphisms in the image of $\mathfrak{g}$ with the natural morphism $\mathscr{F}_1^{\mathscr{R}}(G) \to \mathscr{F}_1^{\mathscr{R}}(G/G_{\text{mtree}})$ is the pushout of the mate of the inclusion $G_{\text{mtree}} \to \mathfrak{u}_2^{\mathscr{R}}(\mathfrak{g})$ under the adjunction $\mathfrak{i}_2^{\mathscr{R}} \dashv \mathfrak{u}_2^{\mathscr{R}}$ along the unique functor between $\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}})$ and the terminal reflexive computad. That is to say, it is the quotient $\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}})$.

Definition 2.9.17. [$f.c.s.$ triple] We say that $(\mathfrak{g}, G_{\text{mtree}}, \mathfrak{h}^{\mathfrak{b}})$ is an *$f.c.s.$ triple* if $\mathfrak{g}$ is a small connected reflexive computad, G_{mtree} is a maximal tree of the underlying graph of $\mathfrak{g}$ and $\mathfrak{h}^{\mathfrak{b}}$ is an $f.c.s.$ of $\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}})$. In this case, we denote by $\check{\mathfrak{h}}^{\mathfrak{b}}$ the reflexive computad

$$\left(\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}})\right)/\mathfrak{h}^{\mathfrak{b}}.$$

Corollary 2.9.18. *Let $(\mathfrak{g}, G_{\text{mtree}}, \mathfrak{h}^{\mathfrak{b}})$ be an $f.c.s.$ triple. The $(2,0)$-category $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ is locally thin if and only if $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\check{\mathfrak{h}}^{\mathfrak{b}})$ is locally thin.*

Proof. By Remark 2.9.7, $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}}))$ is locally thin if and only if $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ is locally thin. By Theorem 2.9.16, the former is locally thin if and only if $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\check{\mathfrak{h}}^{\mathfrak{b}})$ is locally thin. □

As a consequence of Corollary 2.9.18 and Theorem 2.9.12, we get:

Corollary 2.9.19. *Let $(\mathfrak{g}, G_{\text{mtree}}, \mathfrak{h}^{\mathfrak{b}})$ be an $f.c.s.$ triple. The $(2,0)$-category $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ is locally thin if and only if $\pi_2\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g})$ is trivial.*

Proof. By Theorem 2.9.12, $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\check{\mathfrak{h}}^{\mathfrak{b}})$ is isomorphic to $\Sigma^2\pi_2\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\check{\mathfrak{h}}^{\mathfrak{b}})$. Therefore, by Corollary 2.9.18 we conclude that $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g})$ is locally thin if and only if $\Sigma^2\pi_2\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\check{\mathfrak{h}}^{\mathfrak{b}})$ is trivial.

To complete the proof, it remains to prove that $\pi_2\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\check{\mathfrak{h}}^{\mathfrak{b}}) \cong \pi_2\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{h})$. Indeed, since $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}$ preserves colimits and the terminal reflexive computad, we get that

$$\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}})) \cong \mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g})/\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}})$$

and, since $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}}) \to \mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g})$ is a cofibration which is an inclusion of a contractible space, we conclude that $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}}))$ has the same homotopy type of $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g})$. Analogously, we conclude that $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\check{\mathfrak{h}}^{\mathfrak{b}})$ has the same homotopy type of $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}}))$, since $\mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{h}^{\mathfrak{b}}) \to \mathscr{F}_{\mathsf{Top}_2}^{\mathscr{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathscr{R}}(G_{\text{mtree}}))$ is a cofibration which is an inclusion of a contractible space. □

Remark 2.9.20. The study of possible higher dimensional analogues of the isomorphisms given in Remark 2.5.10 and in Theorem 2.5.11 would depend on the study of notions of higher fundamental groupoids, higher homotopy groupoids and higher van Kampen theorems [12, 13, 38]. This is outside of the scope of this paper.

2.10 Presentations of 2-categories

As 2-computads give presentations of categories with equations between 1-cells, $(n+1)$-computads give presentations of n-categories with equations between n-cells. Contrarily to the case of presentations of categories via computads, it is clear that, for $n > 1$, there are n-categories that do not admit presentations via $(n+1)$-computads.

Definition 2.10.1. [Presentation of n-categories via $(n+1)$-computads] Given $n \in \mathbb{N}$, an $(n+1)$-computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to n\text{-}\mathsf{Cat}$ as the n-computad diagram of 2.8.12 *presents the n-category X* if the coequalizer of $\mathfrak{g}$ in n-Cat is isomorphic to X. There is a functor $\mathscr{P}_n : (n+1)\text{-}\mathsf{Cmp} \to n\text{-}\mathsf{Cat}$ which, for each $(n+1)$-computad $\mathfrak{g}$, gives the category $\mathscr{P}_n(\mathfrak{g})$ presented by $\mathfrak{g}$.

The underlying $(n-1)$-category of every n-category that admits a presentation via a $(n+1)$-computad is a free $(n-1)$-category. Thereby:

Proposition 2.10.2. *Let X be an n-category in n-*Cat*. If the underlying $(n-1)$-category of X is not free, then X does not admit a presentation via an $(n+1)$-computad.*

In this section, as the title suggests, our scope is restricted to presentations of 2-categories. Similarly to the 1-dimensional case, we are mainly interested on presentations of locally thin 2-categories, $(2,1)$-categories or $(2,0)$-categories.

We consider (reflexive) small $(2,0)$-categorical and $(2,1)$-categorical (reflexive) small 3-computads which are 3-dimensional analogues of groupoidal computads, called respectively $(3,0,\mathscr{R})$-computads and $(3,1,\mathscr{R})$-computads. More precisely, for each $m \in \{0,1\}$, we define the category of $(3,m,\mathscr{R})$-computads by the comma category $(3,2,m)\text{-}\mathsf{Rcmp} := (- \times \mathscr{L}_{(2,m)}(\mathfrak{G}_2)/\mathscr{L}_{(2,m)}\mathscr{F}_2^{\mathscr{R}})$ in which

$$(- \times \mathscr{L}_{(2,m)}(\mathfrak{G}_2)) : \mathsf{Set} \to (2,m)\text{-}\mathsf{cat}, \quad Y \mapsto Y \times \mathscr{L}_{(2,m)}(\mathfrak{G}_2).$$

Whenever $2 > m \geq 0$, we have a functor $\mathscr{P}^{\mathscr{R}}_{(2,m)} : (3,2,m)\text{-}\mathsf{Rcmp} \to (2,m)\text{-}\mathsf{cat}$ that gives the $(2,m)$-category presented by each $(3,2,m,\mathscr{R})$-computad. More precisely, for each $2 > m \geq 0$, a $(3,2,m,\mathscr{R})$-computad is a functor $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (2,m)\text{-}\mathsf{cat}$

$$\mathfrak{g}_2 \times \mathscr{L}_{(2,m)}(2_2) \rightrightarrows \mathscr{L}_{(2,m)}\mathscr{F}_2^{\mathscr{R}}(G) \tag{$(3,2,m,\mathscr{R})$-computad diagram}$$

and $\mathscr{P}^{\mathscr{R}}_{(2,m)}(\mathfrak{g})$ is the coequalizer of $\mathfrak{g}$ in $(2,m)$-cat. For short, by abuse of language, by i_3 the functors $\mathsf{Rcmp} \to (3,2,m)\text{-}\mathsf{Rcmp}$ induced by i_3.

Theorem 2.10.3. *Assume that $G^{\mathfrak{b}}$ is an f.c.s. of the small reflexive 2-computad G. If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small $(3,2,0,\mathscr{R})$-computad, then the following statements are equivalent:*

– $\mathscr{P}^{\mathscr{R}}_{(2,0)}\left(\mathfrak{g}/\mathrm{i}_3(G^{\mathfrak{b}})\right)$ *is locally thin;*

– $\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{g})$ *is locally thin.*

Proof. We have that $\mathscr{P}^{\mathscr{R}}_{(2,0)}\mathrm{i}_3(G^\flat) \cong \mathscr{L}_{(2,0)}\mathscr{F}_2(G^\flat)$ is locally thin. Therefore

$$\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{g}) \text{ and } \mathscr{P}^{\mathscr{R}}_{(2,0)}\left(\mathfrak{g}/\mathrm{i}_3(G^\flat)\right) \cong \mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{g})\,/\,\mathscr{P}^{\mathscr{R}}_{(2,0)}\left(\mathrm{i}_3(G^\flat)\right)$$

are biequivalent. Thereby the result follows. □

In the setting of the result above, since we are assuming that the 2-computad G has only one 0-cell, we get that there is a $(3,2,0,\mathscr{R})$-computad $(\mathfrak{g},\mathfrak{g}_2,G)$ such that $|\mathfrak{g}_2|$ is precisely the number of 2-cells of $G/G^\flat$ and $\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{g})$ is locally thin.

Theorem 2.10.4. *Assume that* $G^\flat$ *is an f.c.s. of a* 2*-computad* G *in* Rcmp*. There is a* $(3,2,0,\mathscr{R})$*-computad* $(\mathfrak{g},\mathfrak{g}_2,G)$ *such that* $\mathfrak{g}_2 = G_2 - G_2^\flat$ *and* $\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{g})$ *is locally thin. In other words,* $\mathfrak{g}$ *presents the locally thin* $(2,0)$*-category* $\overline{\mathscr{M}_2\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}(G)$ *freely generated by* G*.*

Proof. Recall, by Theorem 2.9.16, that we can consider that $(G/G^\flat)_2 = G_2 - G_2^\flat$. Also, by hypothesis, for each nontrivial 1-cell f of G, $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(G^\flat)$ has a unique 2-cell $\beta_f : f \Rightarrow \mathrm{id}$ or $\beta_f : \mathrm{id} \Rightarrow f$.

We define the $(3,2,0,\mathscr{R})$-computad $(\mathfrak{g},\mathfrak{g}_2,G)$

$$\begin{tikzcd} \mathfrak{g}_2 \times \mathscr{L}_{(2,0)}(2_2) \arrow[r, shift left] \arrow[r, shift right] & \mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(G). \end{tikzcd}$$

For each $\alpha \in \mathfrak{g}_2 = G_2 - G_2^\flat$, we put $\mathfrak{g}(d^1)(\alpha,\widehat{\kappa}) := \alpha : f \Rightarrow g$ and $\mathfrak{g}(d^0)(\alpha,\widehat{\kappa}) := \dot{\alpha}$ in which $\dot{\alpha}$ is the composition of (possibly the inverse) of β_f and (possibly the inverse) of β_g in $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(G)$, that is to say, in other words, $\dot{\alpha}$ is the unique 2-cell with same domain and codomain of α in $\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(G^\flat)$.

It is clear that $\mathscr{P}^{\mathscr{R}}_{(2,0)}\left(\mathfrak{g}/\mathrm{i}_3(G^\flat)\right)$ is locally thin. Therefore the result follows from Theorem 2.10.3. □

Corollary 2.10.5. *Let* $(G,\mathtt{T},H^\flat)$ *be an f.c.s. triple. There is a* $(3,2,0,\mathscr{R})$*-computad* $(\mathfrak{h},\mathfrak{h}_2,G)$ *such that* $\mathfrak{h}_2 = \check{H}^{\flat}{}_2 = G_2 - H_2^\flat = (G/\mathtt{T})_2 - H_2^\flat$ *and* $\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{h})$ *is locally thin.*

Proof. We denote $G/\mathrm{i}_2^{\mathscr{R}}(\mathtt{T})$ by H. Consider the $(3,2,0,\mathscr{R})$-computad $\mathfrak{g} = (\mathfrak{g},\mathfrak{g}_2,H)$ as constructed in Theorem 2.10.4. Since each 2-cell of H corresponds to a unique 2-cell of G, we can lift $\mathfrak{g}$ to a $(3,2,0,\mathscr{R})$-computad $(\mathfrak{g},\mathfrak{g}_2,G)$. We get this lifting $(\mathfrak{g},\mathfrak{g}_2,G)$

$$\begin{tikzcd} \mathfrak{g}_2 \times \mathscr{L}_{(2,0)}(2_2) \arrow[r, shift left] \arrow[r, shift right] & \mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(H) \arrow[r] & \mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(G) \end{tikzcd}$$

after composing each morphism in the image of $\mathfrak{g}$ with $\overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}(H) \simeq \overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}(G)$. Moreover, since

$$\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{g}) \cong \mathscr{P}^{\mathscr{R}}_{(2,0)}\left(\mathfrak{h}/\mathrm{i}_3\mathrm{i}_2^{\mathscr{R}}(\mathtt{T})\right) \cong \mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{h})\,/\,\mathscr{P}^{\mathscr{R}}_{(2,0)}\left(\mathrm{i}_3\mathrm{i}_2^{\mathscr{R}}(\mathtt{T})\right)$$

is locally thin, the result follows from Remark 2.9.7. □

Analogously to Definition 2.6.21, we have:

Definition 2.10.6. [Lifting of 3-Computads] We denote by $(3,2,1)$-$\mathsf{Rcmp}_{\mathsf{lift}}$ the pseudopullback (iso-comma category) of $\mathscr{P}^{\mathscr{R}}_{(2,0)}$ along $\mathscr{L}_{(2,0)}\mathscr{U}_{(2,1)}\mathscr{P}^{\mathscr{R}}_{(2,1)}$. A $(3,2,1,\mathscr{R})$-computad $\mathfrak{g}$ is called a *lifting of* *the* $(3,2,0,\mathscr{R})$*-computad* $\mathfrak{g}'$ if there is an object $\zeta^{\mathfrak{g}}_{\mathfrak{g}'}$ of $(3,2,1)$-$\mathsf{Rcmp}_{\mathsf{lift}}$ such that the images of this object by the functors

$$(3,2,1)\text{-}\mathsf{Rcmp}_{\mathsf{lift}} \to (3,2,0)\text{-}\mathsf{Rcmp}, \qquad (3,2,1)\text{-}\mathsf{Rcmp}_{\mathsf{lift}} \to (3,2,1)\text{-}\mathsf{Rcmp}$$

are respectively $\mathfrak{g}'$ and $\mathfrak{g}$. Analogously, we say that a (reflexive) 3-computad $\mathfrak{h}$ is a lifting of a $(3,2,m,\mathscr{R})$-computad $\mathfrak{h}'$ if $\mathscr{L}_{(2,m)}\mathscr{U}_{(2,m)}\mathscr{P}^{\mathscr{R}}_{(2,m)}(\mathfrak{h}') \cong \mathscr{P}^{\mathscr{R}}_{2}(\mathfrak{h})$.

Proposition 2.10.7. *If a* $(3,2,1,\mathscr{R})$*-computad* $\mathfrak{g}$ *is a lifting of a* $(3,2,0,\mathscr{R})$*-computad* $\mathfrak{g}'$ *such that* $\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{g}')$ *is locally thin, then* $\mathscr{P}^{\mathscr{R}}_{(2,1)}(\mathfrak{g})$ *is locally thin provided that* $\mathscr{P}^{\mathscr{R}}_{(2,1)}(\mathfrak{g})$ *satisfies the* $(2,0)$*-cancellation law.*

Analogously, if a 3*-computad* $\mathfrak{h}$ *is a lifting of a* $(3,2,m,\mathscr{R})$*-computad* $\mathfrak{h}'$ *and* $\mathscr{P}^{\mathscr{R}}_{(2,m)}(\mathfrak{h}')$ *is locally thin, then* $\mathscr{P}^{\mathscr{R}}_{2}(\mathfrak{h})$ *is locally thin provided that* $\mathscr{P}^{\mathscr{R}}_{2}(\mathfrak{g})$ *satisfies the* $(2,m)$*-cancellation law.*

Proof. By hypothesis, $\mathscr{P}^{\mathscr{R}}_{(2,1)}(\mathfrak{g}) \cong \mathscr{L}_{(2,0)}\mathscr{U}_{(2,1)}\mathscr{P}^{\mathscr{R}}_{(2,1)}(\mathfrak{g})$ and $\mathscr{U}_{(2,1)}\mathscr{P}^{\mathscr{R}}_{(2,1)}$ satisfies the $(2,0)$-cancellation law. Therefore $\mathscr{P}^{\mathscr{R}}_{(2,1)}(\mathfrak{g})$ is locally thin. □

2.10.8 The bicategorical replacement of the truncated category of ordinals

In [77–79], we consider 2-dimensional versions of the subcategory $\dot{\Delta}'_3$ of $\dot{\Delta}_3$. For instance, the bicategorical replacement of the category $\dot{\Delta}'_3$. Here, we study the presentations of this locally thin $(2,1)$-category, including the application of our results to the presentation of the bicategorical replacement of $\dot{\Delta}_2$. Following the terminology of [79] (which is Chapter 3), we have:

Definition 2.10.9. The 2-computad $\dot{\mathfrak{d}}_{\mathrm{Str}} = (\mathfrak{g}^{\dot{\Delta}_3}, \mathfrak{g}_2^{\dot{\Delta}_3}, G_{\dot{\Delta}_3})$ is defined by the graph

$$\begin{tikzcd}
\mathsf{0} \arrow[r, "d"] & \mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description] \arrow[r, shift left=2, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=2, "\partial^2"'] & \mathsf{3}
\end{tikzcd}$$

with the 2-cells:

$$\begin{array}{rlcrl}
\sigma_{01}: & \partial^1 d^0 \Rightarrow \partial^0 d^0 & \qquad & n_0: & s^0 d^0 \Rightarrow \mathrm{id}_1 \\
\sigma_{02}: & \partial^2 d^0 \Rightarrow \partial^0 d^1 & & n_1: & \mathrm{id}_1 \Rightarrow s^0 d^1 \\
\sigma_{12}: & \partial^2 d^1 \Rightarrow \partial^1 d^1 & & \vartheta: & d^1 d \Rightarrow d^0 d
\end{array}$$

We denote by $\dot{\Delta}_{\mathrm{Str}}$ the locally thin $(2,1)$-category $\overline{\mathscr{M}_2}\,\overline{\mathscr{L}_2}\mathscr{F}_2(\dot{\mathfrak{d}}_{\mathrm{Str}})$ freely generated by the 2-computad $\dot{\mathfrak{d}}_{\mathrm{Str}}$. We also define the subcomputad $\mathfrak{d}_{\mathrm{Str}}$ of $\dot{\mathfrak{d}}_{\mathrm{Str}}$ such that $\Delta_{\mathrm{Str}} = \overline{\mathscr{M}_2}\,\overline{\mathscr{L}_2}\mathscr{F}_2(\mathfrak{d}_{\mathrm{Str}})$ is the full sub-2-category of $\dot{\Delta}_{\mathrm{Str}}$ and $\mathrm{obj}(\Delta_{\mathrm{Str}}) = \{\mathsf{1},\mathsf{2},\mathsf{3}\}$.

Lemma 2.10.10. *Let* $\mathfrak{g}^{\Delta_2} = (\mathfrak{g}^{\Delta_2}, \mathfrak{g}_2^{\Delta_2}, G_{\Delta_2})$ *be the full subcomputad of* $\dot{\mathfrak{d}}_{\mathrm{Str}}$ *defined by*

$$\begin{tikzcd}
\mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0" description]
\end{tikzcd}$$

with the 2*-cells:* $n_0 : s^0 d^0 \Rightarrow \mathrm{id}_1$, $n_1 : \mathrm{id}_1 \Rightarrow s^0 d^1$. *The* $(2,0)$*-category freely generated by* $\mathfrak{g}^{\Delta_2}$ *is locally thin. In particular, the full sub-*2*-category* $\Delta_{\mathrm{Str}_2} := \overline{\mathscr{M}_2}\,\overline{\mathscr{L}_2}\mathscr{F}_2\left(\mathfrak{g}^{\Delta_2}\right)$ *of the* 2*-category* $\dot{\Delta}_{\mathrm{Str}}$ *is isomorphic to* $\overline{\mathscr{L}_2}\mathscr{F}_2\left(\mathfrak{g}^{\Delta_2}\right)$.

Proof. We should prove that $\overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2(\mathfrak{g}^{\Delta_2})$ is locally thin. By abuse of language, we denote by $\mathscr{E}_{\mathsf{cmp}}(\mathfrak{g}^{\Delta_2})$ the 2-computad $\mathfrak{g}^{\Delta_2}$. We, then, take the maximal tree of the underlying graph of $\mathfrak{g}^{\Delta_2}$ defined by $2 \xrightarrow{s^0} 1$ and denote it by G_{s^0}.

By Remark 2.9.7, $\overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}\left(\mathfrak{g}^{\Delta_2}/\mathrm{i}_2^{\mathscr{R}}(G_{s^0})\right)$ is locally thin if and only if $\overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}\left(\mathfrak{g}^{\Delta_2}\right)$ is locally thin. The quotient $\mathfrak{g}^{\Delta_2}/\mathrm{i}_2^{\mathscr{R}}(G_{s^0})$ is a computad with 1-cells $\widetilde{d^0}$, $\widetilde{d^1}$ and 2-cells $\widetilde{n_0} : \widetilde{d^0} \Rightarrow \mathrm{id}$ and $\widetilde{n_1} : \mathrm{id} \Rightarrow \widetilde{d^1}$. It is clear, then, that $\mathfrak{g}^{\Delta_2}/\mathrm{i}_2^{\mathscr{R}}(G_{s^0})$ is an $f.c.s$ of itself. Thereby the proof is complete. □

Furthermore, the full sub-2-category Δ_{Str} of $\dot{\Delta}_{\mathrm{Str}}$ is a free $(2,1)$-category as proved in:

Theorem 2.10.11 (Δ_{Str}). *There is an isomorphism of* 2*-categories* $\Delta_{\mathrm{Str}} \cong \overline{\mathscr{L}_2}\mathscr{F}_2(\mathfrak{d}_{\mathrm{Str}})$.

Proof. Since Δ_{Str_2} is a full sub-2-category and locally thin, it is enough to prove that $\Delta_{\mathrm{Str}}(1,3)$ and $\Delta_{\mathrm{Str}}(2,3)$ are thin.

It is clear that the nontrivial 2-cells of $\Delta_{\mathrm{Str}}(1,3)$ are horizontal compositions of 2-cells of $\Delta_{\mathrm{Str}}(1,1)$ with σ_{01}, σ_{02} and σ_{12}. More precisely, the set of nontrivial 2-cells of $\Delta_{\mathrm{Str}}(1,3)$ is equal to

$$\{\sigma_{01} * \alpha, \sigma_{02} * \alpha, \sigma_{12} * \alpha \,|\, (\alpha : f \Rightarrow g : 1 \to 1) \in \Delta_{\mathrm{Str}}(1,1)\}.$$

This proves that $\Delta_{\mathrm{Str}}(1,3)$ is thin. Moreover, since the set of 2-cells of $\Delta_{\mathrm{Str}}(2,1) = \Delta_{\mathrm{Str}_2}(2,1)$ is equal to $\left\{\alpha * \mathrm{id}_{s^0} \,|\, (\alpha : f \Rightarrow g : 1 \to 1) \in \Delta_{\mathrm{Str}}(1,1)\right\}$, it follows that the set of 2-cells of $\Delta_{\mathrm{Str}}(2,3)$ is equal to $\left\{\beta * \mathrm{id}_{s^0} \,|\, (\beta : f \Rightarrow g : 1 \to 3) \in \Delta_{\mathrm{Str}}(1,3)\right\}$. Since we already proved that $\Delta_{\mathrm{Str}}(1,3)$ is thin, we conclude that $\Delta_{\mathrm{Str}}(2,3)$ is thin. Hence, as $\Delta_{\mathrm{Str}}(3,2)$ is the initial (empty) category, the proof is complete. □

As proved in Proposition 2.9.5, $\overline{\mathscr{L}_2}\mathscr{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ is not locally thin. We prove below that $\dot{\Delta}_{\mathrm{Str}_2} := \overline{\mathscr{M}_2}\,\overline{\mathscr{L}_2}\mathscr{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ can be presented by a 3-computad with only one 3-cell that corresponds to the equation given in the identity descent diagram.

Theorem 2.10.12 ($\dot{\Delta}_{\mathrm{Str}_2}$). *The* 3*-computad* $\mathfrak{h}^{\dot{\Delta}_2}$ *defined by the* 2*-computad* $\mathfrak{g}^{\dot{\Delta}_2}$ *with only the* 3*-cell*

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[rd, phantom, "\overset{\vartheta}{\Longleftarrow}"] & 1 \arrow[d, "d^1"] \arrow[rdd, bend left=40, equal] \arrow[rdd, phantom, "\overset{n_1}{\Longleftarrow}"] & & & 0 \arrow[dd, bend right=30, "d"'] \arrow[dd, bend left=30, "d"] \arrow[dd, phantom, "="] \\
1 \arrow[r, "d^0"'] \arrow[rd, bend right=30, equal] \arrow[rd, phantom, "\overset{n_0}{\Longleftarrow}"] & 2 \arrow[rd, "s^0"] & \Longrightarrow & & \\
& & 1 & & 1
\end{tikzcd}$$

(identity descent 3-cell)

presents the locally thin $(2,1)$*-category* $\dot{\Delta}_{\mathrm{Str}_2}$. *In other words,* $\overline{\mathscr{L}_2}\,\mathscr{P}_2(\mathfrak{h}^{\dot{\Delta}_2}) \cong \dot{\Delta}_{\mathrm{Str}_2}$.

Proof. By abuse of language, we denote $\mathscr{E}_{\mathsf{cmp}}(\mathfrak{g}^{\dot{\Delta}_2})$ by $\mathfrak{g}^{\dot{\Delta}_2}$. We denote by $\mathtt{T}$ the maximal tree

$$\begin{tikzcd}
0 \arrow[r, "d"] & 1 & 2 \arrow[l, "s^0"']
\end{tikzcd}$$

of the underlying graph of $\mathfrak{g}^{\dot{\Delta}_2}$.

The (reflexive) 2-computad $\mathfrak{g}^{\dot{\Delta}_2}/\mathrm{i}_2^{\mathscr{R}}(\mathtt{T})$ is defined by the 1-cells $\widetilde{d^0}$, $\widetilde{d^1}$ and 2-cells $\widetilde{\vartheta} : \widetilde{d^1} \Rightarrow \widetilde{d^0}$, $\widetilde{n_0} : \widetilde{d^0} \Rightarrow \mathrm{id}$ and $\widetilde{n_1} : \mathrm{id} \Rightarrow \widetilde{d^1}$, while the 2-computad $\mathfrak{g}^{\dot{\Delta}_2}_{fcs} := \mathfrak{g}^{\dot{\Delta}_2}/\mathrm{i}_2^{\mathscr{R}}(G_{s^0})$, defined in the proof of Lemma 2.10.10, is an $f.c.s.$ of $\mathfrak{g}^{\dot{\Delta}_2}/\mathrm{i}_2^{\mathscr{R}}(\mathtt{T})$.

By the proof of Theorem 2.10.4, we get a presentation of $\overline{\mathscr{M}_2}\,\overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\dot{\Delta}_2}/\mathrm{i}_2^{\mathscr{R}}(\mathtt{T}))$ by a $(3,2,0,\mathscr{R})$-computad $\mathfrak{j}'$ such that $\mathfrak{j}'_2 = \mathfrak{g}_2^{\dot{\Delta}_2} - \mathfrak{g}^{\dot{\Delta}_2}_{fcs_2}$. This $(3,2,0,\mathscr{R})$-computad is defined by the 2-computad $\mathfrak{g}^{\dot{\Delta}_2}/\mathrm{i}_2^{\mathscr{R}}(\mathtt{T})$ with the 3-cell $\widetilde{\vartheta} \Longrightarrow \widetilde{n_0}^{-1} \cdot \widetilde{n_1}^{-1}$.

Thereby $\mathscr{U}_{(2,0)}\mathscr{P}^{\mathscr{R}}_{(2,0)}(\mathfrak{h}') \cong \overline{\mathscr{M}_2}\,\overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\dot{\Delta}_2}/\mathrm{i}_2^{\mathscr{R}}(\mathtt{T}))$. Furthermore, by Corollary 2.10.5, composing each morphism in the image of $\mathfrak{j}'$ with the equivalence

$$\mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\dot{\Delta}_2}) \cong \mathscr{L}_{(2,0)}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\dot{\Delta}_2}/\mathrm{i}_2^{\mathscr{R}}(\mathtt{T})),$$

we get a $(3,2,0,\mathscr{R})$-computad $\mathfrak{j}$ which presents $\overline{\mathscr{M}_2}\,\overline{\mathscr{L}_{(2,0)}}\mathscr{F}_2^{\mathscr{R}}(\mathfrak{g}^{\dot{\Delta}_2})$. This $(3,2,0,\mathscr{R})$-computad $\mathfrak{j}$ is defined by the 2-computad $\mathfrak{g}^{\dot{\Delta}_2}$ with the 3-cell

$$\mathrm{id}_{s^0} * \vartheta \Longrightarrow \left(n_0^{-1} \cdot n_1^{-1}\right) * \mathrm{id}_{d}.$$

It is clear that the (reflexive) computad $\mathfrak{g}^{\dot{\Delta}_2}$ together with the identity descent 3-cell define a (reflexive) 3-computad $\mathfrak{h}'$ which is a lifting of $\mathfrak{j}$. Since $\mathscr{L}_2\mathscr{P}_2^{\mathscr{R}}(\mathfrak{h})$ clearly satisfies the $(2,0)$-cancellation law, this completes the proof. $\square$

Theorem 2.10.13 ($\dot{\Delta}_{\mathrm{Str}}$). *The* 3*-computad* $\mathfrak{h}^{\dot{\Delta}}$ *defined by the* 2*-computad* $\mathfrak{g}^{\dot{\Delta}_2}$ *with the* 3*-cell identity descent* 3*-cell and the* 3*-cell below*

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Rightarrow}"] & 1 \arrow[r, "d^0"] \arrow[d, "d^0"] \arrow[dr, phantom, "\overset{\sigma_{01}}{\Rightarrow}"] & 2 \arrow[d, "\partial^0"] & & 3 & 2 \arrow[l, "\partial^0"'] \arrow[r, equal] \arrow[ddr, phantom, "\overset{\vartheta}{\Rightarrow}"] & 2 \\
1 \arrow[r, "d^1"] \arrow[d, "d^1"'] \arrow[drr, phantom, "\overset{\sigma_{12}}{\Rightarrow}"] & 2 \arrow[r, "\partial^1"] & 3 \arrow[d, "\mathrm{id}_3"] & \Longrightarrow & 2 \arrow[u, "\partial^2"] & 1 \arrow[l, "d^0"'] \arrow[u, "d^1"'] \arrow[ul, phantom, "\overset{\sigma_{02}}{\Rightarrow}"] & \\
2 \arrow[rr, "\partial^2"'] & & 3 & & 1 \arrow[u, "d^1"] & 0 \arrow[l, "d"] \arrow[r, "d"'] \arrow[u, "d"'] \arrow[ul, phantom, "\overset{\vartheta}{\Rightarrow}"] & 1 \arrow[uu, "d^0"']
\end{tikzcd}$$

(associativity descent 3-cell)

presents the locally thin $(2,1)$*-category* $\dot{\Delta}_{\mathrm{Str}}$. *That is to say,* $\overline{\mathscr{L}_2}\,\mathscr{P}_2(\mathfrak{h}^{\dot{\Delta}}) \cong \dot{\Delta}_{\mathrm{Str}}$.

Proof. Recall that $\dot{\Delta}_{\mathrm{Str}_2} \to \dot{\Delta}_{\mathrm{Str}}$ is a full inclusion of a locally thin 2-category and $\dot{\Delta}_{\mathrm{Str}}(3,\mathsf{n})$ is thin for any object n of $\dot{\Delta}_{\mathrm{Str}}$. Hence it only remains to prove that $\dot{\Delta}_{\mathrm{Str}}(0,3)$ is thin.

Since the set of 2-cells of $\dot{\Delta}_{\mathrm{Str}}(0,2)$ is given by

$$\{\vartheta\} \cup \left\{\mathrm{id}_{d^i} * \alpha | i \in \{0,1\} \text{ and } (\alpha : f \Rightarrow g : 0 \to 1) \in \dot{\Delta}_{\mathrm{Str}}(0,1)\right\},$$

we conclude that $\dot{\Delta}_{\mathrm{Str}}(0,3)$ is the thin groupoid freely generated by the graph $\mathbb{S}$ defined by the morphisms $0 \to 3$ as objects and the set of arrows (2-cells) $\mathbb{T} \cup \mathbb{T}'$ in which

$$\mathbb{T}' := \left\{\sigma_{ij} * \alpha | i,j \in \{0,1\}, i < j \text{ and } (\alpha : f \Rightarrow g : 0 \to 1) \in \dot{\Delta}_{\mathrm{Str}}(0,1)\right\}$$

and $\mathbb{T} := \left\{\sigma_{ij} * \mathrm{id}_{d} | i,j \in \{0,1\} \text{ and } i<j\right\} \cup \{\mathrm{id}_{\partial^i} * \vartheta | i \in \{0,1,2\}\}$.

We consider the full subgraph of $\mathbb{S}$ with objects in the set

$$\mathbb{O} = \left\{\partial^i \cdot d^j \cdot d | i,j \in \{0,1,2\} \text{ and } j \neq 2\right\}.$$

The set of arrows of $\mathbb{S}$ is precisely $\mathbb{T}$ and, by abuse of language, we also denote the graph by $\mathbb{T}$.

The set of the arrows (2-cells) $\mathbb{T}'$ defines a subgroupoid of $\dot{\Delta}_{\mathsf{Str}}(0,3)$, also denoted by $\mathbb{T}'$. Since $\dot{\Delta}_{\mathsf{Str}}(0,1)$ is thin, it is clear that $\mathbb{T}'$ is thin. Moreover, it is clear that $\mathbb{T}'$ is the coproduct of $\mathbb{T}'_{12}$, $\mathbb{T}'_{02}$ and $\mathbb{T}'_{01}$ which are respectively the subgroupoids defined by the sets of 2-cells $\{\sigma_{12} * \alpha | (\alpha : f \Rightarrow g) \in \Delta_{\mathsf{Str}}(0,1)\}$, $\{\sigma_{02} * \alpha | (\alpha : f \Rightarrow g) \in \Delta_{\mathsf{Str}}(0,1)\}$ and $\{\sigma_{01} * \alpha | (\alpha : f \Rightarrow g) \in \Delta_{\mathsf{Str}}(0,1)\}$. In particular, there is not any 2-cell in $\mathbb{T}'$ between any object of $\mathbb{T}'_{ij}$ and any object $\mathbb{T}'_{xy}$ whenever $(i,j) \neq (x,y)$. For instance, there is no arrows (2-cells) $f \Rightarrow \partial^2 \cdot d^1 \cdot d$, $g \Rightarrow \partial^2 \cdot d^0 \cdot d \Rightarrow$ and $h \Rightarrow \partial^1 \cdot d^0 \cdot d \Rightarrow$ in $\mathbb{T}$ for every f,g,h objects of $\mathbb{T}$ such that f is outside $\mathbb{T}'_{12}$, g is outside $\mathbb{T}'_{02}$ and h is outside $\mathbb{T}'_{01}$.

Therefore, it is enough to study the thin groupoid freely generated by $\mathbb{T}$. More precisely, we have only to observe that the equation given by the 3-cell associativity descent 3-cell indeed presents the thin groupoid freely generated by the graph:

$$\begin{tikzcd}
\partial^0 \cdot d^0 \cdot d & \partial^0 \cdot d^1 \cdot d \arrow[l, "\mathrm{id}_{\partial^0} * \vartheta"'] \\
\partial^1 \cdot d^0 \cdot d \arrow[u, "\sigma_{01} * \mathrm{id}_d"] & \partial^2 \cdot d^0 \cdot d \arrow[u, "\sigma_{02} * \mathrm{id}_d"'] \\
\partial^1 \cdot d^1 \cdot d \arrow[u, "\mathrm{id}_{\partial^1} * \vartheta"] & \partial^2 \cdot d^1 \cdot d \arrow[l, "\sigma_{12} * \mathrm{id}_d"] \arrow[u, "\mathrm{id}_{\partial^2} * \vartheta"']
\end{tikzcd} \tag{$\mathbb{T}$}$$

□

2.10.14 Topology

Analogously to the 1-dimensional case, we denote by $\widehat{\mathfrak{G}_2}$ the 2-computad such that $\mathscr{F}_2(\widehat{\mathfrak{G}_2}) \cong \mathfrak{G}_2$. We also have higher dimensional analogues for Theorem 2.4.3. This isomorphism gives an embedding $(n+1)\text{-}\mathsf{cmp} \to \mathscr{P}\mathrm{re}(\overline{\mathscr{F}_n})$ which shows that $(n+1)$-computads are indeed $\overline{\mathscr{F}_n}$-presentations.

If we denote $\mathrm{i}_{1!} = \mathrm{i}_1$ and $\mathrm{i}_{(n+1)!} = \mathrm{i}_{(n+1)}\mathrm{i}_{n!}$, we have:

Theorem 2.10.15. *More generally, there is an isomorphism* $(n+1)\text{-}\mathsf{cmp} \cong (\mathrm{i}_{n!}(-) \times \widehat{\mathfrak{G}_n}/\overline{\mathscr{F}_n})$ *in which*

$$\mathrm{i}_{n!}(-) \times \widehat{\mathfrak{G}_n} : \mathsf{Set} \to \mathsf{cmp}, \quad Y \mapsto \mathrm{i}_{n!}(Y) \times \widehat{\mathfrak{G}_n}.$$

In particular, there is an isomorphism $3\text{-}\mathsf{cmp} \cong (\mathrm{i}_2\mathrm{i}_1(-) \times \widehat{\mathfrak{G}_2}/\overline{\mathscr{F}_2})$.

Observe that, analogously to the 2-dimensional case presented in 2.5.12, we have a homeomorphism

$$\mathfrak{g}_2 \times \mathrm{cir}_2 : D(\mathfrak{g}_2) \times S^2 \to \mathscr{F}_{\mathsf{Top}_2}(\mathrm{i}_2\mathrm{i}_1(\mathfrak{g}_2) \times \widehat{\mathfrak{G}_2}).$$

for each set $\mathfrak{g}_2$.

There are higher dimensional analogues of the association of each small computad with a CW-complex given in 2.5.12. Nevertheless, again, analogously to Remark 2.9.20, we do not have higher dimensional analogues of the results given in Remark 2.5.10, Theorem 2.5.11 and Theorem 2.5.14.

We sketch a 2-dimensional version of the natural transformation $\lceil\_\rceil : \overline{\mathscr{F}_1}\mathscr{C}_{\mathsf{Top}_1} \longrightarrow \mathscr{C}_{\mathsf{Top}_1}$ to get the association of each small 3-computad with a 3-dimensional CW-complex.

Given a 2-cell α of $\overline{\mathscr{F}_2}\mathscr{C}_{\mathsf{Top}_2}(E)$, we have that there is a unique way of getting α as pasting of 2-cells of $\mathscr{C}_{\mathsf{Top}_2}(E)$. That is to say, it is a "formal pasting" of homotopies. We can glue these homotopies to get a new homotopy, which is what we define to be $\lceil\underline{\alpha}\rceil^2_E : B^2 \to E$. This defines a natural transformation $\lceil\_\rceil^2 : \overline{\mathscr{F}_2}\mathscr{C}_{\mathsf{Top}_2} \longrightarrow \mathscr{C}_{\mathsf{Top}_2}$. We denote by $\lceil\cdot\rceil^2$ the mate under the adjunction $\mathscr{F}_{\mathsf{Top}_2} \dashv \mathscr{C}_{\mathsf{Top}_2}$ and itself.

Given a small 3-computad, seen as a morphism $\mathfrak{g} : \mathrm{i}_2\mathrm{i}_1(\mathfrak{g}_2) \times \widehat{\mathfrak{G}_2} \to \overline{\mathscr{F}_2}(G)$ of small 2-computads, $\mathscr{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is the pushout of the inclusion $S^2 \times D(\mathfrak{g}_2) \to B^3 \times D(\mathfrak{g}_2)$ along the composition of the morphisms

$$D(\mathfrak{g}_2) \times S^2 \xrightarrow{(\mathrm{cir}_2 \times \mathfrak{g}_2)} \mathscr{F}_{\mathsf{Top}_2}(\mathrm{i}_2\mathrm{i}_1(\mathfrak{g}_2) \times \widehat{\mathfrak{G}_2}) \xrightarrow{\mathscr{F}_{\mathsf{Top}_2}(\mathfrak{g})} \mathscr{F}_{\mathsf{Top}_2}\overline{\mathscr{F}_2}(G) \xrightarrow{\lceil\cdot\rceil^2_G} \mathscr{F}_{\mathsf{Top}_2}(G).$$

The topological space $\mathscr{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is clearly a CW-complex of dimension 3. Furthermore, of course, we have groupoidal and reflexive versions of $\mathscr{F}_{\mathsf{Top}_3}$ as well, such as $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3} : \mathsf{3\text{-}Rcmp} \to \mathsf{Top}$.

Lemma 2.10.16. *If $(\mathfrak{g}, \mathfrak{g}_2, G)$ has only one 0-cell and only one 1-cell and $\pi_2\mathscr{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is not trivial, then $\mathscr{L}_{(2,0)}\mathscr{P}_2(\mathfrak{g}, \mathfrak{g}_2, G)$ is not locally thin.*

Thereby, by Theorem 2.10.3, we get:

Theorem 2.10.17. *Assume that $G^{\flat}$ is an f.c.s. of the small reflexive 2-computad G. If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small (reflexive) 3-computad such that $\pi_2\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is not trivial, $\mathscr{L}_{(2,0)}\mathscr{P}^{\mathscr{R}}_2(\mathfrak{g})$ is not locally thin.*

Proof. It follows from Theorem 2.10.3 and from the fact that $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}\mathrm{i}_3(G^{\flat})$ is contractible and its inclusion in $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is a cofibration. □

Since $\mathscr{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ has the same homotopy type of a wedge of circumferences, 2-dimensional balls, 3-dimensional balls and spheres, we know that Euler characteristic $\chi\left(\mathscr{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right)$ is equal to

$$\chi\left(\mathscr{F}_{\mathsf{Top}_2}(G)\right) - |\mathfrak{g}_2|,$$

whenever both $\chi\left(\mathscr{F}_{\mathsf{Top}_2}(G)\right)$ and $|\mathfrak{g}_2|$ are finite.

Corollary 2.10.18. *Assume that $G^{\flat}$ is an f.c.s. of the small reflexive 2-computad G. If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small (reflexive) 3-computad such that*

$$\mathbb{Z} \ni \chi\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right) > 1,$$

then $\mathscr{L}_{(2,0)}\mathscr{P}_2(\mathfrak{g}, \mathfrak{g}_2, G)$ is not locally thin.

Proof. Recall that

$$\chi\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right) = 1 - \dim \mathrm{H}^1\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right) + \dim \mathrm{H}^2\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right) - \dim \mathrm{H}^3\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right).$$

Since $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)$ is clearly 1-connected, $\dim \mathrm{H}^1\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)\right) = 0$. Therefore, by hypothesis,

$$\dim \mathrm{H}^2\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)\right) > \dim \mathrm{H}^3\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)\right) \geq 0.$$

In particular, we conclude that $\dim \mathrm{H}^2\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)\right) > 0$. By the Hurewicz isomorphism theorem and by the universal coefficient theorem, this fact implies that the fundamental group $\pi_2\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)\right)$ is not trivial. By Theorem 2.10.17, we get that $\mathscr{L}_{(2,0)}\mathscr{P}^{\mathscr{R}}_2(\mathfrak{g})$ is not locally thin. □

Assume that $(\mathfrak{g},\mathfrak{g}_2,G)$ is a small (reflexive) 3-computad such that there is an $f.c.s.$ triple $(G,\mathtt{T},H^{\mathfrak{b}})$. Then $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}\mathrm{i}_3\mathrm{i}_2(\mathtt{T}) \to \mathscr{F}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)$ is an cofibrant inclusion of a contractible space. Thereby, $\pi_2\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)$ is trivial if and only if

$$\pi_2\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g})/\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}\mathrm{i}_3\mathrm{i}_2(\mathtt{T})\right) \cong \pi_2\left(\mathscr{F}_{\mathsf{Top}_3}(\mathfrak{g}/\mathrm{i}_3\mathrm{i}_2(\mathtt{T}))\right)$$

is trivial. Therefore, since $\mathscr{L}_{(2,0)}\mathscr{P}_2(\mathfrak{g}/\mathrm{i}_3\mathrm{i}_2(\mathtt{T}))$ is locally thin if and only if $\mathscr{L}_{(2,0)}\mathscr{P}_2(\mathfrak{g})$ is locally thin, it follows from Theorem 2.10.17 and Corollary 2.10.18 the result below:

Corollary 2.10.19. *Assume that* $(\mathfrak{g},\mathfrak{g}_2,G)$ *is a small (reflexive)* 3*-computad such that there is an* $f.c.s.$ *triple* $(G,\mathtt{T},H^{\mathfrak{b}})$*. If* $\pi_2\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)$ *is not trivial,* $\mathscr{L}_{(2,0)}\mathscr{P}^{\mathscr{R}}_2(\mathfrak{g})$ *is not locally thin. Furthermore,*

$$\mathbb{Z} \ni \chi\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_3}(\mathfrak{g},\mathfrak{g}_2,G)\right) > 1,$$

then $\mathscr{L}_{(2,0)}\mathscr{P}_2(\mathfrak{g},\mathfrak{g}_2,G)$ *is not locally thin.*

In particular, we get that, whenever such a 3*-computad presents a locally thin* $(2,0)$*-category,* $|\mathfrak{g}_2| \geq \chi\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_2}(G)\right) - 1.$

This also works for the $(3,2,0,\mathscr{R})$-version of $\mathscr{F}_{\mathsf{Top}_3}$ which would show that the presentation by $(3,2,0,\mathscr{R})$-computads given in Corollary 2.10.5 is in a sense the best presentation via $(3,2,0,\mathscr{R})$-computads of the locally thin $(2,0)$-category generated by the reflexive computad G if $\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_2}(G)$ has finite Euler characteristic. For instance, by Corollary 2.10.19, since $\chi\left(\mathscr{F}^{\mathscr{R}}_{\mathsf{Top}_2}(\mathfrak{g}^{\dot{\Delta}_2})\right) = 2$, the presentation via 3-computad given in Theorem 2.10.12 has the least number of 3-cells.

Chapter 3

Pseudo-Kan Extensions and Descent Theory

There are two main constructions in classical descent theory: the category of algebras and the descent category, which are known to be examples of weighted bilimits. We give a formal approach to descent theory, employing formal consequences of commuting properties of bilimits to prove classical and new theorems in the context of Janelidze-Tholen "Facets of Descent II", such as Bénabou-Roubaud Theorems, a Galois Theorem, embedding results and formal ways of getting effective descent morphisms. In order to do this, we develop the formal part of the theory on commuting bilimits via pseudomonad theory, studying idempotent pseudomonads and proving a 2-dimensional version of the adjoint triangle theorem. Also, we work out the concept of pointwise pseudo-Kan extension, used as a framework to talk about bilimits, commutativity and the descent object. As a subproduct, this formal approach can be an alternative perspective/guiding template for the development of higher descent theory.

Introduction

Descent theory is a generalization of a solution given by Grothendieck to a problem related to modules over rings [43]. There is a pseudofunctor $\mathsf{Mod} : \mathsf{Ring} \to \mathsf{CAT}$ which associates each ring $\mathscr{R}$ with the category $\mathsf{Mod}(\mathscr{R})$ of right $\mathscr{R}$-modules. The original problem of descent is the following: given a morphism $f : \mathscr{R} \to \mathscr{S}$ of rings, we wish to understand what is the image of $\mathsf{Mod}(f) : \mathsf{Mod}(\mathscr{R}) \to \mathsf{Mod}(\mathscr{S})$. The usual approach to this problem in descent theory is somewhat indirect: firstly, we characterize the morphisms f in Ring such that $\mathsf{Mod}(f)$ is a functor that forgets some "extra structure". Then, we would get an easier problem: verifying which objects of $\mathsf{Mod}(\mathscr{S})$ could be endowed with such extra structure (see, for instance, [53]).

Given a category $\mathscr{C}$ with pullbacks and a pseudofunctor $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$, for each morphism $p : E \to B$ of $\mathscr{C}$, the *descent data* plays the role of such "extra structure" in the basic problem (see [51, 52, 107]). More precisely, in this context, there is a natural construction of a category $\mathscr{D}esc_{\mathscr{A}}(p)$, called descent category, such that the objects of $\mathscr{D}esc_{\mathscr{A}}(p)$ are objects of $\mathscr{A}(E)$ endowed with descent data, which encompasses the 2-dimensional analogue for equality/1-dimensional descent: one invertible 2-cell plus coherence. This construction comes with a comparison functor and a factorization; that is to say, we have the commutative diagram below, in which $\mathscr{D}esc_{\mathscr{A}}(p) \to \mathscr{A}(E)$ is the functor which forgets the descent data (see [52]).

$$\begin{tikzcd}
\mathscr{A}(B) \arrow[r, "\phi_p"] \arrow[rd, "\mathscr{A}(p)"'] & \mathscr{D}esc_{\mathscr{A}}(p) \arrow[d] \\
 & \mathscr{A}(E)
\end{tikzcd} \quad \text{(Descent Factorization)}$$

Therefore the problem is reduced to investigating whether the *comparison* functor ϕ_p is an equivalence. If it is so, p is is said to be of *effective $\mathscr{A}$-descent* and the image of $\mathscr{A}(p)$ are the objects of $\mathscr{A}(E)$ that can be endowed with descent data. Pursuing this strategy, it is also usual to study cases in which ϕ_p is fully faithful or faithful: in these cases, p is said to be, respectively, of *$\mathscr{A}$-descent* or of almost *$\mathscr{A}$-descent*.

Furthermore, we may consider that the descent problem (in dimension 2) is, in a broad context, the characterization of the image (up to isomorphism) of a given functor $F : \mathscr{C} \to \mathbb{D}$. In this case, using the strategy described above, we investigate if $\mathscr{C}$ can be viewed as a category of objects in $\mathbb{D}$ with some extra structure (plus coherence). Thereby, taking into account the original basic problem, we can ask, hence, if F is (co)monadic. Again, we would get a factorization, the Eilenberg-Moore factorization:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\phi"] \arrow[rd, "F"'] & (Co)\mathsf{Alg} \arrow[d] \\
 & \mathbb{D}
\end{tikzcd}$$

And this approach leads to what is called "monadic descent theory". Bénabou and Roubaud proved that, if the functor F is induced by a pseudofunctor $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$ such that every $\mathscr{A}(p)$ has a left adjoint and $\mathscr{A}$ satisfies the Beck-Chevalley condition, then "monadic $\mathscr{A}$-descent theory" coincides with "Grothendieck $\mathscr{A}$-descent theory". More precisely, assuming the hypotheses above, the morphism that induces F is of effective descent if and only if F is monadic [6].

Thereby, in the core of classical descent theory, there are two constructions: the category of algebras and the descent category. These constructions are known to be examples of 2-categorical limits (see [103, 107]). Also, in a 2-categorical perspective, we can say that the general idea of category of objects with "extra structure (plus coherence)" is, indeed, captured by the notion of 2-dimensional limits.

Not contradicting such point of view, Street considered that (higher) descent theory is about the higher categorical notion of limit [107]. Following this posture, we investigate whether pure formal methods and commuting properties of bilimits are useful to prove classical and new theorems in the classical context of descent theory of [39, 51–53].

Willing to give such formal approach, we employ the following perspective: *the problems of descent theory are usually reduced to the study of the image of a (pseudo)monadic (pseudo)functor*. We restrict our attention to idempotent pseudomonads and prove formal results on pseudoalgebra structures, such as a biadjoint triangle theorem and lifting theorems.

In order to apply such formal approach to get theorems on commutativity of bilimits, we employ a bicategorical analogue of the concept of (pointwise) Kan extension: (pointwise) pseudo-Kan extension, introduced in [77] (which corresponds to Chapter 4).

By successive applications of these formal results, we get results within the context of [51, 52], such as the Bénabou-Roubaud theorem, embedding results and theorems on effective descent morphisms of bilimits of categories. We also apply this approach to get results on effective descent morphisms of categories of small enriched categories V-Cat provided that V satisfies suitable hypotheses.

In this direction, the fundamental standpoint on "classical descent theory" of this paper is the following: the "descent object" of a (pseudo)cosimplicial object in a given context is the image of the initial object of the appropriate notion of Kan extension of such cosimplicial object. More precisely, in our context of dimension 2 (which is the same context of [52]), we get the following result (Theorem 3.4.11): *The descent category of a pseudocosimplicial object* $\mathscr{A} : \Delta \to \mathsf{CAT}$ *is equivalent to* $\mathrm{Ps}\mathscr{R}an_{\mathrm{j}}\mathscr{A}(0)$, in which $\mathrm{j} : \Delta \to \dot{\Delta}$ is the full inclusion of the category of finite nonempty ordinals into the category of finite ordinals and order preserving functions, and $\mathrm{Ps}\mathscr{R}an_{\mathrm{j}}\mathscr{A}$ denotes the right pseudo-Kan extension of $\mathscr{A}$ along j. In particular, we show abstract features of the "classical theory of descent" as a theory (of pseudo-Kan extensions) of pseudocosimplicial objects or pseudofunctors $\dot{\Delta} \to \mathsf{CAT}$.

This work was motivated by three main aims. Firstly, to get formal proofs of classical results of descent theory. Secondly, to prove new results in the classical context – for instance, formal ways of getting sufficient conditions for a morphism to be effective descent. Thirdly, to get proofs of descent theorems that could be recovered in other contexts, such as in the development of higher descent theory (see, for instance, the work of Hermida [47] and Street [107] in this direction).

In Section 3.1, we give an idea of our scope within the context of [51, 52]: we show the main results classically used to deal with the problem of characterization of effective descent morphisms and we present classical results, which are proved using results on commutativity in Sections 3.8 and 3.9. Namely, the embedding results (Theorems 3.1.1 and 3.1.2) and the Bénabou-Roubaud Theorem (Theorem 3.1.3). At the end of Section 3.1, we establish a theorem on pseudopullbacks of categories (Theorem 3.1.5) which is proved in Section 3.9.

Section 3.2 contains most of the abstract results of our formal approach to descent via pseudomonad theory. We start by establishing our main setting: the tricategory of 2-categories, pseudofunctors and pseudonatural transformations. In 3.2.7, we define and study basic aspects of idempotent pseudomonads. Then, in 3.2.16, we study pseudoalgebra structures w.r.t. idempotent pseudomonads, proving a Biadjoint Triangle Theorem (Theorem 3.2.18) and giving a result related to the study of pseudoalgebra structures in commutative squares (Corollary 3.2.19).

We deal with the technical situation of considering objects that cannot be endowed with pseudoalgebra structures but have comparison morphisms belonging to a special class of morphisms in 3.2.20.

Section 3.3 explains why we do not use the usual enriched Kan extensions to study commutativity of the 2-dimensional limits related to descent theory: the main point is that we like to have results which works for bilimits in general (not only flexible ones). In 3.3.1, we define pseudo-Kan extensions and, then, we give the associated factorizations in 3.3.2. Particular cases of these factorizations are the Eilenberg-Moore factorization of an adjunction and the descent factorization described above.

We give further background material in 3.3.5, studying weighted bilimits and proving that, similarly to the enriched case, the appropriate notion of pointwise pseudo-Kan extension is actually a pseudo-Kan extension in the presence of weighted bilimits.

In 3.3.17, 3.3.22 and 3.3.26, we fit the study of pseudo-Kan extensions into the perspective of Section 3.2. We apply the results of 3.2 to the special case of weighted bilimits and pseudo-Kan extensions: we get, then, results on commutativity of weighted bilimits/pseudo-Kan extensions and exactness/*(almost/effective) descent diagrams*.

Section 3.4 studies *descent objects*. We prove that the classical descent object (category) is given by the pseudo-Kan extension of a pseudocosimplicial object (as explained above). In particular, this means that descent objects are conical bilimits of pseudocosimplicial objects. We adopt this description as our definition of descent object of a pseudocosimplicial object. We finish Section 3.4 presenting also the strict version of a descent object, which is given by a Kan extension of a special type of 2-diagram. We get, then, the strict factorization of descent theory.

Section 3.5 gives elementary examples of our context of *effective descent diagrams*. Every weighted bilimit can be seen as an example, but we focus in examples that we use in applications. As mentioned above, the most important examples of bilimits in descent theory are descent objects and Eilenberg-Moore objects: thereby, Section 3.6 is dedicated to explain how Eilenberg-Moore objects fit in our context, via the free adjunction 2-category of [97].

In Section 3.7, we study the Beck-Chevalley condition: by doctrinal adjunction [57], this is a condition to guarantee that a pointwise adjunction between pseudoalgebras can be, actually, extended to an adjunction between such pseudoalgebras. We show how it is related to commutativity of weighted bilimits, giving our first version of a Bénabou-Roubaud Theorem (Theorem 3.7.4).

We apply our results to the usual context [51, 52] of descent theory in Section 3.8: we prove a general version (Theorem 3.8.2) of the embedding results (Theorem 3.1.1), we prove another Bénabou-Roubaud Theorem (Theorem 3.8.5) and, finally, we give a weak version of Theorem 3.1.5.

We finish the paper in Section 3.9: there, we give a stronger result on commutativity (Theorem 3.9.2) and we apply our results to descent theory, proving Theorem 3.1.5 and the Galois result of [49] (Theorem 3.9.8). We also apply Theorem 3.1.5 to get effective descent morphisms of the category of enriched categories V-Cat, provided that V satisfies some hypotheses. For instance, we apply this result to Top-Cat and Cat-Cat.

This work was realized during my PhD program at University of Coimbra. I am grateful to my supervisor Maria Manuel Clementino for her precious help, support and attention. I also thank all the speakers of our informal seminar on descent theory for their insightful talks: Maria Manuel Clementino, George Janelidze, Andrea Montoli, Dimitri Chikhladze, Pier Basile and Manuela Sobral. Finally, I wish to thank Stephen Lack for our brief conversations which helped me to understand aspects related to this work about 2-dimensional category theory, Kan extensions and coherence.

3.1 Basic Problem

In the context of [18, 50–53, 72, 96], the very basic problem of descent is the characterization of effective descent morphisms w.r.t. the basic fibration. As a consequence of Bénabou-Roubaud theorem [6], this problem is trivial for suitable categories (for instance, for locally cartesian closed categories).

However there are remarkable examples of nontrivial characterizations. The topological case, solved by Tholen and Reiterman [96] and reformulated by Clementino and Hofmann [17, 21], is an important example.

Below, we present some theorems classically used as a framework to deal with this basic problem. In this paper, we show that most of these theorems are consequences of a formal theorem presented in Section 3.2, while others are consequences of theorems about bilimits.

Firstly, the most fundamental features of descent theory are the descent category and its related factorization. Assuming that $\mathscr{C}$ is a category with pullbacks, if $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$ is a pseudofunctor, the Descent Factorization is described by Janelidze and Tholen in [52].

We show in Section 3.4 that the concept of pseudo-Kan extension encompasses these features. In fact, the comparison functor and the (pseudo)factorization described above come from the unit and the triangular identity of the (bi)adjunction $[\mathrm{t}, \mathsf{CAT}] \dashv (\mathrm{Ps})\mathscr{R}an_{\mathrm{t}}$.

Secondly, for the nontrivial problems, the usual approach to study (basic/universal) effective/almost descent morphisms is the embedding in well behaved categories, in which "well behaved category" means just that we know which are the effective descent morphisms of this category. For this matter, there are some theorems in [51] and [72]. That is to say, the embedding results:

Theorem 3.1.1 ([51]). *Let $U : \mathscr{C} \to \mathbb{D}$ be a pullback preserving functor between categories with pullbacks.*

1. *If U is faithful, then U reflects almost descent morphisms;*
2. *If U is fully faithful, then U reflects descent morphisms.*

Theorem 3.1.2 ([51]). *Let $\mathscr{C}$ and $\mathbb{D}$ be categories with pullbacks. If $U : \mathscr{C} \to \mathbb{D}$ is a fully faithful pullback preserving functor and $U(p)$ is of effective descent in $\mathbb{D}$, then p is of effective descent if and only if it satisfies the following property: whenever the diagram below is a pullback in $\mathbb{D}$, there is an object C in $\mathscr{C}$ such that $U(C) \cong A$.*

$$\begin{tikzcd}
U(P) \arrow[r] \arrow[d] & A \arrow[d] \\
U(E) \arrow[r, "U(p)"'] & U(B)
\end{tikzcd}$$

We show in Section 3.8 that Theorem 3.1.1 is a very easy consequence of formal and commuting properties of pseudo-Kan extensions (Corollary 3.3.24 and Corollary 3.3.28) that follow directly from results of Section 3.2, while we show in Section 3.9 that Theorem 3.1.2 is a consequence of a theorem on bilimits (Theorem 3.9.4) which also implies the generalized Galois Theorem of [49]. It is interesting to note that, since Theorems 3.1.1 and 3.1.2 are just formal properties, they can be applied in other contexts – for instance, for morphisms between pseudofunctors $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$ and $\mathscr{B} : \mathbb{D}^{\mathrm{op}} \to \mathsf{CAT}$, as it is explained in Section 3.8.

Finally, Bénabou-Roubaud Theorem [6, 51] is a celebrated result of Descent Theory which allows us to understand some problems via monadicity: it says that monadic $\mathscr{A}$-descent theory is equivalent to Grothendieck $\mathscr{A}$-descent theory in suitable cases, such as the basic fibration. We demonstrate in Section 3.8 that it is also a corollary of formal results of Section 3.2.

Theorem 3.1.3 (Bénabou-Roubaud [6, 51]). *Let $\mathscr{C}$ be a category with pullbacks. If $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$ is a pseudofunctor such that, for every morphism $p : E \to B$ of $\mathscr{C}$, $A(p)$ has left adjoint $A(p)!$ and the invertible 2-cell induced by $\mathscr{A}$ below satisfies the Beck-Chevalley condition, then the factorization described above is pseudonaturally equivalent to the Eilenberg-Moore factorization. In other words, assuming the hypotheses above, Grothendieck $\mathscr{A}$-descent theory is equivalent to monadic descent theory.*

$$\begin{tikzcd}
\mathscr{A}(B) \arrow[r, "\mathscr{A}(p)"] \arrow[d, "\mathscr{A}(p)"'] \arrow[dr, phantom, "\cong"] & \mathscr{A}(E) \arrow[d] \\
\mathscr{A}(E) \arrow[r] & \mathscr{A}(E \times_p E)
\end{tikzcd}$$

3.1.4 Open problems

Clementino and Hofmann [18] studied the problem of characterization of effective descent morphisms for (T,V)-categories provided that V is a lattice. To deal with this problem, they used the embedding $(T,V)\text{-}\mathsf{Cat} \to (T,V)\text{-}\mathsf{Grph}$ and Theorems 3.1.1 and 3.1.2. However, for more general monoidal categories V, such inclusion is not fully faithful and the characterization of effective descent morphisms still is an open problem even for the simpler case of the category of enriched categories $V\text{-}\mathsf{Cat}$.

As an application, we give some results about effective descent morphisms of $V\text{-}\mathsf{Cat}$. They are consequences of formal results given in this paper on effective descent morphisms of categories constructed from other categories: more precisely, 2-dimensional limits of categories.

More precisely, we prove Theorem 3.1.5 in Section 3.9. We can apply it in some cases of categories of enriched categories: if V is a cartesian closed category satisfying suitable hypotheses, there is a full inclusion $V\text{-}\mathsf{Cat} \to \mathsf{Cat}(V)$, in which $\mathsf{Cat}(V)$ is the category of internal categories. When this happens, we conclude that the inclusion reflects effective descent morphisms by Theorem 3.1.5. Since the characterization of effective descent morphisms for internal categories in this setting was already done by Le Creurer [72], we get effective descent morphisms for enriched categories (provided that V satisfies some properties).

Theorem 3.1.5. *Assume that the diagram of categories with pullbacks*

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "S"] \arrow[d, "Z"'] \arrow[dr, phantom, "\cong"] & \mathscr{C} \arrow[d, "F"] \\
\mathbb{D} \arrow[r, "G"'] & \mathbb{E}
\end{tikzcd}$$

is a pseudopullback such that all the functors are pullback preserving functors. If p is a morphism in $\mathbb{B}$ such that $S(p), Z(p)$ are of effective descent and $FS(p)$ is a descent morphism, then p is of effective descent.

3.2 Formal Results

Our perspective herein is that, instead of considering the problem of understanding the image of a generic (pseudo)functor, the main theorems of descent theory usually deal with the problem of understanding the pseudoalgebras of (fully) property-like (pseudo)monads [60]. It is easier to study these pseudoalgebras: they are just the objects that can be endowed with a unique pseudoalgebra structure (up to isomorphism), or, more appropriately, the effective descent points/objects.

Thereby results on pseudoalgebra structures are in the core of our formal approach. In this section, we give the main results of this paper in this direction, restricting the scope to idempotent pseudomonads. This setting is sufficient to deal with the classical descent problem of [51, 52]. We start by recalling basic results of bicategory theory [5]. Most of them can be found in [104, 105]. To fix notation, we start by giving the definitions of the *tricategory of* 2*-categories, pseudofunctors, pseudonatural transformations and modifications*, denoted by $2\text{-}\mathsf{CAT}$.

Henceforth, in a given 2-category, we always denote by $\cdot$ the vertical composition of 2-cells and by $*$ their horizontal composition.

Definition 3.2.1. [Pseudofunctor] Let $\mathfrak{A}, \mathfrak{B}$ be 2-categories. A *pseudofunctor* $\mathscr{A} : \mathfrak{A} \to \mathfrak{B}$ is a pair $(\mathscr{A}, \mathfrak{a})$ with the following data:

- Function $\mathscr{A} : \mathrm{obj}(\mathfrak{A}) \to \mathrm{obj}(\mathfrak{B})$;
- Functors $\mathscr{A}_{XY} : \mathfrak{A}(X,Y) \to \mathfrak{B}(\mathscr{A}(X), \mathscr{A}(Y))$;
- For each pair $g : X \to Y, h : Y \to Z$ of 1-cells in $\mathfrak{A}$, an invertible 2-cell in $\mathfrak{B}$: $\mathfrak{a}_{hg} : \mathscr{A}(h)\mathscr{A}(g) \Rightarrow \mathscr{A}(hg)$;
- For each object X of $\mathfrak{A}$, an invertible 2-cell $\mathfrak{a}_X : \mathrm{Id}_{\mathscr{A}X} \Rightarrow \mathscr{A}(\mathrm{Id}_X)$ in $\mathfrak{B}$;

subject to *associativity*, *identity* and *naturality* axioms [77].

If $\mathscr{A} = (\mathscr{A}, \mathfrak{a}) : \mathfrak{A} \to \mathfrak{B}$ and $(\mathscr{B}, \mathfrak{b}) : \mathfrak{B} \to \mathfrak{C}$ are pseudofunctors, we define the composition as follows: $\mathscr{B} \circ \mathscr{A} := (\mathscr{B}\mathscr{A}, (\mathfrak{b}\mathfrak{a}))$, in which $(\mathfrak{b}\mathfrak{a})_{hg} := \mathscr{B}(\mathfrak{a}_{hg}) \cdot \mathfrak{b}_{\mathscr{A}(h)\mathscr{A}(g)}$ and $(\mathfrak{b}\mathfrak{a})_X := \mathscr{B}(\mathfrak{a}_X) \cdot \mathfrak{b}_{\mathscr{A}(X)}$. This composition is associative and it has trivial identities. A pseudonatural transformation between pseudofunctors $\mathscr{A} \longrightarrow \mathscr{B}$ is a natural transformation in which the usual (natural) commutative squares are replaced by invertible 2-cells plus coherence.

Definition 3.2.2. [Pseudonatural transformation] If $\mathscr{A}, \mathscr{B} : \mathfrak{A} \to \mathfrak{B}$ are pseudofunctors, a *pseudonatural transformation* $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ is defined by:

- For each object X of $\mathfrak{A}$, a 1-cell $\alpha_X : \mathscr{A}(X) \to \mathscr{B}(X)$ of $\mathfrak{B}$;
- For each 1-cell $g : X \to Y$ of $\mathfrak{A}$, an invertible 2-cell $\alpha_g : \mathscr{B}(g)\alpha_X \Rightarrow \alpha_Y\mathscr{A}(g)$ of $\mathfrak{B}$;

such that axioms of associativity, identity and naturality hold [77].

Firstly, the vertical composition, denoted by $\beta\alpha$, of two pseudonatural transformations $\alpha : \mathscr{A} \Rightarrow \mathscr{B}$, $\beta : \mathscr{B} \Rightarrow \mathscr{C}$ is defined by

$$(\beta\alpha)_W := \beta_W \alpha_W$$

$$\begin{tikzcd}
\mathscr{A}(W) \arrow[r, "\beta_W \alpha_W"] \arrow[d, "\mathscr{A}(f)"'] \arrow[dr, phantom, "\overset{(\beta\alpha)_f}{\Longleftarrow}"] & \mathscr{C}(W) \arrow[d, "\mathscr{C}(f)"] & {} \arrow[d, phantom, ":="] & \mathscr{A}(W) \arrow[r, "\alpha_W"] \arrow[d, "\mathscr{A}(f)"] \arrow[dr, phantom, "\overset{\alpha_f}{\Longleftarrow}"] & \mathscr{B}(W) \arrow[r, "\beta_W"] \arrow[d, "\mathscr{B}(f)"'] \arrow[dr, phantom, "\overset{\beta_f}{\Longleftarrow}"] & \mathscr{C}(W) \arrow[d, "\mathscr{C}(f)"] \\
\mathscr{A}(X) \arrow[r, "\beta_X \alpha_X"'] & \mathscr{C}(X) & {} & \mathscr{A}(X) \arrow[r, "\alpha_X"'] & \mathscr{B}(X) \arrow[r, "\beta_X"'] & \mathscr{C}(X)
\end{tikzcd}$$

Secondly, let $(\mathscr{U}, \mathfrak{u}), (\mathscr{L}, \mathfrak{l}) : \mathfrak{B} \to \mathfrak{C}$ and $\mathscr{A}, \mathscr{B} : \mathfrak{A} \to \mathfrak{B}$ be pseudofunctors. If $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$, $\lambda : \mathscr{U} \longrightarrow \mathscr{L}$ are pseudonatural transformations, then the horizontal composition of $\mathscr{U}$ with α, denoted by $\mathscr{U}\alpha$, is defined by: $(\mathscr{U}\alpha)_W := \mathscr{U}(\alpha_W)$ and $(\mathscr{U}\alpha)_f := \left(\mathfrak{u}_{\alpha_X \mathscr{A}(f)}\right)^{-1} \cdot \mathscr{U}(\alpha_f) \cdot \mathfrak{u}_{\mathscr{B}(f)\alpha_W}$, while the composition $\lambda\mathscr{A}$ is defined trivially. Thereby, we get the (usual) definition of the horizontal composition,

$$(\lambda * \alpha) := (\lambda\mathscr{B})(\mathscr{U}\alpha) \cong (\mathscr{L}\alpha)(\lambda\mathscr{A})$$

Similarly, we get the three types of compositions of modifications.

Definition 3.2.3. [Modification] Let $\mathscr{A}, \mathscr{B} : \mathfrak{A} \to \mathfrak{B}$ be pseudofunctors. If $\alpha, \beta : \mathscr{A} \Rightarrow \mathscr{B}$ are pseudonatural transformations, a *modification* $\Gamma : \alpha \Longrightarrow \beta$ is defined by the following data:

– For each object X of $\mathfrak{A}$, a 2-cell $\Gamma_X : \alpha_X \Rightarrow \beta_X$ of $\mathfrak{B}$ satisfying one axiom of naturality [77].

It is straightforward to verify that 2-CAT is a tricategory which is locally a 2-category. In particular, we denote by $[\mathfrak{A}, \mathfrak{B}]_{PS}$ the 2-category of pseudofunctors $\mathfrak{A} \to \mathfrak{B}$, pseudonatural transformations and modifications. Also, we have the *bicategorical Yoneda lemma* [104] and, hence, the usual Yoneda embedding $\mathscr{Y} : \mathfrak{A} \to [\mathfrak{A}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$ is locally an equivalence (*i.e.* it induces equivalences between the hom-categories).

A pseudofunctor $\mathscr{A} : \mathfrak{A} \to \mathsf{CAT}$ is said to be *birepresentable* if there is an object W of $\mathfrak{A}$ such that $\mathscr{A}$ is pseudonaturally equivalent to $\mathfrak{A}(W, -) : \mathfrak{A} \to \mathsf{CAT}$. In this case, W is called the birepresentation of $\mathscr{A}$. By the bicategorical Yoneda lemma, birepresentations are unique up to equivalence.

If $\mathscr{L} : \mathfrak{A} \to \mathfrak{B}$ is a pseudofunctor and X is an object of $\mathfrak{B}$, a right bireflection of X along $\mathscr{L}$ is, if it exists, a birepresentation of the pseudofunctor $\mathfrak{B}(\mathscr{L}-, X) : \mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT}$. We say that $\mathscr{L}$ is left biadjoint to $\mathscr{U} : \mathfrak{B} \to \mathfrak{A}$ if, for every object X of $\mathfrak{B}$, $\mathscr{U}(X)$ is the right bireflection of X along $\mathscr{L}$. In this case, we say that $\mathscr{U}$ is right biadjoint to $\mathscr{L}$. This definition of biadjunction is equivalent to Definition 3.2.4.

Definition 3.2.4. A pseudofunctor $\mathscr{L} : \mathfrak{A} \to \mathfrak{B}$ is *left biadjoint* to $\mathscr{U} : \mathfrak{B} \to \mathfrak{A}$ if there exist

1. pseudonatural transformations $\eta : \mathrm{Id}_{\mathfrak{A}} \longrightarrow \mathscr{U}\mathscr{L}$ and $\varepsilon : \mathscr{L}\mathscr{U} \longrightarrow \mathrm{Id}_{\mathfrak{B}}$
2. invertible modifications $s : \mathrm{Id}_{\mathscr{L}} \Longrightarrow (\varepsilon\mathscr{L}) \cdot (\mathscr{L}\eta)$ and $t : (\mathscr{U}\varepsilon) \cdot (\eta\mathscr{U}) \Longrightarrow \mathrm{Id}_{\mathscr{U}}$

satisfying coherence equations [77]. In this case, $(\mathscr{L} \dashv \mathscr{U}, \eta, \varepsilon, s, t)$ is a *biadjunction*. Sometimes we omit the invertible modifications, denoting a biadjunction by $(\mathscr{L} \dashv \mathscr{U}, \eta, \varepsilon)$.

By the bicategorical Yoneda lemma, if $\mathscr{L} : \mathfrak{A} \to \mathfrak{B}$ is left biadjoint, its right biadjoint is unique up to pseudonatural equivalence. Furthermore, if $\mathscr{L}$ is left 2-adjoint, it is left biadjoint.

Definition 3.2.5. A pseudofunctor $\mathscr{U}$ is a *local equivalence* if it induces equivalences between the hom-categories.

Lemma 3.2.6. *A right biadjoint $\mathscr{U}$ is a local equivalence if and only if the counit of the biadjunction is a pseudonatural equivalence.*

3.2.7 Idempotent Pseudomonads

Since we deal only with idempotent pseudomonads, we give an elementary approach focusing on them. The main benefit of this approach is that idempotent pseudomonads have only free pseudoalgebras. For this reason, assuming that η is the unit of an idempotent pseudomonad $\mathscr{T}$, an object X can be endowed with a $\mathscr{T}$-pseudoalgebra structure if and only if $\eta_{X} : X \to \mathscr{T}(X)$ is an equivalence.

Recall that a *pseudomonad* $\mathscr{T}$ on a 2-category $\mathfrak{H}$ consists of a sextuple $(\mathscr{T}, \mu, \eta, \Lambda, \rho, \Gamma)$, in which $\mathscr{T} : \mathfrak{H} \to \mathfrak{H}$ is a pseudofunctor, $\mu : \mathscr{T}^2 \longrightarrow \mathscr{T}, \eta : \mathrm{Id}_{\mathfrak{H}} \longrightarrow \mathscr{T}$ are pseudonatural transformations and

$$\begin{tikzcd}
\mathscr{T} \arrow[r, "\eta_{\mathscr{T}}"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\overset{\Lambda}{\Longleftarrow}"] & \mathscr{T}^2 \arrow[d, "\mu"] & \mathscr{T} \arrow[l, "\mathscr{T}\eta"'] \arrow[dl, equal, bend left] \arrow[dl, phantom, "\overset{\rho}{\Longleftarrow}"] \\
 & \mathscr{T} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathscr{T}^3 \arrow[r, "\mathscr{T}\mu"] \arrow[d, "\mu_{\mathscr{T}}"'] \arrow[dr, phantom, "\overset{\Gamma}{\Longleftarrow}"] & \mathscr{T}^2 \arrow[d, "\mu"] \\
\mathscr{T}^2 \arrow[r, "\mu"'] & \mathscr{T}
\end{tikzcd}$$

are invertible modifications satisfying the following coherence equations [77, 84]:

- Identity:

$$\begin{tikzcd}
 & & \mathscr{T}^2 \arrow[dll, "\mathscr{T}\eta\mathscr{T}"'] \arrow[drr, "\mathscr{T}\eta\mathscr{T}"] \arrow[dd, "\mathrm{Id}_{\mathscr{T}^2}" description] & & & & & \mathscr{T}^2 \arrow[d, "\mathscr{T}\eta\mathscr{T}"] & \\
\mathscr{T}^3 \arrow[drr, "\mu\mathscr{T}"'] & \overset{\rho\mathscr{T}}{\Longleftarrow} & & \overset{\widehat{\mathscr{T}\Lambda}}{\Longleftarrow} & \mathscr{T}^3 \arrow[dll, "\mathscr{T}\mu"] & & & \mathscr{T}^3 \arrow[dl, "\mu\mathscr{T}"'] \arrow[dr, "\mathscr{T}\mu"] & \\
 & & \mathscr{T}^2 \arrow[d, "\mu"] & & & = & \mathscr{T}^2 \arrow[dr, "\mu"'] & \overset{\Gamma}{\Longleftarrow} & \mathscr{T}^2 \arrow[dl, "\mu"] \\
 & & \mathscr{T} & & & & & \mathscr{T} &
\end{tikzcd}$$

- Associativity:

$$\begin{tikzcd}
\mathscr{T}^4 \arrow[r, "\mathscr{T}^2\mu"] \arrow[d, "\mu\mathscr{T}^2"'] \arrow[dr, "\mathscr{T}\mu\mathscr{T}" description] & \mathscr{T}^3 \arrow[dr, "\mathscr{T}\mu"] \arrow[d, phantom, "\overset{\widehat{\mathscr{T}\Gamma}}{\Longleftarrow}"] & & & \mathscr{T}^4 \arrow[r, "\mathscr{T}^2\mu"] \arrow[d, "\mu\mathscr{T}^2"'] \arrow[dr, phantom, "\overset{\mu_{\mu}^{-1}}{\Longleftarrow}"] & \mathscr{T}^3 \arrow[d, "\mu\mathscr{T}"] \arrow[dr, "\mathscr{T}\mu"] & \\
\mathscr{T}^3 \arrow[dr, "\mu\mathscr{T}"'] \arrow[r, phantom, "\overset{\Gamma\mathscr{T}}{\Longleftarrow}"] & \mathscr{T}^3 \arrow[r, "\mathscr{T}\mu"] \arrow[d, "\mu\mathscr{T}"] \arrow[dr, phantom, "\overset{\Gamma}{\Longleftarrow}"] & \mathscr{T}^2 \arrow[d, "\mu"] & = & \mathscr{T}^3 \arrow[r, "\mathscr{T}\mu"] \arrow[dr, "\mu\mathscr{T}"'] & \mathscr{T}^2 \arrow[dr] \arrow[r, phantom, "\overset{\Gamma}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\Gamma}{\Longleftarrow}"] & \mathscr{T}^2 \arrow[d, "\mu"] \\
 & \mathscr{T}^2 \arrow[r, "\mu"'] & \mathscr{T} & & & \mathscr{T}^2 \arrow[r, "\mu"'] & \mathscr{T}
\end{tikzcd}$$

in which

$$\widehat{\mathscr{T}\Lambda} := (\mathfrak{t}_{\mathscr{T}})^{-1}(\mathscr{T}\Lambda)\left(\mathfrak{t}_{(\mu)(\eta\mathscr{T})}\right), \qquad \widehat{\mathscr{T}\Gamma} := \left(\mathfrak{t}_{(\mu)(\mu\mathscr{T})}\right)^{-1}(\mathscr{T}\Gamma)\left(\mathfrak{t}_{(\mu)(\mathscr{T}\mu)}\right).$$

Definition 3.2.8. [Idempotent pseudomonad] A pseudomonad $(\mathscr{T},\mu,\eta,\Lambda,\rho,\Gamma)$ is *idempotent* if there is an invertible modification $\eta\mathscr{T}\cong\mathscr{T}\eta$.

Similarly to 1-dimensional monad theory, the name *idempotent pseudomonad* is justified by Lemma 3.2.9, which says that multiplications of idempotent pseudomonads are pseudonatural equivalences.

Lemma 3.2.9. *A pseudomonad $(\mathscr{T},\mu,\eta,\Lambda,\rho,\Gamma)$ is idempotent if and only if the multiplication μ is a pseudonatural equivalence. In this case, $\eta\mathscr{T}$ is a pseudonatural equivalence inverse of μ.*

Proof. Since $\mu(\eta\mathscr{T})\cong\mathrm{Id}_{\mathscr{T}}\cong\mu(\mathscr{T}\eta)$, it is obvious that, if μ is a pseudonatural equivalence, then $\eta\mathscr{T}\cong\mathscr{T}\eta$. Therefore $\mathscr{T}$ is idempotent and $\eta\mathscr{T}$ is an equivalence inverse of μ.

Reciprocally, assume that $\mathscr{T}$ is idempotent. By the definition of pseudomonads, there is an invertible modification $\mu(\eta\mathscr{T})\cong\mathrm{Id}_{\mathscr{T}}$. And, since $\eta\mathscr{T}\cong\mathscr{T}\eta$, we get the invertible modifications

$$(\eta\mathscr{T})\mu\cong(\mathscr{T}\mu)(\eta\mathscr{T}^2)\cong(\mathscr{T}\mu)(\mathscr{T}\eta\mathscr{T})\cong\mathscr{T}(\mu(\eta\mathscr{T}))\cong\mathrm{Id}_{\mathscr{T}^2}$$

which prove that μ is a pseudonatural equivalence and $\eta\mathscr{T}$ is a pseudonatural equivalence inverse. □

The reader familiar with lax-idempotent/KZ-pseudomonads will notice that an idempotent pseudomonad is just a KZ-pseudomonad whose adjunction $\mu\dashv\eta\mathscr{T}$ is actually an adjoint equivalence. Hence, idempotent pseudomonads are fully property-like pseudomonads [60].

Every biadjunction induces a pseudomonad [66, 77]. In fact, we get the multiplication μ from the counit, and the invertible modifications Λ,ρ,Γ come from the invertible modifications of Definition 3.2.4. Of course, a biadjunction $\mathscr{L}\dashv\mathscr{U}$ induces an idempotent pseudomonad if and only if its unit η is such that $\eta\mathscr{U}\mathscr{L}\cong\mathscr{U}\mathscr{L}\eta$. As a consequence of this characterization, we have Lemma 3.2.10 which is necessary to give the Eilenberg-Moore factorization for idempotent pseudomonads.

Lemma 3.2.10. *If a biadjunction $(\mathscr{L}\dashv\mathscr{U},\eta,\varepsilon)$ induces an idempotent pseudomonad, then $\eta\mathscr{U}:\mathscr{U}\longrightarrow\mathscr{U}\mathscr{L}\mathscr{U}$ is a pseudonatural equivalence.*

Proof. By the triangular invertible modifications of Definition 3.2.4, if ε is the counit of the biadjunction $\mathscr{L}\dashv\mathscr{U}$, $(\mathscr{U}\varepsilon)(\eta\mathscr{U})\cong\mathrm{Id}_{\mathscr{U}}$. Also, since $\mathscr{U}\mathscr{L}\eta\cong\eta\mathscr{U}\mathscr{L}$, we have the following invertible modifications

$$(\eta\mathscr{U})\cdot(\mathscr{U}\varepsilon)\cong(\mathscr{U}\mathscr{L}\mathscr{U}\varepsilon)(\eta\mathscr{U}\mathscr{L}\mathscr{U})\cong(\mathscr{U}\mathscr{L}\mathscr{U}\varepsilon)(\mathscr{U}\mathscr{L}\eta\mathscr{U})\cong\mathscr{U}\mathscr{L}(\mathrm{Id}_{\mathscr{U}})\cong\mathrm{Id}_{\mathscr{U}\mathscr{L}\mathscr{U}}$$

Therefore $\eta\mathscr{U}$ is a pseudonatural equivalence. □

We can avoid the coherence equations [66, 77, 84] used to define the 2-category of pseudoalgebras of a pseudomonad $\mathscr{T}$ when assuming that $\mathscr{T}$ is idempotent.

Definition 3.2.11. [Pseudoalgebras] Let $(\mathscr{T},\mu,\eta,\Lambda,\rho,\Gamma)$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$. We define the 2-category of *$\mathscr{T}$-pseudoalgebras* Ps-$\mathscr{T}$-Alg as follows:

- Objects: the objects of $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ are the objects X of $\mathfrak{H}$ such that

  $$\eta_X : X \to \mathscr{T}(X)$$

  is an equivalence;

- The inclusion $\mathrm{obj}(\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}) \to \mathrm{obj}(\mathfrak{H})$ extends to a full inclusion 2-functor

  $$\mathscr{I} : \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathfrak{H}$$

In other words, the inclusion $\mathscr{I} : \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathfrak{H}$ is defined to be final among the full inclusions $\widehat{\mathscr{I}} : \mathfrak{A} \to \mathfrak{H}$ such that $\eta\widehat{\mathscr{I}}$ is a pseudonatural equivalence.

If $\eta_X : X \to \mathscr{T}(X)$ is an equivalence, X can be endowed with a pseudoalgebra structure and the left adjoint $a : \mathscr{T}(X) \to X$ to $\eta_X : X \to \mathscr{T}(X)$ is called a pseudoalgebra structure to X. Because we could describe $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ by means of pseudoalgebras/pseudoalgebra structures, we often denote the objects of $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ by small letters a, b.

Theorem 3.2.12 (Eilenberg-Moore biadjunction). *Let $(\mathscr{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$. There is a unique pseudofunctor $\mathscr{L}^{\mathscr{T}}$ such that*

$$\begin{tikzcd}
\mathfrak{H} \arrow[rr, "\mathscr{T}"] \arrow[dr, "\mathscr{L}^{\mathscr{T}}"'] & & \mathfrak{H} \\
& \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[ur, "\mathscr{I}"'] &
\end{tikzcd}$$

is a commutative diagram. Furthermore, $\mathscr{L}^{\mathscr{T}}$ is left biadjoint to $\mathscr{I}$.

Proof. Firstly, we define $\mathscr{L}^{\mathscr{T}}(X) := \mathscr{T}(X)$. On one hand, it is well defined, since, by Lemma 3.2.9,

$$\eta\mathscr{T} : \mathscr{T} \longrightarrow \mathscr{T}^2$$

is a pseudonatural equivalence. On the other hand, the uniqueness of $\mathscr{L}^{\mathscr{T}}$ is a consequence of the fact that $\mathscr{I}$ is a monomorphism.

Now, it remains to show that $\mathscr{L}^{\mathscr{T}}$ is left biadjoint to $\mathscr{I}$. By abuse of language, if a is an object of $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$, we denote by a its pseudoalgebra structure (of Definition 3.2.11). Then we define the equivalences inverses below

$$\begin{array}{cc}
\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}(\mathscr{T}(X), b) \to \mathfrak{H}(X, \mathscr{I}(b)) & \mathfrak{H}(X, \mathscr{I}(b)) \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}(\mathscr{T}(X), b) \\
f \mapsto f\eta_X & g \mapsto bT(g) \\
\alpha \mapsto \alpha * \mathrm{Id}_{\eta_X} & \beta \mapsto \mathrm{Id}_b * T(\beta)
\end{array}$$

It completes the proof that $\mathscr{L}^{\mathscr{T}} \dashv \mathscr{I}$. □

Theorem 3.2.13 shows that this biadjunction $\mathscr{L}^{\mathscr{T}} \dashv \mathscr{I}$ satisfies the expected universal property [66] of the 2-category of pseudoalgebras, which is the Eilenberg-Moore factorization. In other words, we prove that our definition of $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ for idempotent pseudomonads $\mathscr{T}$ agrees with the usual definition [66, 73, 84, 104] of pseudoalgebras for a pseudomonad.

Theorem 3.2.13 (Eilenberg-Moore)**.** *If $\mathscr{L} \dashv \mathscr{U}$ is a biadjunction which induces an idempotent pseudomonad $(\mathscr{T}, \mu, \eta, \Lambda, \rho, \Gamma)$, then we have a unique comparison pseudofunctor $\mathscr{K} : \mathfrak{B} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ such that*

$$\begin{tikzcd} \mathfrak{B} \arrow[r, "\mathscr{K}"] \arrow[dr, "\mathscr{U}"'] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[d, "\mathscr{J}"] \\ & \mathfrak{A} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{A} \arrow[r, "\mathscr{L}^{\mathscr{T}}"] \arrow[dr, "\mathscr{L}"'] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \\ & \mathfrak{B} \arrow[u, "\mathscr{K}"'] \end{tikzcd}$$

commute.

Proof. It is enough to define $\mathscr{K}(X) = \mathscr{U}(X)$ and $\mathscr{K}(f) = \mathscr{U}(f)$. This is well defined, since, by Lemma 3.2.10, $\eta\mathscr{U} : \mathscr{U} \longrightarrow \mathscr{T}\mathscr{U}$ is a pseudonatural equivalence. ☐

Actually, in 2-CAT, every biadjunction $\mathscr{L} \dashv \mathscr{U}$ induces a comparison pseudofunctor and an Eilenberg-Moore factorization [73] as above, in which $\mathscr{T} = \mathscr{U}\mathscr{L}$ denotes the induced pseudomonad. When the comparison pseudofunctor $\mathscr{K}$ is a biequivalence, we say that $\mathscr{U}$ is pseudomonadic. Although there is the Beck's theorem for pseudomonads [47, 73, 77], the setting of idempotent pseudomonads is simpler.

Theorem 3.2.14. *Let $\mathscr{L} \dashv \mathscr{U}$ be a biadjunction. The pseudofunctor $\mathscr{U}$ is a local equivalence (or, equivalently, the counit is a pseudonatural equivalence) if and only if $\mathscr{U}$ is pseudomonadic and the induced pseudomonad is idempotent.*

Proof. Firstly, if the counit ε of the biadjunction of $\mathscr{L} \dashv \mathscr{U}$ is a pseudonatural equivalence, then $\mu := \mathscr{U}\varepsilon\mathscr{L}$ is a pseudonatural equivalence as well. And, thereby, the induced pseudomonad is idempotent. Now, if $a : \mathscr{T}(X) \to X$ is a pseudoalgebra structure to X, we have that

$$\mathscr{K}(\mathscr{L}(X)) = \mathscr{T}(X) \xrightarrow[\simeq]{a} X.$$

Thereby $\mathscr{U}$ is pseudomonadic.

Reciprocally, if $\mathscr{L} \dashv \mathscr{U}$ induces an idempotent pseudomonad and $\mathscr{U}$ is pseudomonadic, then we have that $\mathscr{J} \circ \mathscr{K} = \mathscr{U}$, $\mathscr{K}$ is a biequivalence and $\mathscr{J}$ is a local equivalence. Thereby $\mathscr{U}$ is a local equivalence and ε is a pseudonatural equivalence. ☐

In descent theory, one needs conditions to decide if a given object can be endowed with a pseudoalgebra structure. Idempotent pseudomonads provide the following simplification.

Theorem 3.2.15. *Let $\mathscr{T} = (\mathscr{T}, \mu, \eta, \Lambda, \rho, \Gamma)$ be an idempotent pseudomonad on $\mathfrak{H}$. Given an object X of $\mathfrak{H}$, the following conditions are equivalent:*

1. *The object X can be endowed with a $\mathscr{T}$-pseudoalgebra structure;*
2. *$\eta_X : X \to \mathscr{T}(X)$ is a pseudosection, i.e. there is $a : \mathscr{T}(X) \to X$ such that $a\eta_X \cong \mathrm{Id}_X$;*
3. *$\eta_X : X \to \mathscr{T}(X)$ is an equivalence.*

Proof. Assume that $\eta_X : X \to \mathscr{T}(X)$ is a pseudosection. By hypothesis, there is $a : \mathscr{T}(X) \to X$ such that $a\eta_X \cong \mathrm{Id}_X$. Thereby

$$\eta_X a \cong \mathscr{T}(a)\eta_{\mathscr{T}(X)} \cong \mathscr{T}(a)\mathscr{T}(\eta_X) \cong \mathscr{T}(a\eta_X) \cong \mathrm{Id}_{\mathscr{T}(X)}.$$

Hence η_X is an equivalence. □

3.2.16 Biadjoint Triangle Theorem

The main result of this formal approach is somehow related to distributive laws of pseudomonads [84, 85]. However, we choose a more direct approach, avoiding some technicalities of distributive laws unnecessary to our setting. To give such direct approach, we use the Biadjoint Triangle Theorem 3.2.18.

Precisely, we give a bicategorical analogue (for idempotent pseudomonads) of an adjoint triangle theorem [2, 30, 92]. It is important to note that this bicategorical version holds for pseudomonads in general, so that our restriction to the idempotent version is due to our scope.

Lemma 3.2.17. *Let* $(\mathscr{L} \dashv \mathscr{U}, \eta, \varepsilon)$ *and* $(\widehat{\mathscr{L}} \dashv \widehat{\mathscr{U}}, \widehat{\eta}, \widehat{\varepsilon})$ *be biadjunctions. Assume that* $\widehat{\mathscr{L}} \dashv \widehat{\mathscr{U}}$ *induces an idempotent pseudomonad and that there is a pseudonatural equivalence*

$$\begin{tikzcd}
\mathfrak{A} & & \mathfrak{B} \arrow[ll, "\mathtt{E}" description] \\
& \mathfrak{C} \arrow[ul, "\mathscr{L}" description] \arrow[ur, "\widehat{\mathscr{L}}" description] \arrow[u, phantom, "\simeq"] &
\end{tikzcd}$$

If η_X *is a pseudosection, then* $\widehat{\eta}_X$ *is an equivalence.*

Proof. Let X be an object of $\mathfrak{C}$ such that $\eta_X : X \to \mathscr{U}\mathscr{L}(X)$ is pseudosection. By Theorem 3.2.15, it is enough to prove that $\widehat{\eta}_X$ is a pseudosection, because the pseudomonad induced by $\widehat{\mathscr{L}} \dashv \widehat{\mathscr{U}}$ is idempotent.

To prove that $\widehat{\eta}_X$ is a pseudosection, we construct a pseudonatural transformation $\alpha : \widehat{\mathscr{U}}\widehat{\mathscr{L}} \longrightarrow \mathscr{U}\mathscr{L}$ such that there is an invertible modification

$$\begin{tikzcd}
& \mathrm{Id}_{\mathfrak{C}} \arrow[dl, "\widehat{\eta}"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\cong"] & \\
\widehat{\mathscr{U}}\widehat{\mathscr{L}} \arrow[rr, "\alpha" description] & & \mathscr{U}\mathscr{L}
\end{tikzcd}$$

Without losing generality, we assume that $\mathtt{E} \circ \widehat{\mathscr{L}} = \mathscr{L}$. Then we define $\alpha := (\mathscr{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathscr{L}})(\eta\widehat{\mathscr{U}}\widehat{\mathscr{L}})$. Indeed,

$$\alpha\widehat{\eta} = (\mathscr{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathscr{L}})\left(\eta\widehat{\mathscr{U}}\widehat{\mathscr{L}}\right)(\widehat{\eta}) \cong (\mathscr{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathscr{L}})(\mathscr{U}\mathscr{L}\widehat{\eta})(\eta) \cong (\mathscr{U}\mathtt{E}\widehat{\varepsilon}\widehat{\mathscr{L}})\left(\mathscr{U}\mathtt{E}\widehat{\mathscr{L}}\widehat{\eta}\right)(\eta) \cong \eta$$

Therefore, if η_X is a pseudosection, so is $\widehat{\eta}_X$. And, as mentioned, by Theorem 3.2.15, if $\widehat{\eta}_X$ is a pseudosection, it is an equivalence. □

Let $\widehat{\mathscr{T}}$ be the idempotent pseudomonad induced by $\widehat{\mathscr{L}} \dashv \widehat{\mathscr{U}}$ and $\mathscr{T}$ the pseudomonad induced by $\mathscr{L} \dashv \mathscr{U}$. Then Lemma 3.2.17 could be written as follows:

If X is an object of $\mathfrak{C}$ that can be endowed with a $\mathscr{T}$-pseudoalgebra structure, then X can be endowed with a $\widehat{\mathscr{T}}$-pseudoalgebra structure, provided that there is a pseudonatural equivalence $\mathtt{E}\widehat{\mathscr{L}} \simeq \mathscr{L}$.

Theorem 3.2.18. *Let $(\mathscr{L} \dashv \mathscr{U}, \eta, \varepsilon)$ and $(\widehat{\mathscr{L}} \dashv \widehat{\mathscr{U}}, \widehat{\eta}, \widehat{\varepsilon})$ be biadjunctions such that their right biadjoints are local equivalences. If there is a pseudonatural equivalence*

$$\begin{tikzcd}
\mathfrak{A} & & \mathfrak{B} \arrow[ll, "\mathtt{E}" description] \\
& \mathfrak{C} \arrow[ul, "\mathscr{L}" description] \arrow[ur, "\widehat{\mathscr{L}}" description] \arrow[u, phantom, "\simeq"] &
\end{tikzcd}$$

then $\mathtt{E}$ is left biadjoint to a pseudofunctor $\mathtt{R}$ which is a local equivalence.

Proof. It is enough to define $\mathtt{R} := \widehat{\mathscr{L}}\mathscr{U}$. By Lemma 3.2.17, $(\widehat{\eta}\mathscr{U}) : \mathscr{U} \longrightarrow \widehat{\mathscr{U}}\widehat{\mathscr{L}}\mathscr{U} = \widehat{\mathscr{U}}\mathtt{R}$ is a pseudonatural equivalence. Thereby we get

$$\mathfrak{A}(\mathtt{E}(b),a) \simeq \mathfrak{A}(\mathtt{E}\widehat{\mathscr{L}}\widehat{\mathscr{U}}(b),a) \simeq \mathfrak{A}(\mathscr{L}\widehat{\mathscr{U}}(b),a) \simeq \mathfrak{C}(\widehat{\mathscr{U}}(b),\mathscr{U}(a)) \simeq \mathfrak{C}(\widehat{\mathscr{U}}(b),\widehat{\mathscr{U}}\mathtt{R}(a)) \simeq \mathfrak{B}(b,\mathtt{R}(a)).$$

This completes the proof that $\mathtt{R}$ is right biadjoint to $\mathtt{E}$. ☐

Assume that $\mathscr{A} : \mathfrak{A} \to \mathfrak{B}$ and $\mathscr{B} : \mathfrak{B} \to \mathfrak{C}$ are pseudomonadic pseudofunctors, and their induced pseudomonads are idempotent. Then it is obvious that $\mathscr{B} \circ \mathscr{A} : \mathfrak{A} \to \mathfrak{C}$ is also pseudomonadic and induces an idempotent pseudomonad. Indeed, by Theorem 3.2.14, this statement is equivalent to: *compositions of right biadjoint local equivalences are right biadjoint local equivalences as well.*

Corollary 3.2.19. *Assume that there is a pseudonatural equivalence*

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, phantom, "\simeq" near end, shift right=0ex, draw=none] & \mathfrak{H} \arrow[l, "\mathtt{E}" description] \\
\mathfrak{B} \arrow[u, "\mathscr{L}_{\mathscr{A}}" description] & \mathfrak{C} \arrow[l, "\mathscr{L}_{\mathscr{B}}" description] \arrow[u, "\mathscr{L}_{\mathscr{C}}" description]
\end{tikzcd}$$

such that $\mathscr{L}_{\mathscr{A}} \dashv \mathscr{A}$, $\mathscr{L}_{\mathscr{B}} \dashv \mathscr{B}$ and $\mathscr{L}_{\mathscr{C}} \dashv \mathscr{C}$ are pseudomonadic biadjunctions inducing idempotent pseudomonads $\mathscr{T}_{\mathscr{A}}, \mathscr{T}_{\mathscr{B}}, \mathscr{T}_{\mathscr{C}}$. Then $\mathtt{E} \dashv \mathtt{R}$ and $\mathtt{R}$ is a local equivalence.

In particular, if (X,a) is a $\mathscr{T}_{\mathscr{B}}$-pseudoalgebra that can be endowed with a $\mathscr{T}_{\mathscr{A}}$-pseudoalgebra structure, then X can be endowed with a $\mathscr{T}_{\mathscr{C}}$-pseudoalgebra structure as well.

Lemma 3.2.17 and Corollary 3.2.19 are results on our formal approach to descent theory, *i.e.* they give conditions to decide whether a given object can be endowed with a pseudoalgebra structure. In fact, most of the theorems proved in this paper are consequences of successive applications of these results, including Bénabou-Roubaud theorem and other theorems within the context of [51, 52]. However it does not deal with the technical “almost descent” aspects, which follow from the results on $\mathfrak{F}$-comparisons below.

3.2.20 Comparisons inside special classes of morphisms

Instead of restricting attention to objects that can be endowed with a pseudoalgebra structure, we often are interested in almost descent and descent objects as well. In the context of idempotent pseudomonads, these are objects that possibly do not have pseudoalgebra structure but have comparison 1-cells belonging to special classes of morphisms.

In this subsection, every 2-category $\mathfrak{H}$ is assumed to be endowed with a special subclass of morphisms $\mathfrak{F}_{\mathfrak{H}}$ satisfying the following properties:

- Every equivalence of $\mathfrak{H}$ belongs to $\mathfrak{F}_{\mathfrak{H}}$;

- $\mathfrak{F}_{\mathfrak{H}}$ is closed under compositions and under isomorphisms;

- If fg and f belongs to $\mathfrak{F}_{\mathfrak{H}}$, g is also in $\mathfrak{F}_{\mathfrak{H}}$.

If f is a morphism of $\mathfrak{H}$ that belongs to $\mathfrak{F}_{\mathfrak{H}}$, we say that f is an $\mathfrak{F}_{\mathfrak{H}}$*-morphism*.

Definition 3.2.21. Let $(\mathscr{T},\mu,\eta,\Lambda,\rho,\Gamma)$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$. An object X is an $(\mathfrak{F}_{\mathfrak{H}},\mathscr{T})$*-object* if the comparison $\eta_X : X \to \mathscr{T}(X)$ is an $\mathfrak{F}_{\mathfrak{H}}$-morphism.

We say that a pseudofunctor $\mathscr{E} : \mathfrak{H} \to \mathfrak{H}$ *preserves* $(\mathfrak{F}_{\mathfrak{H}},\mathscr{T})$*-objects* if it takes $(\mathfrak{F}_{\mathfrak{H}},\mathscr{T})$-objects to $(\mathfrak{F}_{\mathfrak{H}},\mathscr{T})$-objects.

Theorem 3.2.22 is a commutativity result for $(\mathfrak{F}_{\mathfrak{H}},\mathscr{T})$-objects. Similarly to Corollary 3.2.19, it follows from the construction given in the proof of Lemma 3.2.17, although it requires some extra hypotheses.

Theorem 3.2.22. *Let*

$$\begin{tikzcd}
\mathfrak{A} & \mathfrak{H} \arrow[l, "\mathrm{E}"'] \\
\mathfrak{B} \arrow[u, "\mathscr{L}_{\mathscr{A}}"] \arrow[ur, phantom, "\simeq"] & \mathfrak{C} \arrow[l, "\mathscr{L}_{\mathscr{B}}"] \arrow[u, "\mathscr{L}_{\mathscr{C}}"']
\end{tikzcd}$$

be a pseudonatural equivalence such that $\mathscr{L}_{\mathscr{A}} \dashv \mathscr{A}$*,* $\mathscr{L}_{\mathscr{B}} \dashv \mathscr{B}$ *and* $\mathscr{L}_{\mathscr{C}} \dashv \mathscr{C}$ *are biadjunctions inducing pseudomonads* $\mathscr{T}_{\mathscr{A}}, \mathscr{T}_{\mathscr{B}}, \mathscr{T}_{\mathscr{C}}$*. Also, we denote by* $\mathscr{T}$ *the pseudomonad induced by the biadjunction* $\mathscr{L}_{\mathscr{A}}\mathscr{L}_{\mathscr{B}} \dashv \mathscr{B}\mathscr{A}$*.*

Assume that all the right biadjoints are local equivalences, $\mathscr{B}$ *takes* $\mathfrak{F}_{\mathfrak{B}}$*-morphisms to* $\mathfrak{F}_{\mathfrak{C}}$*-morphisms and* $\mathscr{T}_{\mathscr{C}}$ *preserves* $(\mathfrak{F}_{\mathfrak{C}},\mathscr{T})$*-objects. If* X *is a* $(\mathfrak{F}_{\mathfrak{C}},\mathscr{T}_{\mathscr{B}})$*-object of* $\mathfrak{C}$ *and* $\mathscr{L}_{\mathscr{B}}(X)$ *is a* $(\mathfrak{F}_{\mathfrak{B}},\mathscr{T}_{\mathscr{A}})$*-object, then* X *is a* $(\mathfrak{F}_{\mathfrak{C}},\mathscr{T}_{\mathscr{C}})$*-object as well.*

Proof. By the proof of Lemma 3.2.17, there is a pseudonatural transformation $\alpha : \mathscr{T}_{\mathscr{C}} \longrightarrow \mathscr{T}$ such that there is an invertible modification

$$\begin{tikzcd}
 & X \arrow[dl, "\eta^{\mathscr{C}}"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\cong"] & \\
\mathscr{T}_{\mathscr{C}} \arrow[rr, "\alpha"] & {} & \mathscr{T}
\end{tikzcd}$$

In particular, if X is an object of $\mathfrak{C}$ satisfying the hypotheses of the theorem, we get an isomorphism

$$\begin{tikzcd}
 & X \arrow[dl, "\eta^{\mathscr{C}}_X"'] \arrow[dr, "\eta_X"] \arrow[d, phantom, "\cong"] & \\
\mathscr{T}_{\mathscr{C}}(X) \arrow[rr, "\alpha_X"] & {} & \mathscr{T}(X)
\end{tikzcd}$$

in which, by the hypotheses, we conclude that $\eta_X \cong \left(\mathscr{B}\eta^{\mathscr{A}}\mathscr{L}_{\mathscr{B}}\right)_X \cdot \eta^{\mathscr{B}}_X$ is an $\mathfrak{F}_{\mathfrak{C}}$-morphism.

By the properties of the subclass $\mathfrak{F}_{\mathfrak{C}}$, it remains to prove that α_X is an $\mathfrak{F}_{\mathfrak{C}}$-morphism. Recall that α_X is defined by $\alpha_X := (\mathscr{B}\mathscr{A}\varepsilon^{\mathscr{C}}\mathscr{L}_{\mathscr{C}})_X \cdot (\eta\mathscr{T}_{\mathscr{C}})_X$, in which $\varepsilon^{\mathscr{C}}$ is the counit of the biadjunction $\mathscr{L}_{\mathscr{C}} \dashv \mathscr{C}$.

Since $(\mathscr{B}\mathscr{A}\varepsilon^{\mathscr{C}}\mathscr{L}_{\mathscr{C}})_X$ is an equivalence and, by hypothesis, $(\eta\mathscr{T}_{\mathscr{C}})_X$ is a $\mathfrak{F}_{\mathfrak{C}}$-morphism, it follows that α_X is a $\mathfrak{F}_{\mathfrak{C}}$-morphism.

This completes the proof that $\eta^{\mathscr{C}}_X$ is also an $\mathfrak{F}_{\mathfrak{C}}$-morphism. □

3.3 Pseudo-Kan Extensions

It is known that the descent category and the category of algebras are 2-categorical limits (see, for instance, [103, 104]). Thereby, our standpoint is to deal with the context of [52] strictly guided by bilimits results.

For the sake of this aim, we focus our study on the pseudomonads coming from bicategorical analogue of the notion of right Kan extension. Actually, since the concept of "right Kan extension" plays the leading role in this work, "Kan extension" means always right Kan extension, while we always make the word "left" explicit when we refer to the dual notion.

We explain below why we need to use a pseudo notion of Kan extension, instead of employing the fully developed theory of enriched Kan extensions: the natural place of (classical) descent theory is 2-CAT. Although we can construct the bilimits related to descent theory as (enriched/strict) Kan extensions of 2-functors in the 3-category of 2-categories, 2-functors, 2-natural transformations and modifications (see [103, 105]), the necessary replacements [67, 77] do not make computations and formal manipulations any easier.

Further, most of the transformations between 2-functors that are necessary in the development of the theory are pseudonatural. Thus, to work within the "strict world" without employing repeatedly coherence theorems (such as the general coherence result of [67]), we would need to add hypotheses to assure that usual Kan extensions of pseudonaturally equivalent diagrams are pseudonaturally equivalent. This is not true in most of the cases: it is easy to construct examples of pseudonaturally isomorphic diagrams such that their usual Kan extensions are not pseudonaturally equivalent. For instance, consider the 2-category $\mathfrak{A}$ below.

$$\begin{tikzcd}
d \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & c
\end{tikzcd}$$

The 2-category $\mathfrak{A}$ has no nontrivial 2-cells. Assume that $\mathfrak{B}$ is the 2-category obtained from $\mathfrak{A}$ adding an initial object s, with inclusion $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$. Now, if $*$ is the terminal category and $\nabla 2$ is the category with two objects and one isomorphism between them (*i.e.* $\nabla 2$ is the localization of the preorder 2 w.r.t.

all morphisms), then there are two 2-natural isomorphism classes of diagrams $\mathfrak{A} \to \mathsf{CAT}$ of the type below, while all such diagrams are pseudonaturally isomorphic.

$$* \rightrightarrows \nabla 2$$

These 2-natural isomorphism classes give pseudonaturally nonequivalent Kan extensions along t. More precisely, if $\mathscr{X}, \mathscr{Y} : \mathfrak{A} \to \mathsf{CAT}$ are such that $\mathscr{X}(d) = \mathscr{Y}(d) = *$, $\mathscr{X}(c) = \mathscr{Y}(c) = \nabla 2$, $\mathscr{X}(\alpha) \neq \mathscr{X}(\beta)$ and $\mathscr{Y}(\alpha) = \mathscr{Y}(\beta)$; then $\mathscr{R}an_{\mathrm{t}}\mathscr{X}(s) = \emptyset$, while $\mathscr{R}an_{\mathrm{t}}\mathscr{Y}(s) = *$. Therefore $\mathscr{R}an_{\mathrm{t}}\mathscr{X}$ and $\mathscr{R}an_{\mathrm{t}}\mathscr{Y}$ are not pseudonaturally equivalent, while $\mathscr{X}$ is pseudonaturally isomorphic to $\mathscr{Y}$.

The usual Kan extensions behave well if we add extra hypotheses related to flexible diagrams (see [8, 9, 67, 77]). However, we do not give such restrictions and technicalities. Thereby we deal with the problems natively in the tricategory 2-CAT, without employing further coherence results. The first step is, hence, to understand the appropriate notion of Kan extension in this tricategory.

3.3.1 The Definition

In a given tricategory, if $\mathrm{t} : a \to b$, $f : a \to c$ are 1-cells, we might consider that the formal right Kan extension of f along t is the right 2-reflection of f along the 2-functor $[\mathrm{t}, c] : [b, c] \to [a, c]$. That is to say, if it exists for all $f : a \to c$, the (formal) global Kan extension along $\mathrm{t} : a \to b$ would be a 2-functor $[a, c] \to [b, c]$ right 2-adjoint to $[\mathrm{t}, c] : [b, c] \to [a, c]$. But, in important cases, such concept is very restrictive, because it does not take into account the bicategorical structure of the hom-2-categories of the tricategory. Hence, it is possible to consider other notions of Kan extension, corresponding to the two other important notions of adjunction between 2-categories [41], that is to say, lax adjunction and biadjunction. For instance, Gray [42] studied the notion of lax-Kan extension.

We also consider an alternative notion of Kan extension in our tricategory 2-CAT, that is to say, the notion of pseudo-Kan extension, introduced in [77]. In our case, the need of this concept comes from the fact that, even with many assumptions, the (formal) Kan extension of a pseudofunctor may not exist. Furthermore, we prove in Section 3.4 that the descent object (descent category) and the Eilenberg-Moore object (Eilenberg-Moore category) can be easily described using our language.

Henceforth, $\mathfrak{A}, \mathfrak{B}$ *always denote small* 2*-categories.* If $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ and $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ are pseudofunctors, the (right) *pseudo-Kan extension* of $\mathscr{A}$ along t, denoted by $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}\mathscr{A}$, is, if it exists, a right bireflection of $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ along the pseudofunctor

$$[\mathrm{t}, \mathfrak{H}]_{PS} : [\mathfrak{B}, \mathfrak{H}]_{PS} \to [\mathfrak{A}, \mathfrak{H}]_{PS}.$$

A *global pseudo-Kan extension* along $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ is, hence, a right biadjoint of $[\mathrm{t}, \mathfrak{H}]_{PS}$, provided that it exists. That is to say, a pseudofunctor $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}} : [\mathfrak{A}, \mathfrak{H}]_{PS} \to [\mathfrak{B}, \mathfrak{H}]_{PS}$ such that $[\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}$. Of course, right pseudo-Kan extensions are unique up to pseudonatural equivalence.

Herein, the expression *Kan extension* refers to the usual notion of Kan extension in CAT-enriched category theory. That is to say, if $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ and $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ are 2-functors, the (right) Kan extension of $\mathscr{A}$ along t, denoted by $\mathscr{R}an_{\mathrm{t}}\mathscr{A} : \mathfrak{B} \to \mathfrak{H}$, is (if it exists) the right 2-reflection of $\mathscr{A}$ along the 2-functor $[\mathrm{t}, \mathfrak{H}]$. And the global Kan extension is a right 2-adjoint of $[\mathrm{t}, \mathfrak{H}] : [\mathfrak{B}, \mathfrak{H}] \to [\mathfrak{A}, \mathfrak{H}]$, in which $[\mathfrak{B}, \mathfrak{H}]$ denotes the 2-category of 2-functors $\mathfrak{B} \to \mathfrak{H}$, CAT-natural transformations and modifications.

If $\mathscr{R}an_{\mathrm{t}}\mathscr{A}$ exists, it is not generally true that $\mathscr{R}an_{\mathrm{t}}\mathscr{A}$ is pseudonaturally equivalent to $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}\mathscr{A}$. This is a coherence problem, related to flexible diagrams [8, 9, 67, 77] and to the construction of bilimits via strict 2-limits [103, 104]. For instance, in particular, using the results of [77], we can easily prove, as a corollary of coherence results [9, 67, 77], that, for a given pseudofunctor $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ and a 2-functor $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$, we can replace $\mathscr{A}$ by a pseudonaturally equivalent 2-functor $\mathscr{A}' : \mathfrak{A} \to \mathfrak{H}$ such that $\mathscr{R}an_{\mathrm{t}}\mathscr{A}'$ is equivalent to $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}\mathscr{A}' \simeq \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}\mathscr{A}$, provided that $\mathfrak{H}$ satisfies some completeness conditions (for instance, if $\mathfrak{H}$ is CAT-complete).

In Section 3.4 we show that the descent category, as defined in [52, 105], of a pseudocosimplicial object $D : \Delta \to \mathsf{CAT}$ is equivalent to $\mathrm{Ps}\mathscr{R}an_{\mathrm{j}}D(0)$, in which $\mathrm{j} : \Delta \to \dot{\Delta}$ is the inclusion of the category of nonempty finite ordinals into the category of finite ordinals. Observe that the Kan extension of a cosimplicial object does not give the descent object: it gives an equalizer (which is the notion of descent for dimension 1), although we might give the descent object via a Kan extension after replacing the (pseudo)cosimplicial objects by suitable strict versions of pseudocosimplicial objects as it is done in 3.4.12.

3.3.2 Factorization

Our setting often reduces to the study of right pseudo-Kan extensions of pseudofunctors $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ along t, in which $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is the full inclusion of a small 2-category $\mathfrak{A}$ into a small 2-category $\dot{\mathfrak{A}}$ which has only one extra object a.

Definition 3.3.3. [a-inclusion] A 2-functor $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is called an a-*inclusion*, if t is an inclusion of a small 2-category $\mathfrak{A}$ into a small 2-category $\dot{\mathfrak{A}}$ in which

$$\mathrm{obj}(\dot{\mathfrak{A}}) = \mathrm{obj}(\mathfrak{A}) \cup \{\mathsf{a}\}$$

is a disjoint union.

In this setting, we have factorizations for pseudo-Kan extensions along a-inclusions, which follow formally from the biadjunction $[\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}$.

Theorem 3.3.4 (Factorization). *Assume that* $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ *is an* a*-inclusion and* $([\mathrm{t}, \mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}, \eta, \varepsilon)$ *is a biadjunction. If* $\mathscr{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ *is a pseudofunctor,* $\mathsf{a} \neq b$ *and* $f : b \to \mathsf{a}$, $g : \mathsf{a} \to b$ *are morphisms of* $\dot{\mathfrak{A}}$*, we get induced "factorizations" (actually, invertible* 2*-cells):*

$$\begin{tikzcd}
\mathscr{A}(b) \arrow[rr, "\mathscr{A}(f)"] \arrow[dr, "f_{\mathscr{A}}"'] & \cong & \mathscr{A}(\mathsf{a}) \arrow[dl, "\eta^{\mathsf{a}}_{\mathscr{A}}"] \\
& \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A} \circ \mathrm{t})(\mathsf{a}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A}(\mathsf{a}) \arrow[rr, "\mathscr{A}(g)"] \arrow[dr, "\eta^{\mathsf{a}}_{\mathscr{A}}"'] & \cong & \mathscr{A}(b) \\
& \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A} \circ \mathrm{t})(\mathsf{a}) \arrow[ur, "g_{\mathscr{A}}"'] &
\end{tikzcd}$$

in which

$$f_{\mathscr{A}} := \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A} \circ \mathrm{t})(f) \circ \eta^{b}_{\mathscr{A}} \qquad\qquad g_{\mathscr{A}} := \varepsilon^{b}_{(\mathscr{A} \circ \mathrm{t})} \circ \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A} \circ \mathrm{t})(g)$$

and $\eta^{\mathsf{a}}_{\mathscr{A}}$, $\varepsilon^{b}_{(\mathscr{A} \circ \mathrm{t})}$ *are the* 1*-cells induced by the components of* η *and* ε.

Proof. By the (triangular) invertible modifications of Definition 3.2.4,

$$g_{\mathscr{A}} \circ \eta^{\mathsf{a}}_{\mathscr{A}} = \varepsilon^{b}_{(\mathscr{A}\circ \mathrm{t})} \circ \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A}\circ \mathrm{t})(g) \circ \eta^{\mathsf{a}}_{\mathscr{A}} \cong \varepsilon^{b}_{(\mathscr{A}\circ \mathrm{t})} \circ \eta^{b}_{\mathscr{A}} \circ \mathscr{A}(g) \cong \mathscr{A}(g)$$

The factorization involving $\mathscr{A}(f)$ follows from the pseudonaturality of η. □

3.3.5 Bilimits and pseudo-Kan extensions

Similarly to the usual approach for (enriched) Kan extensions, we define what should be called *pointwise (right) pseudo-Kan extension*. Then, we prove that, whenever such pointwise pseudo-Kan extensions exist, they are (equivalent to) the pseudo-Kan extensions.

Pointwise right pseudo-Kan extensions are defined via weighted bilimits, the bicategorical analogue of (enriched) weighted limits [77, 104, 105]. Thereby we list some needed results on weighted bilimits.

Definition 3.3.6. [Weighted bilimit] Let $\mathscr{W} : \mathfrak{A} \to \mathsf{CAT}$, $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ be pseudofunctors. The *(weighted) bilimit* of $\mathscr{A}$ with weight $\mathscr{W}$, denoted by $\{\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}}$, if it exists, is the birepresentation of the pseudofunctor

$$\mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT} : \qquad X \mapsto [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathfrak{H}(X, \mathscr{A}-))$$

That is to say, if it exists, a weighted bilimit is an object $\{\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}}$ of $\mathfrak{H}$ endowed with a pseudonatural equivalence (in X) $\mathfrak{H}(X, \{\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}}) \simeq [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathfrak{H}(X, \mathscr{A}-))$. Since, by the bicategorical Yoneda lemma, $\{\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}}$ is unique up to equivalence, we refer to it as *the* bilimit.

It is clear that we have the dual notion, called *weighted bicolimit*. If it exists, we denote by $\mathscr{W} *_{\mathrm{bi}} \mathscr{A}$ the weighted bicolimit of $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ weighted by $\mathscr{W} : \mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT}$, which means that there is a pseudonatural equivalence (in X)

$$[\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathfrak{H}(\mathscr{A}-, X)) \simeq \mathfrak{H}(\mathscr{W} \underset{\mathrm{bi}}{*} \mathscr{A}, X).$$

Remark 3.3.7. [Conical Bilimit] Analogously to the enriched case, if $\top = \mathscr{W} : \mathfrak{A} \to \mathsf{CAT}$ is the terminal weight, $\{\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}}$ is the *conical bilimit* of $\mathscr{A}$.

The 2-category CAT is bicategorically complete, that is to say, it has all (small) weighted bilimits. Indeed, if $\mathscr{W}, \mathscr{A} : \mathfrak{A} \to \mathsf{CAT}$ are pseudofunctors, we have that $\{\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}} \simeq [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathscr{A})$. Moreover, from the bicategorical Yoneda lemma of [104], we get the strong bicategorical Yoneda lemma.

Lemma 3.3.8 (Yoneda Lemma)**.** *Let $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ be a pseudofunctor. There is a pseudonatural equivalence (in X) $\{\mathfrak{A}(X, -), \mathscr{A}\}_{\mathrm{bi}} \simeq \mathscr{A}(X)$.*

There is one important notion remaining: we define the *(pseudo)end* of a pseudofunctor $T : \mathfrak{A} \times \mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT}$ by

$$\int_{\mathfrak{A}} T := [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathfrak{A}(-,-), T).$$

We get then some expected results: they are all analogous to the results of the enriched context of [58].

Proposition 3.3.9. *Let $\mathscr{A},\mathscr{B}:\mathfrak{A}\to\mathfrak{H}$ be pseudofunctors. There is a pseudonatural equivalence*

$$\int_{\mathfrak{A}}\mathfrak{H}(\mathscr{A}-,\mathscr{B}-)\simeq[\mathfrak{A},\mathfrak{H}]_{PS}(\mathscr{A},\mathscr{B}).$$

Proof. Firstly, observe that a pseudonatural transformation

$$\alpha:\mathfrak{A}(-,-)\longrightarrow\mathfrak{H}(\mathscr{A}-,\mathscr{B}-)$$

corresponds to a collection of 1-cells $\alpha_{(W,X)}:\mathfrak{A}(W,X)\to\mathfrak{H}(\mathscr{A}(W),\mathscr{B}(X))$ and collections of invertible 2-cells

$$\begin{aligned}\alpha_{(Y,f)}:\mathfrak{H}(\mathscr{A}(Y),\mathscr{B}(f))\alpha_{(Y,W)}&\cong\ \alpha_{(Y,X)}\mathfrak{A}(Y,f)\\ \alpha_{(f,Y)}:\mathfrak{H}(\mathscr{A}(f),\mathscr{B}(Y))\alpha_{(X,Y)}&\cong\ \alpha_{(W,Y)}\mathfrak{A}(f,Y)\end{aligned}$$

such that, for each object Y of $\mathfrak{A}$, $\alpha_{(Y,-)}$ and $\alpha_{(-,Y)}$ (with the invertible 2-cells above) are pseudonatural transformations. In other words, pseudonatural transformations are transformations which are pseudonatural in each variable.

By the bicategorical Yoneda lemma, we get what we want: such a pseudonatural transformation corresponds (up to isomorphism) to a collection of 1-cells

$$\gamma_W:=\alpha_{W,W}(\mathrm{Id}_W):\mathscr{A}(W)\to\mathscr{B}(W)$$

with (coherent) invertible 2-cells $\mathscr{B}(f)\circ\gamma_W\cong\gamma_W\circ\mathscr{A}(f)$. □

Hence, the original bicategorical Yoneda lemma may be reinterpreted: assume that $\mathscr{A}:\mathfrak{A}\to\mathsf{CAT}$ is a pseudofunctor, then we have the pseudonatural equivalence (in X):

$$\int_{\mathfrak{A}}\mathsf{CAT}(\mathfrak{A}(X,-),\mathscr{A}-)\simeq\mathscr{A}(X).$$

We also need Theorem 3.3.10 to prove that the "pointwise" pseudo-Kan extension is, indeed, a pseudo-Kan extension. This theorem is the bicategorical analogue to the Fubini theorem in the enriched context.

Theorem 3.3.10 (Fubini's Theorem)**.** *Assume that $T:\mathfrak{A}^{\mathrm{op}}\times\mathfrak{B}^{\mathrm{op}}\times\mathfrak{B}\times\mathfrak{A}\to\mathsf{CAT}$ is a pseudofunctor. Then there are pseudofunctors $T^{\mathfrak{B}}:\mathfrak{A}^{\mathrm{op}}\times\mathfrak{A}\to\mathsf{CAT}$ and $T^{\mathfrak{A}}:\mathfrak{B}^{\mathrm{op}}\times\mathfrak{B}\to\mathsf{CAT}$ such that*

$$\int_{\mathfrak{B}}T:=T^{\mathfrak{B}}(A,B)\cong\int_{\mathfrak{B}}T(A,X,X,B)\quad\textit{and}\quad\int_{\mathfrak{A}}T:=T^{\mathfrak{A}}(X,Y)\cong\int_{\mathfrak{A}}T(A,X,Y,A).$$

Furthermore,

$$\int_{\mathfrak{A}\times\mathfrak{B}}T\simeq\int_{\mathfrak{A}}\int_{\mathfrak{B}}T\simeq\int_{\mathfrak{B}}\int_{\mathfrak{A}}T.$$

Before defining pointwise pseudo-Kan extension, the following result, which is mainly used in Section 3.4, already gives a glimpse of the relation between weighted bilimits and pseudo-Kan extensions.

Theorem 3.3.11. *Let* $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$, $\mathscr{W} : \mathfrak{A} \to \mathsf{CAT}$ *be pseudofunctors. If the left pseudo-Kan extension* $\mathrm{Ps}\mathscr{L}an_{\mathrm{t}}\mathscr{W}$ *exists and* $\mathscr{A} : \mathfrak{B} \to \mathfrak{H}$ *is a pseudofunctor, then there is an equivalence*

$$\{\mathscr{W}, \mathscr{A} \circ \mathrm{t}\}_{\mathrm{bi}} \simeq \{\mathrm{Ps}\mathscr{L}an_{\mathrm{t}}\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}}$$

whenever one of the weighted bilimits exists.

Proof. Let X be an object of $\mathfrak{H}$. Assuming the existence of $\{\mathscr{W}, \mathscr{A} \circ \mathrm{t}\}_{\mathrm{bi}}$,

$$\mathfrak{H}(X, \{\mathscr{W}, \mathscr{A} \circ \mathrm{t}\}_{\mathrm{bi}}) \simeq [\mathfrak{B}, \mathfrak{H}]_{PS}(\mathscr{W}, \mathfrak{H}(X, \mathscr{A} \circ \mathrm{t}-)) \simeq [\mathfrak{A}, \mathfrak{H}]_{PS}(\mathrm{Ps}\mathscr{L}an_{\mathrm{t}}\mathscr{W}, \mathfrak{H}(X, \mathscr{A}-))$$

are pseudonatural equivalences (in X). Thereby

$$\{\mathscr{W}, \mathscr{A} \circ \mathrm{t}\}_{\mathrm{bi}} \simeq \{\mathrm{Ps}\mathscr{L}an_{\mathrm{t}}\mathscr{W}, \mathscr{A}\}_{\mathrm{bi}}.$$

The proof of the converse is analogous. □

If we consider the full 2-subcategory $\mathfrak{H}_{\mathscr{Y}}$ of $[\mathfrak{B}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$ such that the objects of $\mathfrak{H}_{\mathscr{Y}}$ are the birepresentable pseudofunctors of a 2-category $\mathfrak{H}$, the Yoneda embedding $\mathscr{Y} : \mathfrak{H} \to \mathfrak{H}_{\mathscr{Y}}$ is a biequivalence: that is to say, we can choose a pseudofunctor $I : \mathfrak{H}_{\mathscr{Y}} \to \mathfrak{H}$ and pseudonatural equivalences $\mathscr{Y}I \simeq \mathrm{Id}$ and $I\mathscr{Y} \simeq \mathrm{Id}$.

Therefore if $\mathfrak{H}$ is a bicategorically complete 2-category, given a pseudofunctor $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$, there is a pseudofunctor $\{-, \mathscr{A}\}_{\mathrm{bi}} : [\mathfrak{A}, \mathsf{CAT}]^{\mathrm{op}}_{PS} \to \mathfrak{H}$ which is unique up to pseudonatural equivalence and which gives the bilimits of $\mathscr{A}$ [77, 103]. More precisely, since we assume that $\mathfrak{H}$ has all weighted bilimits of $\mathscr{A}$, we are assuming that the pseudofunctor $L : [\mathfrak{A}, \mathsf{CAT}]^{\mathrm{op}}_{PS} \to [\mathfrak{H}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$, in which

$$L(\mathscr{W}) : \mathfrak{B}^{\mathrm{op}} \to \mathsf{CAT} : \qquad X \mapsto [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathfrak{H}(X, \mathscr{A}-))$$

is such that $L(\mathscr{W})$ has a birepresentation for every weight $\mathscr{W} : \mathfrak{A} \to \mathsf{CAT}$. Therefore L can be seen as a pseudofunctor $L : [\mathfrak{A}, \mathsf{CAT}]^{\mathrm{op}}_{PS} \to \mathfrak{H}_{\mathscr{Y}}$. Hence we can take $\{-, \mathscr{A}\}_{\mathrm{bi}} := IL$.

Definition 3.3.12. [Pointwise pseudo-Kan extension] Let $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$, $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ be pseudofunctors. The pointwise pseudo-Kan extension is defined by

$$\begin{aligned} \mathscr{R}AN_{\mathrm{t}}\mathscr{A} : \mathfrak{B} & \to \mathfrak{H} \\ X & \mapsto \{\mathfrak{B}(X, \mathrm{t}(-)), \mathscr{A}\}_{\mathrm{bi}}, \end{aligned}$$

provided that the weighted bilimit $\{\mathfrak{B}(X, \mathrm{t}(-)), \mathscr{A}\}_{\mathrm{bi}}$ exists in $\mathfrak{H}$ for every object X of $\mathfrak{B}$.

We prove below that the pointwise pseudo-Kan extension is, actually, a pseudo-Kan extension; that is to say, we have a pseudonatural equivalence

$$[\mathfrak{A}, \mathfrak{H}]_{PS}(- \circ \mathrm{t}, \mathscr{A}) \simeq [\mathfrak{B}, \mathfrak{H}]_{PS}(-, \mathscr{R}AN_{\mathrm{t}}\mathscr{A}).$$

Theorem 3.3.13. *Assume that* $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}, \mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ *are pseudofunctors. If the pointwise right pseudo-Kan extension* $\mathscr{R}AN_{\mathrm{t}}\mathscr{A}$ *is well defined, then* $\mathscr{R}AN_{\mathrm{t}}\mathscr{A} \simeq \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}\mathscr{A}$.

Proof. By the propositions presented in this section and by the definition of a pointwise Kan extension, we have the following pseudonatural equivalences (in S):

$$\begin{aligned}
[\mathfrak{B},\mathfrak{H}]_{PS}(S,\mathscr{R}AN_{\mathrm{t}}\mathscr{A}) &\simeq \int_{\mathfrak{B}} \mathfrak{H}(S(b),\mathscr{R}AN_{\mathrm{t}}\mathscr{A}(b))\\
&\simeq \int_{\mathfrak{B}} \mathfrak{H}(S(b),\{\mathfrak{B}(b,\mathrm{t}(-)),\mathscr{A}\}_{\mathrm{bi}})\\
&\simeq \int_{\mathfrak{B}} [\mathfrak{A},\mathsf{CAT}]_{PS}(\mathfrak{B}(b,\mathrm{t}(-)),\mathfrak{H}(S(b),\mathscr{A}-))\\
&\simeq \int_{\mathfrak{B}}\int_{\mathfrak{A}} \mathsf{CAT}(\mathfrak{B}(b,\mathrm{t}(a)),\mathfrak{H}(S(b),\mathscr{A}(a)))\\
&\simeq \int_{\mathfrak{A}}\int_{\mathfrak{B}} \mathsf{CAT}(\mathfrak{B}(b,\mathrm{t}(a)),\mathfrak{H}(S(b),\mathscr{A}(a)))\\
&\simeq \int_{\mathscr{A}} \mathfrak{H}(S\circ\mathrm{t}(a),\mathscr{A}(a))\\
&\simeq [\mathscr{A},\mathfrak{H}]_{PS}(S\circ\mathrm{t},\mathscr{A}).
\end{aligned}$$

More precisely, the first, fourth, sixth and seventh pseudonatural equivalences come from the fundamental equivalence of ends, while the second and third are, respectively, the definitions of the pointwise pseudo-Kan extension and the definition of bilimit. The remaining pseudonatural equivalence follows from Fubini's theorem. □

Remark 3.3.14. It is clear that Theorem 3.3.13 has a dual. That is to say, $\mathrm{Ps}\mathscr{L}an_{\mathrm{t}}\mathscr{A}(b) \simeq \mathfrak{A}(-,b) *_{\mathrm{bi}} \mathscr{A}$ whenever the weighted bicolimit $\mathfrak{A}(-,b) *_{\mathrm{bi}} \mathscr{A}$ exists.

Remark 3.3.15. By Remark 3.3.7 and Theorem 3.3.13, if $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ is a pseudofunctor, the conical bilimit of $\mathscr{A}$ is equivalent to $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A})(\mathsf{a}) \simeq \left\{\dot{\mathfrak{A}}(\mathsf{a},\mathrm{t}-),\mathscr{A}\right\}_{\mathrm{bi}}$ in which $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is the a-inclusion such that a is the initial object added to $\mathfrak{A}$.

In this paper, for simplicity, we always assume that $\mathfrak{H}$ *is a bicategorically complete* 2*-category*, or at least $\mathfrak{H}$ has enough bilimits to construct the considered (right) pseudo-Kan extensions as pointwise pseudo-Kan extensions.

Remark 3.3.16. The pointwise pseudo-Kan extension was studied originally in [77] using the Biadjoint Triangle Theorem proved therein. The construction presented above is similar to the usual approach of the enriched case [31, 58], while the argument via biadjoint triangles of [77] is not.

3.3.17 The pseudomonads induced by right pseudo-Kan extensions

Let $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ be a local equivalence (between small 2-categories) and $\mathscr{A} : \mathfrak{A} \to \mathfrak{H}$ a pseudofunctor. By the (bicategorical) Yoneda lemma, if the pseudo-Kan extension $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}\mathscr{A}$ exists, it is actually a pseudoextension. More precisely:

Theorem 3.3.18. *If* $\mathrm{t} : \mathfrak{A} \to \mathfrak{B}$ *is a local equivalence and there is a biadjunction* $[\mathrm{t},\mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}$*, its counit is a pseudonatural equivalence. Thereby* $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}} : [\mathfrak{A},\mathfrak{H}]_{PS} \to \left[\dot{\mathfrak{A}},\mathfrak{H}\right]_{PS}$ *is pseudomonadic (a local equivalence) and the induced pseudomonad, denoted by* $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(-\circ\mathrm{t})$*, is idempotent.*

Proof. It follows from the (bicategorical) Yoneda lemma. By Lemma 3.3.8, if X is an object of $\mathfrak{A}$, $\{\mathfrak{B}(\mathrm{t}(X),\mathrm{t}-),\mathscr{A}\}_{\mathrm{bi}} \simeq \{\mathfrak{A}(X,-),\mathscr{A}\}_{\mathrm{bi}} \simeq \mathscr{A}(X)$. □

Our interest is to study the objects of $[\mathfrak{B},\mathfrak{H}]_{PS}$ that can be endowed with $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(-\circ\mathrm{t})$-pseudoalgebra structure, that is to say, the image of the forgetful Eilenberg-Moore 2-functor Ps-$\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(-\circ\mathrm{t})$-$\mathsf{Alg} \to [\mathfrak{B},\mathfrak{H}]_{PS}$.

Definition 3.3.19. [Effective Diagrams] Let $\mathrm{t}:\mathfrak{A}\to\mathfrak{B}, \mathscr{A}:\mathfrak{B}\to\mathfrak{H}$ be pseudofunctors. $\mathscr{A}:\mathfrak{B}\to\mathfrak{H}$ is of *effective* t-*descent* if $\mathscr{A}$ can be endowed with a $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(-\circ\mathrm{t})$-pseudoalgebra structure.

We now can apply the results of Section 3.2 on idempotent pseudomonads. Firstly, by Theorem 3.2.15, we can easily study the $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(-\circ\mathrm{t})$-pseudoalgebra structures on diagrams, using the unit of the biadjunction $[\mathrm{t},\mathfrak{H}]_{PS} \dashv \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}$.

Theorem 3.3.20. *Let* $\mathrm{t}:\mathfrak{A}\to\mathfrak{B}$ *be a local equivalence and* $\mathscr{A}:\dot{\mathfrak{A}}\to\mathfrak{H}$ *a pseudofunctor. The following conditions are equivalent:*

- $\mathscr{A}$ *is of effective* t-*descent;*

- *The component of the unit on* $\mathscr{A}$*/comparison* $\eta_{\mathscr{A}}:\mathscr{A}\to\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A}\circ\mathrm{t})$ *is a pseudonatural equivalence;*

- *The comparison* $\eta_{\mathscr{A}}:\mathscr{A}\to\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A}\circ\mathrm{t})$ *is a pseudonatural pseudosection.*

Moreover, the component of the unit $\eta_{\mathscr{A}}:\mathscr{A}\longrightarrow\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A}\circ\mathrm{t})$ is a pseudonatural equivalence if and only if all components of $\eta_{\mathscr{A}}$ are equivalences. But, by Theorem 3.3.18, assuming that $\mathrm{t}:\mathfrak{A}\to\dot{\mathfrak{A}}$ is an a-inclusion, $\eta_{\mathscr{A}}^{b}$ is an equivalence for all b in $\mathfrak{A}$. Thereby we get:

Lemma 3.3.21. *Let* $\mathrm{t}:\mathfrak{A}\to\dot{\mathfrak{A}}$ *be an* a-*inclusion. If* $\mathscr{A}:\dot{\mathfrak{A}}\to\mathfrak{H}$ *is a pseudofunctor,* $\mathscr{A}$ *is of effective* t-*descent if and only if* $\eta_{\mathscr{A}}^{\mathsf{a}}:\mathscr{A}(\mathsf{a})\to\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A}\circ\mathrm{t})(\mathsf{a})$ *is an equivalence.*

3.3.22 Commutativity

Let $\mathrm{t}:\mathfrak{A}\to\dot{\mathfrak{A}}$ and $\mathrm{h}:\mathfrak{B}\to\dot{\mathfrak{B}}$ be, respectively, an a-inclusion and a b-inclusion. Unless we explicit otherwise, henceforth we always consider right pseudo-Kan extensions along such type of inclusions.

In general, we have that (see [105]): $\left[\dot{\mathfrak{A}}\times\dot{\mathfrak{B}},\mathfrak{H}\right]_{PS}\approx\left[\dot{\mathfrak{A}},\left[\dot{\mathfrak{B}},\mathfrak{H}\right]_{PS}\right]_{PS}\cong\left[\dot{\mathfrak{B}},\left[\dot{\mathfrak{A}},\mathfrak{H}\right]_{PS}\right]_{PS}$. Thereby every pseudofunctor $\mathscr{A}:\dot{\mathfrak{A}}\times\dot{\mathfrak{B}}\to\mathfrak{H}$ can be seen (up to pseudonatural equivalence) as a pseudofunctor $\mathscr{A}:\dot{\mathfrak{A}}\to\left[\dot{\mathfrak{B}},\mathfrak{H}\right]_{PS}$. Also, $\mathscr{A}:\dot{\mathfrak{A}}\to\left[\dot{\mathfrak{B}},\mathfrak{H}\right]_{PS}$ can be seen as a pseudofunctor $\mathscr{A}:\dot{\mathfrak{B}}\to\left[\dot{\mathfrak{A}},\mathfrak{H}\right]_{PS}$.

Applying our formal approach of Section 3.2 to our context of pseudo-Kan extensions, we get theorems on commutativity as we show below.

Theorem 3.3.23. *If* $\mathscr{A}:\dot{\mathfrak{A}}\to\mathfrak{H}$ *is an effective* t-*descent pseudofunctor and* $\mathscr{T}$ *is an idempotent pseudomonad on* $\mathfrak{H}$ *such that* $\mathscr{A}\circ\mathrm{t}$ *can be factorized through* Ps-$\mathscr{T}$-$\mathsf{Alg}\to\mathfrak{H}$*, then* $\mathscr{A}(\mathsf{a})$ *can be endowed with a* $\mathscr{T}$*-pseudoalgebra structure.*

Proof. Let $\mathscr{L} \dashv \mathscr{U}$ be the biadjunction induced by $\mathscr{T}$ and $\widehat{\mathfrak{H}} := \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ (see Definition 3.2.11 and Theorem 3.2.12). Observe that the pseudonatural equivalence

$$\begin{tikzcd}
\left[\mathfrak{A},\widehat{\mathfrak{H}}\right]_{PS} & & \left[\dot{\mathfrak{A}},\widehat{\mathfrak{H}}\right]_{PS} \arrow[ll, "{[\mathrm{t},\widehat{\mathfrak{H}}]_{PS}}"'] \\
 & \simeq & \\
\left[\mathfrak{A},\mathfrak{H}\right]_{PS} \arrow[uu, "{[\mathfrak{A},\mathscr{L}]_{PS}}"] & & \left[\dot{\mathfrak{A}},\mathfrak{H}\right]_{PS} \arrow[ll, "{[\mathrm{t},\mathfrak{H}]_{PS}}"'] \arrow[uu, "{[\dot{\mathfrak{A}},\mathscr{L}]_{PS}}"']
\end{tikzcd}$$

satisfies the hypotheses of Corollary 3.2.19.

If $\mathscr{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is an effective t-descent pseudofunctor such that all the objects of the image of $\mathscr{A} \circ \mathrm{t}$ have $\mathscr{T}$-pseudoalgebra structure, it means that $\mathscr{A}$ satisfies the hypotheses of Corollary 3.2.19. *I.e.* $\mathscr{A}$ is a $\mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(-\circ \mathrm{t})$-pseudoalgebra that can be endowed with a $[\mathfrak{A},\mathscr{T}]_{PS}$-pseudoalgebra structure. Thereby, by Corollary 3.2.19, $\mathscr{A}$ can be endowed with a $\left[\dot{\mathfrak{A}},\mathscr{T}\right]_{PS}$-pseudoalgebra structure. □

Corollary 3.3.24. *Let* $\mathscr{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}},\mathfrak{H}\right]_{PS}$ *be an effective* t*-descent pseudofunctor such that the diagrams in the image of* $\mathscr{A} \circ \mathrm{t}$ *are of effective* h*-descent, then* $\mathscr{A}(\mathsf{a})$ *is of effective* h*-descent as well.*

Corollary 3.3.25. *Assume that the pseudofunctors* $\widehat{\mathscr{A}} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}},\mathfrak{H}\right]_{PS}$ *and* $\bar{\mathscr{A}} : \dot{\mathfrak{B}} \to \left[\dot{\mathfrak{A}},\mathfrak{H}\right]_{PS}$ *are mates such that the diagrams in the image of* $\widehat{\mathscr{A}} \circ \mathrm{t}$ *and* $\bar{\mathscr{A}} \circ \mathrm{h}$ *are respectively of effective* h*- and* t*-descent. We have that* $\widehat{\mathscr{A}}(\mathsf{a})$ *is of effective* h*-descent if and only if* $\bar{\mathscr{A}}(\mathsf{b})$ *is of effective* t*-descent.*

3.3.26 Almost descent pseudofunctors

Recall that a 1-cell in a 2-category $\mathfrak{H}$ is called faithful/fully faithful if its images by the (covariant) representable 2-functors are faithful/fully faithful.

Definition 3.3.27. Let $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ be an a-inclusion. A pseudofunctor $\mathscr{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of *almost* t*-descent*/t*-descent* if $\eta^{\mathsf{a}}_{\mathscr{A}} : \mathscr{A}(\mathsf{a}) \to \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A} \circ \mathrm{t})(\mathsf{a})$ is faithful/fully faithful.

Consider the class $\mathfrak{F}_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}$ of pseudonatural transformations in $\left[\dot{\mathfrak{A}},\mathfrak{H}\right]_{PS}$ whose components are faithful. This class satisfies the properties described in 3.2.20. Also, a pseudofunctor $\mathscr{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of almost descent if and only if $\mathscr{A}$ is a $(\mathfrak{F}_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}, \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A} \circ \mathrm{t}))$-object.

Analogously, if we take the class $\mathfrak{F}'_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}$ of objectwise fully faithful pseudonatural transformations, $\mathscr{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of descent if and only if $\mathscr{A}$ is a $(\mathfrak{F}'_{[\dot{\mathfrak{A}},\mathfrak{H}]_{PS}}, \mathrm{Ps}\mathscr{R}an_{\mathrm{t}}(\mathscr{A} \circ \mathrm{t}))$-object.

Since in our context of right pseudo-Kan extensions along local equivalences the hypotheses of Theorem 3.2.22 hold, we get the corollaries below. Again, we are considering full inclusions $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$, $\mathrm{h} : \mathfrak{B} \to \dot{\mathfrak{B}}$ as in 3.3.22.

Corollary 3.3.28. *Let* $\mathscr{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}},\mathfrak{H}\right]_{PS}$ *be an almost* t*-descent pseudofunctor such that the pseudofunctors in the image of* $\mathscr{A} \circ \mathrm{t}$ *are of almost* h*-descent. In this case,* $\mathscr{A}(\mathsf{a})$ *is also of almost* h*-descent.*

Similarly, if $\mathscr{A}$ *is of* t*-descent and the pseudofunctors of the image of* $\mathscr{A} \circ \mathrm{t}$ *are of* h*-descent, then* $\mathscr{A}(\mathsf{a})$ *is of* h*-descent as well.*

Corollary 3.3.29. *Assume that the mates* $\widehat{\mathscr{A}} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}$ *and* $\bar{\mathscr{A}} : \dot{\mathfrak{B}} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ *are such that the diagrams in the image of* $\widehat{\mathscr{A}} \circ \mathsf{t}$ *and* $\bar{\mathscr{A}} \circ \mathsf{h}$ *are respectively of almost* h- *and* t-*descent. In this case,*

$$\widehat{\mathscr{A}}(\mathsf{a}) \text{ is of almost } \mathsf{h}\text{-descent if and only if } \bar{\mathscr{A}}(\mathsf{b}) \text{ is of almost } \mathsf{t}\text{-descent.}$$

If, furthermore, the pseudofunctors in the image of $\widehat{\mathscr{A}} \circ \mathsf{t}$ *and* $\bar{\mathscr{A}} \circ \mathsf{h}$ *are respectively of* h- *and* t-*descent, then:*

$$\widehat{\mathscr{A}}(\mathsf{a}) \text{ is of } \mathsf{h}\text{-descent if and only if } \bar{\mathscr{A}}(\mathsf{b}) \text{ is of } \mathsf{t}\text{-descent.}$$

3.4 Descent Objects

In this section, we give a description of the descent category, as defined in classical descent theory, via pseudo-Kan extensions. The results of the first part of this section is hence important to fit the context of [51, 52] within our framework.

Let $\mathsf{j} : \Delta \to \dot{\Delta}$ be the full inclusion of the category of finite nonempty ordinals into the category of finite ordinals and order preserving functions. Recall that $\dot{\Delta}$ is generated by its degeneracy and face maps. That is to say, $\dot{\Delta}$ is generated by the diagram

$$\begin{tikzcd}
\mathsf{0} \arrow[r, "d=d^0"] & \mathsf{1} \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & \mathsf{2} \arrow[l, "s^0"'] \arrow[r, shift left=2, "d^0"] \arrow[r, "d^1"] \arrow[r, shift right=2, "d^2"'] & \mathsf{3} \arrow[l, bend right=50, "s^0"'] \arrow[l, bend left=50, "s^1"] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \arrow[l, bend right=50] \arrow[l, bend left=30] \arrow[l, bend left=60]
\end{tikzcd}$$

with the following relations:

$$\begin{aligned}
d^k d^i &= d^i d^{k-1}, \text{ if } i < k; & \qquad d^0 d &= d^1 d; \\
s^k s^i &= s^i s^{k+1}, \text{ if } i \leq k; & s^k d^i &= \mathrm{id}, \text{ if } i = k \text{ and } i = k+1; \\
s^k d^i &= d^i s^{k-1}, \text{ if } i < k; & s^k d^i &= d^{i-1} s^k, \text{ if } i > k+1.
\end{aligned}$$

Remark 3.4.1. The category $\dot{\Delta}$ has an obvious strict monoidal structure $(+, \mathsf{0})$ that turns $(\dot{\Delta}, +, \mathsf{0}, \mathsf{1})$ into the initial object of the category of monoidal categories with a chosen monoid.

Remark 3.4.2. There is a full inclusion $\dot{\Delta} \to \mathsf{CAT}$ such that the image of each n is the corresponding ordinal. This is the reason why we may consider that $\dot{\Delta}$ is precisely the full subcategory of CAT of the finite ordinals (considered as partially ordered sets). In this context, the object n is often confused with its image which is the category

$$\mathsf{0} \to \mathsf{1} \to \mathsf{2} \to \cdots \mathsf{n} - \mathsf{1}.$$

It is important to keep in mind that $\dot{\Delta}$ is a category, but we often consider it inside the tricategory $2\text{-}\mathsf{CAT}$. More precisely, by abuse of language, $\dot{\Delta}$ and Δ denote respectively the images of the categories $\dot{\Delta}$ and Δ by the inclusion $\mathsf{CAT} \to 2\text{-}\mathsf{CAT}$. Hence $\dot{\Delta}$ is locally discrete and is not a full sub-2-category of CAT. In fact, it is clear that $\Delta(\mathsf{1}, \mathsf{n})$ is the image of n by the comonad induced by the right adjoint forgetful functor between the category of small categories and the category of sets, the counit of which is denoted by ε^{d}.

Definition 3.4.3. A pseudofunctor $\mathscr{A} : \Delta \to \mathfrak{H}$ is called a *pseudocosimplicial object* of $\mathfrak{H}$. The *descent object* of such a pseudocosimplicial object $\mathscr{A}$ is $\mathrm{Ps}\mathscr{R}an_{\mathrm{j}}\mathscr{A}(0)$.

Remark 3.4.4. Since 0 is the initial object of $\dot{\Delta}$, the weight $\dot{\Delta}(0,\mathrm{j}-)$ is terminal. By Remark 3.3.7 and Theorem 3.3.13, it implies that the descent object of $\mathscr{A} : \Delta \to \mathfrak{H}$ is its conical bilimit.

Theorem 3.4.11 shows that our definition of descent object agrees with Definition 3.4.6, which is the usual definition of the descent object [51, 105].

Definition 3.4.5. The category $\dot{\Delta}_3$ is generated by the diagram:

$$\begin{tikzcd}
0 \arrow[r, "d"] & 1 \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & 2 \arrow[l, "s^0" description] \arrow[r, shift left=2, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=2, "\partial^2"'] & 3
\end{tikzcd}$$

such that:

$$d^1 d = d^0 d; \qquad \partial^k d^i = \partial^i d^{k-1} \text{ if } \ i < k; \qquad s^0 d^0 = s^0 d^1 = \mathrm{id}.$$

We denote by $\mathrm{j}_3 : \Delta_3 \to \dot{\Delta}_3$ the full inclusion of the subcategory Δ_3 in which $\mathrm{obj}(\Delta_3) = \{1, 2, 3\}$. Still, there are obvious inclusions: $\dot{\mathrm{t}}_3 : \dot{\Delta}_3 \to \dot{\Delta}$ and $\mathrm{t}_3 : \Delta_3 \to \Delta$. Again, $\dot{\Delta}_3$ herein usually denotes the respective locally discrete 2-category.

Definition 3.4.6. We denote by $\mathbb{W} : \Delta_3 \to \mathsf{CAT}$ the weight below (defined in [105]), in which $\nabla\mathsf{n}$ denotes the localization of the category/finite ordinal n w.r.t all the morphisms.

$$\begin{tikzcd}
\nabla 1 \arrow[r, shift left=2] \arrow[r, shift right=2] & \nabla 2 \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \nabla 3
\end{tikzcd}$$

Following [105], if $\mathscr{A} : \Delta \to \mathfrak{H}$ is a pseudofunctor, we define

$$\mathscr{D}esc(\mathscr{A}) := \{\mathbb{W}, \mathscr{A} \circ \mathrm{t}_3\}_{\mathrm{bi}}.$$

Remark 3.4.7. The weight $\mathbb{W}$ is pseudonaturally equivalent to the terminal weight. Therefore, $\mathscr{D}esc(\mathscr{A})$ is by definition (equivalent to) the conical bilimit of $\mathscr{A} \circ \mathrm{t}_3$.

In order to prove Theorem 3.4.10, we need:

Proposition 3.4.8. *Let Y be any category and $\underline{Y} : \Delta_3^{\mathrm{op}} \to \mathsf{CAT}$ the constant 2-functor $\mathsf{n} \mapsto Y$. Given any (strict) 2-functor $\mathscr{B} : \Delta_3^{\mathrm{op}} \to \mathfrak{H}$ and a pseudonatural transformation $\alpha : \mathscr{A} \longrightarrow \underline{Y}$, the following equations hold:*

$$\begin{tikzcd}
 & Y \arrow[rr, leftrightarrow, "\mathrm{Id}_Y"] & & Y & & & & Y & \\
\mathscr{B}(1) \arrow[ur, "\alpha_1"] \arrow[r, phantom, "\overset{\alpha^{-1}_{d^1}}{\Longrightarrow}"] & \arrow[r, phantom, "\overset{\alpha_{d^0}}{\Longrightarrow}"] & \mathscr{B}(1) \arrow[ul, "\alpha_1"'] \arrow[ur, "\alpha_1"] \arrow[u, phantom, "="] \arrow[d, phantom, "="] \arrow[r, phantom, "\overset{\alpha^{-1}_{d^1}}{\Longrightarrow}"] & \arrow[r, phantom, "\overset{\alpha_{d^0}}{\Longrightarrow}"] & \mathscr{B}(1) \arrow[ul, "\alpha_1"'] \arrow[r, phantom, "="] & & \mathscr{B}(1) \arrow[ur, "\alpha_1"] \arrow[r, phantom, "\overset{\alpha^{-1}_{d^1}}{\Longrightarrow}"] & \arrow[r, phantom, "\overset{\alpha_{d^0}}{\Longrightarrow}"] & \mathscr{B}(1) \arrow[ul, "\alpha_1"'] \\
 & \mathscr{B}(2) \arrow[ul, "\mathscr{B}(d^1)"] \arrow[uu, "\alpha_2" description] \arrow[ur, "\mathscr{B}(d^0)"'] & \mathscr{B}(3) \arrow[l, "\mathscr{B}(\partial^2)"] \arrow[r, "\mathscr{B}(\partial^0)"'] & \mathscr{B}(2) \arrow[ul, "\mathscr{B}(d^1)"] \arrow[uu, "\alpha_2" description] \arrow[ur, "\mathscr{B}(d^0)"'] & & & & \mathscr{B}(2) \arrow[ul, "\mathscr{B}(d^1)"] \arrow[uu, "\alpha_2" description] \arrow[ur, "\mathscr{B}(d^0)"'] & \mathscr{B}(3) \arrow[l, "\mathscr{B}(\partial^1)"]
\end{tikzcd}$$

(associativity codescent equation)

$$\begin{tikzcd}
 & & Y & & \\
\mathscr{B}(1) \arrow[urr, "\alpha_1"] & \overset{\alpha^{-1}_{d^1}}{\Longrightarrow} & & \overset{\alpha_{d^0}}{\Longrightarrow} & \mathscr{B}(1) \arrow[ull, "\alpha_1"'] \\
 & & \mathscr{B}(2) \arrow[ull, "\mathscr{B}(d^1)"] \arrow[urr, "\mathscr{B}(d^0)"'] \arrow[uu, "\alpha_2"] & & \\
 & & \mathscr{B}(1) \arrow[u, "\mathscr{B}(s^0)"] & &
\end{tikzcd}
=
\begin{tikzcd}
Y \\
\\
\\
\mathscr{B}(1) \arrow[uuu, bend left, "\alpha_1"] \arrow[uuu, bend right, "\alpha_1"'] \arrow[uuu, phantom, "="]
\end{tikzcd}
\qquad \text{(identity of codescent)}$$

Proof. We start by proving the identity of codescent. Indeed, by Definition 3.2.2 of pseudonatural transformation (see [77]), since $d^0s^0 = d^1s^0 = \mathrm{id}_1$, $\mathscr{B}$ is a 2-functor and $\underline{Y}$ is constant equal to Y, we have that $\alpha_{d^0s^0} = \mathrm{Id}_{\alpha_1} = \alpha_{d^1s^0}$ which implies in particular that

$$\begin{tikzcd}
\mathscr{B}(1) \arrow[rr, "\mathscr{B}(s^0)"] \arrow[ddrr, "\alpha_1"'] & & \mathscr{B}(2) \arrow[rr, "\mathscr{B}(d^0)"] \arrow[dd, "\alpha_2"] & & \mathscr{B}(1) \arrow[ddll, "\alpha_1"] \\
 & \overset{\alpha_{s^0}}{\Longrightarrow} & & \overset{\alpha_{d^0}}{\Longrightarrow} & \\
 & & Y & &
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{B}(1) \arrow[dd, bend right, "\alpha_1"'] \arrow[dd, bend left, "\alpha_1"] \arrow[dd, phantom, "="] \\
\\
Y
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{B}(1) \arrow[rr, "\mathscr{B}(s^0)"] \arrow[ddrr, "\alpha_1"'] & & \mathscr{B}(2) \arrow[rr, "\mathscr{B}(d^1)"] \arrow[dd, "\alpha_2"] & & \mathscr{B}(1) \arrow[ddll, "\alpha_1"] \\
 & \overset{\alpha_{s^0}}{\Longrightarrow} & & \overset{\alpha_{d^1}}{\Longrightarrow} & \\
 & & Y & &
\end{tikzcd}$$

and therefore:

$$\begin{tikzcd}
\mathscr{B}(1) \arrow[ddrr, "\alpha_1"'] & & \mathscr{B}(2) \arrow[ll, "\mathscr{B}(d^1)"'] \arrow[dd, "\alpha_2"] & & \mathscr{B}(1) \arrow[ll, "\mathscr{B}(s^0)"'] \arrow[rr, "\mathscr{B}(s^0)"] \arrow[ddll, "\alpha_1"] \arrow[ddrr, "\alpha_1"'] & & \mathscr{B}(2) \arrow[rr, "\mathscr{B}(d^0)"] \arrow[dd, "\alpha_2"] & & \mathscr{B}(1) \arrow[ddll, "\alpha_1"] \\
 & \overset{\alpha^{-1}_{d^1}}{\Longrightarrow} & & \overset{\alpha^{-1}_{s^0}}{\Longrightarrow} & = & \overset{\alpha_{s^0}}{\Longrightarrow} & & \overset{\alpha_{d^0}}{\Longrightarrow} & \\
 & & Y \arrow[rrrr, leftrightarrow, "\mathrm{Id}_Y"'] & & & & Y & &
\end{tikzcd}
=
\begin{tikzcd}
 & & \mathscr{B}(2) \arrow[dll, "\mathscr{B}(d^1)"'] \arrow[drr, "\mathscr{B}(d^0)"] \arrow[dd, "\alpha_2"] & & \mathscr{B}(1) \arrow[ll, "\mathscr{B}(s^0)"'] \\
\mathscr{B}(1) \arrow[drr, "\alpha_1"'] & \overset{\alpha^{-1}_{d^1}}{\Longrightarrow} & & \overset{\alpha_{d^0}}{\Longrightarrow} & \mathscr{B}(1) \arrow[dll, "\alpha_1"] \\
 & & Y & &
\end{tikzcd}$$

is equal to the identity on α_1. This proves that the identity of codescent holds.

It remains to prove that the associativity codescent equation holds. Since, by the definition of pseudonatural transformation, we have that

$$\left(\alpha_{d^0} * \mathrm{Id}_{\mathscr{B}(\partial^2)}\right) \cdot \alpha_{\partial^2} = \alpha_{d^0\partial^2} = \alpha_{d^1\partial^0} = \left(\alpha_{d^1} * \mathrm{Id}_{\mathscr{B}(\partial^0)}\right) \cdot \alpha_{\partial^0},$$

we conclude that

$$\begin{tikzcd}
Y \arrow[rrrr, leftrightarrow, "\mathrm{Id}_Y"] & & & & Y \\
 & & = & & \\
 & \overset{\alpha_{d^0}}{\Longrightarrow} & \mathscr{B}(1) \arrow[uull, "\alpha_1"] \arrow[uurr, "\alpha_1"'] & \overset{\alpha^{-1}_{d^1}}{\Longrightarrow} & \\
 & & = & & \\
\mathscr{B}(2) \arrow[uuuu, "\alpha_2"] \arrow[uurr, "\mathscr{B}(d^0)"] & & \mathscr{B}(3) \arrow[ll, "\mathscr{B}(\partial^2)"] \arrow[rr, "\mathscr{B}(\partial^0)"'] & & \mathscr{B}(2) \arrow[uuuu, "\alpha_2"'] \arrow[uull, "\mathscr{B}(d^1)"']
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[rrrr, "\mathrm{Id}_Y"] & & & & Y \\
 & \overset{\alpha^{-1}_{\partial^2}}{\Longrightarrow} & = & \overset{\alpha_{\partial^0}}{\Longrightarrow} & \\
\mathscr{B}(2) \arrow[uu, "\alpha_2"] & & \mathscr{B}(3) \arrow[ll, "\mathscr{B}(\partial^2)"] \arrow[rr, "\mathscr{B}(\partial^0)"'] \arrow[uull, bend left, "\alpha_3"] \arrow[uurr, bend right, "\alpha_3"'] & & \mathscr{B}(2) \arrow[uu, "\alpha_2"']
\end{tikzcd}$$

holds. Since $\alpha_{d^0\partial^0} = \alpha_{d^1\partial^0}$, $\alpha_{d^1\partial^2} = \alpha_{d^1\partial^1}$, by the equality above, the left side of the associativity codescent equation is equal to

$$\begin{tikzcd}
 & & Y \arrow[rrrr, "\mathrm{Id}_Y"] & & & & Y & & & & & & Y \arrow[rrrr, "\mathrm{Id}_Y"] & & & & Y & & \\
\mathscr{B}(1) \arrow[urr, "\alpha_1"] & \overset{\alpha_{d^1}^{-1}}{\Longrightarrow} & & \overset{\alpha_{\partial^2}^{-1}}{\Longrightarrow} & = & \overset{\alpha_{\partial^0}}{\Longrightarrow} & & \overset{\alpha_{d^0}}{\Longrightarrow} & \mathscr{B}(1) \arrow[ull, "\alpha_1"'] & = & \mathscr{B}(1) \arrow[urr, "\alpha_1"] & \overset{\alpha_{d^1}^{-1}}{\Longrightarrow} & & \overset{\alpha_{\partial^1}^{-1}}{\Longrightarrow} & = & \overset{\alpha_{\partial^1}}{\Longrightarrow} & & \overset{\alpha_{d^0}}{\Longrightarrow} & \mathscr{B}(1) \arrow[ull, "\alpha_1"'] \\
 & & \mathscr{B}(2) \arrow[uu, "\alpha_2"] \arrow[ull, "\mathscr{B}(d^1)"] & & \mathscr{B}(3) \arrow[ll, "\mathscr{B}(\partial^2)"] \arrow[rr, "\mathscr{B}(\partial^0)"'] \arrow[uull, bend left, "\alpha_3"] \arrow[uurr, bend right, "\alpha_3"'] & & \mathscr{B}(2) \arrow[uu, "\alpha_2"] \arrow[urr, "\mathscr{B}(d^0)"'] & & & & & & \mathscr{B}(2) \arrow[uu, "\alpha_2"] \arrow[ull, "\mathscr{B}(d^1)"] & & \mathscr{B}(3) \arrow[ll, "\mathscr{B}(\partial^1)"] \arrow[rr, "\mathscr{B}(\partial^1)"'] \arrow[uull, bend left, "\alpha_3"] \arrow[uurr, bend right, "\alpha_3"'] & & \mathscr{B}(2) \arrow[uu, "\alpha_2"] \arrow[urr, "\mathscr{B}(d^0)"'] & &
\end{tikzcd}$$

which is clearly equal to the right side of the associativity codescent equation. □

Remark 3.4.9. One important difference between (pointwise) pseudo-Kan extensions (weighted bilimits) and (pointwise) Kan extensions (strict 2-limits) is the following: if we consider the inclusion $\mathsf{t}_2 : \Delta_2 \to \Delta$ of the full subcategory with only 1 and 2 as objects into the category Δ, then $\mathscr{L}an_{\mathsf{t}_2}\top \cong \top$ while $\mathrm{Ps}\mathscr{L}an_{\mathsf{t}_2}\top \not\simeq \top$, where, by abuse of language, $\top$ always denotes the appropriate 2-functor constantly equal to the terminal category. Actually, $\mathrm{Ps}\mathscr{L}an_{\mathsf{t}_2}\top(3)$ is equivalent to the category with only one object and one nontrivial automorphism.

Theorem 3.4.10. *Let* $\top : \Delta_3 \to \mathsf{CAT}$ *and* $\top : \Delta \to \mathsf{CAT}$ *be the terminal weights. We have that* $\mathrm{Ps}\mathscr{L}an_{\mathsf{t}_3}\top \simeq \top$.

Proof. We prove below that, given a constant 2-functor $\underline{Y} : \Delta_3 \to \mathsf{CAT}$,

$$[\Delta_3, \mathsf{CAT}]_{PS}(\Delta(\mathsf{t}_3 -, \mathsf{n}), \underline{Y}) \simeq \mathsf{CAT}(\nabla\mathsf{n}, Y)$$

which, by the dual of Theorem 3.3.13 given in Remark 3.3.14, completes our argument since it proves that $\Delta(\mathsf{t}_3 -, \mathsf{n}) *_{\mathrm{bi}} \top \simeq \nabla\mathsf{n} \simeq \top(\mathsf{n})$.

Let ε^{d} be the counit of the discrete comonad on the category of small categories (see 3.4.2), we define the functor

$$\mathsf{CAT}(\nabla\mathsf{n}, Y) \to [\Delta_3, \mathsf{CAT}]_{PS}(\Delta(\mathsf{t}_3 -, \mathsf{n}), \underline{Y}), \quad A \mapsto \xi^A, \quad (\mathfrak{x} : A \to B) \mapsto (\xi^{\mathfrak{x}} : \xi^A \Longrightarrow \xi^B)$$

in which, given a functor $A : \nabla\mathsf{n} \to Y$ and a natural transformation $\mathfrak{x} : A \to B$, ξ^A and $\xi^{\mathfrak{x}}$ are defined by:

$$\begin{array}{lll}
\xi^A_1 := A \circ \varepsilon^{\mathrm{d}}_{\mathsf{n}}, & \xi^A_{d^1} := \mathrm{Id}_{\xi_2}, & \left(\xi^A_{d^0}\right)_{f:2\to\mathsf{n}} := A(f(0) \leq f(1)), \\
\xi^A_2 := A \circ \varepsilon^{\mathrm{d}}_{\mathsf{n}} \circ \Delta(\mathsf{t}_3(d^1), \mathsf{n}), & \xi^A_{s^0} := \mathrm{Id}_{\xi_2}, & \xi^A_{\partial^0} := \mathrm{Id}_{\Delta(\mathsf{t}_3(\partial^2), \mathsf{n})} * \xi^A_{d^0}. \\
\xi^A_3 := A \circ \varepsilon^{\mathrm{d}}_{\mathsf{n}} \circ \Delta(\mathsf{t}_3(d^1\partial^2), \mathsf{n}), & \xi^A_{\partial^1} := \mathrm{Id}_{\xi_2}, &
\end{array}$$

$$\xi^{\mathfrak{x}}_1 := \mathfrak{x} * \mathrm{Id}_{\varepsilon^{\mathrm{d}}_{\mathsf{n}}}, \quad \xi^{\mathfrak{x}}_2 := \mathfrak{x} * \mathrm{Id}_{\varepsilon^{\mathrm{d}}_{\mathsf{n}} \circ \Delta(\mathsf{t}_3(d^1), \mathsf{n})}, \quad \xi^{\mathfrak{x}}_3 := \mathfrak{x} * \mathrm{Id}_{\varepsilon^{\mathrm{d}}_{\mathsf{n}} \circ \Delta(\mathsf{t}_3(d^1\partial^2), \mathsf{n})}.$$

We prove that this functor is actually an equivalence. Firstly, we define the inverse equivalence

$$[\Delta_3, \mathsf{CAT}]_{PS}(\Delta(\mathsf{t}_3 -, \mathsf{n}), \underline{Y}) \to \mathsf{CAT}(\nabla\mathsf{n}, Y), \quad \alpha \mapsto \wp^{\alpha}, \quad (\mathfrak{y} : \alpha \Longrightarrow \beta) \mapsto \left(\wp^{\mathfrak{y}} : \wp^{\alpha} \Longrightarrow \wp^{\beta}\right)$$

where $(\wp^{\mathfrak{y}})_{\mathsf{j}} := (\mathfrak{y}_1)_{\mathsf{j}}$ and $\wp^{\alpha}(\mathsf{i} \leq \mathsf{j})$ is the component of the natural transformation below on the object $(\mathsf{i},\mathsf{j}) : \mathsf{2} \to \mathsf{n}$ of $\Delta(\mathsf{t}_3(\mathsf{2}),\mathsf{n})$.

$$\begin{tikzcd}
 & Y & \\
\Delta(\mathsf{t}_3(\mathsf{1}),\mathsf{n}) \arrow[ur, "\alpha_1"] \arrow[r, phantom, "\overset{\alpha_{d^1}^{-1}}{\Longrightarrow}"] & {} \arrow[r, phantom, "\overset{\alpha_{d^0}}{\Longrightarrow}"] & \Delta(\mathsf{t}_3(\mathsf{1}),\mathsf{n}) \arrow[ul, "\alpha_1"'] \\
 & \Delta(\mathsf{t}_3(\mathsf{2}),\mathsf{n}) \arrow[uu, "\alpha_2" near start] \arrow[ul, "\Delta(d^1{,}\mathsf{n})"] \arrow[ur, "\Delta(d^0{,}\mathsf{n})"'] &
\end{tikzcd}$$

It remains to show that $\wp^{\alpha}$ defines a functor $\nabla\mathsf{n} \to Y$. Indeed, this follows from the associativity codescent equation and the identity of codescent of Proposition 3.4.8. More precisely, α satisfies the equations of this proposition, since $\Delta(\mathsf{t}_3-,\mathsf{n})$ is a 2-functor. Given $\mathsf{i} \leq \mathsf{j} \leq \mathsf{k}$ of $\nabla\mathsf{n}$, by the definition of $\wp^{\alpha}$, $\wp^{\alpha}(\mathsf{j} \leq \mathsf{k})\wp^{\alpha}(\mathsf{i} \leq \mathsf{j})$ is the component of the natural transformation of the left side of the associativity codescent equation on $(\mathsf{i},\mathsf{j},\mathsf{k}) : \mathsf{3} \to \mathsf{n}$, while the component of the right side on $(\mathsf{i},\mathsf{j},\mathsf{k})$ is equal to $\wp^{\alpha}(\mathsf{i} \leq \mathsf{k})$. Analogously, the identity of codescent implies that $\wp^{\alpha}(\mathrm{id}_{\mathsf{i}}) = \mathrm{id}_{\wp^{\alpha}(\mathsf{i})}$.

Finally, since it is clear that $\wp^{\xi^{(-)}} = \mathrm{Id}_{\mathsf{CAT}(\nabla\mathsf{n},Y)}$, the proof is completed by showing the natural isomorphism

$$\Gamma : \xi^{\wp^{(-)}} \Longrightarrow \mathrm{Id}_{[\Delta_3,\mathsf{CAT}]_{PS}(\Delta(\mathsf{t}_3-,\mathsf{n}),\underline{Y})}$$

where each component is the invertible modification defined by:

$$(\Gamma_{\alpha})_1 := \mathrm{Id}_{\alpha_1}, \quad (\Gamma_{\alpha})_2 := \alpha_{d^1}, \quad (\Gamma_{\alpha})_3 := \alpha_{d^1\partial^2}.$$

□

Theorem 3.4.11 (Descent Objects)**.** *Let* $\mathscr{A} : \Delta \to \mathfrak{H}$ *be a pseudofunctor. We have that* $\mathscr{D}esc(\mathscr{A}) \simeq \mathrm{Ps}\mathscr{R}an_{\mathsf{j}}\mathscr{A}(\mathsf{0})$.

Proof. By Remarks 3.4.4 and 3.4.7, we need to prove that the conical bilimit of $\mathscr{A}$ is equivalent to the conical bilimit of $\mathscr{A} \circ \mathsf{t}_3$. Indeed, by Theorems 3.3.11 and 3.4.10,

$$\{\top, \mathscr{A} \circ \mathsf{t}_3\}_{\mathsf{bi}} \simeq \left\{\mathrm{Ps}\mathscr{L}an_{\mathsf{t}_3}\top, \mathscr{A}\right\}_{\mathsf{bi}} \simeq \{\top, \mathscr{A}\}_{\mathsf{bi}}.$$

□

Observe that, by Theorem 3.4.11, if $\mathscr{A} : \dot{\Delta} \to \mathfrak{H}$ is a pseudofunctor, then $\mathscr{A}$ is of (almost/effective) j-descent if and only if $\mathscr{A} \circ \dot{\mathsf{t}}_3$ is of (almost/effective) j_3-descent.

3.4.12 Strict Descent Objects

To finish this section, we show how we can see descent objects via (strict/enriched) Kan extensions of 2-diagrams. Although this construction gives a few strict features of descent theory (such as the strict factorization), we do not use the results of this part in the rest of the paper (since, as explained in Section 3.3, we avoid coherence technicalities).

Clearly, unlike the general viewpoint of this paper, we have to deal closely with coherence theorems. Most of the coherence replacements used here follow from the 2-monadic approach to general coherence results [9, 67, 77]. Also, to formalize some observations of free 2-categories, we use the concept of computad, defined in [103].

The first step is actually older than the general coherence results: the strictification of a bicategory. We take the strictification of the 2-category $\dot{\Delta}_3$ and denote it by $\dot{\Delta}_{\mathrm{Str}}$. More precisely, this is defined herein as follows:

Definition 3.4.13. We denote by $\dot{\Delta}_{\mathrm{Str}}$ the locally preordered 2-category freely generated by the diagram

$$\begin{tikzcd} 0 \arrow[r, "d"] & 1 \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & 2 \arrow[l, "s^0" description] \arrow[r, shift left=2, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=2, "\partial^2"'] & 3 \end{tikzcd}$$

with the invertible 2-cells:

$$\begin{aligned} \sigma_{01} &: \ \partial^1 d^0 \cong \partial^0 d^0 & n_0 &: \ s^0 d^0 \cong \mathrm{Id}_1 \\ \sigma_{02} &: \ \partial^2 d^0 \cong \partial^0 d^1 & n_1 &: \ \mathrm{Id}_1 \cong s^0 d^1 \\ \sigma_{12} &: \ \partial^2 d^1 \cong \partial^1 d^1 & \vartheta &: \ d^1 d \cong d^0 d \end{aligned}$$

We consider the full inclusion $\mathrm{j}_{\mathrm{Str}} : \Delta_{\mathrm{Str}} \to \dot{\Delta}_{\mathrm{Str}}$ in which $\mathrm{obj}(\Delta_{\mathrm{Str}}) = \{1,2,3\}$.

Remark 3.4.14. Observe that the diagram and the invertible 2-cells described above define a computad [103] which we denote by $\mathfrak{d}$. Thereby Definition 3.4.13 is precise in the following sense: there is a forgetful functor between the category of locally groupoidal and preordered 2-categories and the category of computads. This forgetful functor has a left adjoint which gives the locally preordered and groupoidal 2-categories freely generated by each computad. The (locally groupoidal) 2-category $\dot{\Delta}_{\mathrm{Str}}$ is, by definition, the image of the computad $\mathfrak{d}$ by this left adjoint functor.

Remark 3.4.15. [[80]] Δ_{Str} is the locally groupoidal 2-category freely generated by the corresponding diagram and invertible 2-cells σ_{01}, σ_{02}, σ_{12}, n_0, n_1, since there are no equations involving just these 2-cells.

Indeed, $\dot{\Delta}_{\mathrm{Str}}$ and Δ_{Str} are strict replacements of our 2-categories $\dot{\Delta}_3$ and Δ_3 respectively. Actually, $\mathrm{j}_{\mathrm{Str}}$ is the strictification of j_3. By the construction of $\dot{\Delta}_{\mathrm{Str}}$, we get the desired main coherence result of this subsection:

Proposition 3.4.16. *There are obvious biequivalences* $\Delta_{\mathrm{Str}} \approx \Delta_3$ *and* $\dot{\Delta}_{\mathrm{Str}} \approx \dot{\Delta}_3$ *which are bijective on objects. Also, if* $\mathfrak{H}$ *is any* 2*-category,* $[\Delta_{\mathrm{Str}}, \mathfrak{H}] \to [\Delta_{\mathrm{Str}}, \mathfrak{H}]_{PS}$ *is essentially surjective.*

Moreover, for any 2*-functor* $\mathscr{C} : \Delta_{\mathrm{Str}} \to \mathsf{CAT}$*, we have an equivalence*

$$[\Delta_{\mathrm{Str}}, \mathsf{CAT}]\,(\dot{\Delta}_{\mathrm{Str}}(0, \mathrm{j}_{\mathrm{Str}}(-)), \mathscr{C}) \simeq [\Delta_{\mathrm{Str}}, \mathsf{CAT}]_{PS}\,(\dot{\Delta}_{\mathrm{Str}}(0, \mathrm{j}_{\mathrm{Str}}(-)), \mathscr{C}).$$

Corollary 3.4.17. *If* $\mathscr{A} : \Delta_{\mathrm{Str}} \to \mathfrak{H}$ *is a* 2*-functor,*

$$\mathrm{Ps}\mathscr{R}an_{\mathrm{j}_3} \check{\mathscr{A}} \simeq \mathrm{Ps}\mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}} \mathscr{A} \simeq \mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}} \mathscr{A}$$

provided that the pointwise Kan extension $\mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}}\check{\mathscr{A}}$ *exists, in which* $\check{\mathscr{A}}$ *is the composition of* $\mathscr{A}$ *with the biequivalence* $\Delta_3 \approx \Delta_{\mathrm{Str}}$.

Assuming that the pointwise Kan extension $\mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathscr{A}$ exists, $\mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathscr{A}(0)$ is called the *strict descent diagram* of $\mathscr{A}$. By the last result, the descent object of $\mathscr{A}$ is equivalent to its strict descent object provided that $\mathscr{A}$ has a strict descent object.

Remark 3.4.18. Using the strict descent object, we can construct the "strict" factorization described in Section 3.1. If $\mathscr{A} : \dot{\Delta}_{\mathrm{Str}} \to \mathfrak{H}$ is a 2-functor and $\mathfrak{H}$ has strict descent objects, we get the factorization from the universal property of the right Kan extension of $\mathscr{A} \circ \mathrm{j}_{\mathrm{Str}} : \Delta_{\mathrm{Str}} \to \mathfrak{H}$ along $\mathrm{j}_{\mathrm{Str}}$. More precisely, since $\mathrm{j}_{\mathrm{Str}}$ is fully faithful, we can consider that $\mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}}\mathscr{A} \circ \mathrm{j}_{\mathrm{Str}}$ is actually a strict extension of $\mathscr{A} \circ \mathrm{j}_{\mathrm{Str}}$. Thereby we get the factorization

$$\begin{tikzcd}
 & \mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}}(\mathscr{A} \circ \mathrm{j}_{\mathrm{Str}})(0) \arrow[d, "\mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}}(\mathscr{A} \circ \mathrm{j}_{\mathrm{Str}})(d)"] \\
\mathscr{A}(0) \arrow[ur, "\eta^{0}_{\mathscr{A}}"] \arrow[r, "\mathscr{A}(d)"'] & \mathscr{A}(1)
\end{tikzcd}$$

in which $\eta^{0}_{\mathscr{A}}$ is the comparison induced by the unit/comparison $\eta_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathscr{R}an_{\mathrm{j}_{\mathrm{Str}}}(\mathscr{A} \circ \mathrm{j}_{\mathrm{Str}})$.

Remark 3.4.19. As observed in Section 3.3.1, the Kan extension of a 2-functor $\mathscr{A} : \Delta \to \mathfrak{H}$ along j gives the equalizer of $\mathscr{A}(d^0)$ and $\mathscr{A}(d^1)$. This is a consequence of the isomorphism $\mathscr{L}an_{\mathrm{t}_2}\top \cong \top$ of Remark 3.4.9.

We get a glimpse of the explicit nature of the (strict) descent object at Theorem 3.4.20 which gives a presentation to $\dot{\Delta}_{\mathrm{Str}}$. We denote by $\dot{\mathfrak{d}}$ the locally groupoidal 2-category freely generated by the diagram and 2-cells described in Definition 3.4.13. It is important to note that $\dot{\mathfrak{d}}$ is not locally preordered. Moreover, there is an obvious 2-functor $\dot{\mathfrak{d}} \to \dot{\Delta}_{\mathrm{Str}}$, induced by the unit of the adjunction between the category of locally groupoidal 2-categories and the category of locally groupoidal and preordered 2-categories.

Theorem 3.4.20 ([80]). *Let* $\mathfrak{H}$ *be a* 2*-category. There is a bijection between* 2*-functors* $\mathscr{A} : \dot{\Delta}_{\mathrm{Str}} \to \mathfrak{H}$ *and* 2*-functors* $\mathsf{A} : \dot{\mathfrak{d}} \to \mathfrak{H}$ *satisfying the following equations:*

- *Associativity:*

$$\begin{tikzcd}
\mathsf{A}(0) \arrow[r, "\mathsf{A}(d)"] \arrow[d, "\mathsf{A}(d)"'] \arrow[dr, phantom, "\overset{\mathsf{A}(\vartheta)}{\Longrightarrow}"] & \mathsf{A}(1) \arrow[r, "\mathsf{A}(d^0)"] \arrow[d, "\mathsf{A}(d^0)"] \arrow[dr, phantom, "\overset{\mathsf{A}(\sigma_{01})}{\Longrightarrow}"] & \mathsf{A}(2) \arrow[d, "\mathsf{A}(\partial^0)"] & & \mathsf{A}(3) & \mathsf{A}(2) \arrow[l, "\mathsf{A}(\partial^0)"'] \arrow[r, equal] & \mathsf{A}(2) \\
\mathsf{A}(1) \arrow[r, "\mathsf{A}(d^1)"] \arrow[d, "\mathsf{A}(d^1)"'] \arrow[drr, phantom, "\overset{\mathsf{A}(\sigma_{12})}{\Longrightarrow}"] & \mathsf{A}(2) \arrow[r, "\mathsf{A}(\partial^1)"] & \mathsf{A}(3) \arrow[d, "\mathsf{A}(\mathrm{id}_3)"] & = & \mathsf{A}(2) \arrow[u, "\mathsf{A}(\partial^2)"] \arrow[ur, phantom, "\overset{\mathsf{A}(\sigma_{02})}{\Longrightarrow}"] & \mathsf{A}(1) \arrow[l, "\mathsf{A}(d^0)"'] \arrow[u, "\mathsf{A}(d^1)"] \arrow[r, phantom, "\overset{\mathsf{A}(\vartheta)}{\Longrightarrow}"] & {} \\
\mathsf{A}(2) \arrow[rr, "\mathsf{A}(\partial^2)"'] & & \mathsf{A}(3) & & \mathsf{A}(1) \arrow[u, "\mathsf{A}(d^1)"] \arrow[ur, phantom, "\overset{\mathsf{A}(\vartheta)}{\Longrightarrow}"] & \mathsf{A}(0) \arrow[l, "\mathsf{A}(d)"] \arrow[r, "\mathsf{A}(d)"'] \arrow[u, "\mathsf{A}(d)"] & \mathsf{A}(1) \arrow[uu, "\mathsf{A}(d^0)"']
\end{tikzcd}$$

– *Identity:*

$$\begin{tikzcd}
\mathsf{A}(0) \arrow[r, "\mathsf{A}(d)"] \arrow[d, "\mathsf{A}(d)"'] & \mathsf{A}(1) \arrow[d, "\mathsf{A}(d^1)"] \arrow[ddr, bend left, equal] \arrow[dl, phantom, "\overset{\mathsf{A}(\vartheta)}{\Longleftarrow}"] & & \mathsf{A}(0) \arrow[dd, bend right, "\mathsf{A}(d)"'] \arrow[dd, bend left, "\mathsf{A}(d)"] \arrow[dd, phantom, "="] \\
\mathsf{A}(1) \arrow[r, "\mathsf{A}(d^0)"] \arrow[drr, bend right, equal] & \mathsf{A}(2) \arrow[dr, "\mathsf{A}(s^0)"] \arrow[r, phantom, "\overset{\mathsf{A}(n_1)}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\mathsf{A}(n_0)}{\Longleftarrow}"] & = & \\
& & \mathsf{A}(1) & \mathsf{A}(1)
\end{tikzcd}$$

Remark 3.4.21. [[77]] The 2-category CAT is CAT-complete. In particular, CAT has strict descent objects. More precisely, if $\mathscr{A} : \Delta_{\mathrm{Str}} \to \mathsf{CAT}$ is a 2-functor, then

$$\left\{\dot{\Delta}_{\mathrm{Str}}(0, \mathrm{j}_{\mathrm{Str}}(-)), \mathscr{A}\right\} \cong [\Delta_{\mathrm{Str}}, \mathsf{CAT}]\left(\dot{\Delta}_{\mathrm{Str}}(0, \mathrm{j}_{\mathrm{Str}}(-)), \mathscr{A}\right).$$

Thereby, we can describe the category the strict descent object of $\mathscr{A} : \Delta_{\mathrm{Str}} \to \mathsf{CAT}$ explicitly as follows:

1. Objects are 2-natural transformations $\mathsf{W} : \dot{\Delta}_{\mathrm{Str}}(0, -) \longrightarrow \mathscr{A}$. We have a bijective correspondence between such 2-natural transformations and pairs (W, ρ_{W}) in which W is an object of $\mathscr{A}(1)$ and $\rho_{\mathsf{W}} : \mathscr{A}(d^1)(W) \to \mathscr{A}(d^0)(W)$ is an isomorphism in $\mathscr{A}(2)$ satisfying the following equations:
   - Associativity:
   $$\left(\mathscr{A}(\partial^0)(\rho_{\mathsf{W}})\right)\left(\mathscr{A}(\sigma_{02})_W\right)\left(\mathscr{A}(\partial^2)(\rho_{\mathsf{W}})\right)\left(\mathscr{A}(\sigma_{12})_W^{-1}\right) = \left(\mathscr{A}(\sigma_{01})_W\right)\left(\mathscr{A}(\partial^1)(\rho_{\mathsf{W}})\right)$$
   - Identity:
   $$\left(\mathscr{A}(n_0)_W\right)\left(\mathscr{A}(s^0)(\rho_{\mathsf{W}})\right)\left(\mathscr{A}(n_1)_W\right) = \mathrm{id}_W$$

   If $\mathsf{W} : \dot{\Delta}(0, -) \longrightarrow \mathscr{A}$ is a 2-natural transformation, we get such pair by the correspondence $\mathsf{W} \mapsto (\mathsf{W}_1(d), \mathsf{W}_2(\vartheta))$.
2. The morphisms are modifications. In other words, a morphism $\mathsf{m} : \mathsf{W} \to \mathsf{X}$ is determined by a morphism $\mathfrak{m} : W \to X$ such that $\mathscr{A}(d^0)(\mathfrak{m})\rho_{\mathsf{W}} = \rho_X \mathscr{A}(d^1)(\mathfrak{m})$.

3.5 Elementary Examples

We use some particular elementary examples of inclusions $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ for which we can study the Ps-$\mathscr{R}an_{\mathrm{t}}(- \circ \mathrm{t})$-pseudoalgebras/effective t-descent diagrams in the setting of Section 3.2. These examples are given herein.

Let $\mathfrak{H}$ be a 2-category with enough bilimits to construct our pseudo-Kan extensions as global pointwise pseudo-Kan extensions. The most simple example is taking the final category 1 and the inclusion $0 \to 1$ of the empty category/empty ordinal. In this case, a pseudofunctor $\mathscr{A} : 1 \to \mathfrak{H}$ is of effective descent if and only if this pseudofunctor (which corresponds to an object of $\mathfrak{H}$) is equivalent to the pseudofinal object of $\mathfrak{H}$.

If, instead, we take the inclusion $d^0 : 1 \to 2$ of the ordinal 1 into the ordinal 2 such that d^0 is the inclusion of the codomain object, then a pseudofunctor $\mathscr{A} : 2 \to \mathfrak{H}$ corresponds to a 1-cell of $\mathfrak{H}$ and $\mathscr{A}$ is of effective d^0-descent if and only if its image is an equivalence 1-cell. Moreover, $\mathscr{A}$ is almost

d^0-descent/d^0-descent if and only if its image is faithful/fully faithful. Precisely, the comparison morphism would be the image $\mathscr{A}(0 \xrightarrow{d} 1)$ of the only nontrivial 1-cell of 2.

Furthermore, we may consider the following 2-categories $\dot{\mathfrak{B}}$. The first one corresponds to the bilimit notion of lax-pullback, while the second corresponds to the notion of pseudopullback.

$$\begin{tikzcd} \mathsf{b} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & e \arrow[d] \\ c \arrow[r] & o \end{tikzcd} \qquad \begin{tikzcd} \mathsf{b} \arrow[r] \arrow[d] & e \arrow[d] \\ c \arrow[r] & o \end{tikzcd}$$

As explained in Remark 3.3.15, the examples above are all conical bilimits: it is clear that we can get every conical bilimit via pseudo-Kan extension. Actually, we can study the exactness of any weighted bilimit in our setting. More precisely, if $\mathscr{W} : \mathfrak{A} \to \mathsf{CAT}$ is a weight, we can define $\dot{\mathfrak{A}}$ adding an extra object a and defining

$$\dot{\mathfrak{A}}(\mathsf{a},\mathsf{a}) := * \qquad \dot{\mathfrak{A}}(\mathsf{a},b) := \mathscr{W}(b) \qquad \dot{\mathfrak{A}}(b,\mathsf{a}) := \emptyset$$

for each object b of $\mathfrak{A}$. Hence, it remains just to define the unique nontrivial composition, that is to say, we define the functor composition $\circ : \dot{\mathfrak{A}}(b,c) \times \dot{\mathfrak{A}}(\mathsf{a},b) \to \dot{\mathfrak{A}}(\mathsf{a},c)$ for each pair of objects b,c of $\mathfrak{A}$ to be the "mate" of

$$\mathscr{W}_{bc} : \dot{\mathfrak{A}}(b,c) \to \mathsf{CAT}(\mathscr{W}(b), \mathscr{W}(c)).$$

Thereby, a pseudofunctor $\mathscr{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of effective t-descent/t-descent/almost t-descent if the canonical comparison 1-cell $\mathscr{A}(\mathsf{a}) \to \{\mathscr{W}, \mathscr{A} \circ \mathsf{t}\}_{\mathrm{bi}}$ is an equivalence/fully faithful/faithful.

3.6 Eilenberg-Moore Objects

Let $\mathfrak{H}$ be a 2-category as in the last sections. The 2-category Adj such that an adjunction in a 2-category corresponds to a 2-functor $\mathsf{Adj} \to \mathfrak{H}$ is described in [97]. There is a full inclusion $\mathrm{m} : \mathsf{Mnd} \to \mathsf{Adj}$ such that monads of $\mathfrak{H}$ correspond to 2-functors $\mathsf{Mnd} \to \mathfrak{H}$. We describe this 2-category below, and we show how it (still) works in our setting. The 2-category Adj has two objects: alg and b. The hom-categories are defined as follows:

$$\mathsf{Adj}(\mathsf{b},\mathsf{b}) := \dot{\Delta} \qquad \mathsf{Adj}(\mathsf{alg},\mathsf{b}) := \Delta_{-} \qquad \mathsf{Adj}(\mathsf{alg},\mathsf{alg}) := \Delta_{-}^{+} \qquad \mathsf{Adj}(\mathsf{b},\mathsf{alg}) := \Delta^{+}$$

in which Δ_{-} denotes the subcategory of Δ with the same objects such that its morphisms preserve initial objects and, analogously, Δ_{+} is the subcategory of Δ with the same objects and last-element-preserving arrows. Finally, Δ_{-}^{+} is just the intersection of both Δ_{-} and Δ^{+}.

Then the composition of Adj is such that $\mathsf{Adj}(\mathsf{b},w) \times \mathsf{Adj}(c,\mathsf{b}) \to \mathsf{Adj}(c,w)$ is given by the usual "ordinal sum" $+$ (given by the usual strict monoidal structure of Δ) for every objects c,w of Adj and

$$\begin{array}{ll} \mathsf{Adj}(\mathsf{alg},w) \times \mathsf{Adj}(c,\mathsf{alg}) & \to \mathsf{Adj}(c,w) \\ (x,y) & \mapsto x+y-1 \\ (\phi : x \to x', \upsilon : y \to y') & \mapsto \phi \oplus \upsilon \end{array}$$

in which

$$\phi \oplus \upsilon(i) := \begin{cases} \upsilon(i), & \text{if } i < y \\ \phi(i-m) - 1 + y' & \text{otherwise.} \end{cases}$$

It is straightforward to verify that Adj is a 2-category. We denote by u the 1-cell $1 \in \mathsf{Adj}(\mathsf{alg}, \mathsf{b})$ and by l the 1-cell $1 \in \mathsf{Adj}(\mathsf{b}, \mathsf{alg})$. Also, we consider the following 2-cells

$$\dot{\Delta}(0,1) \ni n : \mathrm{id}_{\mathsf{b}} \Rightarrow ul, \qquad\qquad \Delta_{-}^{+}(1,2) \ni e : lu \Rightarrow \mathrm{id}_{\mathsf{alg}}.$$

The 2-category Mnd is defined to be the full sub-2-category of Adj with the unique object b. As mentioned above, we denote its full inclusion by $\mathrm{m} : \mathsf{Mnd} \to \mathsf{Adj}$.

Firstly, observe that $(l \dashv u, n, e)$ is an adjunction in Adj, therefore the image of $(l \dashv u, n, e)$ by a 2-functor is an adjunction. Also, if $(L \dashv U, \eta, \varepsilon)$ is an adjunction in $\mathfrak{H}$, then there is a unique 2-functor $\mathscr{A} : \mathsf{Adj} \to \mathfrak{H}$ such that $\mathscr{A}(u) := U$, $\mathscr{A}(l) := L$, $\mathscr{A}(e) := \varepsilon$ and $\mathscr{A}(u) := \eta$. Thereby, it gives a bijection between adjunctions in $\mathfrak{H}$ and 2-functors $\mathsf{Adj} \to \mathfrak{H}$ [97].

Secondly, as observed in [97], there is a similar bijection between 2-functors $\mathsf{Mnd} \to \mathfrak{H}$ and monads in the 2-category $\mathfrak{H}$. Also, if the pointwise (enriched) Kan extension of a 2-functor $\mathsf{Mnd} \to \mathfrak{H}$ along m exists, it gives the usual Eilenberg-Moore adjunction. Moreover, given a 2-functor $\mathscr{A} : \mathsf{Adj} \to \mathfrak{H}$, if the pointwise Kan extension $\mathscr{R}an_{\mathrm{m}} (\mathscr{A} \circ \mathrm{m})$ exists, the usual comparison $\mathscr{A}(\mathsf{alg}) \to \mathscr{R}an_{\mathrm{m}} (\mathscr{A} \circ \mathrm{m}) (\mathsf{alg})$ is the Eilenberg-Moore comparison 1-cell.

If, instead, $\mathscr{A} : \mathsf{Adj} \to \mathfrak{H}$ is a pseudofunctor, we also get that $\mathscr{A}(l) \dashv \mathscr{A}(u)$ and

$$\left(\mathscr{A}(l) \dashv \mathscr{A}(u), \mathfrak{a}_{ul}^{-1} \mathscr{A}(n) \mathfrak{a}_{\mathsf{b}}, \mathfrak{a}_{\mathsf{alg}}^{-1} \mathscr{A}(e) \mathfrak{a}_{lu}\right)$$

is an adjunction in $\mathfrak{H}$. The unique 2-functor $\mathscr{A}'$ corresponding to this adjunction is pseudonaturally isomorphic to $\mathscr{A}$. Furthermore, the Eilenberg-Moore object is a flexible limit as it is shown in [8].

Proposition 3.6.1 ([8])**.** *If $\mathfrak{H}$ is any 2-category, $[\mathsf{Adj}, \mathfrak{H}] \to [\mathsf{Adj}, \mathfrak{H}]_{PS}$ is essentially surjective. Moreover, for any 2-functor $\mathscr{C} : \mathsf{Adj} \to \mathsf{CAT}$, we have an equivalence*

$$[\mathsf{Adj}, \mathsf{CAT}]\,(\mathsf{Adj}(\mathsf{alg}, \mathrm{m}(-)), \mathscr{C}) \simeq [\mathsf{Adj}, \mathsf{CAT}]_{PS}\,(\mathsf{Adj}(\mathsf{alg}, \mathrm{m}(-)), \mathscr{C}).$$

Corollary 3.6.2. *If $\mathscr{A} : \mathsf{Mnd} \to \mathfrak{H}$ is a pseudofunctor,*

$$\mathrm{Ps}\mathscr{R}an_{\mathrm{j}_3} \mathscr{A} \simeq \mathrm{Ps}\mathscr{R}an_{\mathrm{m}} \check{\mathscr{A}} \simeq \mathscr{R}an_{\mathrm{m}} \check{\mathscr{A}}$$

provided that the pointwise Kan extension $\mathscr{R}an_{\mathrm{m}} \check{\mathscr{A}}$ exists, in which $\check{\mathscr{A}}$ is a 2-functor pseudonaturally isomorphic to $\mathscr{A}$.

Therefore, if $\mathfrak{H}$ has Eilenberg-Moore objects, a pseudofunctor $\mathscr{A} : \mathsf{Adj} \to \mathfrak{H}$ is of effective m-descent/m-descent/almost m-descent if and only if $\mathscr{A}(u)$ is monadic/premonadic/almost monadic. Also, the “factorizations”

$$\begin{tikzcd}
\mathscr{A}(\mathsf{b}) \arrow[rr, "\mathscr{A}(l)"] \arrow[dr, "l_{\mathscr{A}}"'] & \cong & \mathscr{A}(\mathsf{alg}) \\
& \mathrm{Ps}\mathscr{R}an_{\mathrm{m}}(\mathscr{A} \circ \mathrm{m})(\mathsf{alg}) \arrow[ur, "\eta^{\mathsf{alg}}_{\mathscr{A}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A}(\mathsf{alg}) \arrow[rr, "\mathscr{A}(u)"] \arrow[dr, "\eta^{\mathsf{alg}}_{\mathscr{A}}"'] & \cong & \mathscr{A}(\mathsf{b}) \\
& \mathrm{Ps}\mathscr{R}an_{\mathrm{m}}(\mathscr{A} \circ \mathrm{m})(\mathsf{alg}) \arrow[ur, "u_{\mathscr{A}}"'] &
\end{tikzcd}$$

described in Theorem 3.3.4 are pseudonaturally equivalent to the usual Eilenberg-Moore factorizations. Henceforth, these factorizations are called Eilenberg-Moore factorizations (even if the 2-category $\mathfrak{H}$ does not have the strict version of it).

3.7 The Beck-Chevalley Condition

With this elementary examples, we already can give generalizations of Theorems 3.8.2 and 3.1.1. We keep our setting in which $\mathsf{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is an a-inclusion as in 3.3.22.

Let $\mathcal{T}$ be an idempotent pseudomonad over the 2-category $\mathfrak{H}$. The most obvious consequence of the commutativity results of Section 3.2 is the following: if an object X of $\mathfrak{H}$ can be endowed with a $\mathcal{T}$-pseudoalgebra structure and there is an equivalence $X \to W$, then W can be endowed with a $\mathcal{T}$-pseudoalgebra as well.

In the case of pseudo-Kan extensions, we have the following: let $\mathscr{A}, \mathscr{B} : \dot{\mathfrak{A}} \to \mathfrak{H}$ be pseudofunctors. A pseudonatural transformation $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ can be seen as a pseudofunctor $\mathscr{C}_\alpha : 2 \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$. By Corollaries 3.3.24 and 3.3.28, we get the following: if $\mathscr{C}_\alpha(1)$ is of effective t-descent/t-descent/almost t-descent and the images of the mate $\dot{\mathfrak{A}} \to [2, \mathfrak{H}]_{PS}$ of $\mathscr{C}_\alpha$ are of effective d^0-descent/d^0-descent/almost d^0-descent as well, then $\mathscr{C}_\alpha(0)$ is also of effective t-descent/t-descent/almost t-descent. In Section 3.8, we show that Theorem 3.1.1 is a particular case of:

Proposition 3.7.1. *Let $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ be a pseudonatural transformation. If $\mathscr{B}$ is of effective* t*-descent/*t*-descent/almost* t*-descent and α is a pseudonatural equivalence/objectwise fully faithful/objectwise faithful, then $\mathscr{A}$ is of effective* t*-descent/*t*-descent/almost* t*-descent as well.*

Definition 3.7.2. [Beck-Chevalley condition] A pseudonatural transformation $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ satisfies the *Beck-Chevalley condition* if every 1-cell component of α is left adjoint and, for each 1-cell $f : w \to c$ of the domain of $\mathscr{A}$, the mate of the invertible 2-cell $\alpha_{_f} : \mathscr{B}(f)\alpha_{_w} \Rightarrow \alpha_{_c}\mathscr{A}(f)$ w.r.t. the adjunctions $\widehat{\alpha}^{^w} \dashv \alpha_{_w}$ and $\widehat{\alpha}^{^c} \dashv \alpha_{_c}$ is invertible.

By doctrinal adjunction [57], $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ satisfies the Beck-Chevalley condition if and only if α is itself a right adjoint in the 2-category $\left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$. In other words, we get:

Lemma 3.7.3. *Let $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ be a pseudonatural transformation and $\mathscr{C}_\alpha : 2 \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ the corresponding pseudofunctor. Consider the inclusion $\mathrm{u} : 2 \to \mathsf{Adj}$ of the morphism u. There is a pseudofunctor $\widehat{\mathscr{C}_\alpha} : \mathsf{Adj} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ such that $\widehat{\mathscr{C}_\alpha} \circ \mathrm{u} = \mathscr{C}_\alpha$ if and only if α satisfies the Beck-Chevalley condition.*

Thereby, as straightforward consequences of Corollaries 3.3.25 and 3.3.29, using the terminology of Lemma 3.7.3, we get what can be called a generalized version of Bénabou-Roubaud Theorem:

Theorem 3.7.4. *Assume that $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ is a pseudonatural transformation satisfying the Beck-Chevalley condition and all components of $\alpha\mathsf{t} = \alpha * Id_{_\mathsf{t}}$ are monadic.*

- *If $\mathscr{B}$ is of almost* t*-descent, then: $\alpha_{_\mathsf{a}}$ is of almost* m*-descent if and only if $\mathscr{A}$ is of almost* t*-descent;*

- *If $\mathscr{B}$ is of* t*-descent, then: $\alpha_{_\mathsf{a}}$ is premonadic if and only if $\mathscr{A}$ is of* t*-descent;*

– *If $\mathscr{B}$ is of effective* t*-descent, then:* α_{a} *is monadic if and only if $\mathscr{A}$ is of effective* t*-descent.*

Proof. Indeed, by the hypotheses, for each item, there is a pseudofunctor $\widehat{\mathscr{C}_\alpha} : \mathsf{Adj} \to \left[\dot{\mathfrak{A}}, \mathfrak{H}\right]_{PS}$ satisfying the hypotheses of Corollary 3.3.25 or Corollary 3.3.29. □

Remark 3.7.5. It is important to observe that the hypothesis of the theorem obviously does not include the monadicity of α_{a}, since $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$ is an a-inclusion.

3.8 Descent Theory

In this section, we establish the setting of [52] and prove all the classical results mentioned in Section 3.1 for pseudocosimplicial objects, except Theorem 3.1.2 which is postponed to Section 3.9.

Henceforth, let $\mathscr{C}, \mathbb{D}$ be categories with pullbacks and $\mathfrak{H}$ be a 2-category with the weighted bilimits whenever needed as in the last sections. In the context of [52], given a pseudofunctor $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathfrak{H}$, the morphism $p : E \to B$ of $\mathscr{C}$ is of *effective $\mathscr{A}$-descent/$\mathscr{A}$-descent/almost $\mathscr{A}$-descent* if $\mathscr{A}_p : \dot{\Delta} \to \mathfrak{H}$ is of effective j-descent/j-descent/almost j-descent, where $\mathscr{A}_p$ is the composition of the diagram

$$\mathscr{D}_p : \dot{\Delta}^{\mathrm{op}} \to \mathscr{C}$$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & E \times_p E \times_p E \arrow[l, bend right=40] \arrow[l, bend right=25] \arrow[l, bend left=30] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & E \times_p E \arrow[l, bend right=30] \arrow[l, bend left=30] \arrow[r, shift left=2] \arrow[r, shift right=2] & E \arrow[l] \arrow[r, "p"] & B
\end{tikzcd}$$

with the pseudofunctor $\mathscr{A}$, in which the diagram above is given by the pullbacks of p along itself, its projections and diagonal morphisms. By the results of Section 3.4, for $\mathfrak{H} = \mathsf{CAT}$, this definition of effective $\mathscr{A}$-descent morphism coincides with the classical one in the context of [51, 52].

We get the usual factorizations of (Grothendieck) $\mathscr{A}$-descent theory [52] from Theorem 3.3.4, although the usual strict factorization comes from Remark 3.4.18. More precisely, if $p : E \to B$ is a morphism of $\mathscr{C}$, we get:

$$\begin{tikzcd}
\mathscr{A}_p(0) = \mathscr{A}(B) \arrow[rr, "\mathscr{A}(p)" description] \arrow[dr, "\eta^0_{\mathscr{A}\circ\mathscr{D}_p}" description] & \cong & \mathscr{A}_p(1) = \mathscr{A}(E) \\
& \mathscr{D}esc_{\mathscr{A}}(p) \simeq \mathrm{Ps}\mathscr{R}an_{\mathrm{j}}(\mathscr{A}_p \circ \mathrm{j})(0) \arrow[ur, "d_{\mathscr{A}_p}" description] &
\end{tikzcd}$$

In descent theory, a *morphism* (U, α) between pseudofunctors $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathscr{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ is a pullback preserving functor $U : \mathscr{C} \to \mathbb{D}$ with a pseudonatural transformation $\alpha : \mathscr{A} \longrightarrow \mathscr{B} \circ U$. Such a morphism is called *faithful/fully faithful* if α is objectwise faithful/fully faithful.

For each morphism $p : E \to B$ of $\mathscr{C}$, a morphism (U, α) between pseudofunctors $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathscr{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ induces a pseudonatural transformation $\alpha^p : \mathscr{A}_p \longrightarrow \mathscr{B}_{U(p)}$. Of course, α^p is objectwise faithful/fully faithful if (U, α) is faithful/fully faithful.

We say that such a morphism (U, α) between pseudofunctors $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathscr{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ reflects almost descent/descent/effective descent morphisms if, whenever $U(p)$ is of almost $\mathscr{B}$-descent/$\mathscr{B}$-descent/effective $\mathscr{B}$-descent, p is of almost $\mathscr{A}$-descent/$\mathscr{A}$-descent/effective $\mathscr{A}$-descent.

Remark 3.8.1. Consider the pseudofunctor given by the basic fibration $(\ )^* : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$ in which

$$(p)^* : \mathscr{C}/B \to \mathscr{C}/E$$

is the change of base functor, given by the pullback along $p : E \to B$. For short, we say that a morphism $p : E \to B$ is of effective descent if p is of effective $(\ )^*$-descent.

In this case, a pullback preserving functor $U : \mathscr{C} \to \mathbb{D}$ induces a morphism (U, u) between the basic fibrations $(\ )^* : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$ and $(\ )^* : \mathbb{D}^{\mathrm{op}} \to \mathsf{CAT}$ in which, for each object B of $\mathscr{C}$, $\mathrm{u}_B : \mathscr{C}/B \to \mathbb{D}/U(B)$ is given by the evaluation of U. If U is faithful/fully faithful, so is the induced morphism (U, u) between the basic fibrations.

We study pseudocosimplicial objects $\mathscr{A} : \dot{\Delta} \to \mathfrak{H}$ and verify the obvious implications within the setting described above. We start with the embedding results (which are particular cases of 3.7.1):

Theorem 3.8.2 (Embedding Results)**.** *Let $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ be a pseudonatural transformation. If α is objectwise faithful and $\mathscr{B}$ is of almost* j*-descent, then so is $\mathscr{A}$. Furthermore, if $\mathscr{B}$ is of* j*-descent and α is objectwise fully faithful, then $\mathscr{A}$ is of* j*-descent as well.*

Of course, we have that, if $\mathscr{A} \simeq \mathscr{B}$, then $\mathscr{A}$ is of almost j-descent/j-descent/effective j-descent if and only if $\mathscr{B}$ is of almost j-descent/j-descent/effective j-descent as well.

Corollary 3.8.3. *Let (U, α) be a morphism between the pseudofunctors $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathfrak{H}$ and $\mathscr{B} : \mathbb{D}^{\mathrm{op}} \to \mathfrak{H}$ (as defined above).*

- *If (U, α) is faithful, it reflects almost descent morphisms;*
- *If (U, α) is fully faithful, it reflects descent morphisms;*
- *If α is a pseudonatural equivalence, (U, α) reflects and preserves effective descent morphisms, descent morphisms and almost descent morphisms.*

We finish this section by proving Bénabou-Roubaud theorems. A functor F is a *pseudosection* if there is G such that $G \circ F$ is naturally isomorphic to the identity. We use the following straightforward result:

Lemma 3.8.4 (Monadicity of pseudosections)**.** *If a pseudosection is right adjoint, then it is monadic. In particular, if $\mathscr{A}$ is a pseudocosimplicial object, then $\mathscr{A}(d^i : \mathsf{n} \to \mathsf{n}+1)$ is monadic whenever it is right adjoint.*

Proof. Assume that $G \circ F$ is isomorphic to the identity. Given an absolute colimit diagram $F \circ D$, it follows that $G \circ F \circ D \cong D$ is an absolute colimit diagram. The result follows, then, from the monadicity theorem [4].

The second part of the lemma follows from the fact that d^i is a retraction and, hence, since $\mathscr{A}$ is a pseudofunctor, $\mathscr{A}(d^0 : \mathsf{n} \to \mathsf{n}+1)$ is a pseudosection for any $i \leq \mathsf{n}$. □

Recall that 1 is a monoid of $\dot{\Delta}$, as explained in Remark 3.4.1. On one hand, the monad induced by this monoid, considered, for instance, in [104] and [71], is denoted by $\mathrm{suc} := (- + 1)$ on $\dot{\Delta}$. On the other hand, this monad induces a pseudomonad $\mathrm{Suc} := [\mathrm{suc}, \mathfrak{H}]_{PS}$ on the 2-category $[\mathrm{suc}, \mathfrak{H}]_{PS}$ of pseudocosimplicial objects of $\mathfrak{H}$. This is the 2-dimensional (dual) analogue of the notion of décalage of simplicial sets as in [32].

In particular, for each $\mathscr{A} : \dot{\Delta} \to \mathfrak{H}$ the component of the unit of Suc on $\mathscr{A}$ gives a pseudonatural transformation $\mathrm{Suc}^{\mathscr{A}} : \mathscr{A} \longrightarrow \mathscr{A} \circ \mathrm{Suc}$ whose correspondent pseudofunctor is denoted by $\mathscr{C}_{\mathscr{A}} : 2 \to \left[\dot{\Delta}, \mathfrak{H}\right]_{PS}$.

Observe that, $\mathscr{C}_{\mathscr{A}} : 2 \to \left[\dot{\Delta}, \mathfrak{H}\right]_{PS}$ is given by the mate of $\mathscr{A} \circ \mathrm{n} : 2 \times \dot{\Delta} \to \mathfrak{H}$, in which n is the mate of the unit of suc viewed as a functor $2 \to \left[\dot{\Delta}, \dot{\Delta}\right]$, defined by

$$\mathrm{n} : 2 \times \dot{\Delta} \to \dot{\Delta}$$

$$(a,b) \mapsto b+a \qquad\qquad (d, \mathrm{id}_b) \mapsto \left(d^0 : b \to (b+1)\right)$$

$$(\mathrm{id}_a, d^i) \mapsto \begin{cases} d^i : b \to (b+1), & \text{if } a = 0 \\ d^{i+1} : (b+1) \to (b+2), & \text{otherwise} \end{cases}$$

$$(\mathrm{id}_a, s^i) \mapsto \begin{cases} s^i : b \to (b+1), & \text{if } a = 0 \\ s^{i+1} : (b+1) \to (b+2), & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}[column sep=huge, row sep=large]
0 \arrow[r, "d"] \arrow[d, "d"'] & 1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "d^0"'] & 2 \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "d^0"'] & 3 \arrow[l, bend right=20, "s^0" description] \arrow[l, bend right=40, "s^1" description] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d, "d^0"'] & \cdots \arrow[l, shift right=4, bend right=5, "s^0" description] \arrow[l, bend right=25, "s^1" description] \arrow[l, bend right=45, "s^2" description] \arrow[d, "d^0"'] \\
1 \arrow[r, "d^1"'] & 2 \arrow[r, shift left=3, "d^1" description] \arrow[r, shift right=3, "d^2" description] & 3 \arrow[l, "s^1" description] \arrow[r, shift left=3, "d^1" description] \arrow[r, "d^2" description] \arrow[r, shift right=3, "d^3" description] & 4 \arrow[l, bend left=20, "s^1" description] \arrow[l, bend left=40, "s^2" description] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \arrow[l, shift left=4, bend left=5, "s^1" description] \arrow[l, bend left=25, "s^2" description] \arrow[l, bend left=45, "s^3" description]
\end{tikzcd}$$

We say that a pseudofunctor $\mathscr{A} : \dot{\Delta} \to \mathfrak{H}$ satisfies the descent shift property (or just *shift property* for short) if $\mathscr{A} \circ \mathrm{Suc}$ is of effective j-descent. We get, then, a version of Bénabou-Roubaud Theorem for pseudocosimplicial objects:

Theorem 3.8.5. *Let $\mathscr{A} : \dot{\Delta} \to \mathfrak{H}$ be a pseudofunctor satisfying the shift property. If the pseudonatural transformation* $\mathrm{Suc}^{\mathscr{A}}$ *satisfies the Beck-Chevalley condition, then the Eilenberg-Moore factorization of $\mathscr{A}(d)$ is pseudonaturally equivalent to its usual factorization of* j*-descent theory. In particular,*

- *$\mathscr{A}$ is of effective* j*-descent iff $\mathscr{A}(d)$ is monadic;*
- *$\mathscr{A}$ is of* j*-descent iff $\mathscr{A}(d)$ is premonadic;*
- *$\mathscr{A}$ is of almost* j*-descent iff the $\mathscr{A}(d)$ is almost monadic.*

Proof. By Lemma 3.8.4, the components of $\mathrm{Suc}^{\mathscr{A}}\mathrm{j} = (\mathrm{Suc}^{\mathscr{A}}) * \mathrm{Id}_{\mathrm{j}}$ are monadic. □

It is known that in the context of [52] introduced in this section, the natural morphism $E \times_p E \to E$ is always of effective $\mathscr{A}$-descent. It follows from this fact that $\mathscr{A}_p$ always satisfies the shift property. More precisely:

Lemma 3.8.6. *Let $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$ be a pseudofunctor, in which $\mathscr{C}$ is a category with pullbacks. If p is a morphism of $\mathscr{C}$, $\mathscr{A}_p$ (defined above as $\mathscr{A}_p := \mathscr{A} \circ \mathscr{D}_p$) satisfies the shift property.*

Proof. This follows from the fact that, for any pseudofunctor $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathsf{CAT}$, given a morphism $p : E \to B$ of $\mathscr{C}$, the natural morphism $E \times_p E \to E$ between the pullback of p along p and E (being a split epimorphism) is of effective $\mathscr{A}$-descent. In particular, $\mathscr{A}_p \circ \mathrm{Suc} \simeq \mathscr{A}_{E \times_B E \to E}$ is of effective j-descent. □

Thereby, by Theorem 3.8.5, the usual Bénabou-Roubaud Theorem (Theorem 3.1.3) follows from Theorem 3.8.5, as it is shown below.

Proof. Assuming that $\mathscr{A} : \mathscr{C}^{\mathrm{op}} \to \mathfrak{H}$ satisfies the hypotheses of Theorem 3.1.3, we have that, given a morphism p of $\mathscr{C}$, the Beck Chevalley condition of the theorem implies, in particular, that $\mathrm{Suc}^{\mathscr{A}_p}$ satisfies the Beck Chevalley condition. Therefore, since $\mathscr{A}_p$ satisfies the shift property, $\mathscr{A}_p(d) = \mathscr{A}(p)$ is monadic/premonadic/almost monadic iff $\mathscr{A}$ is of effective j-descent/j-descent/almost j-descent. □

Finally, the most obvious consequence of the commutativity properties is that bilimits of effective j-descent diagrams are effective j-descent diagrams. For instance, taking into account Remark 3.8.1 and realizing that pseudopullbacks of functors induce pseudopullback of overcategories we already get a weak version of Theorem 3.1.5.

Next section, we study stronger results on bilimits and apply them to descent theory.

3.9 Further on Bilimits and Descent

Henceforth, let $\mathsf{t} : \mathfrak{A} \to \dot{\mathfrak{A}}, \mathsf{h} : \mathfrak{B} \to \dot{\mathfrak{B}}$ be inclusions as in 3.3.22 and let $\mathfrak{H}$ be a bicategorically complete 2-category.

Definition 3.9.1. [Pure Structure] A morphism $f : \mathsf{a} \to b$ of $\dot{\mathfrak{A}}$ is called a t-*irreducible* morphism if $b \neq \mathsf{a}$ and f is not in the image of

$$\circ : \dot{\mathfrak{A}}(c,b) \times \dot{\mathfrak{A}}(\mathsf{a},c) \to \dot{\mathfrak{A}}(\mathsf{a},b),$$

for every $b \neq c$ in $\mathfrak{A}$.

An object c of $\mathfrak{A}$ is called a t-*pure structure object* if each 1-cell g of $\dot{\mathfrak{A}}(\mathsf{a},c)$ can be factorized through some t-irreducible morphism $f : \mathsf{a} \to b$ such that $b \neq c$. That is to say, c is a t-pure structure object if, for all $g \in \dot{\mathfrak{A}}(\mathsf{a},c)$, there are a morphism g' and a t-irreducible morphism f such that $g'f = g$.

The full sub-2-category of the t-pure structure objects of $\mathfrak{A}$ is denoted by $\mathfrak{S}_{\mathsf{t}}$, while the full sub-2-category of $\dot{\mathfrak{A}}$ of the objects that are not in $\mathfrak{S}_{\mathsf{t}}$ (including a) is denoted by $\mathfrak{I}_{\mathsf{t}}$. We have the full inclusion $\mathrm{i}_{\mathsf{t}} : \mathfrak{I}_{\mathsf{t}} \to \dot{\mathfrak{A}}$.

In particular, if $f : \mathsf{a} \to b$ is a t-irreducible morphism of $\dot{\mathfrak{A}}$, then b is an object of $\mathfrak{I}_{\mathsf{t}}$. We denote by $\mathfrak{g}_{\mathsf{t}} : \overline{\mathfrak{I}_{\mathsf{t}} \times 2} \to \mathfrak{I}_{\mathsf{t}} \times 2$ the full inclusion in which

$$\mathrm{obj}\left(\overline{\mathfrak{I}_{\mathsf{t}} \times 2}\right) := \mathrm{obj}\left(\mathfrak{I}_{\mathsf{t}} \times 2\right) - \{(\mathsf{a},0)\}.$$

Theorem 3.9.2. *Let* $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ *be an objectwise fully faithful pseudonatural transformation such that* $\mathscr{B}$ *is of effective* t*-descent. We consider the mate of* α*, denoted by* $\mathscr{C}_\alpha : \dot{\mathfrak{A}} \times 2 \to \mathfrak{H}$*. The* *pseudofunctor* $\mathcal{A}$ *is of effective* t*-descent if and only if* $\mathcal{C}_\alpha \circ (\mathrm{i}_{\mathrm{t}} \times \mathrm{Id}_2) : \mathfrak{J}_{\mathrm{t}} \times 2 \to \mathfrak{H}$ *is of effective* $\mathfrak{g}_{\mathrm{t}}$*-descent.*

Proof. Without losing generality, we prove it to $\mathfrak{H} = \mathsf{CAT}$ and get the general result via representable 2-functors. We just need to prove that $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}\mathcal{A} \circ \mathrm{t}(\mathsf{a})$ is equivalent to $\mathrm{Ps}\mathcal{R}an_{\mathfrak{g}_{\mathrm{t}}} \left(\mathcal{C}_\alpha \circ (\mathrm{i}_{\mathrm{t}} \times \mathrm{Id}_2) \circ \mathfrak{g}_{\mathrm{t}}\right)(\mathsf{a}, 0)$.

The category of pseudonatural transformations $\rho' : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \to \mathcal{A} \circ \mathrm{t}$ is equivalent to the category of pseudonatural transformations $\rho : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \longrightarrow \mathcal{B} \circ \mathrm{t}$ that can be factorized through $\alpha\mathrm{t}$, since $\alpha\mathrm{t}$ is objectwise fully faithful. Also, given $\rho : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \longrightarrow \mathcal{B} \circ \mathrm{t}$, there exists $\rho' : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \to \mathcal{A} \circ \mathrm{t}$ such that $\rho \cong (\alpha\mathrm{t})\rho'$ if and only if the image of $(\alpha\mathrm{t})_b$ is essentially surjective in the image of ρ_b for every b of $\mathfrak{A}$. Also, if such ρ' exists, it is unique up to isomorphism: it is the pseudopullback of ρ along $(\alpha\mathrm{t})$.

Actually, we claim that, for the existence of such ρ', it is (necessary and) sufficient $(\alpha\mathrm{t})_b$ be essentially surjective onto the image of ρ_b for every object b of $\mathfrak{J}_{\mathrm{t}}$. That is to say, we just need to verify the lifting property for the objects in $\mathfrak{J}_{\mathrm{t}}$.

Indeed, assume that $\rho\mathrm{i}_{\mathrm{t}}$ can be lifted by $\alpha\mathrm{t}\mathrm{i}_{\mathrm{t}}$. Given an object c of $\mathfrak{S}_{\mathrm{t}}$ and a morphism $g : \mathsf{a} \to c$, we prove that $\rho_c(g)$ is in the image of $(\alpha\mathrm{t})_c$ up to isomorphism. Actually, there is a t-irreducible morphism $f : \mathsf{a} \to b$ such that $g'f = f$ for some $g' : b \to c$ morphism of $\mathfrak{A}$, and, by hypothesis, there is an object u of $\mathcal{A}(b)$ such that $(\alpha\mathrm{t})_b(u) \cong \rho_b(f)$, thereby:

$$\rho_c(g) = \rho_c \cdot \left(\dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(g'))\right)(f) \cong \mathcal{B}(g')\rho_b(f) \cong \mathcal{B}(g')\,(\alpha\mathrm{t})_b\,(u) \cong (\alpha\mathrm{t})_c\,(\mathcal{A}(g')(u)).$$

This completes the proof that it is enough to test the lifting property for the objects in $\mathfrak{J}_{\mathrm{t}}$. Now, one should observe that, since $\mathcal{B}$ is of effective t-descent, a pseudonatural transformation

$$\mathfrak{J}_{\mathrm{t}} \times 2((\mathsf{a}, 0), \mathfrak{g}_{\mathrm{t}}-) \longrightarrow \mathcal{C}_\alpha \circ (\mathrm{i}_{\mathrm{t}} \times \mathrm{Id}_2) \circ \mathfrak{g}_{\mathrm{t}}$$

is precisely determined (up to isomorphism) by a pseudonatural transformation

$$\rho : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \longrightarrow \mathcal{B} \circ \mathrm{t}.$$

(*i.e.*, an object of $\mathcal{B}(\mathsf{a})$), such that $\rho\mathrm{i}_{\mathrm{t}}$ can be lifted by $\alpha\mathrm{t}\mathrm{i}_{\mathrm{t}}$. That is to say, as we proved, this is just a pseudonatural transformation

$$\rho' : \dot{\mathfrak{A}}(\mathsf{a}, \mathrm{t}(-)) \to \mathcal{A} \circ \mathrm{t}.$$

□

Remark 3.9.3. Definition 3.9.1 and Theorem 3.9.2 are part of a general perspective over generalizations of classical theorems of cubes and pullbacks. The exhaustive exposition of such is outside the scope of this paper.

We return to the context of Section 3.2. Let $\mathcal{T}$ be an idempotent pseudomonad on a 2-category $\mathfrak{H}$ and X be an object of $\mathfrak{H}$. We say that X is of $\mathcal{T}$*-descent* if the comparison $\eta_X : X \to \mathcal{T}(X)$ is fully faithful. It is important to note that, if $\mathcal{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of t-descent (following Definition 3.3.27), then $\mathcal{A}$ is of $\mathrm{Ps}\mathcal{R}an_{\mathrm{t}}(- \circ \mathrm{t})$-descent.

Corollary 3.9.4. *Let $\mathscr{T}$ be an idempotent pseudomonad on $\mathfrak{H}$ and $\mathscr{A} : \dot{\mathfrak{A}} \to \mathfrak{H}$ a pseudofunctor such that all the objects in the image of $\mathscr{A} \circ \mathsf{t}$ are $\mathscr{T}$-descent objects. Assume that both $\mathscr{A}, \mathscr{T} \circ \mathscr{A}$ are of effective* t*-descent. We assume that $\mathscr{A}(b)$ can be endowed with a $\mathscr{T}$-pseudoalgebra structure for every object $b \notin \mathfrak{S}_{\mathsf{t}}$ in $\mathfrak{A}$. Then $\mathscr{A}(\mathsf{a})$ can be endowed with a $\mathscr{T}$-pseudoalgebra structure.*

Corollary 3.9.5. *Let $\mathscr{A} : \dot{\mathfrak{A}} \to \left[\dot{\mathfrak{B}}, \mathfrak{H}\right]_{PS}$ be an effective* t*-descent pseudofunctor such that all the pseudofunctors in the image of $\mathscr{A} \circ \mathsf{t}$ are of* h*-descent. Furthermore, we assume that $\mathscr{A}(b)$ is of effective* h*-descent for every $b \notin \mathfrak{S}_{\mathsf{t}}$ in $\mathfrak{A}$. Then $\mathscr{A}(\mathsf{a})$ is of effective* h*-descent.*

Recall the following full inclusion of 2-categories $\mathsf{h} : \mathfrak{B} \to \dot{\mathfrak{B}}$ described in Section 3.5.

$$\begin{tikzcd}
 & e \arrow[d] & \mapsto & \mathsf{b} \arrow[r] \arrow[d] & e \arrow[d] \\
c \arrow[r] & o & & c \arrow[r] & o
\end{tikzcd} \tag{$\mathfrak{P}$}$$

As explained there, a diagram $\dot{\mathfrak{B}} \to \mathfrak{H}$ is of effective h-descent if and only if it is a pseudopullback. In this case, the unique object in $\mathfrak{S}_{\mathsf{h}}$ is o. Thereby we get:

Corollary 3.9.6. *Assume that $\mathscr{A} : \dot{\mathfrak{B}} \to [\dot{\mathfrak{A}}, \mathfrak{H}]_{PS}$ is a pseudopullback diagram. If $\mathscr{A}(c), \mathscr{A}(e) : \dot{\mathfrak{A}} \to \mathfrak{H}$ are of effective* t*-descent and $\mathscr{A}(o) : \dot{\mathfrak{A}} \to \mathfrak{H}$ is of* t*-descent, then $\mathscr{A}(\mathsf{b})$ is of effective* t*-descent.*

Taking into account Remark 3.8.1 and realizing that pseudopullbacks of functors induce pseudopullback of overcategories, we get Theorem 3.1.5 as a corollary.

3.9.7 Applications

In this subsection, we finish the paper giving applications of our results and proving the remaining theorems presented in Section 3.1. Firstly, considering our inclusion $\mathsf{j} : \Delta \to \dot{\Delta}$, it is important to observe that $1 \notin \mathfrak{S}_{\mathsf{j}}$, while all the other objects of Δ belong to $\mathfrak{S}_{\mathsf{j}}$. We start proving Theorem 4.2 of [49], which is presented therein as a generalized Galois Theorem.

Theorem 3.9.8 (Galois)**.** *Let $\mathscr{A}, \mathscr{B} : \dot{\Delta} \to \mathsf{CAT}$ be pseudofunctors and $\alpha : \mathscr{A} \longrightarrow \mathscr{B}$ be an objectwise fully faithful pseudonatural transformation. We assume that $\mathscr{B}$ is of effective* j*-descent. The pseudofunctor $\mathscr{A}$ is also of effective* j*-descent if and only if the diagram below is a pseudopullback.*

$$\begin{tikzcd}
\mathscr{A}(0) \arrow[r, "\mathscr{A}(d)"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\overset{\alpha_d}{\Longrightarrow}"] & \mathscr{A}(1) \arrow[d, "\alpha_1"] \\
\mathscr{B}(0) \arrow[r, "\mathscr{B}(d)"'] & \mathscr{B}(1)
\end{tikzcd}$$

Proof. Since, in this case, $\mathfrak{I}_{\mathsf{j}} = 2$ and the inclusion $\mathfrak{g}_{\mathsf{j}} : \overline{\mathfrak{I}_{\mathsf{j}} \times 2} \to \mathfrak{I}_{\mathsf{j}} \times 2$ is precisely equal to the inclusion described in the diagram $\mathfrak{P}$, by Theorem 3.9.2, the proof is complete. □

As a consequence of Theorem 3.9.8, we get a generalization of Theorem 3.1.2. More precisely, in the context of Section 3.8 and using the definitions presented there, we get:

Corollary 3.9.9. *Let* (U,α) *be a fully faithful morphism between pseudofunctors* $\mathscr{A}:\mathscr{C}^{\mathrm{op}}\to\mathfrak{H}$ *and* $\mathscr{B}:\mathbb{D}^{\mathrm{op}}\to\mathfrak{H}$*, in which* $\mathscr{C}$ *and* $\mathbb{D}$ *are categories with pullbacks. Assume that* $U(p)$ *is an effective* $\mathscr{B}$*-descent morphism of* $\mathbb{D}$*. Then* $p:E\to B$ *is of effective* $\mathscr{A}$*-descent if and only if, whenever there are* $u\in\mathscr{B}(B), v\in\mathscr{A}(E)$ *such that* $\alpha_1^p(u)\cong\mathscr{B}_{U(p)}(d)(v)$*, there is* $w\in\mathscr{A}(B)$ *such that* $\alpha_0^p(w)\cong u$*.*

Proof. Recall the definitions of $\mathscr{A}_p,\mathscr{B}_{U(p)},\alpha^p$. Since we already know that $\mathscr{A}_p$ is j-descent, the condition described is precisely the condition necessary and sufficient to conclude that the diagram of Theorem 3.9.8 is a pseudopullback. □

Indeed, taking into account Remark 3.8.1, we conclude that Theorem 3.1.2 is actually a immediate consequence of last corollary.

Given a category with pullbacks V, we denote by $\mathsf{Cat}(V)$ the category of internal categories in V. If V is a category with products, we denote by V-Cat the category of small categories enriched over V. We give a simple application of the Theorem 3.1.5 below.

Lemma 3.9.10. *If* $(V,\times,I)$ *is an infinitary lextensive category such that*

$$\begin{aligned} J:\mathsf{Set} &\to V\\ A &\mapsto \sum_{a\in A} I_a\end{aligned}$$

is fully faithful, then the pseudopullback of the projection of the object of objects $U_0:\mathsf{Cat}(V)\to V$ *along* J *is the category* V-Cat*.*

Proof. We denote by *Span*(V) the usual bicategory of objects of V and spans between them and by V-Mat the usual bicategory of sets and V-matrices between them. Let $\mathit{Span}_{\mathsf{Set}}(V)$ be the full sub-bicategory of *Span*(V) in which the objects are in the image of Set.

Assuming our hypotheses, we have that $\mathit{Span}_{\mathsf{Set}}(V)$ is biequivalent to V-Mat. Indeed, we define "identity" on the objects and, if A,B are sets, take a matrix $M:A\times B\to\mathrm{obj}(V)$ to the obvious span given by the coproduct $\sum\limits_{(x,y)\in A\times B} M(x,y)$, that is to say, the morphism $\sum_{(x,y)\in A\times B}M(x,y)\to A$ is induced by the morphisms $M(x,y)\to I_x$ and the morphism $\sum_{(x,y)\in A\times B}M(x,y)\to B$ is analogously defined.

Since V is lextensive, this defines a biequivalence. Thereby this completes our proof. □

Corollary 6.2.5 of [72] says in particular that, for lextensive categories, effective descent morphisms of $\mathsf{Cat}(V)$ are preserved by the projection $U_0:\mathsf{Cat}(V)\to V$ to the objects of objects. Thereby, by Theorem 3.1.5, we get:

Theorem 3.9.11. *If* $(V,\times,I)$ *is an infinitary lextensive category such that each arrow of* V *can be factorized as a regular epimorphism followed by a monomorphism and*

$$\begin{aligned} J:\mathsf{Set} &\to V\\ A &\mapsto \sum_{a\in A} I_a\end{aligned}$$

is fully faithful, then $I:V$-$\mathsf{Cat}\to\mathsf{Cat}(V)$ *reflects effective descent morphisms.*

Proof. We denote by $U : V\text{-}\mathsf{Cat} \to \mathsf{Set}$ the forgetful functor and by $U_0 : \mathsf{Cat}(V) \to V$ the projection defined above. We have that U_0, U, J and I are pullback preserving functors.

If $p : E \to B$ is a morphism of V-Cat such that $I(p)$ is of effective descent, then $U_0 I(p)$ is of descent (by Corollary 5.2.1 of [72]). Therefore $JU(p)$ is of descent.

Since J is fully faithful, by Theorem 3.8.3, $U(p)$ is of descent. Therefore, since descent morphisms of Set are of effective descent, we conclude that $U(p)$ is of effective descent. This completes the proof. $\square$

For instance, Theorem 6.2.8 of [72] and Proposition 3.9.11 can be applied to the cases of $V = \mathsf{Cat}$ or $V = \mathsf{Top}$:

Corollary 3.9.12. *A* 2*-functor* F *between* Cat*-categories is of effective descent in* Cat-Cat*, if*

- *F is surjective on objects;*
- *F is surjective on composable triples of* 2*-cells;*
- *F induces a functor surjective on composable pairs of* 2*-cells between the categories of composable pairs of* 1*-cells;*
- *F induces a functor surjective on* 2*-cells between the categories of composable triples of* 1*-cells.*

Corollary 3.9.13. *A* Top*-functor* F *between* Top*-categories is of effective descent in* Top-Cat*, if* F *induces*

- *effective descent morphisms between the discrete spaces of objects and between the spaces of morphisms in* Top*;*
- *a descent continuous map between the spaces of composable pairs of morphisms in* Top*;*
- *an almost descent continuous map between the spaces of composable triples of morphisms in* Top.

Since the characterization of (effective/almost) descent morphisms in Top is known [17, 21, 96], the result above gives effective descent morphisms of Top-Cat.

Remark 3.9.14. We can give further formal results on (basic) effective descent morphisms (context of Remark 3.8.1). The main technique in this case is to understand our overcategory as a bilimit of other overcategories.

For instance, we study below the categories of morphisms of a given category $\mathscr{C}$ with pullbacks. Consider the full inclusion of 2-categories $\mathrm{t} : \mathfrak{A} \to \dot{\mathfrak{A}}$

$$\begin{tikzcd}
\mathsf{0} \arrow[d, "d"] & & \mathsf{a} \arrow[rr, "pro_1"] \arrow[dr, "pro_0"', ""{name=P}] & & \mathsf{0} \arrow[dl, "d", ""'{name=D}] \\
\mathsf{1} & \mapsto & & \mathsf{1} &
\arrow[from=P, to=D, Rightarrow, "\xi", shorten=3mm]
\end{tikzcd}$$

Given a morphism of $\mathscr{C}$, *i.e.* a functor $F : 2 \to \mathscr{C}$, we take the overcategory $Fun(2,\mathscr{C})/F$ and define $\mathscr{A} : \dot{\mathfrak{A}} \to \mathsf{CAT}$ in which

$$\mathscr{A}(\mathsf{a}) := Fun(2,\mathscr{C})/F, \qquad \mathscr{A}(0) := \mathscr{C}/F(1), \qquad \mathscr{A}(1) := \mathscr{C}/F(0).$$

Finally, $\mathscr{A}(pro_0), \mathscr{A}(pro_1)$ are given by the obvious projections, $\mathscr{A}(d) := F(d)^*$ and the component $\mathscr{A}(\xi)$ in a morphism $\varpi : H \to F$ is given by the induced morphism from $H(0)$ to the pullback.

Observe that $\mathscr{A}$ is of effective t-descent, that is to say, we have that the overcategory $Fun(2,\mathscr{C})/F$ is a bilimit constructed from overcategories $\mathscr{C}/F(0)$ and $\mathscr{C}/F(1)$. Also, given a natural transformation $\varpi : F \to G$ between functors $2 \to \mathscr{C}$, *i.e.* a morphism of $Fun(2,\mathscr{C})$, taking Remark 3.8.1, we can extend $\mathscr{A}$ to a 2-functor $\overline{\mathscr{A}} : \dot{\mathfrak{A}} \to [\dot{\Delta}, \mathsf{CAT}]$ in which $\overline{\mathscr{A}}(\mathsf{a}) := (\ )^*_{\varpi}$, $\overline{\mathscr{A}}(0) := (\ )^*_{\varpi_1}$ and $\overline{\mathscr{A}}(1) := (\ )^*_{\varpi_0}$.

The 2-functor $\overline{\mathscr{A}}$ is also of effective t-descent. Therefore, by our results, we conclude that, *if the components* ϖ_1, ϖ_0 *are of (basic) effective descent, so is* ϖ. Analogously, considering the category of spans in $\mathscr{C}$, the morphisms between spans which are objectwise of effective descent are of effective descent.

Chapter 4

On Biadjoint Triangles

We prove a biadjoint triangle theorem and its strict version, which are 2-dimensional analogues of the adjoint triangle theorem of Dubuc. Similarly to the 1-dimensional case, we demonstrate how we can apply our results to get the pseudomonadicity characterization (due to Le Creurer, Marmolejo and Vitale). Furthermore, we study applications of our main theorems in the context of the 2-monadic approach to coherence. As a direct consequence of our strict biadjoint triangle theorem, we give the construction (due to Lack) of the left 2-adjoint to the inclusion of the strict algebras into the pseudoalgebras. In the last section, we give two brief applications on lifting biadjunctions and pseudo-Kan extensions.

Introduction

Assume that $E : \mathbb{A} \to \mathbb{C}$, $J : \mathbb{A} \to \mathbb{B}$, $L : \mathbb{B} \to \mathbb{C}$ are functors such that there is a natural isomorphism

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "J"] \arrow[dr, "E"'] & \cong & \mathbb{B} \arrow[dl, "L"] \\
 & \mathbb{C} &
\end{tikzcd}$$

Dubuc [30] proved that if $L : \mathbb{B} \to \mathbb{C}$ is precomonadic, $E : \mathbb{A} \to \mathbb{C}$ has a right adjoint and $\mathbb{A}$ has some needed equalizers, then J has a right adjoint. In this paper, we give a 2-dimensional version of this theorem, called the biadjoint triangle theorem. More precisely, let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be 2-categories and assume that

$$E : \mathfrak{A} \to \mathfrak{C}, J : \mathfrak{A} \to \mathfrak{B}, L : \mathfrak{B} \to \mathfrak{C}$$

are pseudofunctors such that L is pseudoprecomonadic and E has a right biadjoint. We prove that, if we have the pseudonatural equivalence below, then J has a right biadjoint G, provided that $\mathfrak{A}$ has some needed descent objects.

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "E"'] & \simeq & \mathfrak{B} \arrow[dl, "L"] \\
 & \mathfrak{C} &
\end{tikzcd}$$

We also give sufficient conditions under which the unit and the counit of the obtained biadjunction are pseudonatural equivalences, provided that E and L induce the same pseudocomonad. Moreover, we prove a strict version of our main theorem on biadjoint triangles. That is to say, we show that, under suitable conditions, it is possible to construct (strict) right 2-adjoints.

Similarly to the 1-dimensional case [30], the biadjoint triangle theorem can be applied to get the pseudo(co)monadicity theorem due to Le Creurer, Marmolejo and Vitale [73]. Also, some of the constructions of biadjunctions related to two-dimensional monad theory given by Blackwell, Kelly and Power [9] are particular cases of the biadjoint triangle theorem.

Furthermore, Lack [67] proved what may be called a general coherence result: his theorem states that the inclusion of the strict algebras into the pseudoalgebras of a given 2-monad $\mathscr{T}$ on a 2-category $\mathfrak{C}$ has a left 2-adjoint and the unit of this 2-adjunction is a pseudonatural equivalence, provided that $\mathfrak{C}$ has and $\mathscr{T}$ preserves strict codescent objects. This coherence result is also a consequence of the biadjoint triangle theorems proved in Section 4.4.

Actually, although the motivation and ideas of the biadjoint triangle theorems came from the original adjoint triangle theorem [30, 109] and its enriched version stated in Section 4.1, Theorem 4.4.3 may be seen as a generalization of the construction, given in [67], of the right biadjoint to the inclusion of the 2-category of strict coalgebras into the 2-category of pseudocoalgebras.

In Section 4.1, we give a slight generalization of Dubuc's theorem, in its enriched version (Proposition 4.1.1). This version gives the 2-adjoint triangle theorem for 2-pre(co)monadicity, but it lacks applicability for biadjoint triangles and pseudopre(co)monadicity. Then, in Section 4.2 we change our setting: we recall some definitions and results of the tricategory 2-CAT of 2-categories, pseudofunctors, pseudonatural transformations and modifications. Most of them can be found in Street's articles [104, 105].

Section 4.3 gives definitions and results related to descent objects [104, 105], which is a very important type of 2-categorical limit in 2-dimensional universal algebra. Within our established setting, in Section 4.4 we prove our main theorems (Theorem 4.4.3 and Theorem 4.4.6) on biadjoint triangles, while, in Section 4.5, we give consequences of such results in terms of pseudoprecomonadicity (Corollary 4.5.10), using the characterization of pseudoprecomonadic pseudofunctors given by Proposition 4.5.7, that is to say, Corollary 4.5.9.

In Section 4.6, we give results (Theorem 4.6.3 and Theorem 4.6.5) on the counit and unit of the obtained biadjunction $J \dashv G$ in the context of biadjoint triangles, provided that E and L induce the same pseudocomonad. Moreover, we demonstrate the pseudoprecomonadicity characterization of [73] as a consequence of our Corollary 4.5.9.

In Section 4.7, we show how we can apply our main theorem to get the pseudocomonadicity characterization [47, 73] and we give a corollary of Theorem 4.6.5 on the counit of the biadjunction $J \dashv G$ in this context. Furthermore, in Section 4.8 we show that the theorem of [67] on the inclusion $\mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ is a direct consequence of the theorems presented herein, giving a brief discussion on consequences of the biadjoint triangle theorems in the context of the 2-(co)monadic approach to coherence. Finally, we discuss a straightforward application on lifting biadjunctions in Section 4.9.

Since our main application in Section 4.9 is about construction of right biadjoints, we prove theorems for pseudoprecomonadic functors instead of proving theorems on pseudopremonadic functors. But, for instance, to apply the results of this work in the original setting of [9], or to get the construction of the left biadjoint given in [67], we should, of course, consider the dual version: the Biadjoint Triangle Theorem 4.4.4.

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4.1 Enriched Adjoint Triangles

Consider a cocomplete, complete and symmetric monoidal closed category V. Assume that $L : \mathscr{B} \to \mathscr{C}$ is a V-functor and $(L \dashv U, \eta, \varepsilon)$ is a V-adjunction. We denote by

$$\chi : \mathscr{C}(L-,-) \cong \mathscr{B}(-,U-)$$

its associated V-natural isomorphism, that is to say, for every object X of $\mathscr{B}$ and every object Z of $\mathscr{C}$, $\chi_{(X,Z)} = \mathscr{B}(\eta_X, UZ) \circ U_{LX,Z}$.

Proposition 4.1.1 (Enriched Adjoint Triangle Theorem)**.** *Let* $(L \dashv U, \eta, \varepsilon)$*,* $(E \dashv R, \rho, \mu)$ *be* V*-adjunctions such that*

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "J"] \arrow[rd, "E"'] & & \mathscr{B} \arrow[ld, "L"] \\
& \mathscr{C} &
\end{tikzcd}$$

is a commutative triangle of V*-functors. Assume that, for each pair of objects* $(A \in \mathscr{A}, Y \in \mathscr{B})$*, the induced diagram*

$$\begin{tikzcd}
\mathscr{B}(JA,Y) \arrow[r, "L_{JA,Y}"] & \mathscr{C}(EA,LY) \arrow[r, shift left, "L_{JA,ULY} \circ \chi_{(JA,LY)}"] \arrow[r, shift right, "\mathscr{C}(EA{,}L(\eta_Y))"'] & \mathscr{C}(EA,LULY)
\end{tikzcd}$$

is an equalizer in V*. The* V*-functor* J *has a right* V*-adjoint* G *if and only if, for each object* Y *of* $\mathscr{B}$*, the* V*-equalizer of*

$$\begin{tikzcd}
RLY \arrow[r, shift left, "RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}"] \arrow[r, shift right, "RL(\eta_Y)"'] & RLULY
\end{tikzcd}$$

exists in the V*-category* $\mathscr{A}$*. In this case, this equalizer gives the value of* GY*.*

Proof. For each pair of objects $(A \in \mathscr{A}, Y \in \mathscr{B})$, the V-natural isomorphism $\mathscr{C}(E-,-) \cong \mathscr{A}(-,R-)$ gives the components of the natural isomorphism

$$\begin{tikzcd}
\mathscr{B}(JA,Y) \arrow[r, "L_{JA,Y}"] \arrow[d, equal] & \mathscr{C}(EA,LY) \arrow[r, shift left, "L_{JA,ULY} \circ \chi_{(JA,LY)}"] \arrow[r, shift right, "\mathscr{C}(EA{,}L(\eta_Y))"'] \arrow[d, "\cong"] & \mathscr{C}(EA,LULY) \arrow[d, "\cong"] \\
\mathscr{B}(JA,Y) \arrow[r] & \mathscr{A}(A,RLY) \arrow[r, shift left, "\mathscr{A}(A{,}r_Y)"] \arrow[r, shift right, "\mathscr{A}(A{,}q_Y)"'] & \mathscr{A}(A,RLULY)
\end{tikzcd}$$

in which $q_Y = RL(\eta_Y)$ and $r_Y = RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}$. Thereby, since, by hypothesis, the top row is an equalizer, $\mathscr{B}(JA,Y)$ is the equalizer of $(\mathscr{A}(A,q_Y),\mathscr{A}(A,r_Y))$.

Assuming that the pair (q_Y,r_Y) has a V-equalizer GY in $\mathscr{A}$ for every Y of $\mathscr{B}$, we have that $\mathscr{A}(A,GY)$ is also an equalizer of $(\mathscr{A}(A,q_Y),\mathscr{A}(A,r_Y))$. Therefore we get a V-natural isomorphism $\mathscr{A}(-,GY)\cong\mathscr{B}(J-,Y)$.

Reciprocally, if G is right V-adjoint to J, since $\mathscr{A}(-,GY)\cong\mathscr{B}(J-,Y)$ is an equalizer of

$$(\mathscr{A}(-,q_Y),\mathscr{A}(-,r_Y)),$$

GY is the V-equalizer of (q_Y,r_Y). This completes the proof that the V-equalizers of q_Y, r_Y are also necessary. ☐

The results on (co)monadicity in V-CAT are similar to those of the classical context of CAT (see, for instance, [31, 100]). Actually, some of those results of the enriched context can be seen as consequences of the classical theorems because of Street's work [100].

Our main interest is in Beck's theorem for V-precomonadicity. More precisely, it is known that the 2-category V-CAT admits construction of coalgebras [100]. Therefore every left V-adjoint $L:\mathscr{B}\to\mathscr{C}$ comes with the corresponding Eilenberg-Moore factorization.

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "\phi"] \arrow[dr, "L"'] & \mathsf{CoAlg} \arrow[d] \\
& \mathscr{C}
\end{tikzcd}$$

If $V = \mathsf{Set}$, Beck's theorem asserts that ϕ is fully faithful if and only if the diagram below is an equalizer for every object Y of $\mathscr{B}$. In this case, we say that L is precomonadic.

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] & ULY \arrow[r, shift left, "\eta_{ULY}"] \arrow[r, shift right, "UL(\eta_Y)"'] & ULULY
\end{tikzcd}$$

With due adaptations, this theorem also holds for enriched categories. That is to say, ϕ is V-fully faithful if and only if the diagram above is a V-equalizer for every object Y of $\mathscr{B}$. This result gives what we need to prove Corollary 4.1.2, which is the enriched version for Dubuc's theorem [30].

Corollary 4.1.2. *Let* $(L\dashv U,\eta,\varepsilon)$, $(E\dashv R,\rho,\mu)$ *be* V*-adjunctions and* J *be a* V*-functor such that*

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "J"] \arrow[dr, "E"'] & & \mathscr{B} \arrow[dl, "L"] \\
& \mathscr{C} &
\end{tikzcd}$$

commutes and L is V-precomonadic. The V-functor J has a right V-adjoint G if and only if, for each object Y of $\mathscr{B}$*, the V-equalizer of*

$$\begin{tikzcd}
RLY \arrow[r, shift left, "RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}"] \arrow[r, shift right, "RL(\eta_Y)"'] & RLULY
\end{tikzcd}$$

exists in the V-category $\mathscr{A}$. In this case, these equalizers give the value of the right adjoint G.

Proof. The isomorphisms induced by the V-natural isomorphism $\chi : \mathscr{C}(L-,-) \cong \mathscr{B}(-,U-)$ are the components of the natural isomorphism

$$\begin{tikzcd}
\mathscr{B}(JA,Y) \arrow[r, "L_{JA,Y}"] \arrow[d, equal] & \mathscr{C}(EA,LY) \arrow[r, shift left, "L_{JA,ULY} \circ \chi_{(JA,LY)}"] \arrow[r, shift right, "\mathscr{C}(EA,L(\eta_Y))"'] \arrow[d, "\chi_{(JA,LY)}"] & \mathscr{C}(EA,LULY) \arrow[d, "\chi_{(JA,LULY)}"] \\
\mathscr{B}(JA,Y) \arrow[r, "\mathscr{B}(JA,\eta_Y)"] & \mathscr{B}(JA,ULY) \arrow[r, shift left, "\mathscr{B}(JA,\eta_{ULY})"] \arrow[r, shift right, "\mathscr{B}(JA,UL(\eta_Y))"'] & \mathscr{B}(JA,ULULY)
\end{tikzcd}$$

Since L is V-precomonadic, by the previous observations, the top row of the diagram above is an equalizer. Thereby, for every object A of $\mathscr{A}$ and every object Y of $\mathscr{B}$, the bottom row, which is the diagram D_Y^{JA}, is an equalizer. By Proposition 4.1.1, this completes the proof. $\square$

Proposition 4.1.1 applies to the case of CAT-enriched category theory. But it does not give results about pseudomonad theory. For instance, the construction above does not give the right biadjoint constructed in [9, 67]

$$\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \to \mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}}.$$

Thereby, to study pseudomonad theory properly, we study biadjoint triangles, which cannot be dealt with only CAT-enriched category theory. Yet, a 2-dimensional version of the perspective given by Proposition 4.1.1 is what enables us to give the construction of (strict) right 2-adjoint functors in Subsection 4.4.5.

4.2 Bilimits

We denote by $2\text{-}\mathsf{CAT}$ the tricategory of 2-categories, pseudofunctors (homomorphisms), pseudonatural transformations (strong transformations) and modifications. Since this is our main setting, we recall some results and concepts related to $2\text{-}\mathsf{CAT}$. Most of them can be found in [104], and a few of them are direct consequences of results given there.

Firstly, to fix notation, we set the tricategory $2\text{-}\mathsf{CAT}$, defining pseudofunctors, pseudonatural transformations and modifications. Henceforth, in a given 2-category, we always denote by $\cdot$ the vertical composition of 2-cells and by $*$ their horizontal composition.

Definition 4.2.1. [Pseudofunctor] Let $\mathfrak{B}, \mathfrak{C}$ be 2-categories. A *pseudofunctor* $L : \mathfrak{B} \to \mathfrak{C}$ is a pair $(L, \mathfrak{l})$ with the following data:

- Function $L : \mathrm{obj}(\mathfrak{B}) \to \mathrm{obj}(\mathfrak{C})$;
- For each pair (X,Y) of objects in $\mathfrak{B}$, functors $L_{X,Y} : \mathfrak{B}(X,Y) \to \mathfrak{C}(LX, LY)$;
- For each pair $g : X \to Y, h : Y \to Z$ of 1-cells in $\mathfrak{B}$, an invertible 2-cell of $\mathfrak{C}$:

$$\mathfrak{l}_{hg} : L(h)L(g) \Rightarrow L(hg);$$

- For each object X of $\mathfrak{B}$, an invertible 2-cell in $\mathfrak{C}$:

$$\mathfrak{l}_X : \mathrm{id}_{LX} \Rightarrow L(\mathrm{id}_X);$$

such that, if $\hat{g}, g : X \to Y, \hat{h}, h : Y \to Z, f : W \to X$ are 1-cells of $\mathfrak{B}$, and $\mathfrak{x} : g \Rightarrow \hat{g}, \mathfrak{y} : h \Rightarrow \hat{h}$ are 2-cells of $\mathfrak{B}$, the following equations hold:

1. Associativity:

$$\begin{tikzcd}
LW \arrow[rrr, "L(f)"] \arrow[ddd, "L(hgf)"'] \arrow[dddrrr, "L(gf)" description] & & & LX \arrow[ddd, "L(g)"] & & LW \arrow[rrr, "L(f)"] \arrow[ddd, "L(hgf)"'] & & & LX \arrow[ddd, "L(g)"] \arrow[dddlll, "L(hg)" description] \\
& & \overset{\mathfrak{l}_{gf}}{\Longleftarrow} & & & & \overset{\mathfrak{l}_{(hg)f}}{\Longleftarrow} & & \\
& \overset{\mathfrak{l}_{h(gf)}}{\Longleftarrow} & & & = & & & \overset{\mathfrak{l}_{hg}}{\Longleftarrow} & \\
LZ & & & LY \arrow[lll, "L(h)"] & & LZ & & & LY \arrow[lll, "L(h)"]
\end{tikzcd}$$

2. Identity:

$$\begin{tikzcd}
LW \arrow[r, "L(f)"] \arrow[d, "L(\mathrm{id}_X f)"'] \arrow[from=1-1, to=2-2, phantom, "\overset{\mathfrak{l}_{\mathrm{id}_X f}}{\Longleftarrow}"] & LX \arrow[d, bend right=50, "L(\mathrm{id}_X)"'] \arrow[d, bend left=50, "\mathrm{id}_{LX}"] \arrow[d, phantom, "\overset{\mathfrak{l}_X}{\Longleftarrow}"] & = & LW \arrow[r, equal] \arrow[d, "L(f\mathrm{id}_W)"'] \arrow[from=1-4, to=2-5, phantom, "\overset{\mathfrak{l}_{f\mathrm{id}_W}}{\Longleftarrow}"] & LW \arrow[d, bend right=50, "L(\mathrm{id}_W)"'] \arrow[d, bend left=50, "\mathrm{id}_{LW}"] \arrow[d, phantom, "\overset{\mathfrak{l}_W}{\Longleftarrow}"] & = & LW \arrow[d, bend right=40, "L(f)"'] \arrow[d, bend left=40, "L(f)"] \arrow[d, phantom, "="] \\
LX \arrow[r, equal] & LX & & LX & LW \arrow[l, "L(f)"] & & LX
\end{tikzcd}$$

3. Naturality:

$$\begin{tikzcd}
LX \arrow[r, equal] \arrow[dd, "L(\hat{h}\hat{g})"'] \arrow[from=1-1, to=3-2, phantom, "\overset{\mathfrak{l}_{\hat{h}\hat{g}}}{\Longleftarrow}"] & LX \arrow[r, equal] \arrow[d, "L(\hat{g})"'] \arrow[from=1-2, to=2-3, phantom, "\overset{L(\mathfrak{x})}{\Longleftarrow}"] & LX \arrow[d, "L(g)"] & & LX \arrow[r, equal] \arrow[dd, "L(\hat{h}\hat{g})"'] \arrow[from=1-5, to=3-6, phantom, "\overset{L(\mathfrak{y}*\mathfrak{x})}{\Longleftarrow}"] & LX \arrow[r, "L(g)"] \arrow[dd, "L(hg)"'] \arrow[from=1-6, to=3-7, phantom, "\overset{\mathfrak{l}_{hg}}{\Longleftarrow}"] & LY \arrow[dd, "L(h)"] \\
& LY \arrow[r, equal] \arrow[d, "L(\hat{h})"'] \arrow[from=2-2, to=3-3, phantom, "\overset{L(\mathfrak{y})}{\Longleftarrow}"] & LY \arrow[d, "L(h)"] & = & & & \\
LZ \arrow[r, equal] & LZ \arrow[r, equal] & LZ & & LZ \arrow[r, equal] & LZ \arrow[r, equal] & LZ
\end{tikzcd}$$

The composition of pseudofunctors is easily defined. Namely, if $(J, \mathfrak{j}) : \mathfrak{A} \to \mathfrak{B}, (L, \mathfrak{l}) : \mathfrak{B} \to \mathfrak{C}$ are pseudofunctors, we define the composition by $L \circ J := (LJ, (\mathfrak{l}\mathfrak{j}))$, in which $(\mathfrak{l}\mathfrak{j})_{hg} := L(\mathfrak{j}_{hg}) \cdot \mathfrak{l}_{J(h)J(g)}$ and $(\mathfrak{l}\mathfrak{j})_X := L(\mathfrak{j}_X) \cdot \mathfrak{l}_{JX}$. This composition is associative and it has trivial identities.

Furthermore, recall that a 2-functor $L : \mathfrak{B} \to \mathfrak{C}$ is just a pseudofunctor $(L, \mathfrak{l})$ such that its invertible 2-cells $\mathfrak{l}_f$ (for every morphism f) and $\mathfrak{l}_X$ (for every object X) are identities.

Definition 4.2.2. [Pseudonatural transformation] If $L, E : \mathfrak{B} \to \mathfrak{C}$ are pseudofunctors, a *pseudonatural transformation* $\alpha : L \longrightarrow E$ is defined by:

- For each object X of $\mathfrak{B}$, a 1-cell $\alpha_X : LX \to EX$ of $\mathfrak{C}$;

- For each 1-cell $g : X \to Y$ of $\mathfrak{B}$, an invertible 2-cell $\alpha_g : E(g)\alpha_X \Rightarrow \alpha_Y L(g)$ of $\mathfrak{C}$;

such that, if $g, \hat{g} : X \to Y, f : W \to X$ are 1-cells of $\mathfrak{A}$, and $\mathfrak{x} : g \Rightarrow \hat{g}$ is a 2-cell of $\mathfrak{A}$, the following equations hold:

1. Associativity:

$$\begin{tikzcd}
LW \arrow[rr, equal] \arrow[dddd, "L(gf)"'] & & LW \arrow[rr, "\alpha_W"] \arrow[dd, "L(f)"'] & & EW \arrow[dd, "E(f)"] \\
& & & \overset{\alpha_f}{\Leftarrow} & \\
& \overset{\mathfrak{i}_{gf}}{\Leftarrow} & LX \arrow[rr, "\alpha_X"'] \arrow[dd, "L(g)"'] & & EX \arrow[dd, "E(g)"] \\
& & & \overset{\alpha_g}{\Leftarrow} & \\
LY \arrow[rr, equal] & & LY \arrow[rr, "\alpha_Y"'] & & EY
\end{tikzcd}
=
\begin{tikzcd}
LW \arrow[rr, "\alpha_W"] \arrow[dd, "L(gf)"'] & & EW \arrow[rr, "E(f)"] \arrow[dd, "E(gf)"'] & & EX \arrow[dd, "E(g)"] \\
& \overset{\alpha_{gf}}{\Leftarrow} & & \overset{\mathfrak{e}_{gf}}{\Leftarrow} & \\
LY \arrow[rr, "\alpha_Y"'] & & EY \arrow[rr, equal] & & EY
\end{tikzcd}$$

2. Identity:

$$\begin{tikzcd}
LW \arrow[rr, "\alpha_W"] \arrow[dd, bend right=60, "L(\mathrm{id}_W)"'] \arrow[dd, bend left=60, "\mathrm{id}_{LW}"] \arrow[dd, phantom, "\overset{\mathfrak{i}_W}{\Leftarrow}"] & & EW \arrow[dd, "\mathrm{id}_{EW}"] \\
& = & \\
LW \arrow[rr, "\alpha_W"'] & & EW
\end{tikzcd}
=
\begin{tikzcd}
LW \arrow[rr, "\alpha_W"] \arrow[dd, "L(\mathrm{id}_W)"'] & & EW \arrow[dd, bend right=60, "E(\mathrm{id}_W)"'] \arrow[dd, bend left=60, "\mathrm{id}_{EW}"] \arrow[dd, phantom, "\overset{\mathfrak{e}_W}{\Leftarrow}"] \\
& \overset{\alpha_{\mathrm{id}_W}}{\Leftarrow} & \\
LW \arrow[rr, "\alpha_W"'] & & EW
\end{tikzcd}$$

3. Naturality:

$$\begin{tikzcd}
LX \arrow[rr, "\alpha_X"] \arrow[dd, bend right=60, "L(\hat{g})"'] \arrow[dd, bend left=60, "L(g)"] \arrow[dd, phantom, "\overset{L(\mathfrak{x})}{\Leftarrow}"] & & EX \arrow[dd, "E(g)"] \\
& \overset{\alpha_g}{\Leftarrow} & \\
LY \arrow[rr, "\alpha_Y"'] & & EY
\end{tikzcd}
=
\begin{tikzcd}
LX \arrow[rr, "\alpha_X"] \arrow[dd, "L(\hat{g})"'] & & EX \arrow[dd, bend right=60, "E(\hat{g})"'] \arrow[dd, bend left=60, "E(g)"] \arrow[dd, phantom, "\overset{E(\mathfrak{x})}{\Leftarrow}"] \\
& \overset{\alpha_{\hat{g}}}{\Leftarrow} & \\
LY \arrow[rr, "\alpha_Y"'] & & EY
\end{tikzcd}$$

Firstly, we define the vertical composition, denoted by $\beta\alpha$, of two pseudonatural transformations $\alpha : L \longrightarrow E, \beta : E \longrightarrow U$ by

$$(\beta\alpha)_W := \beta_W \alpha_W$$

$$\begin{tikzcd}
LW \arrow[rr, "\beta_W \alpha_W"] \arrow[dd, "L(f)"'] & & UW \arrow[dd, "U(f)"] \\
& \overset{(\beta\alpha)_f}{\Leftarrow} & \\
LX \arrow[rr, "\beta_X \alpha_X"'] & & UX
\end{tikzcd}
:=
\begin{tikzcd}
LW \arrow[rr, "\alpha_W"] \arrow[dd, "L(f)"] & & EW \arrow[rr, "\beta_W"] \arrow[dd, "E(f)"'] & & UW \arrow[dd, "U(f)"] \\
& \overset{\alpha_f}{\Leftarrow} & & \overset{\beta_f}{\Leftarrow} & \\
LX \arrow[rr, "\alpha_X"'] & & EX \arrow[rr, "\beta_X"'] & & UX
\end{tikzcd}$$

Secondly, assume that $L, E : \mathfrak{B} \to \mathfrak{C}$ and $G, J : \mathfrak{A} \to \mathfrak{B}$ are pseudofunctors. We define the horizontal composition of two pseudonatural transformations $\alpha : L \longrightarrow E, \lambda : G \longrightarrow J$ by $(\alpha * \lambda) := (\alpha J)(L\lambda)$, in which αJ is trivially defined and $(L\lambda)$ is defined by: $(L\lambda)_W := L(\lambda_W)$ and $(L\lambda)_f := \left(\mathfrak{l}_{\lambda_X G(f)}\right)^{-1} \cdot L(\lambda_f) \cdot \mathfrak{l}_{J(f)\lambda_W}$.

Also, recall that a 2-natural transformation is just a pseudonatural transformation $\alpha : L \longrightarrow E$ such that its components $\alpha_g : E(g)\alpha_X \Rightarrow \alpha_Y L(g)$ are identities (for all morphisms g).

Definition 4.2.3. [Modification] Let $L, E : \mathfrak{B} \to \mathfrak{C}$ be pseudofunctors. If $\alpha, \beta : L \longrightarrow E$ are pseudonatural transformations, a *modification* $\Gamma : \alpha \Longrightarrow \beta$ is defined by the following data:

- For each object X of $\mathfrak{B}$, a 2-cell $\Gamma_X : \alpha_X \Rightarrow \beta_X$ of $\mathfrak{C}$;

such that: if $f : W \to X$ is a 1-cell of $\mathfrak{B}$, the equation below holds.

$$\begin{tikzcd}
LW \arrow[r, "L(f)"] \arrow[d, bend right=60, "\alpha_W"'] \arrow[d, bend left=60, "\beta_W"] \arrow[d, phantom, "\overset{\Gamma_W}{\Longrightarrow}"] & LX \arrow[d, "\beta_X"] \arrow[dl, phantom, "\overset{\beta_f}{\Longrightarrow}"] & & LW \arrow[r, "L(f)"] \arrow[d, "\alpha_W"'] & LX \arrow[d, bend right=60, "\alpha_X"'] \arrow[d, bend left=60, "\beta_X"] \arrow[d, phantom, "\overset{\Gamma_X}{\Longrightarrow}"] \arrow[dl, phantom, "\overset{\alpha_f}{\Longrightarrow}"] \\
EW \arrow[r, "E(f)"'] & EX & = & EW \arrow[r, "E(f)"'] & EX
\end{tikzcd}$$

The three types of compositions of modifications are defined in the obvious way. Thereby, it is straightforward to verify that, indeed, 2-CAT is a tricategory, lacking strictness/2-functoriality of the whiskering. In particular, we denote by $[\mathfrak{A}, \mathfrak{B}]_{PS}$ the 2-category of pseudofunctors $\mathfrak{A} \to \mathfrak{B}$, pseudonatural transformations and modifications.

The bicategorical Yoneda Lemma [104] says that there is a pseudonatural equivalence

$$[\mathfrak{S}, \mathsf{CAT}]_{PS}(\mathfrak{S}(a, -), \mathscr{D}) \simeq \mathscr{D}a$$

given by the evaluation at the identity.

Lemma 4.2.4 (Yoneda Embedding [104])**.** *The Yoneda* 2*-functor* $\mathscr{Y} : \mathfrak{A} \to [\mathfrak{A}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$ *is locally an equivalence (i.e. it induces equivalences between the hom-categories).*

Considering pseudofunctors $L : \mathfrak{B} \to \mathfrak{C}$ and $U : \mathfrak{C} \to \mathfrak{B}$, we say that U is right biadjoint to L if we have a pseudonatural equivalence $\mathfrak{C}(L-, -) \simeq \mathfrak{B}(-, U-)$. This concept can be also defined in terms of unit and counit as it is done at Definition 4.2.5.

Definition 4.2.5. Let $U : \mathfrak{C} \to \mathfrak{B}, L : \mathfrak{B} \to \mathfrak{C}$ be pseudofunctors. L is *left biadjoint* to U if there exist

1. pseudonatural transformations $\eta : \mathrm{Id}_{\mathfrak{B}} \longrightarrow UL$ and $\varepsilon : LU \longrightarrow \mathrm{Id}_{\mathfrak{C}}$

2. invertible modifications $s : \mathrm{id}_L \Longrightarrow (\varepsilon L)(L\eta)$ and $t : (U\varepsilon)(\eta U) \Longrightarrow \mathrm{id}_U$

such that the following 2-cells are identities [41]:

$$\begin{tikzcd}
\mathrm{Id}_{\mathfrak{B}} \arrow[r, "\eta"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\overset{\eta_{(\eta)}}{\Longrightarrow}"] & UL \arrow[d, "\eta UL"'] \arrow[ddr, equal, bend left] & \\
UL \arrow[r, "UL\eta"'] \arrow[drr, equal, bend right] \arrow[rr, phantom, shift right=6ex, "\overset{\mathfrak{u}^{-1}_{(L\eta)(\varepsilon L)}(Us)\mathfrak{u}_L}{\Longrightarrow}"] & ULUL \arrow[dr, "U\varepsilon L"] \arrow[r, phantom, "\overset{tL}{\Rightarrow}"] & \\
 & & UL
\end{tikzcd}
\qquad
\begin{tikzcd}
LU \arrow[dr, "L\eta U"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] \arrow[rr, phantom, shift right=3ex, "\overset{\mathfrak{l}^{-1}_U(Lt)\mathfrak{l}_{(U\varepsilon)(\eta U)}}{\Longrightarrow}"] \arrow[dd, phantom, shift left=3ex, "\overset{sU}{\Rightarrow}"] & & \\
 & LULU \arrow[r, "LU\varepsilon"] \arrow[d, "\varepsilon LU"] \arrow[dr, phantom, "\overset{\varepsilon_{(\varepsilon)}}{\Longrightarrow}"] & LU \arrow[d, "\varepsilon"] \\
 & LU \arrow[r, "\varepsilon"'] & \mathrm{Id}_{\mathfrak{C}}
\end{tikzcd}$$

Remark 4.2.6. By definition, if a pseudofunctor L is left biadjoint to U, there is at least one associated data $(L \dashv U, \eta, \varepsilon, s, t)$ as described above. Such associated data is called a *biadjunction*.

Also, every biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$ has an associated pseudonatural equivalence $\chi : \mathfrak{C}(L-,-) \simeq \mathfrak{B}(-,U-)$, in which

$$\begin{array}{lll}
\chi_{(X,Z)} : \mathfrak{C}(LX,Z) & \to \mathfrak{B}(X,UZ) \\
f & \mapsto U(f)\eta_X \\
\mathfrak{m} & \mapsto U(\mathfrak{m}) * \mathrm{id}_{\eta_X}
\end{array}
\qquad
\left(\chi_{(g,h)}\right)_f := \left(\mathfrak{u}_{(hf)Lg} * \mathrm{id}_{\eta_X}\right) \cdot \left(\mathfrak{u}_{hf} * \eta_g^{-1}\right)$$

Reciprocally, such a pseudonatural equivalence induces a biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$.

Remark 4.2.7. Similarly to the 1-dimensional case, if $(L \dashv U, \eta, \varepsilon, s, t)$ is a biadjunction, the counit $\varepsilon : LU \longrightarrow \mathrm{id}_{\mathfrak{C}}$ is a pseudonatural equivalence if and only if, for every pair (X,Y) of objects of $\mathfrak{C}$, $U_{X,Y} : \mathfrak{C}(X,Y) \to \mathfrak{B}(UX,UY)$ is an equivalence (that is to say, U is *locally an equivalence*).

The proof is also analogous to the 1-dimensional case. Indeed, given a pair (X,Y) of objects in $\mathfrak{B}$, the composition of functors

$$\mathfrak{B}(X,Y) \xrightarrow{\mathfrak{B}(\varepsilon_X, Y)} \mathfrak{B}(LUX,Y) \xrightarrow{\chi_{(UX,Y)}} \mathfrak{B}(LX,LY)$$

is obviously isomorphic to $U_{X,Y} : \mathfrak{C}(X,Y) \to \mathfrak{B}(UX,UY)$. Since $\chi_{(UX,Y)}$ is an equivalence, ε_X is an equivalence for every object X (that is to say, it is a pseudonatural equivalence) if and only if U is locally an equivalence. Dually, the unit of this biadjunction is a pseudonatural equivalence if and only if L is locally an equivalence.

Remark 4.2.8. Recall that, if the modifications s,t of a biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$ are identities, L, U are 2-functors and η, ε are 2-natural transformations, then L is left 2-adjoint to U and $(L \dashv U, \eta, \varepsilon)$ is a 2-adjunction.

If it exists, a birepresentation of a pseudofunctor $\mathscr{U} : \mathfrak{C} \to \mathsf{CAT}$ is an object X of $\mathfrak{C}$ endowed with a pseudonatural equivalence $\mathfrak{C}(X,-) \simeq \mathscr{U}$. When $\mathscr{U}$ has a birepresentation, we say that $\mathscr{U}$ is birepresentable. Moreover, in this case, by Lemma 4.2.4, its birepresentation is unique up to equivalence.

Lemma 4.2.9 ([104]). *Assume that* $\mathscr{U} : \mathfrak{C} \to [\mathfrak{B}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$ *is a pseudofunctor such that, for each object* X *of* $\mathfrak{C}$, $\mathscr{U}X$ *has a birepresentation* $e_X : \mathscr{U}X \simeq \mathfrak{B}(-, UX)$. *Then there is a pseudofunctor* $U : \mathfrak{C} \to \mathfrak{B}$ *such that the pseudonatural equivalences* e_X *are the components of a pseudonatural equivalence* $\mathscr{U} \simeq \mathfrak{B}(-, U-)$, *in which* $\mathfrak{B}(-, U-)$ *denotes the pseudofunctor*

$$\mathfrak{C} \to [\mathfrak{B}^{\mathrm{op}}, \mathsf{CAT}]_{PS} : \ X \mapsto \mathfrak{B}(-, UX)$$

As a consequence, a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ has a right biadjoint if and only if, for each object X of $\mathfrak{C}$, the pseudofunctor $\mathfrak{C}(L-, X)$ is birepresentable. *Id est*, for each object X, there is an object UX of $\mathfrak{B}$ endowed with a pseudonatural equivalence $\mathfrak{C}(L-, X) \simeq \mathfrak{B}(-, UX)$.

The natural notion of limit in our context is that of (weighted) bilimit [104, 105]. Namely, assuming that $\mathfrak{S}$ is a small 2-category, if $\mathscr{W} : \mathfrak{S} \to \mathsf{CAT}, \mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ are pseudofunctors, the (weighted) bilimit, denoted herein by $\{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}$, when it exists, is a birepresentation of the 2-functor

$$\mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT} : \ X \mapsto [\mathfrak{S}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathfrak{A}(X, \mathscr{D}-)).$$

Since, by the (bicategorical) Yoneda Lemma, $\{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}$ is unique up to equivalence, we sometimes refer to it as *the* (weighted) bilimit.

Finally, if $\mathscr{W}$ and $\mathscr{D}$ are 2-functors, recall that the (strict) weighted limit $\{\mathscr{W}, \mathscr{D}\}$ is, when it exists, a 2-representation of the 2-functor $X \mapsto [\mathfrak{S}, \mathsf{CAT}](\mathscr{W}, \mathfrak{A}(X, \mathscr{D}-))$, in which $[\mathfrak{S}, \mathsf{CAT}]$ is the 2-category of 2-functors $\mathfrak{S} \to \mathsf{CAT}$, 2-natural transformations and modifications [103].

It is easy to see that CAT is bicategorically complete. More precisely, if $\mathscr{W} : \mathfrak{S} \to \mathsf{CAT}$ and $\mathscr{D} : \mathfrak{S} \to \mathsf{CAT}$ are pseudofunctors, then

$$\{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}} \simeq [\mathfrak{S}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathscr{D}).$$

Moreover, from the bicategorical Yoneda Lemma of [104], we get the (strong) bicategorical Yoneda Lemma.

Lemma 4.2.10 ((Strong) Yoneda Lemma). *Let* $\mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ *be a pseudofunctor between* 2*-categories. There is a pseudonatural equivalence* $\{\mathfrak{S}(a, -), \mathscr{D}\}_{\mathrm{bi}} \simeq \mathscr{D}a$.

Proof. By the bicategorical Yoneda Lemma, we have a pseudonatural equivalence (in X and a)

$$[\mathfrak{S}, \mathsf{CAT}]_{PS}(\mathfrak{S}(a, -), \mathfrak{A}(X, \mathscr{D}-)) \simeq \mathfrak{A}(X, \mathscr{D}a).$$

Therefore $\mathscr{D}a$ is *the* bilimit $\{\mathfrak{S}(a, -), \mathscr{D}\}_{\mathrm{bi}}$. □

Recall that the usual (enriched) Yoneda embedding $\mathfrak{A} \to [\mathfrak{A}^{\mathrm{op}}, \mathsf{CAT}]$ preserves and reflects weighted limits. In the 2-dimensional case, we get a similar result.

Lemma 4.2.11. *The Yoneda embedding* $\mathscr{Y} : \mathfrak{A} \to [\mathfrak{A}^{\mathrm{op}}, \mathsf{CAT}]_{PS}$ *preserves and reflects weighted bilimits.*

Proof. By definition, a weighted bilimit $\{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}$ exists if and only if, for each object X of $\mathfrak{A}$,

$$\mathfrak{A}(X, \{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}) \simeq [\mathfrak{A}, \mathsf{CAT}]_{PS}(\mathscr{W}, \mathfrak{A}(X, \mathscr{D}-)) \simeq \{\mathscr{W}, \mathfrak{A}(X, \mathscr{D}-)\}_{\mathrm{bi}}.$$

By the pointwise construction of weighted bilimits, this means that $\{\mathscr{W},\mathscr{D}\}_{\mathrm{bi}}$ exists if and only if $\mathscr{Y}\{\mathscr{W},\mathscr{D}\}_{\mathrm{bi}} \simeq \{\mathscr{W},\mathscr{Y}\circ\mathscr{D}\}_{\mathrm{bi}}$. This proves that $\mathscr{Y}$ reflects and preserves weighted bilimits. □

Remark 4.2.12. Let $\mathfrak{S}$ be a small 2-category and $\mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ be a pseudofunctor. Consider the pseudofunctor

$$[\mathfrak{S},\mathfrak{C}]_{PS} \to [\mathfrak{A}^{\mathrm{op}},\mathsf{CAT}]_{PS} : \mathscr{W} \mapsto \mathbb{D}_{\mathscr{W}}$$

in which the 2-functor $\mathbb{D}_{\mathscr{W}}$ is given by $X \mapsto [\mathfrak{S},\mathsf{CAT}]_{PS}(\mathscr{W},\mathfrak{A}(X,\mathscr{D}-))$. By Lemma 4.2.9, we conclude that it is possible to get a pseudofunctor $\{-,\mathscr{D}\}_{\mathrm{bi}}$ defined in a full sub-2-category of $[\mathfrak{S},\mathsf{CAT}]_{PS}$ of weights $\mathscr{W} : \mathfrak{S} \to \mathsf{CAT}$ such that $\mathfrak{A}$ has the bilimit $\{\mathscr{W},\mathscr{D}\}_{\mathrm{bi}}$.

4.3 Descent Objects

In this section, we describe the 2-categorical limits called *descent objects*. We need both constructions, strict descent objects and descent objects [105]. Our domain 2-category, denoted by Δ, is the dual of that defined at Definition 2.1 in [73].

Definition 4.3.1. We denote by $\dot{\Delta}$ the 2-category generated by the diagram

$$\begin{tikzcd}
0 \arrow[r, "d"] & 1 \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & 2 \arrow[l, "s^0" description] \arrow[r, shift left=3, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=3, "\partial^2"'] & 3
\end{tikzcd}$$

with the invertible 2-cells $\sigma_{12} : \partial^2 d^1 \cong \partial^1 d^1$, $\sigma_{02} : \partial^2 d^1 \cong \partial^0 d^1$, $\sigma_{01} : \partial^1 d^0 \cong \partial^0 d^0$, $n_0 : s^0 d^0 \cong \mathrm{id}_1$ $n_1 : \mathrm{id}_1 \cong s^0 d^1$ and $\vartheta : d^1 d \cong d^0 d$ satisfying the equations below:

- Associativity:

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Rightarrow}"] & 1 \arrow[r, "d^0"] \arrow[d, "d^0"] \arrow[dr, phantom, "\overset{\sigma_{01}}{\Rightarrow}"] & 2 \arrow[d, "\partial^0"] & & 3 & 2 \arrow[l, "\partial^0"'] \arrow[r, equal] \arrow[ddr, phantom, "\overset{\vartheta}{\Rightarrow}"] & 2 \\
1 \arrow[r, "d^1"] \arrow[d, "d^1"'] \arrow[drr, phantom, "\overset{\sigma_{12}}{\Rightarrow}"] & 2 \arrow[r, "\partial^1"] & 3 \arrow[d, "\mathrm{id}_3"] & = & 2 \arrow[u, "\partial^2"] \arrow[ur, phantom, "\overset{\sigma_{02}}{\Rightarrow}"] & 1 \arrow[l, "d^0"'] \arrow[u, "d^1"'] & \\
2 \arrow[rr, "\partial^2"'] & & 3 & & 1 \arrow[u, "d^1"] \arrow[ur, phantom, "\overset{\vartheta}{\Rightarrow}"] & 0 \arrow[l, "d"] \arrow[r, "d"'] \arrow[u, "d"'] & 1 \arrow[uu, "d^0"']
\end{tikzcd}$$

- Identity:

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Leftarrow}"] & 1 \arrow[d, "d^1"] \arrow[ddr, bend left, equal] \arrow[ddr, phantom, "\overset{n_1}{\Leftarrow}"] & & & 0 \arrow[dd, bend right, "d"'] \arrow[dd, bend left, "d"] \arrow[dd, phantom, "="] \\
1 \arrow[r, "d^0"'] \arrow[drr, bend right, equal] \arrow[drr, phantom, "\overset{n_0}{\Leftarrow}"] & 2 \arrow[dr, "s^0"] & & = & \\
& & 1 & & 1
\end{tikzcd}$$

The 2-category Δ is, herein, the full sub-2-category of $\dot{\Delta}$ with objects $1,2,3$. We denote the inclusion by $\mathrm{j} : \Delta \to \dot{\Delta}$.

Remark 4.3.2. In fact, the 2-category $\dot{\Delta}$ is the locally preordered 2-category freely generated by the diagram and 2-cells described above. Moreover, Δ is the 2-category freely generated by the corresponding diagram and the 2-cells $\sigma_{01}, \sigma_{02}, \sigma_{12}, n_0, n_1$.

Let $\mathfrak{A}$ be a 2-category and $\mathscr{A} : \Delta \to \mathfrak{A}$ be a 2-functor. If the weighted bilimit $\left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{A}\right\}_{\mathrm{bi}}$ exists, we say that $\left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{A}\right\}_{\mathrm{bi}}$ is the *descent object* of $\mathscr{A}$. Analogously, when it exists, we call the (strict) weighted 2-limit $\left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{A}\right\}$ the *strict descent object* of $\mathscr{A}$.

Assuming that $\mathscr{D} : \dot{\Delta} \to \mathfrak{A}$ is a pseudofunctor, we have a pseudonatural transformation $\dot{\Delta}(0,\mathrm{j}-) \longrightarrow \mathfrak{A}(\mathscr{D}0, \mathscr{D}\circ\mathrm{j}-)$ given by the evaluation of $\mathscr{D}$. By the definition of weighted bilimit, if $\mathscr{D}\circ\mathrm{j}$ has a descent object, this pseudonatural transformation induces a comparison 1-cell

$$\mathscr{D}0 \to \left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{D}\circ\mathrm{j}\right\}_{\mathrm{bi}}.$$

Analogously, if $\mathscr{D}$ is a 2-functor, we get a comparison $\mathscr{D}0 \to \left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{D}\circ\mathrm{j}\right\}$, provided that the strict descent object of $\mathscr{D}\circ\mathrm{j}$ exists.

Definition 4.3.3. [Effective Descent Diagrams] We say that a 2-functor $\mathscr{D} : \dot{\Delta} \to \mathfrak{A}$ is of *effective descent* if $\mathfrak{A}$ has the descent object of $\mathscr{D}\circ\mathrm{j}$ and the comparison $\mathscr{D}0 \to \left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{D}\circ\mathrm{j}\right\}_{\mathrm{bi}}$ is an equivalence.

We say that $\mathscr{D}$ is of *strict descent* if $\mathfrak{A}$ has the strict descent object of $\mathscr{D}\circ\mathrm{j}$ and the comparison $\mathscr{D}0 \to \left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{D}\circ\mathrm{j}\right\}$ is an isomorphism.

Lemma 4.3.4. *Strict descent objects are descent objects. Thereby, strict descent diagrams are of effective descent as well.*

Also, if $\mathfrak{A}$ has strict descent objects, a 2-functor $\mathscr{D} : \dot{\Delta} \to \mathfrak{A}$ is of effective descent if and only if the comparison $\mathscr{D}0 \to \left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{D}\circ\mathrm{j}\right\}$ is an equivalence.

Lemma 4.3.5. *Assume that $\mathscr{A}, \mathscr{B}, \mathscr{D} : \dot{\Delta} \to \mathfrak{A}$ are 2-functors. If there are a 2-natural isomorphism $\mathscr{A} \longrightarrow \mathscr{B}$ and a pseudonatural equivalence $\mathscr{B} \longrightarrow \mathscr{D}$, then*

- *$\mathscr{A}$ is of strict descent if and only if $\mathscr{B}$ is of strict descent;*
- *$\mathscr{B}$ is of effective descent if and only if $\mathscr{D}$ is of effective descent.*

We say that an effective descent diagram $\mathscr{D} : \dot{\Delta} \to \mathfrak{B}$ is *preserved* by a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ if $L\circ\mathscr{D}$ is of effective descent. Also, $\mathscr{D} : \dot{\Delta} \to \mathfrak{B}$ is said to be an *absolute effective descent* diagram if $L\circ\mathscr{D}$ is of effective descent for any pseudofunctor L.

In this setting, a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ is said to *reflect absolute effective descent diagrams* if, whenever a 2-functor $\mathscr{D} : \dot{\Delta} \to \mathfrak{B}$ is such that $L\circ\mathscr{D}$ is an absolute effective descent diagram, $\mathscr{D}$ is of effective descent. Moreover, we say herein that a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ *creates absolute effective descent diagrams* if L reflects absolute effective descent diagrams and, whenever a diagram $\mathscr{A} : \Delta \to \mathfrak{B}$ is such that $L\circ\mathscr{A} \simeq \mathscr{D}\circ\mathrm{j}$ for some absolute effective descent diagram $\mathscr{D} : \dot{\Delta} \to \mathfrak{C}$, there is a diagram $\mathscr{B} : \dot{\Delta} \to \mathfrak{B}$ such that $L\circ\mathscr{B} \simeq \mathscr{D}$ and $\mathscr{B}\circ\mathrm{j} = \mathscr{A}$.

Recall that right 2-adjoints preserve strict descent diagrams and right biadjoints preserve effective descent diagrams. Also, the usual (enriched) Yoneda embedding $\mathfrak{A} \to [\mathfrak{A}^{\mathrm{op}}, \mathsf{CAT}]$ preserves and reflects strict descent diagrams, and, from Lemma 4.2.11, we get:

Lemma 4.3.6. *The Yoneda embedding* $\mathscr{Y}:\mathfrak{A}\to[\mathfrak{A}^{\mathrm{op}},\mathsf{CAT}]_{PS}$ *preserves and reflects effective descent diagrams.*

Remark 4.3.7. The dual notion of descent object is that of codescent object, described by Lack [67] and Le Creurer, Marmolejo, Vitale [73]. It is, of course, the descent object in the opposite 2-category.

Remark 4.3.8. The 2-category CAT is CAT-complete. In particular, CAT has strict descent objects. More precisely, if $\mathscr{A}:\Delta\to\mathsf{CAT}$ is a 2-functor, then

$$\left\{\dot{\Delta}(0,\mathsf{j}-),\mathscr{A}\right\}\cong[\Delta,\mathsf{CAT}]\left(\dot{\Delta}(0,\mathsf{j}-),\mathscr{A}\right).$$

Thereby, we can describe the strict descent object of $\mathscr{A}:\Delta\to\mathsf{CAT}$ explicitly as follows:

1. Objects are 2-natural transformations $\mathsf{f}:\dot{\Delta}(0,\mathsf{j}-)\longrightarrow\mathscr{A}$. We have a bijective correspondence between such 2-natural transformations and pairs (f,ρ_{f}) in which f is an object of $\mathscr{A}1$ and $\rho_{\mathsf{f}}:\mathscr{A}(d^1)f\to\mathscr{A}(d^0)f$ is an isomorphism in $\mathscr{A}2$ satisfying the following equations:

   - Associativity:
   $$\left(\mathscr{A}(\partial^0)(\rho_{\mathsf{f}})\right)\left(\mathscr{A}(\sigma_{02})_f\right)\left(\mathscr{A}(\partial^2)(\rho_{\mathsf{f}})\right)\left(\mathscr{A}(\sigma_{12})_f^{-1}\right)=\left(\mathscr{A}(\sigma_{01})_f\right)\left(\mathscr{A}(\partial^1)(\rho_{\mathsf{f}})\right)$$
   - Identity:
   $$\left(\mathscr{A}(n_0)_f\right)\left(\mathscr{A}(s^0)(\rho_{\mathsf{f}})\right)\left(\mathscr{A}(n_1)_f\right)=\mathrm{id}_f$$

   If $\mathsf{f}:\dot{\Delta}(0,\mathsf{j}-)\longrightarrow\mathscr{A}$ is a 2-natural transformation, we get such pair by the correspondence $\mathsf{f}\mapsto(\mathsf{f}_1(d),\mathsf{f}_2(\vartheta))$.

2. The morphisms are modifications. In other words, a morphism $\mathsf{m}:\mathsf{f}\to\mathsf{h}$ is determined by a morphism $\mathfrak{m}:f\to h$ such that $\mathscr{A}(d^0)(\mathfrak{m})\rho_{\mathsf{f}}=\rho_h\mathscr{A}(d^1)(\mathfrak{m})$.

4.4 Biadjoint Triangles

In this section, we give our main theorem on biadjoint triangles, Theorem 4.4.3, and its strict version, Theorem 4.4.6. Let $L:\mathfrak{B}\to\mathfrak{C}$ and $U:\mathfrak{C}\to\mathfrak{B}$ be pseudofunctors, and $(L\dashv U,\eta,\varepsilon,s,t)$ be a biadjunction. We denote by $\chi:\mathfrak{C}(L-,-)\simeq\mathfrak{B}(-,U-)$ its associated pseudonatural equivalence as described in Remark 4.2.6.

Definition 4.4.1. In this setting, for every pair (X,Y) of objects of $\mathfrak{B}$, we have an induced diagram $\mathscr{D}_Y^X:\dot{\Delta}\to\mathsf{CAT}$

$$\begin{tikzcd}
\mathfrak{B}(X,Y) \arrow[d, mapsto, "L_{X,Y}"'] & & \\
\mathfrak{C}(LX,LY) \arrow[r, shift left=6, "L_{X,ULY}\circ\chi_{(X,LY)}"] \arrow[r, shift right=6, "\mathfrak{C}(LX,L(\eta_Y))"'] & \mathfrak{C}(LX,LULY) \arrow[l, "\mathfrak{C}(LX,\varepsilon_{LY})" description] \arrow[r, shift left=6, "L_{X,(UL)^2Y}\circ\chi_{(X,LULY)}"] \arrow[r, "\mathfrak{C}(LX,L(\eta_{ULY}))" description] \arrow[r, shift right=6, "\mathfrak{C}(LX,LUL(\eta_Y))"'] & \mathfrak{C}(LX,L(UL)^2Y)
\end{tikzcd} \qquad (\mathscr{D}_Y^X)$$

in which the images of the 2-cells of $\dot{\Delta}$ by $\mathscr{D}^X_Y : \dot{\Delta} \to \mathsf{CAT}$ are defined as:

$$
\begin{array}{llll}
\mathscr{D}^X_Y(\vartheta)_g & := L\left(\eta_g^{-1}\right) \cdot \mathfrak{l}_{\eta_Y g} & \mathscr{D}^X_Y(\sigma_{01})_f & := \mathfrak{l}_{UL(U(f)\eta_X)\eta_X} \cdot (L\eta)^{-1}_{U(f)\eta_X} \\
\mathscr{D}^X_Y(\sigma_{12})_f & := (L\eta)_{\eta_Y} * \mathrm{id}_f & \mathscr{D}^X_Y(\sigma_{02})_f & := L\left(\mathfrak{u}_{L(\eta_Y)f} * \mathrm{id}_{\eta_X}\right) \cdot \mathfrak{l}_{UL(\eta_Y)L(U(f)\eta_X)} \\
\mathscr{D}^X_Y(n_1)_f & := s_Y * \mathrm{id}_f & \mathscr{D}^X_Y(n_0)_f & := \left(\mathrm{id}_f * s_X^{-1}\right) \cdot \left(\varepsilon_f^{-1} * \mathrm{id}_{\eta_X}\right) \cdot \left(\mathrm{id}_{\varepsilon_{LY}} * \mathfrak{l}^{-1}_{U(f)\eta_X}\right)
\end{array}
$$

We claim that $\mathscr{D}^X_Y$ *is well defined.* In fact, by the axioms of naturality and associativity of Definition 4.2.2 (of pseudonatural transformation), for every morphism $g \in \mathfrak{B}(X,Y)$, we have the equality

$$
\begin{tikzcd}
LX \arrow[rr, "L(g)"] \arrow[d, "L(\eta_X)"'] \arrow[rrd, phantom, "\overset{\gamma}{\Leftarrow}"] & & LY \arrow[d, "L(\eta_Y)"] \arrow[rrd, "L(\eta_Y)"] & & \\
LULX \arrow[rr, "LUL(g)"'] \arrow[rrd, "LUL(\eta_X)"'] & & LULY \arrow[rrd, "LUL(\eta_Y)"] \arrow[rr, phantom, "\overset{(L\eta)^{-1}_{\eta_Y}}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\widehat{LU(\gamma)}}{\Longleftarrow}"] & & LULY \arrow[d, "L(\eta_{ULY})"] \\
 & & LULULX \arrow[rr, "LULUL(g)"'] & & LULULY
\end{tikzcd}
=
\begin{tikzcd}
LX \arrow[rr, "L(g)"] \arrow[d, "L(\eta_X)"'] \arrow[rrd, "L(\eta_X)"] & & LY \arrow[rrd, "L(\eta_Y)"] \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] & & \\
LULX \arrow[rrd, "LUL(\eta_X)"'] \arrow[rr, phantom, "\overset{(L\eta)^{-1}_{\eta_X}}{\Longleftarrow}"] & & LULX \arrow[rr, "LUL(g)"] \arrow[d, "L(\eta_{ULX})"] \arrow[rrd, phantom, "\overset{(L\eta)^{-1}_{UL(g)}}{\Longleftarrow}"] & & LULY \arrow[d, "L(\eta_{ULY})"] \\
 & & LULULX \arrow[rr, "LULUL(g)"'] & & LULULY
\end{tikzcd}
$$

in which:

$$
\gamma := \mathfrak{l}^{-1}_{UL(g)\eta_X} \cdot \mathscr{D}^X_Y(\vartheta)_g = (L\eta)^{-1}_g, \qquad \widehat{LU(\gamma)} := (\mathfrak{lu})^{-1}_{LUL(g)L(\eta_X)} \cdot LU(\gamma) \cdot (\mathfrak{lu})_{L(\eta_X)L(g)}.
$$

By the definition of $\mathscr{D}^X_Y$ given above, this is the same as saying that the equation

$$
\begin{tikzcd}
\mathscr{D}^X_Y 3 \arrow[r, equal] & \mathscr{D}^X_Y 3 & \mathscr{D}^X_Y 2 \arrow[l, "\mathscr{D}^X_Y(\partial^0)"'] \arrow[r, equal] & \mathscr{D}^X_Y 2 \\
\arrow[r, phantom, "\overset{\mathscr{D}^X_Y(\sigma_{12})^{-1}}{\Longrightarrow}"] & \mathscr{D}^X_Y 2 \arrow[u, "\mathscr{D}^X_Y(\partial^2)"] \arrow[ru, phantom, "\overset{\mathscr{D}^X_Y(\sigma_{02})}{\Longrightarrow}"] & \mathscr{D}^X_Y 1 \arrow[l, "\mathscr{D}^X_Y(d^0)"] \arrow[u, "\mathscr{D}^X_Y(d^1)"'] \arrow[r, phantom, "\overset{\mathscr{D}^X_Y(\vartheta)}{\Longrightarrow}"] & \\
\mathscr{D}^X_Y 2 \arrow[uu, "\mathscr{D}^X_Y(\partial^1)"] & \mathscr{D}^X_Y 1 \arrow[l, "\mathscr{D}^X_Y(d^1)"] \arrow[u, "\mathscr{D}^X_Y(d^1)"] \arrow[ru, phantom, "\overset{\mathscr{D}^X_Y(\vartheta)}{\Longrightarrow}"] & \mathscr{D}^X_Y 0 \arrow[l, "\mathscr{D}^X_Y(d)"] \arrow[r, "\mathscr{D}^X_Y(d)"'] \arrow[u, "\mathscr{D}^X_Y(d)"'] & \mathscr{D}^X_Y 1 \arrow[uu, "\mathscr{D}^X_Y(d^0)"']
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{D}^X_Y 0 \arrow[r, "\mathscr{D}^X_Y(d)"] \arrow[d, "\mathscr{D}^X_Y(d)"'] \arrow[rd, phantom, "\overset{\mathscr{D}^X_Y(\vartheta)}{\Longrightarrow}"] & \mathscr{D}^X_Y 1 \arrow[r, "\mathscr{D}^X_Y(d^0)"] \arrow[d, "\mathscr{D}^X_Y(d^0)"] \arrow[rd, phantom, "\overset{\mathscr{D}^X_Y(\sigma_{01})}{\Longrightarrow}"] & \mathscr{D}^X_Y 2 \arrow[d, "\mathscr{D}^X_Y(\partial^0)"] \\
\mathscr{D}^X_Y 1 \arrow[r, "\mathscr{D}^X_Y(d^1)"'] & \mathscr{D}^X_Y 2 \arrow[r, "\mathscr{D}^X_Y(\partial^1)"'] & \mathscr{D}^X_Y 3
\end{tikzcd}
$$

holds, which is equivalent to the usual equation of associativity given in Definition 4.3.1. Also, by the naturality of the modification $s : \mathrm{id}_L \Longrightarrow (\varepsilon L)(L\eta)$ (see Definition 4.2.3), for every morphism $g \in \mathfrak{B}(X,Y)$, the pasting of 2-cells

$$
\begin{tikzcd}
LX \arrow[rrr, "L(g)"] \arrow[dd, equal] \arrow[rd, "L(\eta_X)"] & & & LY \arrow[dd, equal] \arrow[ld, "L(\eta_Y)"'] \\
\arrow[r, phantom, "\overset{s_X^{-1}}{\Longleftarrow}"] & LULX \arrow[r, "LUL(g)"] \arrow[ld, "\varepsilon_{LX}"] \arrow[u, phantom, "\overset{(L\eta)^{-1}_g}{\Longleftarrow}"'] \arrow[d, phantom, "\overset{(\varepsilon L)^{-1}_g}{\Longleftarrow}"] & LULY \arrow[rd, "\varepsilon_{LY}"'] \arrow[r, phantom, "\overset{s_Y}{\Longleftarrow}"] & \\
LX \arrow[rrr, "L(g)"'] & & & LY
\end{tikzcd}
$$

is equal to the identity $L(g) \Rightarrow L(g)$ in $\mathfrak{C}$. This is equivalent to say that

$$\begin{tikzcd}
\mathscr{D}^X_Y 0 \arrow[r, "\mathscr{D}^X_Y(d)"] \arrow[d, "\mathscr{D}^X_Y(d)"'] \arrow[rd, phantom, "\overset{\mathscr{D}^X_Y(\vartheta)}{\Longleftarrow}"] & \mathscr{D}^X_Y 1 \arrow[d, "\mathscr{D}^X_Y(d^1)"] \arrow[rdd, equal, bend left] \arrow[rd, phantom, "\overset{\mathscr{D}^X_Y(n_1)}{\Longleftarrow}"] & & & \mathscr{D}^X_Y 0 \arrow[dd, bend right, "\mathscr{D}^X_Y(d)"'] \arrow[dd, bend left, "\mathscr{D}^X_Y(d)"] \arrow[dd, phantom, "="] \\
\mathscr{D}^X_Y 1 \arrow[r, "\mathscr{D}^X_Y(d^0)"] \arrow[rrd, equal, bend right] \arrow[rd, phantom, "\overset{\mathscr{D}^X_Y(n_0)}{\Longleftarrow}"] & \mathscr{D}^X_Y 2 \arrow[rd, "\mathscr{D}^X_Y(s^0)"] & & = & \\
 & & \mathscr{D}^X_Y 1 & & \mathscr{D}^X_Y 1
\end{tikzcd}$$

holds, which is the usual identity equation of Definition 4.3.1. Thereby it completes the proof that indeed $\mathscr{D}^X_Y$ is well defined.

As in the enriched case, we also need to consider another special 2-functor induced by a biadjoint triangle.

Definition 4.4.2. Let $(E \dashv R, \rho, \mu, v, w)$ and $(L \dashv U, \eta, \varepsilon, s, t)$ be biadjunctions such that we have a commutative triangle of pseudofunctors $LJ = E$. In this setting, for each object Y of $\mathfrak{B}$, we define the 2-functor $\mathscr{A}_Y : \Delta \to \mathfrak{A}$,

$$\begin{tikzcd}
RLY \arrow[r, shift left=2, "RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}"] \arrow[r, shift right=2, "RL(\eta_Y)"'] & RLULY \arrow[l, "R(\varepsilon_{LY})" description] \arrow[r, shift left=2, "RL(U(\mu_{LULY})\eta_{JRLULY})\rho_{RLULY}"] \arrow[r, "RL(\eta_{ULY})" description] \arrow[r, shift right=2, "RLUL(\eta_Y)"'] & RLULULY
\end{tikzcd} \qquad (\mathscr{A}_Y)$$

in which:

$$\begin{aligned}
\mathscr{A}_Y(\sigma_{12}) &:= (RL\eta)_{\eta_Y} \quad \mathscr{A}_Y(n_1) := \mathfrak{r}^{-1}_{\varepsilon_{LY} \cdot L(\eta_Y)} R(s_Y) \cdot \mathfrak{r}_{LY} \\
\mathscr{A}_Y(n_0) &:= (w_{LY}) \\
&\quad \cdot \left(\mathrm{id}_{R(\mu_{LY})} * \left(\mathfrak{r}^{-1}_{ERLY} \cdot R(s^{-1}_{JRLY}) \cdot \mathfrak{r}_{\varepsilon_{ERLY} L(\eta_{JRLY})}\right) \cdot \mathrm{id}_{\rho_{RLY}}\right) \\
&\quad \cdot \left((R\varepsilon)^{-1}_{\mu_{LY}} * \mathrm{id}_{RL(\eta_{JRLY})\rho_{RLY}}\right) \\
&\quad \cdot \left(\mathrm{id}_{R(\varepsilon_{LY})} * (\mathfrak{r}\mathfrak{l})^{-1}_{U(\mu_{LY})\eta_{JRLY}} * \mathrm{id}_{\rho_{RLY}}\right) \\
\mathscr{A}_Y(\sigma_{02}) &:= \left((\mathfrak{r}\mathfrak{l})_{U(\mu_{RLULY})\eta_{JRLULY}} * \mathrm{id}_{\rho_{RLULY} RL(\eta_Y)}\right) \cdot \left((RLU\mu L)\,(RL\eta JRL)\,(\rho RL)\right)_{\eta_Y} \\
&\quad \cdot \left(\mathrm{id}_{RLUL(\eta_Y)} * (\mathfrak{r}\mathfrak{l})^{-1}_{U(\mu_{LY})\eta_{JRLY}} * \mathrm{id}_{\rho_{RLY}}\right) \\
\mathscr{A}_Y(\sigma_{01}) &:= \left((\mathfrak{r}\mathfrak{l})_{U(\mu_{LULY})\eta_{JRLULY}} * \rho_{RL(U(\mu_{LY})\eta_{JRLY})} * \mathrm{id}_{\rho_{RLY}}\right) \\
&\quad \cdot \left(((RLU\mu L)(RL\eta JRL))_{U(\mu_{LY})\eta_{JRLY}} * \rho_{\rho_{RLY}}\right) \\
&\quad \cdot \left(\mathrm{id}_{RLUL(U(\mu_{LY})\eta_{JRLY})RLU(\mu_{ERLY})} * (RL\eta J)_{\rho_{RLY}} * \mathrm{id}_{\rho_{RLY}}\right) \\
&\quad \cdot \left(\mathrm{id}_{RLUL(U(\mu_{LY})\eta_{JRLY})} * \left((\mathfrak{r}\mathfrak{l}\mathfrak{u})^{-1}_{\mu_{ERLY} E(\rho_{RLY})} \cdot RLU(v_{RLY}) \cdot (\mathfrak{r}\mathfrak{l}\mathfrak{u})_{ERLY}\right) * \mathrm{id}_{RL(\eta_{JRLY})\rho_{RLY}}\right) \\
&\quad \cdot \left((RL\eta)^{-1}_{U(\mu_{LY})\eta_{JRLY}} * \mathrm{id}_{\rho_{RLY}}\right).
\end{aligned}$$

Theorem 4.4.3 (Biadjoint Triangle)**.** *Let* $(E \dashv R, \rho, \mu, v, w)$ *and* $(L \dashv U, \eta, \varepsilon, s, t)$ *be biadjunctions such that*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "E"'] & & \mathfrak{B} \arrow[dl, "L"] \\
& \mathfrak{C} &
\end{tikzcd}$$

is a commutative triangle of pseudofunctors. Assume that, for each pair of objects $(Y \in \mathfrak{B}, A \in \mathfrak{A})$*, the* 2*-functor*

$$\begin{tikzcd}[column sep=huge]
\mathfrak{B}(JA,Y) \arrow[d, "L_{JA,Y}"'] & & & & (\mathscr{D}_Y^{JA}) \\
\mathfrak{C}(LJA,LY) \arrow[r, shift left=3, "L_{JA,ULY} \circ \chi_{(JA,LY)}"] \arrow[r, shift right=3, "\mathfrak{C}(LJA,L(\eta_Y))"'] & \mathfrak{C}(LJA,LULY) \arrow[l, "\mathfrak{C}(LJA,\varepsilon_{LY})" description] \arrow[r, shift left=3, "L_{JA,(UL)^2Y} \circ \chi_{(JA,LULY)}"] \arrow[r, "\mathfrak{C}(LJA,L(\eta_{ULY}))" description] \arrow[r, shift right=3, "\mathfrak{C}(LJA,LUL(\eta_Y))"'] & \mathfrak{C}(LJA,L(UL)^2Y)
\end{tikzcd}$$

is of effective descent. The pseudofunctor J has a right biadjoint if and only if, for every object Y of $\mathfrak{B}$*, the descent object of the diagram* $\mathscr{A}_Y : \Delta \to \mathfrak{A}$ *exists in* $\mathfrak{A}$*. In this case, J is left biadjoint to G, defined by* $GY := \left\{\dot{\Delta}(0,\mathrm{j}-), \mathscr{A}_Y\right\}_{\mathrm{bi}}$.

Proof. We denote by $\xi : \mathfrak{C}(E-,-) \simeq \mathfrak{A}(-,R-)$ the pseudonatural equivalence associated to the biadjunction $(E \dashv R, \rho, \mu, v, w)$ (see Remark 4.2.6). For each object A of $\mathfrak{A}$ and each object Y of $\mathfrak{B}$, the components of ξ induce a pseudonatural equivalence

$$\psi : \mathscr{D}_Y^{JA} \circ \mathrm{j} \longrightarrow \mathfrak{A}(A, \mathscr{A}_Y -)$$

in which:

$$\begin{aligned}
\psi_1 &:= \xi_{(A,LY)} : \mathfrak{C}(EA,LY) \to \mathfrak{A}(A,RLY) \\
\psi_2 &:= \xi_{(A,LULY)} : \mathfrak{C}(EA,LULY) \to \mathfrak{A}(A,RLULY) \\
\psi_3 &:= \xi_{(A,LULULY)} : \mathfrak{C}(EA,LULULY) \to \mathfrak{A}(A,RLULULY)
\end{aligned}$$

$$\begin{aligned}
\left(\psi_{s^0}\right)_f &:= \mathfrak{r}_{\varepsilon_{LY} f} * \mathrm{id}_{\rho_A} & \qquad \left(\psi_{\partial^1}\right)_f &:= \mathfrak{r}_{L(\eta_{ULY})f} * \mathrm{id}_{\rho_A} \\
\left(\psi_{d^1}\right)_f &:= \mathfrak{r}_{L(\eta_Y)f} * \mathrm{id}_{\rho_A} & \qquad \left(\psi_{\partial^2}\right)_f &:= \mathfrak{r}_{LUL(\eta_Y)f} * \mathrm{id}_{\rho_A}
\end{aligned}$$

$$\begin{aligned}
\left(\psi_{d^0}\right)_f := & \left((\mathfrak{r}\mathfrak{l})_{U(f)\eta_{JA}} * \mathrm{id}_{\rho_A}\right) \\
& \cdot \left(\mathrm{id}_{RLU(f)} * \left((\mathfrak{r}\mathfrak{l}\mathfrak{u})^{-1}_{EA} \cdot RLU(v_A^{-1}) \cdot (\mathfrak{r}\mathfrak{l}\mathfrak{u})_{\mu_{EA} E(\rho_A)}\right) * \mathrm{id}_{RLU(\eta_{JA})\rho_A}\right) \\
& \cdot \left(\mathrm{id}_{RLU(f)RL(\mu_{EA})} * ((RL\eta J)\rho)^{-1}_{\rho_A}\right) \cdot \left(((RLU\mu)(RL\eta JR)(\rho R))^{-1}_f * \mathrm{id}_{\rho_A}\right) \\
& \cdot \left((\mathfrak{r}\mathfrak{l})^{-1}_{U(\mu_{LY})\eta_{JRLY}} * \mathrm{id}_{\rho_{RLY} R(f)\rho_A}\right)
\end{aligned}$$

$$\begin{aligned}\left(\psi_{\partial^0}\right)_f \quad := \quad & \left((\mathfrak{r}\mathfrak{l})_{U(f)\eta_{JA}} * \mathrm{id}_{\rho_A}\right)\\ & \cdot \left(\mathrm{id}_{RLU(f)} * \left((\mathfrak{r}\mathfrak{l}\mathfrak{u})^{-1}_{EA} \cdot RLU(v^{-1}_{A}) \cdot (\mathfrak{r}\mathfrak{l}\mathfrak{u})_{\mu_{EA}E(\rho_A)}\right) * \mathrm{id}_{RLU(\eta_{JA})\rho_A}\right)\\ & \cdot \left(\mathrm{id}_{RLU(f)RL(\mu_{EA})} * ((RL\eta J)\rho)^{-1}_{\rho_A}\right) \cdot \left(((RLU\mu)(RL\eta JR)(\rho R))^{-1}_{f} * \mathrm{id}_{\rho_A}\right)\\ & \cdot \left((\mathfrak{r}\mathfrak{l})^{-1}_{U(\mu_{LULY})\eta_{JRLULY}} * \mathrm{id}_{\rho_{RLULY}R(f)\rho_A}\right).\end{aligned}$$

First of all, we assume that $\mathscr{D}^{JA}_{Y}$ is of effective descent for every object A of $\mathfrak{A}$ and every object Y of $\mathfrak{B}$. Then the descent object of $\mathscr{D}^{JA}_{Y} \circ \mathrm{j} \simeq \mathfrak{A}(A, \mathscr{A}_Y -)$ is $\mathscr{D}^{JA}_{Y}\mathsf{0}$. Moreover, since this is true for all objects A of $\mathfrak{A}$, we conclude that the descent object of $\mathscr{Y} \circ \mathscr{A}_Y$ is $\mathfrak{C}(J-, Y) : \mathfrak{A}^{\mathrm{op}} \to \mathsf{CAT}$.

If, furthermore, $\mathfrak{A}$ has the descent object of $\mathscr{A}_Y$, we get that $\mathscr{Y}\left\{\dot{\Delta}(\mathsf{0}, \mathrm{j}-), \mathscr{A}_Y\right\}_{\mathrm{bi}}$ is also a descent object of $\mathscr{Y} \circ \mathscr{A}_Y$. Therefore we get a pseudonatural equivalence

$$\mathfrak{C}(J-, Y) \simeq \mathfrak{A}\left(-, \left\{\dot{\Delta}(\mathsf{0}, \mathrm{j}-), \mathscr{A}_Y\right\}_{\mathrm{bi}}\right).$$

This proves that J is left biadjoint to G, provided that the descent object of $\mathscr{A}_Y$ exists for every object Y of $\mathfrak{B}$.

Reciprocally, if J is left biadjoint to a pseudofunctor G, since $\mathfrak{C}(-, GY) \simeq \mathfrak{C}(J-, Y)$ is the descent object of $\mathfrak{A}(-, \mathscr{A}_Y -)$, we conclude that GY is the descent object of $\mathscr{A}_Y$. ☐

We establish below the obvious dual version of Theorem 4.4.3, which is the relevant theorem to the usual context of pseudopremonadicity [73]. For being able to give such dual version, we have to employ the observations given in Remark 4.3.7 on codescent objects. Also, if $(L \dashv U, \eta, \varepsilon, s, t)$ is a biadjunction, we need to consider its associated pseudonatural equivalence $\tau : \mathfrak{C}(-, U-) \to \mathfrak{B}(L-, -)$. In particular,

$$\tau_{(X,Z)} : \mathfrak{C}(X, UZ) \to \mathfrak{B}(LX, Z) : \qquad f \mapsto \varepsilon_Z L(f); \qquad \mathfrak{m} \mapsto \mathrm{id}_{\varepsilon_Z} * L(\mathfrak{m}).$$

Theorem 4.4.4 (Biadjoint Triangle)**.** *Let* $(E \dashv R, \rho, \mu, v, w)$ *and* $(L \dashv U, \eta, \varepsilon, s, t)$ *be biadjunctions such that*

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "J"] \arrow[dr, "R"'] & & \mathfrak{B} \arrow[dl, "U"] \\ & \mathfrak{C} & \end{tikzcd}$$

is a commutative triangle of pseudofunctors. Assume that, for each pair of objects $(Y \in \mathfrak{B}, A \in \mathfrak{A})$*, the* 2*-functor*

$$\dot{\Delta} \to \mathsf{CAT}$$

$$\begin{tikzcd} \mathfrak{B}(Y, JA) \arrow[d, "U_{Y,JA}"] & & & & \\ \mathfrak{C}(UY, UJA) & & \mathfrak{C}(ULUY, UJA) \arrow[ll, "\mathfrak{C}(\eta_{UY}, UJA)" description] \arrow[rr, "\mathfrak{C}(U(\varepsilon_{LUY}), UJA)" description] \arrow[rr, shift left=3, "\mathfrak{C}(ULU(\varepsilon_Y), UJA)"] \arrow[rr, shift right=3, "U_{(LU)^2Y,JA} \circ \tau_{(ULUY,JA)}"'] & & \mathfrak{C}(U(LU)^2Y, UJA) \arrow[from=2-1, to=2-3, shift left=3, "\mathfrak{C}(U(\varepsilon_A), UJA)"] \arrow[from=2-1, to=2-3, shift right=3, "U_{LUY,JA} \circ \tau_{(UY,JA)}"'] \end{tikzcd}$$

(with omitted 2*-cells) is of effective descent. We have that* J *has a left biadjoint if and only if, for every object* Y *of* $\mathfrak{B}$*,* $\mathfrak{A}$ *has the codescent object of the diagram (with the obvious* 2*-cells)*

$$\Delta^{\mathrm{op}} \to \mathfrak{A}$$

$$\begin{tikzcd}[column sep=huge]
EUY \arrow[r, "E(\eta_{UY})" description] & EULUY \arrow[l, shift right=3, "EU(\varepsilon_Y)"'] \arrow[l, shift left=3, "\mu_{EUY}EU(\varepsilon_{JEUY}L(\rho_{UY}))"] & EULULUY \arrow[l, shift right=3, "EULU(\varepsilon_Y)"'] \arrow[l, "EU(\varepsilon_{LUY})" description] \arrow[l, shift left=3, "\mu_{EULUY}EU(\varepsilon_{JEULUY}L(\rho_{ULUY}))"]
\end{tikzcd}$$

4.4.5 Strict Version

The techniques employed to prove strict versions of Theorem 4.4.3 are virtually the same. We just need to repeat the same constructions, but, now, by means of strict descent objects and 2-adjoints. For instance, we have:

Theorem 4.4.6 (Strict Biadjoint Triangle)**.** *Let* $(L \dashv U, \eta, \varepsilon, s, t)$ *be a biadjunction between* 2*-functors and* $(E \dashv R, \rho, \mu)$ *be a* 2*-adjunction such that the triangle of* 2*-functors*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "E"'] & & \mathfrak{B} \arrow[dl, "L"] \\
& \mathfrak{C} &
\end{tikzcd}$$

commutes and $(\eta J) : J \longrightarrow UE$ *is a* 2*-natural transformation. We assume that, for every pair of objects* $(A \in \mathfrak{A}, Y \in \mathfrak{B})$*, the diagram* $\mathscr{D}_Y^{JA} : \dot{\Delta} \to \mathsf{CAT}$ *induced by* $(L \dashv U, \eta, \varepsilon, s, t)$ *is of strict descent. The* 2*-functor* J *has a right* 2*-adjoint if and only if, for every object* Y *of* $\mathfrak{B}$*, the strict descent object of* $\mathscr{A}_Y : \Delta \to \mathfrak{A}$ *exists in* $\mathfrak{A}$*.*

Proof. In particular, we have the setting of Theorem 4.4.3. Therefore, again, we can define $\psi : \mathscr{D}_Y^{JA} \circ \mathsf{j} \longrightarrow \mathfrak{A}(A, \mathscr{A}_Y -)$ as it was done in the proof of Theorem 4.4.3. However, since $(E \dashv R, \rho, \mu)$ is a 2-adjunction, J, E, R, L, U are 2-functors and (ηJ) is a 2-natural transformation, the components ψ_{d0}, ψ_{d1}, ψ_{s0}, $\psi_{\partial 0}$, $\psi_{\partial 1}$, $\psi_{\partial 2}$ are identities. Thereby ψ is a 2-natural transformation. Moreover, since $(E \dashv R, \rho, \mu)$ is a 2-adjunction, ψ is a pointwise isomorphism. Thus it is a 2-natural isomorphism.

Firstly, we assume that $\mathscr{D}_Y^{JA}$ is of strict descent for every object A of $\mathfrak{A}$ and every object Y of $\mathfrak{B}$. Then the strict descent object of $\mathfrak{A}(A, \mathscr{A}_Y -)$ is $\mathscr{D}_Y^{JA}\mathsf{0}$.

If, furthermore, $\mathfrak{A}$ has the strict descent object of $\mathscr{A}_Y$, we get a 2-natural isomorphism

$$\mathfrak{C}(J-, Y) \cong \mathfrak{A}\left(-, \left\{\dot{\Delta}(\mathsf{0}, \mathsf{j}-), \mathscr{A}_Y\right\}\right).$$

This proves that J is left 2-adjoint, provided that the strict descent object of $\mathscr{A}_Y$ exists for every object Y of $\mathfrak{B}$.

Reciprocally, if J is left 2-adjoint to a 2-functor G, since $\mathfrak{C}(-, GY) \cong \mathfrak{C}(J-, Y)$ is the strict descent object of $\mathfrak{A}(-, \mathscr{A}_Y -)$, we conclude that GY is the strict descent object of $\mathscr{A}_Y$. □

4.5 Pseudoprecomonadicity

A pseudomonad [66, 84] is the same as a doctrine, whose definition can be found in page 123 of [104], while a pseudocomonad is the dual notion. Similarly to the 1-dimensional case, for each pseudocomonad $\mathscr{T}$ on a 2-category $\mathfrak{C}$, there is an associated right biadjoint to the forgetful 2-functor $\mathsf{L} : \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \to \mathfrak{C}$, in which $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ is the 2-category of pseudocoalgebras [66] of Definition 4.5.2. Also, every biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$ induces a comparison pseudofunctor and an Eilenberg-Moore factorization [73]

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "\mathscr{K}"] \arrow[dr, "L"'] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \arrow[d] \\
 & \mathfrak{C}
\end{tikzcd}$$

in which $\mathscr{T}$ denotes the induced pseudocomonad. Before proving Corollary 4.5.10 which is a consequence of Theorem 4.4.3 in the context of pseudocomonads, we sketch some basic definitions and known results needed to fix notation and show Lemma 4.5.6. Some of them are related to the formal theory of pseudo(co)monads developed by Lack [66]. There, it is employed the coherence result of tricategories [40] (and, hence, with due adaptations, the formal theory developed therein works for any tricategory).

Definition 4.5.1. [Pseudocomonad] A *pseudocomonad* $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$ on a 2-category $\mathfrak{C}$ is a pseudofunctor $(\mathscr{T}, \mathfrak{t}) : \mathfrak{C} \to \mathfrak{C}$ with

1. Pseudonatural transformations:

$$\varpi : \mathscr{T} \longrightarrow \mathscr{T}^2 \qquad\qquad \varepsilon : \mathscr{T} \longrightarrow \mathrm{id}_{\mathfrak{C}}$$

2. Invertible modifications:

$$\begin{aligned}
\Lambda &: (\varpi\mathscr{T})(\varpi) \Longrightarrow (\mathscr{T}\varpi)(\varpi) \\
\mathsf{s} &: (\varepsilon\mathscr{T})(\varpi) \Longrightarrow \mathrm{id}_{\mathscr{T}} \\
\delta &: \mathrm{id}_{\mathscr{T}} \Longrightarrow (\mathscr{T}\varepsilon)(\varpi)
\end{aligned}$$

such that the following equations hold:

- Associativity:

$$\begin{tikzcd}
\mathscr{T} \arrow[r, "\varpi"] \arrow[d, "\varpi"'] \arrow[dr, phantom, "\overset{\Lambda}{\Leftarrow}"] & \mathscr{T}^2 \arrow[d, "\varpi\mathscr{T}"] \arrow[dr, "\varpi\mathscr{T}"] & & & \mathscr{T} \arrow[r, "\varpi"] \arrow[d, "\varpi"'] \arrow[dr, "\varpi"] & \mathscr{T}^2 \arrow[dr, "\varpi\mathscr{T}"] \arrow[d, phantom, "\overset{\Lambda}{\Leftarrow}"] & \\
\mathscr{T}^2 \arrow[r, "\mathscr{T}\varpi"'] \arrow[dr, "\mathscr{T}\varpi"'] & \mathscr{T}^3 \arrow[dr, "\mathscr{T}\varpi\mathscr{T}"] \arrow[d, phantom, "\overset{\widehat{\mathscr{T}\Lambda}}{\Longleftarrow}"] & \mathscr{T}^3 \arrow[l, phantom, "\overset{\Lambda\mathscr{T}}{\Longleftarrow}"] \arrow[d, "\varpi\mathscr{T}^2"] & = & \mathscr{T}^2 \arrow[dr, "\mathscr{T}\varpi"'] & \mathscr{T}^2 \arrow[l, phantom, "\overset{\Lambda}{\Leftarrow}"] \arrow[r, "\mathscr{T}\varpi"] \arrow[d, "\varpi\mathscr{T}"] \arrow[dr, phantom, "\overset{\varpi_{\varpi}^{-1}}{\Longleftarrow}"] & \mathscr{T}^3 \arrow[d, "\varpi\mathscr{T}^2"] \\
 & \mathscr{T}^3 \arrow[r, "\mathscr{T}^2\varpi"'] & \mathscr{T}^4 & & & \mathscr{T}^3 \arrow[r, "\mathscr{T}^2\varpi"'] & \mathscr{T}^4
\end{tikzcd}$$

- Identity:

$$\begin{tikzcd}
 & \mathscr{T} \arrow[dl, "\varpi"'] \arrow[dr, "\varpi"] & \\
\mathscr{T}^2 \arrow[dr, "\mathscr{T}\varpi"'] & \overset{\Lambda}{\Leftarrow} & \mathscr{T}^2 \arrow[dl, "\varpi\mathscr{T}"] \\
 & \mathscr{T}^3 \arrow[d, "\mathscr{T}\varepsilon\mathscr{T}"] & \\
 & \mathscr{T}^2 &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & \mathscr{T} \arrow[d, "\varpi"] & & \\
 & & \mathscr{T}^2 \arrow[dll, "\mathscr{T}\varpi"'] \arrow[drr, "\varpi\mathscr{T}"] \arrow[dd, equal] & & \\
\mathscr{T}^3 \arrow[drr, "\mathscr{T}\varepsilon\mathscr{T}"'] & \overset{\delta\mathscr{T}}{\Longleftarrow} & & \overset{\widehat{\mathscr{T}\mathsf{s}}}{\Longleftarrow} & \mathscr{T}^3 \arrow[dll, "\mathscr{T}\varepsilon\mathscr{T}"] \\
 & & \mathscr{T}^2 & &
\end{tikzcd}$$

in which $\widehat{\mathscr{T}\mathsf{s}}, \widehat{\mathscr{T}\Lambda}$ denote "corrections" of domain and codomain given by the isomorphisms induced by the pseudofunctor $\mathscr{T}$. That is to say,

$$\widehat{\mathscr{T}\mathsf{s}} := \ \mathsf{t}^{-1}_{(\varepsilon\mathscr{T})(\varpi)}(\mathscr{T}\mathsf{s})\mathsf{t}_{\mathscr{T}^2} \qquad\qquad \widehat{\mathscr{T}\Lambda} := \ \mathsf{t}^{-1}_{(\mathscr{T}\varpi)(\varpi)}(\mathscr{T}\Lambda)\mathsf{t}_{(\varpi\mathscr{T})(\varpi)}$$

Definition 4.5.2. [Pseudocoalgebras] Let $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$ be a pseudocomonad in $\mathfrak{C}$. We define the objects, 1-cells and 2-cells of the 2-category Ps-$\mathscr{T}$-CoAlg as follows:

1. Objects: pseudocoalgebras are defined by $\mathsf{z} = (Z, \rho_{\mathsf{z}}, \varsigma_{\mathsf{z}}, \Omega_{\mathsf{z}})$ in which $\rho_{\mathsf{z}} : Z \to \mathscr{T}Z$ is a morphism in $\mathfrak{C}$ and

$$\varsigma_{\mathsf{z}} : \ \ \mathrm{id}_Z \Rightarrow \varepsilon_Z\rho_{\mathsf{z}} \qquad\qquad \Omega_{\mathsf{z}} : \ \ \varpi_Z\rho_{\mathsf{z}} \Rightarrow \mathscr{T}(\rho_{\mathsf{z}})\rho_{\mathsf{z}}$$

are invertible 2-cells of $\mathfrak{C}$ such that the equations

$$\begin{tikzcd}
Z \arrow[r, "\rho_{\mathsf{z}}"] \arrow[d, "\rho_{\mathsf{z}}"'] \arrow[dr, phantom, "\overset{\Omega_{\mathsf{z}}}{\Longleftarrow}"] & \mathscr{T}Z \arrow[d, "\varpi_Z"] \arrow[dr, "\varpi_Z"] & \\
\mathscr{T}Z \arrow[r, "\mathscr{T}(\rho_{\mathsf{z}})"'] \arrow[dr, "\mathscr{T}(\rho_{\mathsf{z}})"'] & \mathscr{T}^2Z \arrow[r, phantom, "\overset{\Lambda_Z}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\widehat{\mathscr{T}(\Omega_{\mathsf{z}})}}{\Longleftarrow}"] \arrow[dr, "(\mathscr{T}\varpi)_Z"] & \mathscr{T}^2Z \arrow[d, "\varpi_{\mathscr{T}Z}"] \\
 & \mathscr{T}^2Z \arrow[r, "\mathscr{T}^2(\rho_{\mathsf{z}})"'] & \mathscr{T}^3Z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Z \arrow[r, "\rho_{\mathsf{z}}"] \arrow[d, "\rho_{\mathsf{z}}"'] \arrow[dr, "\rho_{\mathsf{z}}"] & \mathscr{T}Z \arrow[dr, "\varpi_Z"] \arrow[d, phantom, "\overset{\Omega_{\mathsf{z}}}{\Longleftarrow}"] & \\
\mathscr{T}Z \arrow[r, phantom, "\overset{\Omega_{\mathsf{z}}}{\Longleftarrow}"] \arrow[dr, "\mathscr{T}(\rho_{\mathsf{z}})"'] & \mathscr{T}Z \arrow[r, "\mathscr{T}(\rho_{\mathsf{z}})"] \arrow[d, "\varpi_Z"] \arrow[dr, phantom, "\overset{\varpi^{-1}_{\rho_{\mathsf{z}}}}{\Longleftarrow}"] & \mathscr{T}^2Z \arrow[d, "\varpi_{\mathscr{T}Z}"] \\
 & \mathscr{T}^2Z \arrow[r, "\mathscr{T}^2(\rho_{\mathsf{z}})"'] & \mathscr{T}^3Z
\end{tikzcd}$$

$$\begin{tikzcd}
 & Z \arrow[dl, "\rho_{\mathsf{z}}"'] \arrow[dr, "\rho_{\mathsf{z}}"] & \\
\mathscr{T}Z \arrow[dr, "\mathscr{T}(\rho_{\mathsf{z}})"'] & \overset{\Omega_{\mathsf{z}}}{\Longleftarrow} & \mathscr{T}Z \arrow[dl, "\varpi_Z"] \\
 & \mathscr{T}^2Z \arrow[d, "(\mathscr{T}\varepsilon)_Z"] & \\
 & \mathscr{T}Z &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & Z \arrow[d, "\rho_{\mathsf{z}}"] & & \\
 & & \mathscr{T}Z \arrow[dll, "\mathscr{T}(\rho_{\mathsf{z}})"'] \arrow[drr, "\varpi_Z"] \arrow[dd, equal] & & \\
\mathscr{T}^2Z \arrow[drr, "(\mathscr{T}\varepsilon)_Z"'] & \overset{\widehat{\mathscr{T}(\varsigma_{\mathsf{z}})}}{\Longleftarrow} & & \overset{\delta_Z}{\Longleftarrow} & \mathscr{T}^2Z \arrow[dll, "(\mathscr{T}\varepsilon)_Z"] \\
 & & \mathscr{T}Z & &
\end{tikzcd}$$

are satisfied, in which

$$\widehat{\mathscr{T}(\varsigma_{\mathsf{z}})} := \ \mathsf{t}^{-1}_{\varepsilon_Z\rho_{\mathsf{z}}}\mathscr{T}(\varsigma_{\mathsf{z}})\mathsf{t}_Z \qquad\qquad \widehat{\mathscr{T}(\Omega_{\mathsf{z}})} := \ \mathsf{t}^{-1}_{\varpi_Z\rho_{\mathsf{z}}}\mathscr{T}(\Omega_{\mathsf{z}})\mathsf{t}_{\mathscr{T}(\rho_{\mathsf{z}})\rho_{\mathsf{z}}}$$

2. Morphisms: $\mathscr{T}$-pseudomorphisms $\mathsf{f} : \mathsf{x} \to \mathsf{z}$ are pairs $\mathsf{f} = (f, \rho_f^{-1})$ in which $f : X \to Z$ is a morphism in $\mathfrak{C}$ and $\rho_{\mathsf{f}} : \mathscr{T}(f)\rho_{\mathsf{x}} \Rightarrow \rho_{\mathsf{z}} f$ is an invertible 2-cell of $\mathfrak{C}$ such that, defining $\widehat{\mathscr{T}(\rho_{\mathsf{f}}^{-1})} := \mathsf{t}^{-1}_{\mathscr{T}(f)\rho_{\mathsf{x}}} \mathscr{T}(\rho_{\mathsf{f}}^{-1}) \mathsf{t}_{\rho_{\mathsf{z}} f}$,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\rho_{\mathsf{x}}"'] \arrow[dr, phantom, "\overset{\rho_{\mathsf{f}}^{-1}}{\Longleftarrow}"] & Z \arrow[d, "\rho_{\mathsf{z}}"] \arrow[dr, "\rho_{\mathsf{z}}"] & \\
\mathscr{T}X \arrow[r, "\mathscr{T}(f)"'] \arrow[dr, "\mathscr{T}(\rho_{\mathsf{x}})"'] & \mathscr{T}Z \arrow[r, phantom, "\overset{\Omega_{\mathsf{z}}}{\Longleftarrow}"] \arrow[dr, "\mathscr{T}(\rho_{\mathsf{z}})"] \arrow[d, phantom, "\overset{\widehat{\mathscr{T}(\rho_{\mathsf{f}}^{-1})}}{\Longleftarrow}"] & \mathscr{T}Z \arrow[d, "\varpi_Z"] \\
& \mathscr{T}^2X \arrow[r, "\mathscr{T}^2(f)"'] & \mathscr{T}^2Z
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\rho_{\mathsf{x}}"'] \arrow[dr, "\rho_{\mathsf{x}}"] & Z \arrow[dr, "\rho_{\mathsf{z}}"] \arrow[d, phantom, "\overset{\rho_{\mathsf{f}}^{-1}}{\Longleftarrow}"] & \\
\mathscr{T}X \arrow[r, phantom, "\overset{\Omega_{\mathsf{x}}}{\Longleftarrow}"] \arrow[dr, "\mathscr{T}(\rho_{\mathsf{x}})"'] & \mathscr{T}X \arrow[r, "\mathscr{T}(f)"] \arrow[d, "\varpi_X"] \arrow[dr, phantom, "\overset{\varpi_f^{-1}}{\Longleftarrow}"] & \mathscr{T}Z \arrow[d, "\varpi_Z"] \\
& \mathscr{T}^2X \arrow[r, "\mathscr{T}^2(f)"'] & \mathscr{T}^2Z
\end{tikzcd}$$

holds and the 2-cell below is the identity.

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[dd, equal] \arrow[dr, "\rho_{\mathsf{x}}"] & & & Z \arrow[dd, equal] \arrow[dl, "\rho_{\mathsf{z}}"'] \\
\overset{\varsigma_{\mathsf{x}}}{\Longleftarrow} & \mathscr{T}X \arrow[r, "\mathscr{T}(f)"] \arrow[dl, "\varepsilon_X"'] \arrow[ur, phantom, "\overset{\rho_{\mathsf{f}}}{\Longleftarrow}"] \arrow[dr, phantom, "\overset{(\varepsilon)_f^{-1}}{\Longleftarrow}"] & \mathscr{T}Z \arrow[dr, "\varepsilon_Z"] & \overset{\varsigma_{\mathsf{z}}^{-1}}{\Longleftarrow} \\
X \arrow[rrr, "f"'] & & & Y
\end{tikzcd}$$

3. 2-cells: a $\mathscr{T}$-transformation between $\mathscr{T}$-pseudomorphisms $\mathsf{m} : \mathsf{f} \Rightarrow \mathsf{h}$ is a 2-cell $\mathfrak{m} : f \Rightarrow h$ in $\mathfrak{C}$ such that the equation below holds.

$$\begin{tikzcd}
X \arrow[rr, "\rho_{\mathsf{x}}"] \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "h"] \arrow[d, phantom, "\overset{\mathfrak{m}}{\Rightarrow}"] & \overset{\rho_h}{\Rightarrow} & \mathscr{T}X \arrow[d, "\mathscr{T}(h)"] \\
Z \arrow[rr, "\rho_{\mathsf{z}}"'] & & \mathscr{T}Z
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[rr, "\rho_{\mathsf{x}}"] \arrow[d, "f"'] & \overset{\rho_f}{\Rightarrow} & \mathscr{T}X \arrow[d, bend right=50, "\mathscr{T}(f)"'] \arrow[d, bend left=50, "\mathscr{T}(h)"] \arrow[d, phantom, "\overset{\mathscr{T}(\mathfrak{m})}{\Rightarrow}"] \\
Z \arrow[rr, "\rho_{\mathsf{z}}"'] & & \mathscr{T}Z
\end{tikzcd}$$

Remark 4.5.3. If $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$ is a pseudocomonad on $\mathfrak{C}$, then $\mathscr{T}$ induces a biadjunction $(\mathsf{L} \dashv \mathsf{U}, \rho, \varepsilon, \underline{\mathsf{s}}, \underline{\mathsf{t}})$ in which L, U are defined by

$$\begin{aligned}
\mathsf{L} : \text{Ps-}\mathscr{T}\text{-CoAlg} &\to \mathfrak{C} & \mathsf{U}\ &: \mathfrak{C} \to \text{Ps-}\mathscr{T}\text{-CoAlg} \\
\mathsf{z} = (Z, \rho_{\mathsf{z}}, \varsigma_{\mathsf{z}}, \Omega_{\mathsf{z}}) &\mapsto Z & Z &\mapsto (\mathscr{T}(Z), \varpi_Z, \mathsf{s}_Z, \Lambda_Z) \\
\mathsf{f} = (f, \rho_f^{-1}) &\mapsto f & f &\mapsto \left(\mathscr{T}(f), \varpi_f^{-1}\right) \\
\mathsf{m} &\mapsto \mathfrak{m} & \mathfrak{m} &\mapsto \mathscr{T}(\mathfrak{m})
\end{aligned}$$

Reciprocally, we know that each biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$ induces a pseudocomonad

$$\mathscr{T} = (LU, L\eta U, \varepsilon, (L\eta)^{-1}_{\eta_U}, \widehat{(Lt)}, sU).$$

Lemma 4.5.4 gives some further aspects of these constructions (which follows from calculations on the formal theory of pseudocomonads in 2-CAT).

Lemma 4.5.4. *Let $L : \mathfrak{B} \to \mathfrak{C}$ be a pseudofunctor. A biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$ induces commutative triangles*

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "\mathscr{K}"] \arrow[rd, "L"'] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \arrow[d, "\mathsf{L}"] & & \mathfrak{C} \arrow[r, "U"] \arrow[rd, "\mathsf{U}"'] & \mathfrak{B} \arrow[d, "\mathscr{K}"] \\
 & \mathfrak{C} & & & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}
\end{tikzcd}$$

in which $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$ is the pseudocomonad induced by $(L \dashv U, \eta, \varepsilon, s, t)$, $(\mathsf{L} \dashv \mathsf{U}, \rho, \varepsilon, \underline{\mathsf{s}}, \underline{\mathsf{t}})$ is the biadjunction induced by $\mathscr{T}$ and $\mathscr{K} : \mathfrak{B} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ is the unique (up to pseudonatural isomorphism) comparison pseudofunctor making the triangles above commutative. Namely,

$$\begin{aligned}
\mathscr{K} : \quad & \mathfrak{B} && \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \\
& Y && \mapsto \left(LY, L(\eta_Y), s_Y^{-1}, (L\eta)_{\eta_Y}^{-1}\right) \\
& g && \mapsto \left(L(g), (L\eta)_g^{-1}\right) \\
& \mathfrak{m} && \mapsto L(\mathfrak{m})
\end{aligned}$$

Furthermore, we have the obvious equalities

$$L(\eta_Y) = \quad \rho_{\mathscr{K}Y} \qquad\qquad \varpi_{LY} = \quad (L\eta U)_{LY}.$$

Proposition 4.5.5. *Let $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$ be a pseudocomonad on $\mathfrak{C}$. Given $\mathscr{T}$-pseudocoalgebras*

$$\mathsf{x} = (X, \rho_{\mathsf{x}}, \varsigma_{\mathsf{x}}, \Omega_{\mathsf{x}}), \mathsf{z} = (Z, \rho_{\mathsf{z}}, \varsigma_{\mathsf{z}}, \Omega_{\mathsf{z}}),$$

the category $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}(\mathsf{x}, \mathsf{z})$ is the strict descent object of the diagram $\mathbb{T}_{\mathsf{z}}^{\mathsf{x}} : \Delta \to \mathsf{CAT}$

$$\begin{tikzcd}[column sep=huge]
\mathfrak{C}(\mathsf{Lx}, \mathsf{Lz}) \arrow[r, shift left=3, "\mathfrak{C}(\rho_{\mathsf{x}}, \mathscr{T}\mathsf{Lz}) \circ \mathscr{T}_{(\mathsf{Lx}, \mathsf{Lz})}"] \arrow[r, shift right=3, "\mathfrak{C}(\mathsf{Lx}, \rho_{\mathsf{z}})"'] & \mathfrak{C}(\mathsf{Lx}, \mathscr{T}\mathsf{Lz}) \arrow[l, "\mathfrak{C}(\mathsf{Lx}, \varepsilon_{\mathsf{Lz}})" description] \arrow[r, shift left=3, "\mathfrak{C}(\rho_{\mathsf{x}}, \mathscr{T}\mathsf{Lz}) \circ \mathscr{T}_{(\mathsf{Lx}, \mathscr{T}\mathsf{Lz})}"] \arrow[r, "\mathfrak{C}(\mathsf{Lx}, \varpi_{\mathsf{Lz}})" description] \arrow[r, shift right=3, "\mathfrak{C}(\mathsf{Lx}, \mathscr{T}(\rho_{\mathsf{z}}))"'] & \mathfrak{C}(\mathsf{Lx}, \mathscr{T}^2\mathsf{Lz})
\end{tikzcd} \qquad (\mathbb{T}_{\mathsf{z}}^{\mathsf{x}})$$

such that

$$\begin{aligned}
\mathbb{T}_{\mathsf{z}}^{\mathsf{x}}(\sigma_{02})_f &:= \left(\mathsf{t}_{\rho_{\mathsf{z}} f} * \mathrm{id}_{\rho_{\mathsf{x}}}\right) \\
\mathbb{T}_{\mathsf{z}}^{\mathsf{x}}(\sigma_{12})_f &:= \left(\Omega_{\mathsf{z}}^{-1} * \mathrm{id}_f\right) \\
\mathbb{T}_{\mathsf{z}}^{\mathsf{x}}(n_1)_f &:= \left(\varsigma_{\mathsf{z}}^{-1} * \mathrm{id}_f\right)
\end{aligned} \qquad
\begin{aligned}
\mathbb{T}_{\mathsf{z}}^{\mathsf{x}}(\sigma_{01})_f &:= \left(\mathsf{t}_{\mathscr{T}(f)\rho_{\mathsf{x}}} * \mathrm{id}_{\rho_{\mathsf{x}}}\right) \cdot \left(\mathrm{id}_{\mathscr{T}^2(f)} * \Omega_{\mathsf{x}}\right) \cdot \left(\varpi_f^{-1} * \mathrm{id}_{\rho_{\mathsf{x}}}\right) \\
\mathbb{T}_{\mathsf{z}}^{\mathsf{x}}(n_0)_f &:= \left(\mathrm{id}_f * \varsigma_{\mathsf{x}}\right) \cdot \left(\varepsilon_f^{-1} * \mathrm{id}_{\rho_{\mathsf{x}}}\right)
\end{aligned}$$

Proof. It follows from Definition 4.5.2 and Remark 4.3.8. □

Recall that every biadjunction induces diagrams $\mathscr{D}_Y^X : \dot{\Delta} \to \mathsf{CAT}$ (Definition 4.4.1). Also, for every pseudocomonad $\mathscr{T}$ and objects x, z of $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$, we defined in Proposition 4.5.5 a diagram $\mathbb{T}_{\mathsf{z}}^{\mathsf{x}} : \Delta \to \mathsf{CAT}$ whose strict descent object is $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}(\mathsf{x}, \mathsf{z})$. Now, we give the relation between these two diagrams.

Lemma 4.5.6. *Let $L : \mathfrak{B} \to \mathfrak{C}$ be a pseudofunctor, $(L \dashv U, \eta, \varepsilon, s, t)$ be a biadjunction and $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$ be the induced pseudocomonad. For each pair (X, Y) of objects in $\mathfrak{B}$, $(L \dashv U, \eta, \varepsilon, s, t)$ induces the diagram $\mathscr{D}_Y^X : \dot{\Delta} \to \mathsf{CAT}$ and $\mathscr{T}$ induces the diagram $\mathbb{T}_{\mathscr{K}Y}^{\mathscr{K}X} : \Delta \to \mathsf{CAT}$*

defined in Proposition 4.5.5, in which $\mathscr{K} : \mathfrak{B} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ *is the comparison pseudofunctor. In this setting, there is a pseudonatural isomorphism* $\beta : \mathscr{D}_Y^X \circ \mathsf{j} \longrightarrow \mathbb{T}_{\mathscr{K}Y}^{\mathscr{K}X}$ *for every such pair* (X,Y) *of objects in* $\mathfrak{B}$*. Moreover, if* L *is a* 2*-functor,* β *is actually a* 2*-natural isomorphism.*

Proof. We can write $\mathbb{T}_{\mathscr{K}Y}^{\mathscr{K}X} : \Delta \to \mathsf{CAT}$ as follows

$$\begin{tikzcd}[column sep=huge]
\mathfrak{C}(LX,LY) \arrow[r, shift left=3, "\mathfrak{C}(L(\eta_X)\text{,}LULY)\circ\,(LU)_{LX,LY}"] \arrow[r, shift right=3, "\mathfrak{C}(LX\text{,}L(\eta_Y))"'] & \mathfrak{C}(LX,LULY) \arrow[l, "\mathfrak{C}(LX\text{,}\varepsilon_{LY})" description] \arrow[r, shift left=3, "\mathfrak{C}(L(\eta_X)\text{,}LULY)\circ\,(LU)_{LX,LULY}"] \arrow[r, "\mathfrak{C}(LX\text{,}L(\eta_{LY}))" description] \arrow[r, shift right=3, "\mathfrak{C}(LX\text{,}LUL(\eta_Y))"'] & \mathfrak{C}(LX,LULULY)
\end{tikzcd}$$

Furthermore, by Lemma 4.5.4 and the observations given in this section, we can define a pseudonatural isomorphism

$$\beta : \mathscr{D}_Y^X \circ \mathsf{j} \longrightarrow \mathbb{T}_{\mathscr{K}Y}^{\mathscr{K}X}$$

such that $\beta_1, \beta_2, \beta_3$ are identity functors, $\beta_{_{d^1}}, \beta_{_{\partial^1}}, \beta_{_{\partial^2}}, \beta_{_{s^0}}$ are identity natural transformations, $\left(\beta_{_{d^0}}\right)_f :=$ $\mathfrak{l}_{U(f)\eta_X}$ and $\left(\beta_{_{\partial^0}}\right)_f := \mathfrak{l}_{U(f)\eta_X}$. This completes the proof. $\square$

Let $(L \dashv U, \eta, \varepsilon, s, t)$ be a biadjunction and $\mathscr{T}$ be the induced pseudocomonad. By Lemma 4.3.4, Proposition 4.5.5 and Lemma 4.5.6, $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}(\mathscr{K}X, \mathscr{K}Y)$ is a descent object of $\mathscr{D}_Y^X \circ \mathsf{j}$ for every pair of objects (X,Y) of $\mathfrak{B}$. Moreover, $\mathscr{K}_{_{X,Y}} : \mathfrak{B}(X,Y) \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}(\mathscr{K}X, \mathscr{K}Y)$ is the comparison $\mathscr{D}_Y^X 0 \to \left\{\dot{\Delta}(0,\mathsf{j}-), \mathscr{D}_Y^X \circ \mathsf{j}\right\}$. Thereby we get:

Proposition 4.5.7. *Let* $(L \dashv U, \eta, \varepsilon, s, t)$ *be a biadjunction,* $\mathscr{T}$ *be the induced pseudocomonad and* $\mathscr{K} : \mathfrak{B} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ *be the comparison pseudofunctor. For each pair of objects* (X,Y) *in* $\mathfrak{B}$*,* $\mathscr{D}_Y^X : \dot{\Delta} \to \mathsf{CAT}$ *is of effective descent if and only if*

$$\mathscr{K}_{_{X,Y}} : \mathfrak{B}(X,Y) \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}(\mathscr{K}X, \mathscr{K}Y)$$

is an equivalence. Furthermore, if L *is a* 2*-functor,* $\mathscr{D}_Y^X$ *is of strict descent if and only if* $\mathscr{K}_{_{X,Y}}$ *is an isomorphism.*

4.5.8 Biadjoint Triangles

In this subsection, we reexamine the results of Section 4.4 in the context of pseudocomonad theory. More precisely, we prove Corollary 4.5.10 of our main theorems in Section 4.4, Theorem 4.4.3 and Theorem 4.4.6.

Let $(L, U, \eta, \varepsilon, s, t)$ be a biadjunction and $\mathscr{T}$ be its induced pseudocomonad. We say that $L : \mathfrak{B} \to \mathfrak{C}$ is *pseudoprecomonadic*, if its induced comparison pseudofunctor $\mathscr{K} : \mathfrak{B} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ is locally an equivalence. As a consequence of Proposition 4.5.7, we get a characterization of pseudoprecomonadic pseudofunctors.

Corollary 4.5.9 (Pseudoprecomonadic)**.** *Let* $(L \dashv U, \eta, \varepsilon, s, t)$ *be a biadjunction. The pseudofunctor* $L : \mathfrak{B} \to \mathfrak{C}$ *is pseudoprecomonadic if and only if* $\mathscr{D}_Y^X : \dot{\Delta} \to \mathsf{CAT}$ *is of effective descent for every pair of objects* (X,Y) *of* $\mathfrak{B}$*.*

By Corollary 4.5.9, assuming that $(L \dashv U, \eta, \varepsilon, s, t)$ is a biadjunction and $J : \mathfrak{A} \to \mathfrak{B}$ is a pseudofunctor, if $L : \mathfrak{B} \to \mathfrak{C}$ is pseudoprecomonadic, then, in particular, $\mathscr{D}_Y^{JA} : \dot{\Delta} \to \mathsf{CAT}$ is of effective descent for every object A of $\mathfrak{A}$ and every object Y of $\mathfrak{B}$. Thereby, as a consequence of Theorem 4.4.3, Theorem 4.4.6 and Propostion 4.5.7, we get:

Corollary 4.5.10 (Biadjoint Triangle Theorem)**.** *Assume that* $(E \dashv R, \rho, \mu, v, w), (L \dashv U, \eta, \varepsilon, s, t)$ *are biadjunctions such that the triangle of pseudofunctors*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "E"'] & & \mathfrak{B} \arrow[dl, "L"] \\
& \mathfrak{C} &
\end{tikzcd}$$

is commutative and L is pseudoprecomonadic. Then J has a right biadjoint if and only if, for every object Y of $\mathfrak{B}$*,* $\mathfrak{A}$ *has the descent object of the diagram* $\mathscr{A}_Y : \Delta \to \mathfrak{A}$*. In this case, J is left biadjoint to* $GY := \left\{\dot{\Delta}(0,\mathsf{j}-), \mathscr{A}_Y\right\}_{\mathsf{bi}}$.

If, furthermore, E, R, J, L, U *are* 2*-functors,* $(E \dashv R, \rho, \mu)$ *is a* 2*-adjunction,* (ηJ) *is a* 2*-natural transformation and the comparison* 2*-functor* $\mathscr{K} : \mathfrak{B} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ *induced by the biadjunction* $L \dashv U$ *is locally an isomorphism, then J is left* 2*-adjoint if and only if the strict descent object of* $\mathscr{A}_Y$ *exists for every object Y of* $\mathfrak{B}$*. In this case,* $GY := \left\{\dot{\Delta}(0,\mathsf{j}-), \mathscr{A}_Y\right\}$ *defines the right* 2*-adjoint to J.*

4.6 Unit and Counit

In this section, we show that the pseudoprecomonadicity characterization given in Theorem 3.5 of [73] is a consequence of Corollary 4.5.9. Secondly, we study again biadjoint triangles. Namely, in the context of Corollary 4.5.10, we give necessary and sufficient conditions under which the unit and the counit of the obtained biadjunction $J \dashv G$ are pseudonatural equivalences, provided that E and L induce the same pseudocomonad. In other words, we prove the appropriate analogous versions of Corollary 1 and Corollary 2 of page 76 in [30] within our context of biadjoint triangles.

Again, we need to consider another type of 2-functors induced by biadjunctions. The definition below is given in Theorem 3.5 of [73].

Definition 4.6.1. Assume that $L : \mathfrak{B} \to \mathfrak{C}$ is a pseudofunctor and $(L \dashv U, \eta, \varepsilon, s, t)$ is a biadjunction. For each object Y of $\mathfrak{B}$, we get the 2-functor $\mathscr{V}_Y : \dot{\Delta} \to \mathfrak{B}$

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] & ULY \arrow[r, shift left=3, "\eta_{ULY}"] \arrow[r, leftarrow, "U(\varepsilon_{LY})" description] \arrow[r, shift right=3, "UL(\eta_Y)"'] & ULULY \arrow[r, shift left=3, "\eta_{ULULY}"] \arrow[r, "UL(\eta_{ULY})" description] \arrow[r, shift right=3, "ULUL(\eta_Y)"'] & ULULULY
\end{tikzcd} \qquad (\mathscr{V}_Y)$$

in which $\mathscr{V}_Y(\vartheta) := \left(\eta_{\eta_Y} : UL(\eta_Y)\eta_Y \cong \eta_{ULY}\eta_Y\right)$ is the invertible 2-cell component of the unit η at the morphism η_Y. Analogously, the images of the 2-cells σ_{ik}, n_0, n_1 are defined below.

$$\mathscr{V}_Y(\sigma_{01}) := \eta_{\eta_{ULY}} \qquad \mathscr{V}_Y(n_0) := t_{LY}$$

$$\mathscr{V}_Y(\sigma_{02}) := \eta_{UL(\eta_Y)} \qquad \mathscr{V}_Y(n_1) := \left(\mathfrak{u}_{\varepsilon_{LY}, L(\eta_Y)}\right)^{-1} \cdot U(s_Y) \cdot \mathfrak{u}_{ULY}$$

$$\mathscr{V}_Y(\sigma_{12}) := (UL\eta)_{\eta_Y}$$

We verify below that $\mathscr{V}_Y$ *is well defined.* That is to say, we have to prove that $\mathscr{V}_Y$ satisfies the equations given in Definition 4.3.1. Firstly, the associativity and naturality equations of Definition 4.2.2 give the following equality

$$\begin{tikzcd}
 & \mathscr{V}_Y 0 \arrow[r, "\mathscr{V}_Y(d)"] \arrow[dl, "\mathscr{V}_Y(d)"'] \arrow[d, "\mathscr{V}_Y(d)"] & \mathscr{V}_Y 1 \arrow[d, "\mathscr{V}_Y(d^1)"] \\
\mathscr{V}_Y 1 \arrow[r, phantom, "\xLeftarrow{\mathscr{V}_Y(\vartheta)}"] \arrow[dr, "\mathscr{V}_Y(d^0)"'] & \mathscr{V}_Y 1 \arrow[r, "\mathscr{V}_Y(d^0)"] \arrow[d, "\mathscr{V}_Y(d^1)"] & \mathscr{V}_Y 2 \arrow[d, "\mathscr{V}_Y(\partial^2)"] \\
 & \mathscr{V}_Y 2 \arrow[r, "\mathscr{V}_Y(\partial^0)"] & \mathscr{V}_Y 3
\arrow[from=1-2, to=2-3, phantom, "\xLeftarrow{\mathscr{V}_Y(\vartheta)}"]
\arrow[from=2-2, to=3-3, phantom, "\xLeftarrow{\mathscr{V}_Y(\sigma_{02})}"]
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{V}_Y 0 \arrow[r, "\mathscr{V}_Y(d)"] \arrow[d, "\mathscr{V}_Y(d)"] & \mathscr{V}_Y 1 \arrow[d, "\mathscr{V}_Y(d^1)"] \arrow[dr, "\mathscr{V}_Y(d^1)"] & \\
\mathscr{V}_Y 1 \arrow[r, "\mathscr{V}_Y(d^0)"] \arrow[d, "\mathscr{V}_Y(d^0)"] & \mathscr{V}_Y 2 \arrow[d, "\mathscr{V}_Y(\partial^1)"] \arrow[r, phantom, "\xLeftarrow{\mathscr{V}_Y(\sigma_{12})}"] & \mathscr{V}_Y 2 \arrow[dl, "\mathscr{V}_Y(\partial^2)"] \\
\mathscr{V}_Y 2 \arrow[r, "\mathscr{V}_Y(\partial^0)"] & \mathscr{V}_Y 3 &
\arrow[from=1-1, to=2-2, phantom, "\xLeftarrow{\mathscr{V}_Y(\vartheta)}"]
\arrow[from=2-1, to=3-2, phantom, "\xLeftarrow{\mathscr{V}_Y(\sigma_{01})}"]
\end{tikzcd}$$

which is the associativity equation of Definition 4.3.1. Furthermore, by Definition 4.2.5 of biadjunction, we have that

$$\begin{tikzcd}
\mathscr{V}_Y 0 \arrow[r, "\mathscr{V}_Y(d)=\eta_Y"] \arrow[d, "\mathscr{V}_Y(d)=\eta_Y"'] & \mathscr{V}_Y 1 \arrow[d, "\mathscr{V}_Y(d^1)=UL(\eta_Y)"] \arrow[ddr, bend left, equal] & \\
\mathscr{V}_Y 1) \arrow[r, "\mathscr{V}_Y(d^0)=\eta_{ULY}"'] \arrow[drr, bend right, equal] & \mathscr{V}_Y 2 \arrow[dr, "\mathscr{V}_Y(s^0)"] \arrow[r, phantom, "\xLeftarrow{\mathscr{V}_Y(n_1)}"] & \\
 & \xLeftarrow{\mathscr{V}_Y(n_0)} & \mathscr{V}_Y 1
\arrow[from=1-1, to=2-2, phantom, "\xLeftarrow{\mathscr{V}_Y(\vartheta)}"]
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{V}_Y 0 \arrow[d, bend right=40, "\mathscr{V}_Y(d)"'] \arrow[d, bend left=40, "\mathscr{V}_Y(d)"] \arrow[d, phantom, "="] \\
\mathscr{V}_Y 1
\end{tikzcd}$$

which proves that $\mathscr{V}_Y$ satisfies the identity equation of Definition 4.3.1.

As mentioned before, Corollary 4.6.2 is Theorem 3.5 of [73]. Below, it is proved as a consequence of Corollary 4.5.9.

Corollary 4.6.2 ([73])**.** *Let $(L \dashv U, \eta, \varepsilon, s, t)$ be a biadjunction. The pseudofunctor L is pseudoprecomonadic if and only if, for every object Y of $\mathfrak{B}$, the 2-functor $\mathscr{V}_Y : \dot{\Delta} \to \mathfrak{B}$ is of effective descent.*

Proof. On one hand, by Corollary 4.5.9, L is pseudoprecomonadic if and only if $\mathscr{D}_Y^X : \dot{\Delta} \to \mathsf{CAT}$ is of effective descent for every pair (X,Y) of objects in $\mathfrak{B}$. On the other hand, by Lemma 4.3.6, $\mathscr{V}_Y$ is of effective descent if and only if $\mathfrak{B}(X, \mathscr{V}_Y -) : \dot{\Delta} \to \mathsf{CAT}$ is of effective descent for every object X in $\mathfrak{B}$.

Therefore, by Lemma 4.3.5, to complete our proof, we just need to verify that $\mathscr{D}_Y^X \simeq \mathfrak{B}(X, \mathscr{V}_Y -)$. Indeed, there is a pseudonatural equivalence

$$\iota : \mathscr{D}_Y^X \longrightarrow \mathfrak{B}(X, \mathscr{V}_Y -)$$

induced by $\chi : \mathfrak{C}(L-,-) \simeq \mathfrak{B}(-,U-)$ such that

$$\begin{aligned}
\iota_0 &:= \mathrm{Id}_{\mathfrak{B}(X,Y)} \\
\iota_1 &:= \chi_{(X,LY)} : \mathfrak{C}(LX,LY) \to \mathfrak{B}(X,ULY) \\
\psi_2 &:= \chi_{(X,LULY)} : \mathfrak{C}(LX,LULY) \to \mathfrak{B}(X,ULULY) \\
\psi_3 &:= \chi_{(X,LULULY)} : \mathfrak{C}(LX,LULULY) \to \mathfrak{B}(X,ULULULY)
\end{aligned}$$

$$\left(\iota_{d}\right)_f := \eta_f^{-1} \qquad \left(\iota_{\partial^0}\right)_f := \eta^{-1}_{U(f)\eta_X} \qquad \left(\iota_{\partial^2}\right)_f := \mathfrak{u}_{LUL(\eta_Y)f} * \mathrm{id}_{\eta_X}$$

$$\left(\iota_{d^0}\right)_f := \eta^{-1}_{U(f)\eta_X} \qquad \left(\iota_{\partial^1}\right)_f := \mathfrak{u}_{L(\eta_{ULY})f} * \mathrm{id}_{\eta_X} \qquad \left(\iota_{s^0}\right)_f := \mathfrak{u}_{\varepsilon_{LY}f} * \mathrm{id}_{\eta_X}$$

$$\left(\iota_{d^1}\right)_f := \mathfrak{u}_{L(\eta_Y)f} * \mathrm{id}_{\eta_X}$$

□

We assume the existence of a biadjunction $J \dashv G$ in the commutative triangles below and study its counit and unit, provided that the biadjunctions $(E \dashv R, \rho, \mu, v, w), (L \dashv U, \eta, \varepsilon, s, t)$ induce the same pseudocomonad. We start with the unit.

Theorem 4.6.3 (Unit)**.** *Assume that $(E \dashv R, \rho, \mu, v, w), (L \dashv U, \eta, \varepsilon, s, t), (J \dashv G, \overline{\eta}, \overline{\varepsilon}, \overline{s}, \overline{t})$ are biadjunctions such that the triangles*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "E"'] & & \mathfrak{B} \arrow[dl, "L"] \\
& \mathfrak{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] & & \mathfrak{B} \\
& \mathfrak{C} \arrow[ul, "R"] \arrow[ur, "U"'] &
\end{tikzcd}$$

are commutative. If $(E \dashv R, \rho, \mu, v, w)$, $(L \dashv U, \eta, \varepsilon, s, t)$ induce the same pseudocomonad $\mathscr{T}$, then the following statements are equivalent:

1. *The unit $\overline{\eta} : \mathrm{Id}_{\mathfrak{A}} \longrightarrow GJ$ is a pseudonatural equivalence;*

2. *E is pseudoprecomonadic;*

3. *The following 2-functor is of effective descent for every pair of objects (A, B) in $\mathfrak{A}$*

$$\widehat{\mathscr{D}}^A_B : \dot{\Delta} \to \mathsf{CAT}$$

$$\begin{tikzcd}
\mathfrak{A}(A,B) \arrow[d, "E_{A,B}"] & & \\
\mathfrak{C}(EA,EB) \arrow[r, shift left=2, "E_{A,REB} \circ \xi_{(A,EB)}"] \arrow[r, shift right=2, "\mathfrak{C}(EA,E(\rho_B))"'] & \mathfrak{C}(EA,EREB) \arrow[l, "\mathfrak{C}(EA,\mu_{EB})"] \arrow[r, shift left=3, "E_{A,(RE)^2B} \circ \xi_{(A,EREB)}"] \arrow[r, "\mathfrak{C}(EA,E(\rho_{REB}))"] \arrow[r, shift right=3, "\mathfrak{C}(EA,ERE(\rho_B))"'] & \mathfrak{C}(EA,E(RE)^2B)
\end{tikzcd}$$

in which $\xi : \mathfrak{C}(E-,-) \simeq \mathfrak{A}(-,R-)$ is the pseudonatural equivalence induced by the biadjunction $(E \dashv R, \rho, \mu, v, w)$ described in Remark 4.2.6.

4. *For each object B of $\mathfrak{A}$, the following diagram is of effective descent.*

$$\widehat{\mathscr{V}_{B}} : \dot{\Delta} \to \mathfrak{A}$$

$$\begin{tikzcd}
B \arrow[r, "\rho_B"] & REB \arrow[r, shift left=3, "\rho_{REB}"] \arrow[r, shift right=3, "RE(\rho_B)"'] & REREY \arrow[l, "R(\mu_{EB})"] \arrow[r, shift left=3, "\rho_{REREB}"] \arrow[r, "RE(\rho_{REB})"] \arrow[r, shift right=3, "RERE(\rho_B)"'] & REREREB
\end{tikzcd}$$

Proof. By Remark 4.2.7, the unit $\bar{\eta}$ is a pseudonatural equivalence if and only if J is locally an equivalence. Moreover, by the hypothesis and the universal property of the 2-category of pseudocolagebras, we have the following diagram

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "J"] \arrow[rrd, "E"'] \arrow[rr, bend left=40, "\widetilde{\mathscr{K}}"] \arrow[rr, phantom, bend left=25, "\cong"] & \mathfrak{B} \arrow[r, "\mathscr{K}"] \arrow[rd, "L"] & \text{Ps-}\mathscr{T}\text{-CoAlg} \arrow[d, "\mathsf{L}"] \\
& & \mathfrak{C}
\end{tikzcd}$$

such that $\widetilde{\mathscr{K}}, \mathscr{K}$ are the comparison pseudofunctors.

Since, by hypothesis, we know that $\mathscr{K}$ is locally an equivalence, we conclude that J is locally an equivalence if and only if $\widetilde{\mathscr{K}}$ is locally an equivalence. Thereby, to conclude, we just need to apply the characterizations of pseudoprecomonadic pseudofunctors: that is to say, Corollary 4.5.9 and Corollary 4.6.2. □

Before studying the counit, for future references, we need the following result about the diagram $\mathscr{V}_Y : \dot{\Delta} \to \mathfrak{B}$ in the context of biadjoint triangles.

Lemma 4.6.4. *Let*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[rd, "E"'] & & \mathfrak{B} \arrow[ld, "L"] & \mathfrak{A} \arrow[rr, "J"] & & \mathfrak{B} \\
& \mathfrak{C} & & & \mathfrak{C} \arrow[lu, "R"] \arrow[ru, "U"'] &
\end{tikzcd}$$

be commutative triangles of pseudofunctors such that we have biadjunctions $(E \dashv R, \rho, \varepsilon, v, w)$ *and* $(L \dashv U, \eta, \varepsilon, s, t)$ *inducing the same pseudocomonad* $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$. *We consider the diagram* $\mathscr{A}_Y : \Delta \to \mathfrak{A}$. *Then, for each object* Y *of* $\mathfrak{B}$, *there is a pseudonatural isomorphism*

$$\zeta^{Y} : J \circ \mathscr{A}_Y \longrightarrow \mathscr{V}_Y \circ \mathrm{j}.$$

Proof. Again, we have the same diagram of the proof of Theorem 4.6.3. In particular, for each object Y of $\mathfrak{B}$, there is an invertible 2-cell $\mathfrak{y}_Y : J(\rho_{RLY}) \Rightarrow \eta_{ULY}$. Thereby, we can define $\zeta^{Y} : J \circ \mathscr{A}_Y \longrightarrow \mathscr{V}_Y \circ \mathrm{j}$ such that the components $\zeta_1, \zeta_2, \zeta_3$ are identity 1-cells, the components ζ^{Y}_{d1}, ζ^{Y}_{s0}, $\zeta^{Y}_{\partial 1}$, $\zeta^{Y}_{\partial 2}$ are identity 2-cells, $\left(\zeta^{Y}_{d0}\right) := \mathfrak{y}_Y \cdot J\left(\left((\mathfrak{rl})^{-1}_{LY} \cdot RL(t_{LY})\right) * \mathrm{id}_{\rho_{RLY}}\right)$ and $\left(\zeta^{Y}_{\partial 0}\right) := \mathfrak{y}_{ULY} \cdot J\left(\left((\mathfrak{rl})^{-1}_{LULY} \cdot RL(t_{LY})\right) * \mathrm{id}_{\rho_{RLULY}}\right)$. □

Theorem 4.6.5 (Counit)**.** *Let* $(E \dashv R, \rho, \varepsilon, v, w)$ *and* $(L \dashv U, \eta, \varepsilon, s, t)$ *be biadjunctions inducing the same pseudocomonad* $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon, \Lambda, \delta, \mathsf{s})$ *such that the triangles of pseudofunctors*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[rd, "E"'] & & \mathfrak{B} \arrow[ld, "L"] & \mathfrak{A} \arrow[rr, "J"] & & \mathfrak{B} \\
& \mathfrak{C} & & & \mathfrak{C} \arrow[lu, "R"] \arrow[ru, "U"'] &
\end{tikzcd}$$

commute. We assume that $(J \dashv G, \overline{\eta}, \overline{\varepsilon}, \overline{s}, \overline{t})$ *is a biadjunction and* L *is pseudoprecomonadic. We consider the diagram* $\mathscr{A}_Y : \Delta \to \mathfrak{A}$. *Then* $J\left\{\dot{\Delta}(0, \mathrm{j}-), \mathscr{A}_Y\right\}_{\mathrm{bi}}$ *is the descent object of* $J \circ \mathscr{A}_Y$ *for every object* Y *of* $\mathfrak{B}$ *if and only if the counit* $\overline{\varepsilon} : JG \longrightarrow \mathrm{Id}_{\mathfrak{B}}$ *is a pseudonatural equivalence.*

Proof. Actually, this is a corollary of Lemma 4.6.4, Corollary 4.5.10 and Corollary 4.6.2. More precisely, by Lemma 4.6.4, $J \circ \mathscr{A}_Y \simeq \mathscr{V}_Y \circ \mathrm{j}$. By Corollary 4.6.2, since L is pseudoprecomonadic, $\mathscr{V}_Y$

is of effective descent. Moreover, by the constructions of Theorem 4.4.3 (which proves Corollary 4.5.10), the counit is pointwise defined by the comparison 1-cells

$$J\left\{\dot{\Delta}(0,\mathsf{j}-),\mathscr{A}_Y\right\}_{\mathsf{bi}} \to Y = \mathscr{V}_Y 0 \simeq \left\{\dot{\Delta}(0,\mathsf{j}-),\mathscr{V}_Y\circ\mathsf{j}\right\}.$$

This completes the proof. □

4.7 Pseudocomonadicity

Similarly to the 1-dimensional case, to prove the characterization of pseudocomonadic pseudofunctors employing the biadjoint triangle theorems, we need two results: Lemma 4.7.1 and Proposition 4.7.2, which are proved in [73] in Lemma 2.3 and Proposition 3.2 respectively.

We start with Lemma 4.7.1, which is a basic and known property of the diagram $\mathscr{V}_Y$. It follows from explicit calculations using the definition of descent objects: we give a sketch of the proof below.

Lemma 4.7.1 ([73]). *Let $(L\dashv U,\eta,\varepsilon,s,t)$ be a biadjunction. For each object Y of $\mathfrak{B}$, the diagram $L\circ\mathscr{V}_Y$ is of absolute effective descent.*

Proof. Trivially, given a pseudofunctor $\mathscr{F}:\mathfrak{C}\to\mathfrak{Z}$, we can see $\mathscr{F}\circ L\circ\mathscr{V}_Y$ as a 2-functor, taking, if necessary, the obvious pseudonaturally equivalent version of $\mathscr{F}\circ L\circ\mathscr{V}_Y$. Then, for each objects Z of $\mathfrak{Z}$, by Remark 4.3.8, we can consider the strict descent object of the 2-functor explicitly

$$\mathfrak{Z}(Z,\mathscr{F}\circ L\circ\mathscr{V}_Y\circ\mathsf{j}-):\Delta\to\mathsf{CAT}$$

$$\begin{tikzcd}
\mathfrak{Z}(Z,\mathscr{F}LULY) \arrow[r, shift left=3, "\mathfrak{Z}(Z,\mathscr{F}L(\eta_{ULY}))"] \arrow[r, shift right=3, "\mathfrak{Z}(Z,\mathscr{F}LUL(\eta_Y))"'] & \mathfrak{Z}(Z,\mathscr{F}LULULY) \arrow[l, "\mathfrak{Z}(Z,\mathscr{F}LU(\varepsilon_{LY}))" description] \arrow[r, shift left=4, "\mathfrak{Z}(Z,\mathscr{F}L(\eta_{ULULY}))"] \arrow[r, "\mathfrak{Z}(Z,\mathscr{F}LUL(\eta_{ULY}))" description] \arrow[r, shift right=4, "\mathfrak{Z}(Z,\mathscr{F}LULUL(\eta_Y))"'] & \mathfrak{Z}(Z,\mathscr{F}LULULULY)
\end{tikzcd}$$

Thereby, by straightforward calculations, taking Remark 4.3.8 into account, we conclude that

$$\begin{array}{rcl}
\mathfrak{Z}(Z,\mathscr{F}LY) & \to & \left\{\dot{\Delta}(0,\mathsf{j}-),\mathfrak{Z}(Z,\mathscr{F}\circ L\circ\mathscr{V}_Y\circ\mathsf{j}-)\right\} \\
f & \mapsto & (\mathscr{F}L(\eta_Y)f,(\mathscr{F}L\eta)_{\eta_Y}*\mathrm{id}_f) \\
\mathfrak{m} & \mapsto & \mathrm{id}_{\mathscr{F}L(\eta_Y)}*\mathfrak{m}
\end{array}$$

gives an equivalence of categories (and it is the comparison functor). This completes the proof. □

Proposition 4.7.2 ([73]). *Let $\mathscr{T}=(\mathscr{T},\varpi,\varepsilon,\Lambda,\delta,\mathsf{s})$ be a pseudocomonad on $\mathfrak{C}$. The forgetful pseudofunctor $L:\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}\to\mathfrak{C}$ creates absolute effective descent diagrams.*

In this section, henceforth we work within the following setting (and notation): given a biadjunction $(E\dashv R,\rho,\mu,v,w)$, recall that, by Lemma 4.5.4, it induces a biadjunction, herein denoted by $(L\dashv U,\eta,\varepsilon,s,t)$. We also get commutative triangles

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\mathscr{K}"] \arrow[rd, "E"'] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \arrow[d, "L"] \\
& \mathfrak{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathfrak{C} \arrow[r, "R"] \arrow[rd, "U"'] & \mathfrak{A} \arrow[d, "\mathscr{K}"] \\
& \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}
\end{tikzcd}$$

in which, clearly, the biadjunctions $E \dashv R$, $L \dashv U$ induce the same pseudocomonad $\mathscr{T}$. In this context, if the comparison pseudofunctor $\mathscr{K}$ is a biequivalence, we say that E is *pseudocomonadic*. In other words, we say that E is pseudocomonadic if there is a pseudofunctor $G : \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \to \mathfrak{A}$ such that $G \circ \mathscr{K} \simeq \mathrm{Id}_{\mathfrak{A}}$ and $\mathscr{K} \circ G \simeq \mathrm{Id}_{\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}}$.

Of course, in the triangle above, the forgetful pseudofunctor L is always pseudocomonadic. In particular, L is always pseudoprecomonadic. Therefore the triangle satisfies the basic hypothesis of Corollary 4.5.10.

Observe that, to verify the pseudocomonadicity of a left biadjoint pseudofunctor L, we can do it in three steps:

1. Verify whether $\mathscr{K}$ has a right biadjoint via Corollary 4.5.10;
2. If it does, the next step would be to verify whether the counit of the biadjunction $\mathscr{K} \dashv G$ is a pseudonatural equivalence via Theorem 4.6.5;
3. The final step would be to verify whether the unit of the biadjunction $\mathscr{K} \dashv G$ is a pseudonatural equivalences via Theorem 4.6.3.

These are precisely the steps used below.

Theorem 4.7.3 (Pseudocomonadicity [73])**.** *A left biadjoint pseudofunctor $E : \mathfrak{A} \to \mathfrak{C}$ is pseudocomonadic if and only if it creates absolute effective descent diagrams.*

Proof. By Proposition 4.7.2, pseudocomonadic pseudofunctors create absolute effective descent diagrams. Reciprocally, assume that E creates absolute effective descent diagrams.

1. $\mathscr{K}$ has a right biadjoint G:

In this proof, we take a biadjunction $(E \dashv R, \rho, \varepsilon, v, w)$ and assume that $\mathscr{T}$ is its induced pseudocomonad. Also, we denote by $(L \dashv U, \eta, \varepsilon, s, t)$ the biadjunction induced by $\mathscr{T}$ (as described above).

On one hand, by Lemma 4.6.4 and Lemma 4.7.1, for each object z of $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$, the diagram $\mathscr{A}_{\mathsf{z}} : \Delta \to \mathfrak{A}$ is such that $E \circ \mathscr{A}_{\mathsf{z}} \simeq L \circ \mathscr{V}_{\mathsf{z}} \circ \mathrm{j}$ is an absolute effective descent diagram, in which $\mathscr{V}_{\mathsf{z}} : \dot{\Delta} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ is induced by the biadjunction $L \dashv U$.

Therefore, since E creates absolute effective diagrams, we conclude that there is an effective descent diagram $\mathscr{B}_{\mathsf{z}}$ such that $\mathscr{A}_{\mathsf{z}} = \mathscr{B}_{\mathsf{z}} \circ \mathrm{j}$ and $E \circ \mathscr{B}_{\mathsf{z}} \simeq L \circ \mathscr{V}_{\mathsf{z}}$. Thus, by Corollary 4.5.10, we conclude that $\mathscr{K}$ has a right biadjoint G.

2. The counit of the biadjunction $\mathscr{K} \dashv G$ is a pseudonatural equivalence:

Since $L \circ \mathscr{K} \circ \mathscr{B}_{\mathsf{z}} = E \circ \mathscr{B}_{\mathsf{z}} \simeq L \circ \mathscr{V}_{\mathsf{z}}$ is of absolute effective descent and L creates absolute effective descent diagrams, we conclude that $\mathscr{K} \circ \mathscr{B}_{\mathsf{z}}$ is of effective descent. By Theorem 4.6.5, it completes this second step.

3. The unit of the biadjunction $\mathscr{K} \dashv G$ is a pseudonatural equivalence:

By Lemma 4.7.1, for every object A of $\mathfrak{A}$, $E\circ\widehat{\mathscr{V}_A}:\dot{\Delta}\to\mathfrak{C}$ is of absolute effective descent, in which $\widehat{\mathscr{V}_A}$ is induced by the biadjunction $E\dashv R$. Since E creates absolute effective descent diagrams, we get that $\widehat{\mathscr{V}_A}$ is of effective descent. Therefore, by Corollary 4.6.2, E is pseudoprecomonadic. By Theorem 4.6.3, it completes the proof of the final step. □

As a consequence of Theorem 4.7.3, within the setting of Theorem 4.4.3, if J has a right biadjoint and E,L are pseudocomonadic, then J is pseudocomonadic as well. Furthermore, it is worth to point out that the second step of the proof of Theorem 4.7.3 follows directly from the fact that E preserves the effective descent diagrams $\mathscr{B}_{\mathrm{z}}$ and from the pseudocomonadicity of L. More precisely, as direct consequence of Lemma 4.7.1, Theorem 4.6.5 and Proposition 4.7.2, we get:

Corollary 4.7.4 (Counit)**.** *Let $(E\dashv R,\rho,\mu,v,w),(L\dashv U,\eta,\varepsilon,s,t)$ be biadjunctions inducing the same pseudocomonad $\mathscr{T}=(\mathscr{T},\varpi,\varepsilon,\Lambda,\delta,\mathrm{s})$ such that*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "E"'] & & \mathfrak{B} \arrow[dl, "L"] \\
& \mathfrak{C} &
\end{tikzcd}$$

commutes. Assume that L is pseudocomonadic, $J\circ R=U$ and $(J,G,\overline{\varepsilon},\overline{\eta},\overline{s},\overline{t})$ is a biadjunction. The counit $\overline{\varepsilon}:JG\longrightarrow\mathrm{Id}_{\mathfrak{B}}$ is a pseudonatural equivalence if and only if, for every object Y of $\mathfrak{B}$, E preserves the descent object of $\mathscr{A}_Y:\Delta\to\mathfrak{A}$.

Proof. By Corollary 4.5.10, since J is left biadjoint, for each object Y of $\mathfrak{B}$, there is an effective descent diagram $\mathscr{B}_Y:\dot{\Delta}\to\mathfrak{A}$ such that $\mathscr{B}_Y\circ\mathrm{j}\simeq\mathscr{A}_Y$. By the commutativity of the triangles $L\circ J=E$ and $J\circ R=U$, since $(E\dashv R,\rho,\mu)$ and $(L\dashv U,\eta,\varepsilon,s,t)$ induce the same pseudocomonad, our setting satisfies the hypotheses of Lemma 4.6.4. Thus, for each object Y of $\mathfrak{B}$, there is a pseudonatural equivalence

$$J\circ\mathscr{B}_Y\circ\mathrm{j}\simeq J\circ\mathscr{A}_Y\simeq\mathscr{V}_Y\circ\mathrm{j}.$$

By Theorem 4.6.5, to complete this proof, it is enough to show that $J\circ\mathscr{B}_Y$ is of effective descent if and only if $E\circ\mathscr{B}_Y$ is of effective descent.

Firstly, we assume that $J\circ\mathscr{B}_Y$ is of effective descent. In this case, since $J\circ\mathscr{B}_Y\circ\mathrm{j}\simeq\mathscr{V}_Y\circ\mathrm{j}$ and $\mathscr{V}_Y$ is of effective descent, we conclude that $\mathscr{V}_Y\simeq J\circ\mathscr{B}_Y$. Thus, by Lemma 4.7.1,

$$L\circ\mathscr{V}_Y\simeq L\circ J\circ\mathscr{B}_Y=E\circ\mathscr{B}_Y$$

is, in particular, of effective descent.

Reciprocally, we assume that $E\circ\mathscr{B}_Y$ is of effective descent. Again, since $E\circ\mathscr{B}_Y\circ\mathrm{j}\simeq L\circ\mathscr{V}_Y\circ\mathrm{j}$ and $L\circ\mathscr{V}_Y$ is of absolute effective descent, we conclude that $E\circ\mathscr{B}_Y\simeq L\circ\mathscr{V}_Y$ is of absolute effective descent. Therefore, since L is pseudocomonadic, by Proposition 4.7.2, we conclude that $J\circ\mathscr{B}_Y$ is of effective descent.

□

4.8 Coherence

A 2-(co)monadic approach to *coherence* consists of studying the inclusion of the 2-category of strict (co)algebras into the 2-category of pseudo(co)algebras of a given 2-(co)monad to get general coherence results [9, 67, 93]. More precisely, one is interested, firstly, to understand whether the inclusion of the 2-category of strict coalgebras into the 2-category of pseudocoalgebras has a right 2-adjoint G (what is called a "coherence theorem of the first type" in [67]). Secondly, if there is such a right 2-adjoint, one is interested in investigating whether every pseudocoalgebra z is equivalent to the strict replacement $G(\mathsf{z})$ (what is called a "coherence theorem of the second type" in [67]).

We fix the notation of this section as follows: we have a 2-comonad $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon)$ on a 2-category $\mathfrak{C}$. We denote by $\mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}}$ the 2-category of strict coalgebras, strict morphisms and $\mathscr{T}$-transformations, that is to say, the usual CAT-enriched category of coalgebras of the CAT-comonad $\mathscr{T}$. The 2-adjunction $E \dashv R : \mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \to \mathfrak{C}$ induces the Eilenberg-Moore factorization w.r.t. the pseudocoalgebras:

$$\begin{tikzcd}
\mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \arrow[r, "J"] \arrow[rd, "E"'] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \arrow[d, "L"] \\
& \mathfrak{C}
\end{tikzcd}$$

in which $J : \mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ is the usual inclusion.

Firstly, Corollary 4.5.10 gives, in particular, necessary and sufficient conditions for which a 2-comonad satisfies the "coherence theorem of the first type" and a weaker version of it, that is to say, it also studies when J has a right biadjoint G. Secondly, Corollary 4.7.4 gives necessary and sufficient conditions for getting a stronger version of the "coherence theorem of the second type", that is to say, it studies when the counit of the obtained biadjunction/2-adjunction is a pseudonatural equivalence.

Corollary 4.8.1 (Coherence Theorem)**.** *Let $\mathscr{T} = (\mathscr{T}, \varpi, \varepsilon)$ be a 2-comonad on a 2-category $\mathfrak{C}$. It induces a 2-adjunction $(E \dashv R, \rho, \varepsilon)$ and a biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$ such that*

$$\begin{tikzcd}
\mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \arrow[r, "J"] \arrow[rd, "E"'] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg} \arrow[d, "L"] \\
& \mathfrak{C}
\end{tikzcd}$$

commutes. The inclusion $J : \mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$ has a right biadjoint if and only if $\mathscr{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}}$ has the descent object of

$$\begin{tikzcd}[column sep=huge]
RL\mathsf{z} \arrow[r, shift left=3, "\rho_{RL\mathsf{z}}"] \arrow[r, shift right=3, "RL(\eta_{\mathsf{z}})"'] & R\mathscr{T}L\mathsf{z} \arrow[l, "R(\varepsilon_{L\mathsf{z}})" description] \arrow[r, shift left=3, "\rho_{R\mathscr{T}L\mathsf{z}}"] \arrow[r, "RL(\eta_{UL\mathsf{z}})" description] \arrow[r, shift right=3, "R\mathscr{T}L(\eta_{\mathsf{z}})"'] & R\mathscr{T}^2L\mathsf{z}
\end{tikzcd} \qquad (\mathscr{A}_{\mathsf{z}})$$

for every pseudocoalgebra z of $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}$. In this case, J is left biadjoint to G, given by $G\mathsf{z} := \{\dot{\Delta}(0,\mathsf{j}-), \mathscr{A}_{\mathsf{z}}\}_{\mathrm{bi}}$. Moreover, assuming the existence of the biadjunction $(J \dashv G, \overline{\varepsilon}, \overline{\eta}, \overline{s}, \overline{t})$, the counit $\overline{\varepsilon} : JG \longrightarrow \mathrm{id}_{\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}}$ is a pseudonatural equivalence if and only if E preserves the descent object of $\mathscr{A}_{\mathsf{z}}$ for every pseudocoalgebra z.

Furthermore, J has a genuine right 2-adjoint G if and only if $\mathcal{T}$-$\mathsf{CoAlg}_{\mathsf{s}}$ admits the strict descent object of $\mathscr{A}_{\mathsf{z}}$ for every $\mathcal{T}$-pseudocoalgebra z. In this case, the right 2-adjoint is given by $G\mathsf{z} := \{\dot{\Delta}(0,\mathsf{j}-), \mathscr{A}_{\mathsf{z}}\}$.

Proof. Since $(E \dashv R, \rho, \varepsilon)$ and $(L \dashv U, \eta, \varepsilon, s, t)$ induce the same pseudocomonad and $(\eta J) = (J\rho)$ is a 2-natural transformation, it is enough to apply Corollary 4.7.4 and Corollary 4.5.10 to the triangle $L \circ J = E$. □

We say that a 2-comonad $\mathcal{T}$ satisfies the *main coherence theorem* if there is a right 2-adjoint Ps-$\mathcal{T}$-$\mathsf{CoAlg} \to \mathcal{T}$-$\mathsf{CoAlg}_{\mathsf{s}}$ to the inclusion and the counit of such 2-adjunction is a pseudonatural equivalence.

To get the original statement of [67], we have to employ the following well known result (which is a consequence of a more general result on enriched comonads):

Let $\mathcal{T}$ be a 2-comonad on $\mathfrak{C}$. The forgetful 2-functor $\mathcal{T}$-$\mathsf{CoAlg}_{\mathsf{s}} \to \mathfrak{C}$ creates all those strict descent objects which exist in $\mathfrak{C}$ and are preserved by $\mathcal{T}$ and $\mathcal{T}^2$.

Employing this result and Corollary 4.8.1, we prove Theorem 3.2 and Theorem 4.4 of [67]. For instance, we get:

Corollary 4.8.2 ([67])**.** *Let $\mathcal{T}$ be a 2-comonad on a 2-category $\mathfrak{C}$. If $\mathfrak{C}$ has and $\mathcal{T}$ preserves strict descent objects, then $\mathcal{T}$ satisfies the main coherence theorem.*

4.9 On lifting biadjunctions

One of the most elementary corollaries of the adjoint triangle theorem [30] is about lifting adjunctions to adjunctions between the Eilenberg-Moore categories. In our case, let $\mathcal{T} : \mathfrak{A} \to \mathfrak{A}$ and $\mathcal{S} : \mathfrak{C} \to \mathfrak{C}$ be 2-comonads (with omitted comultiplications and counits), if

$$\begin{tikzcd}
\mathcal{T}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \arrow[r, "J"] \arrow[d, "\widehat{L}"'] & \mathcal{S}\text{-}\mathsf{CoAlg}_{\mathsf{s}} \arrow[d, "L"] \\
\mathfrak{A} \arrow[r, "E"'] & \mathfrak{C}
\end{tikzcd}$$

is a commutative diagram, such that E has a right 2-adjoint R, then Proposition 4.1.1 gives necessary and sufficient conditions to construct a right 2-adjoint to J. Also, of course, as a consequence of Corollary 4.5.10, we have the analogous version for pseudocomonads.

Corollary 4.9.1. *Let $\mathcal{T} : \mathfrak{A} \to \mathfrak{A}$ and $\mathcal{S} : \mathfrak{C} \to \mathfrak{C}$ be pseudocomonads. If the diagram*

$$\begin{tikzcd}
\mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{CoAlg} \arrow[r, "J"] \arrow[d, "\widehat{L}"'] & \mathsf{Ps}\text{-}\mathcal{S}\text{-}\mathsf{CoAlg} \arrow[d, "L"] \\
\mathfrak{A} \arrow[r, "E"'] & \mathfrak{C}
\end{tikzcd}$$

commutes and E has a right biadjoint, then J has a right biadjoint provided that Ps-$\mathcal{T}$-CoAlg has descent objects.

Recall that Ps-$\mathscr{T}$-CoAlg has descent objects if $\mathfrak{A}$ has and $\mathscr{T}$ preserves descent objects. Therefore the pseudofunctor J of the last result has a right biadjoint in this case.

4.9.2 On pseudo-Kan extensions

One simple application of Corollary 4.9.1 is about pseudo-Kan extensions. In the tricategory 2-CAT, the natural notion of Kan extension is that of pseudo-Kan extension. More precisely, a *right pseudo-Kan extension of a pseudofunctor* $\mathscr{D}:\mathfrak{S}\to\mathfrak{A}$ *along a pseudofunctor* $\mathrm{h}:\mathfrak{S}\to\dot{\mathfrak{S}}$*, denoted by* $\mathrm{Ps}\text{-}\mathscr{R}an_{\mathrm{h}}\mathscr{D}$*, is (if it exists) a birepresentation of the pseudofunctor* $\mathscr{W}\mapsto[\mathfrak{S},\mathfrak{A}]_{PS}(\mathscr{W}\circ\mathrm{h},\mathscr{D})$. Recall that birepresentations are unique up to equivalence and, therefore, right pseudo-Kan extensions are unique up to pseudonatural equivalence.

Assuming that $\mathrm{h}:\mathfrak{S}\to\dot{\mathfrak{S}}$ is a pseudofunctor between small 2-categories, in the setting described above, the following are natural problems on pseudo-Kan extensions: (1) investigating the left biadjointness of the pseudofunctor $\mathscr{W}\to\mathscr{W}\circ\mathrm{h}$, namely, investigating whether all right pseudo-Kan extensions along h exist; (2) understanding pointwise pseudo-Kan extensions (that is to say, proving the existence of right pseudo-Kan extensions provided that $\mathfrak{A}$ has all bilimits).

It is shown in [9] that, if $\mathfrak{S}_0$ denotes the discrete 2-category of the objects of $\mathfrak{S}$, the restriction $[\mathfrak{S},\mathfrak{A}]\to[\mathfrak{S}_0,\mathfrak{A}]$ is 2-comonadic, provided that $[\mathfrak{S},\mathfrak{A}]\to[\mathfrak{S}_0,\mathfrak{A}]$ has a right 2-adjoint $\mathscr{R}an_{\mathfrak{S}\to\mathfrak{S}_0}$. It is also shown there that the 2-category of pseudocoalgebras of the induced 2-comonad is $[\mathfrak{S},\mathfrak{A}]_{PS}$. It actually works more generally: $[\mathfrak{S},\mathfrak{A}]_{PS}\to[\mathfrak{S}_0,\mathfrak{A}]_{PS}=[\mathfrak{S}_0,\mathfrak{A}]$ is pseudocomonadic whenever there is a right biadjoint $\mathrm{Ps}\text{-}\mathscr{R}an_{\mathfrak{S}_0\to\mathfrak{S}}:[\mathfrak{S}_0,\mathfrak{A}]_{PS}\to[\mathfrak{S},\mathfrak{A}]_{PS}$ because existing bilimits of $\mathfrak{A}$ are constructed objectwise in $[\mathfrak{S},\mathfrak{A}]_{PS}$ (and, therefore, the hypotheses of the pseudocomonadicity theorem [73] are satisfied). Thus, we get the following commutative square:

$$\begin{tikzcd}
{[\dot{\mathfrak{S}},\mathfrak{A}]_{PS}} \arrow[r, "{[\mathrm{h},\mathfrak{A}]_{PS}}"] \arrow[d] & {[\mathfrak{S},\mathfrak{A}]_{PS}} \arrow[d] \\
{[\dot{\mathfrak{S}}_0,\mathfrak{A}]} \arrow[r, "{[\mathrm{h},\mathfrak{A}]_{PS}}"'] & {[\mathfrak{S}_0,\mathfrak{A}]}
\end{tikzcd}$$

Thereby, Corollary 4.9.1 gives a way to study pseudo-Kan extensions, even in the absence of strict 2-limits. That is to say, on one hand, if the 2-category $\mathfrak{A}$ is complete, our results give pseudo-Kan extensions as descent objects of strict 2-limits. On the other hand, in the absence of strict 2-limits and, in particular, assuming that $\mathfrak{A}$ is bicategorically complete, we can construct the following pseudo-Kan extensions:

$$\begin{array}{rll}
\mathrm{Ps}\text{-}\mathscr{R}an_{\mathfrak{S}_0\to\dot{\mathfrak{S}}_0}: & [\mathfrak{S}_0,\mathfrak{A}] & \to[\dot{\mathfrak{S}}_0,\mathfrak{A}]_{PS} \\
& \mathscr{D} & \mapsto \mathrm{Ps}\text{-}\mathscr{R}an_{\mathfrak{S}_0\to\dot{\mathfrak{S}}_0}\mathscr{D}:\left(x\mapsto\displaystyle\prod_{\mathrm{h}(a)=x}\mathscr{D}a\right)
\end{array}$$

$$\begin{array}{rll}
\mathrm{Ps}\text{-}\mathscr{R}an_{\dot{\mathfrak{S}}_0\to\dot{\mathfrak{S}}}: & [\dot{\mathfrak{S}}_0,\mathfrak{A}] & \to[\dot{\mathfrak{S}},\mathfrak{A}]_{PS} \\
& \mathscr{D} & \mapsto \mathrm{Ps}\text{-}\mathscr{R}an_{\dot{\mathfrak{S}}_0\to\dot{\mathfrak{S}}}\mathscr{D}:\left(x\mapsto\displaystyle\prod_{y\in\dot{\mathfrak{S}}_0}\dot{\mathfrak{S}}(x,y)\pitchfork\mathscr{D}y\right)
\end{array}$$

$$\begin{array}{rlll} \text{Ps-}\mathscr{R}an_{\mathfrak{S}_0\to\mathfrak{S}}: & [\mathfrak{S}_0,\mathfrak{A}] & \to [\mathfrak{S},\mathfrak{A}]_{PS} \\ & \mathscr{D} & \mapsto \text{Ps-}\mathscr{R}an_{\mathfrak{S}_0\to\mathfrak{S}}\mathscr{D}: \left(a \mapsto \prod\limits_{b\in\mathfrak{S}_0} \mathfrak{S}(a,b) \pitchfork \mathscr{D}b\right) \end{array}$$

in which $\prod$ and $\pitchfork$ denote the bilimit versions of the product and cotensor product, respectively. Thereby, by Corollary 4.9.1, the pseudo-Kan extension Ps-$\mathscr{R}an_{\mathrm{h}}$ can be constructed pointwise as descent objects of a diagram obtained from the pseudo-Kan extensions above. Namely, Ps-$\mathscr{R}an_{\mathrm{h}}\mathscr{D}x$ is the descent object of a diagram

$$\begin{tikzcd} \mathfrak{a}_0 \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathfrak{a}_1 \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathfrak{a}_2 \end{tikzcd}$$

in which, by Theorem 4.4.3 and the last observations,

$$\begin{array}{rcl} \mathfrak{a}_0 & = & \prod\limits_{y\in\dot{\mathfrak{S}}_0}\left(\dot{\mathfrak{S}}(x,y)\pitchfork \prod\limits_{\mathrm{h}(a)=y}\mathscr{D}a\right) \simeq \prod\limits_{a\in\mathfrak{S}_0}\left(\dot{\mathfrak{S}}(x,\mathrm{h}(a))\pitchfork\mathscr{D}a\right) \\ \mathfrak{a}_1 & = & \left(\dot{\mathfrak{S}}(x,y)\pitchfork\prod\limits_{\mathrm{h}(a)=y}\left(\prod\limits_{b\in\mathfrak{S}_0}\mathfrak{S}(a,b)\pitchfork\mathscr{D}b\right)\right) \\ & \simeq & \prod\limits_{a\in\mathfrak{S}_0}\left(\dot{\mathfrak{S}}(x,\mathrm{h}(a))\pitchfork\left(\prod\limits_{b\in\mathfrak{S}_0}\mathfrak{S}(a,b)\pitchfork\mathscr{D}b\right)\right) \\ & \simeq & \prod\limits_{(a,b)\in\mathfrak{S}_0\times\mathfrak{S}_0}\left(\left(\mathfrak{S}(a,b)\times\dot{\mathfrak{S}}(x,\mathrm{h}(a))\right)\pitchfork\mathscr{D}b\right) \\ \mathfrak{a}_2 & \simeq & \prod\limits_{(a,b,c)\in\mathfrak{S}_0\times\mathfrak{S}_0\times\mathfrak{S}_0}\left(\left(\mathfrak{S}(b,c)\times\mathfrak{S}(a,b)\times\dot{\mathfrak{S}}(x,\mathrm{h}(a))\right)\pitchfork\mathscr{D}c\right) \end{array}$$

This implies that, indeed, if $\mathfrak{A}$ is bicategorically complete, then Ps-$\mathscr{R}an_{\mathrm{h}}\mathscr{D}$ exists and, once we assume the results of [105] related to the construction of weighted bilimits via descent objects, we conclude that:

Proposition 4.9.3 (Pointwise pseudo-Kan extension)**.** *Let $\mathfrak{S},\dot{\mathfrak{S}}$ be small 2-categories and $\mathfrak{A}$ be a bicategorically complete 2-category. If $\mathrm{h}:\mathfrak{S}\to\dot{\mathfrak{S}}$ is a pseudofunctor, then*

$$\text{Ps-}\mathscr{R}an_{\mathrm{h}}\mathscr{D}x = \left\{\dot{\mathfrak{S}}(x,\mathrm{h}-),\mathscr{D}\right\}_{\mathrm{bi}}$$

Corollary 4.9.4. *If $\mathscr{A}:\Delta\to\mathfrak{A}$ is a pseudofunctor and $\mathfrak{A}$ has the descent object of $\mathscr{A}$, then Ps-$\mathscr{R}an_{\mathrm{j}}\mathscr{A}0$ is the descent object of $\mathscr{A}$.*

Moreover, by the bicategorical Yoneda Lemma, we get:

Corollary 4.9.5. *If $\mathrm{h}:\mathfrak{S}\to\dot{\mathfrak{S}}$ is locally an equivalence and there is a biadjunction $[\mathrm{h},\mathfrak{A}]\dashv$ Ps-$\mathscr{R}an_{\mathrm{h}}$, its counit is a pseudonatural equivalence.*

Finally, let $\mathfrak{A}$ be a 2-category with all descent objects and $\mathscr{T}$ be a pseudocomonad on $\mathfrak{A}$. Recall that, if $\mathscr{T}$ preserves all effective descent diagrams, Ps-$\mathscr{T}$-CoAlg has all descent objects. Therefore, if $\mathrm{h}:\mathfrak{S}\to\dot{\mathfrak{S}}$ is a pseudofunctor, in this setting, the commutative diagram below satisfies the hypotheses of Corollary 4.9.1 (and, thereby, it can be used to lift pseudo-Kan extensions to pseudocoalgebras).

$$\begin{tikzcd}
{[\dot{\mathfrak{S}}, \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}]_{PS}} \arrow[r] \arrow[d] & {[\mathfrak{S}, \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{CoAlg}]_{PS}} \arrow[d] \\
{[\dot{\mathfrak{S}}, \mathfrak{A}]_{PS}} \arrow[r] & {[\mathfrak{S}, \mathfrak{A}]_{PS}}
\end{tikzcd}$$

Remark 4.9.6. Assume that $\mathrm{h} : \mathfrak{S} \to \dot{\mathfrak{S}}$ is a pseudofunctor, in which $\mathfrak{S}, \dot{\mathfrak{S}}$ are small 2-categories. There is another way of proving Proposition 4.9.3. Firstly, we define the bilimit version of *end*. That is to say, if $T : \mathfrak{S} \times \mathfrak{S}^{\mathrm{op}} \to \mathsf{CAT}$ is a pseudofunctor, we define

$$\int_{\mathfrak{S}} T := [\mathfrak{A} \times \mathfrak{A}^{\mathrm{op}}, \mathsf{CAT}]_{PS}(\mathfrak{A}(-,-), T)$$

From this definition, it follows Fubini's theorem (up to equivalence). And, if $\mathscr{B}, \mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ are pseudofunctors, the following equivalence holds:

$$\int_{\mathfrak{S}} \mathfrak{A}(\mathscr{B}a, \mathscr{D}a) \simeq [\mathfrak{S}, \mathfrak{A}]_{PS}(\mathscr{B}, \mathscr{D})$$

Therefore, if $\mathrm{h} : \mathfrak{S} \to \dot{\mathfrak{S}}$ is a pseudofunctor and we define $\mathrm{Ps}\mathscr{R}an_{\mathrm{h}}\mathscr{D}x = \left\{ \dot{\mathfrak{S}}(x, \mathrm{h}-), \mathscr{D} \right\}_{\mathrm{bi}}$, we have the pseudonatural equivalences (analogous to the enriched case [58])

$$\begin{aligned}
\left[\dot{\mathfrak{S}}, \mathfrak{A}\right]_{PS}(\mathscr{W}, \mathrm{Ps}\mathscr{R}an_{\mathrm{h}}\mathscr{D}) &\simeq \int_{\dot{\mathfrak{S}}} \mathfrak{A}(\mathscr{W}x, \mathrm{Ps}\mathscr{R}an_{\mathrm{h}}\mathscr{D}x) \\
&\simeq \int_{\dot{\mathfrak{S}}} \mathfrak{A}(\mathscr{W}x, \left\{\dot{\mathfrak{S}}(x, \mathrm{h}-), \mathscr{D}\right\}_{\mathrm{bi}}) \\
&\simeq \int_{\dot{\mathfrak{S}}} [\mathfrak{S}, \mathsf{CAT}]_{PS}(\dot{\mathfrak{S}}(x, \mathrm{h}-), \mathfrak{A}(\mathscr{W}x, \mathscr{D}-)) \\
&\simeq \int_{\dot{\mathfrak{S}}} \int_{\mathfrak{S}} \mathsf{CAT}(\dot{\mathfrak{S}}(x, \mathrm{h}(a)), \mathfrak{A}(\mathscr{W}x, \mathscr{D}a)) \\
&\simeq \int_{\mathfrak{S}} \int_{\dot{\mathfrak{S}}} \mathsf{CAT}(\dot{\mathfrak{S}}(x, \mathrm{h}(a)), \mathfrak{A}(\mathscr{W}x, \mathscr{D}a)) \\
&\simeq \int_{\mathfrak{S}} \left[\dot{\mathfrak{S}}^{\mathrm{op}}, \mathsf{CAT}\right]_{PS}(\dot{\mathfrak{S}}(-, \mathrm{h}(a)), \mathfrak{A}(\mathscr{W}-, \mathscr{D}a)) \\
&\simeq \int_{\mathfrak{S}} \mathfrak{A}(\mathscr{W} \circ \mathrm{h}(a), \mathscr{D}a) \\
&\simeq [\mathfrak{S}, \mathfrak{A}]_{PS}(\mathscr{W} \circ \mathrm{h}, \mathscr{D})
\end{aligned}$$

This completes the proof that if the pointwise right pseudo-Kan extension $\mathrm{Ps}\mathscr{R}an_{\mathrm{h}}$ exists, it is a right pseudo-Kan extension. Within this setting and assuming this result, the original argument used to prove Proposition 4.9.3 using biadjoint triangles gets the construction via descent objects of weighted bilimits originally given in [105].

Chapter 5

On lifting of biadjoints and lax algebras

By the biadjoint triangle theorem, given a pseudomonad $\mathcal{T}$ on a 2-category $\mathfrak{B}$, if a right biadjoint $\mathfrak{A} \to \mathfrak{B}$ has a lifting to the pseudoalgebras $\mathfrak{A} \to \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$ then this lifting is also right biadjoint provided that $\mathfrak{A}$ has codescent objects. In this paper, we give general results on lifting of biadjoints. As a consequence, we get a *biadjoint triangle theorem* which, in particular, allows us to study triangles involving the 2-category of lax algebras, proving analogues of the result described above. In the context of lax algebras, denoting by $\ell : \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell$ the inclusion, if $R : \mathfrak{A} \to \mathfrak{B}$ is right biadjoint and has a lifting $J : \mathfrak{A} \to \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}$, then $\ell \circ J$ is right biadjoint as well provided that $\mathfrak{A}$ has some needed weighted bicolimits. In order to prove such result, we study *descent objects* and *lax descent objects*. At the last section, we study direct consequences of our theorems in the context of the 2-*monadic approach to coherence*.

Introduction

This paper has three main theorems. One of them (Theorem 5.2.3) is about lifting of biadjoints: a generalization of Theorem 4.4 of [77]. The others (Theorem 5.5.2 and Theorem 5.5.3) are consequences of the former on lifting biadjoints to the 2-category of lax algebras. These results can be seen as part of what is called *two-dimensional universal algebra*, or, more precisely, *two-dimensional monad theory*: for an idea of the scope of this field (with applications), see for instance [9, 11, 47, 60, 67, 77, 79, 93, 95].

There are several theorems about lifting of adjunctions in the literature [1, 10, 55, 108, 111], including, for instance, adjoint triangle theorems [2, 30]. Although some of these results can be proved for enriched categories or more general contexts [77, 92], they often are not enough to deal with problems within 2-dimensional category theory. The reason is that these problems involve concepts that are not of (strict/usual) Cat-enriched category theory nature, as it is explained in [9, 68].

For example, in 2-dimensional category theory, the enriched notion of monad, the 2-monad, gives rise to the 2-category of (strict/enriched) algebras, but it also gives rise to the 2-category of pseudoalgebras and the 2-category of lax algebras. The last two types of 2-categories of algebras (and full sub-2-categories of them) are usually of the most interest despite the fact that they are not “strict” notions.

In short, most of the aspects of 2-dimensional universal algebra are not covered by the usual Cat-enriched category theory of [31, 58] or by the *formal theory of monads* of [100]. Actually, in the context of pseudomonad theory, the appropriate analogue of the formal theory of monads is the formal theory (and definition) of pseudomonads of [66, 83]. In this direction, the problem of lifting biadjunctions is the appropriate analogue of the problem of lifting adjunctions.

Some results on lifting of biadjunctions are consequences of the biadjoint triangle theorems proved in [77]. One of these consequences is the following: let $\mathscr{T}$ be a pseudomonad on a 2-category $\mathfrak{B}$. Assume that $R : \mathfrak{A} \to \mathfrak{B}$, $J : \mathfrak{A} \to \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ are pseudofunctors such that we have the pseudonatural equivalence below. If R is right biadjoint then J is right biadjoint as well provided that $\mathfrak{A}$ has some needed codescent objects.

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "R"'] & \simeq & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[dl, "U"] \\
& \mathfrak{B} &
\end{tikzcd}$$

One simple application of this result is, for instance, within the 2-monadic approach to coherence [77]: roughly, the 2-monadic approach to coherence is the study of biadjunctions and 2-adjunctions between the many types of 2-categories of algebras rising from a given 2-monad. This allows us to prove "general coherence results" [9, 67, 93] which encompass many coherence results – such as the strict replacement of monoidal categories, the strict/flexible replacement of bicategories [67, 69], the strict/flexible replacement of pseudofunctors [9] and so on.

If $\mathscr{T}'$ is a 2-monad, the result described above gives the construction of the left biadjoint to the inclusion

$$\mathscr{T}'\text{-}\mathsf{Alg}_{\mathsf{s}} \to \mathsf{Ps}\text{-}\mathscr{T}'\text{-}\mathsf{Alg}$$

subject to the existence of some codescent objects in $\mathscr{T}'\text{-}\mathsf{Alg}_{\mathsf{s}}$. The strict version of the biadjoint triangle theorem of [77] shows when we can get a genuine left 2-adjoint to this inclusion (and also studies when the unit is a pseudonatural equivalence), getting the coherence results of [67] w.r.t. pseudoalgebras.

In this paper, we prove Theorem 5.2.3 which is a generalization of Theorem 4.3 of [77] on biadjoint triangles. Our result allows us to study lifting of biadjunctions to lax algebras. Hence, we prove the analogue of the result described above for lax algebras. More precisely, let $\mathscr{T}$ be a pseudomonad on a 2-category $\mathfrak{B}$ and let $\ell : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell}$ be the locally full inclusion of the 2-category of lax $\mathscr{T}$-algebras and $\mathscr{T}$-pseudomorphisms into the 2-category of lax $\mathscr{T}$-algebras and lax $\mathscr{T}$-morphisms. Assuming that

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "R"'] & \simeq & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[dl] \\
& \mathfrak{B} &
\end{tikzcd}$$

is a pseudonatural equivalence in which R is right biadjoint, we prove that J is right biadjoint as well, provided that $\mathfrak{A}$ has some needed codescent objects. Moreover, $\ell \circ J$ is right biadjoint if and only if $\mathfrak{A}$ has lax codescent objects of some special diagrams. Still, we study when we can get strict left 2-adjoints to J and $\ell \circ J$, provided that J is a 2-functor.

As an immediate application, we also prove general coherence theorems related to the work of [67]: we get the construction of the left biadjoints of the inclusions

$$\mathscr{T}'\text{-}\mathsf{Alg}_{\mathrm{s}} \to \mathsf{Lax}\text{-}\mathscr{T}'\text{-}\mathsf{Alg}_{\ell} \qquad\qquad \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell}$$

provided that $\mathscr{T}'$ is a 2-monad, $\mathscr{T}$ is a pseudomonad and $\mathscr{T}'\text{-}\mathsf{Alg}_{\mathrm{s}}$, $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ have some needed lax codescent objects.

We start in Section 5.1 establishing our setting: we recall basic results and definitions, such as weighted bicolimits and computads. In Section 5.2, we give our main theorems on lifting of biadjoints: these are simple but pretty general results establishing basic techniques to prove theorems on lifting of biadjoints. These techniques apply to the context of [77] but also apply to the study of other biadjoint triangles, such as our main application - which is the lifting of biadjoints to the 2-category of lax algebras.

Then, we restrict our attention to 2-dimensional monad theory: in order to do so, we present the weighted bicolimits called lax codescent objects and codescent objects in Section 5.3. Our approach to deal with descent objects is more general than the approach of [77, 79, 105], since it allows us to study descent objects of more general diagrams. Thanks to this approach, in Section 5.4, after defining pseudomonads and lax algebras, we show how we can get the category of pseudomorphisms between two lax algebras as a descent object at Proposition 5.4.5. This result also shows how we can get the category of lax morphisms between two lax algebras as a lax descent object.

In Section 5.5, we prove our main results on lax algebras: Theorem 5.5.2 and Theorem 5.5.3. They are direct consequences of the results of Section 5.2 and Section 5.4, but we also give explicit calculations of the weighted bicolimits/weighted 2-colimits needed in $\mathfrak{A}$ to get the left biadjoints/left 2-adjoints. We finish the paper in Section 5.6 giving straightforward applications of our results within the context of the 2-monadic approach to coherence explained above.

This work was realized in the course of my PhD studies at University of Coimbra. I wish to thank my supervisor Maria Manuel Clementino for her support, attention and useful feedback.

5.1 Preliminaries

In this section, we recall some basic results related to our setting, which is the tricategory $2\text{-}\mathsf{CAT}$ of 2-categories, pseudofunctors, pseudonatural transformations and modifications. Most of what we need was originally presented in [5, 103–105]. Also, for elements of enriched category theory, see [58]. We use the notation established in Section 2 of [77] for pseudofunctors, pseudonatural transformations and modifications.

We start with considerations about size. Let $\mathsf{cat} = \mathsf{int}(\mathsf{Set})$ be the cartesian closed *category of small categories*. Also, assume that $\mathsf{Cat}, \mathsf{CAT}$ are cartesian closed categories of categories in two different universes such that cat is an internal category of the subcategory of discrete categories of Cat, while Cat is itself an internal category of the subcategory of discrete categories of CAT. Since these three categories of categories are complete and cartesian closed, they are enriched over themselves and they are cocomplete and complete in the enriched sense.

Henceforth, Cat-*category* is a Cat-enriched category such that its collection of objects is a discrete category of CAT. Thereby, we have that Cat-categories can be seen as internal categories of CAT such that their categories of objects are discrete. In other words, there is a full inclusion $\mathsf{Cat}\text{-}\mathsf{CAT} \to \mathsf{int}(\mathsf{CAT})$ in which $\mathsf{Cat}\text{-}\mathsf{CAT}$ denotes the category of Cat-categories. Moreover, since there is a forgetful functor $\mathsf{int}(\mathsf{CAT}) \to \mathsf{CAT}$, there is a forgetful functor $\mathsf{Cat}\text{-}\mathsf{CAT} \to \mathsf{CAT}$.

So, we adopt the following terminology: firstly, a 2-*category* is a Cat-category. Secondly, a *possibly (locally) large* 2-*category* is an internal category of CAT such that its category of objects is discrete. Finally, a *small* 2-*category* is a 2-category which can be seen as an internal category of cat.

Let $\mathscr{W} : \mathfrak{S} \to \mathsf{Cat}, \mathscr{W}' : \mathfrak{S}^{\mathrm{op}} \to \mathsf{Cat}$ and $\mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ be 2-functors with small domains. If it exists, we denote the *weighted limit* of $\mathscr{D}$ with weight $\mathscr{W}$ by $\{\mathscr{W}, \mathscr{D}\}$. Dually, we denote by $\mathscr{W}' * \mathscr{D}$ the *weighted colimit* provided that it exists.

Remark 5.1.1. Consider the category, denoted in this remark by $\mathfrak{S}_{\mathrm{ist}}$ with two objects and two parallel arrows between them. We can define the weight

$$\mathscr{W}_{\mathrm{insert}} : \mathfrak{S}_{\mathrm{ist}} \to \mathsf{Cat}$$

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[r, mapsto] & I \arrow[r, shift left, "\text{codomain}"] \arrow[r, shift right, "\text{domain}"'] & 2 \end{tikzcd}$$

in which 2 is the category with two objects and only one morphism between them and I is the terminal category. The colimits with this weight are called *coinserters* (see [59]).

The *bicategorical Yoneda Lemma* says that there is a pseudonatural equivalence

$$[\mathfrak{S}, \mathsf{Cat}]_{PS}(\mathfrak{S}(a,-), \mathscr{D}) \simeq \mathscr{D}a$$

given by the evaluation at the identity, in which $[\mathfrak{S}, \mathsf{Cat}]_{PS}$ is the possibly large 2-category of pseudofunctors, pseudonatural transformations and modifications $\mathfrak{S} \to \mathsf{Cat}$. As a consequence, the Yoneda embedding $\mathscr{Y}_{\mathfrak{S}^{\mathrm{op}}} : \mathfrak{S}^{\mathrm{op}} \to [\mathfrak{S}, \mathsf{Cat}]_{PS}$ is locally an equivalence (*i.e.* it induces equivalences between the hom-categories).

If $\mathscr{W} : \mathfrak{S} \to \mathsf{Cat}, \mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ are pseudofunctors with a small domain, recall that the *weighted bilimit*, when it exists, is an object $\{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}$ of $\mathfrak{A}$ endowed with a pseudonatural equivalence (in X) $\mathfrak{A}(X, \{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}) \simeq [\mathfrak{S}, \mathsf{Cat}]_{PS}(\mathscr{W}, \mathfrak{A}(X, \mathscr{D}-))$.

The dual concept is that of weighted bicolimit: if $\mathscr{W}' : \mathfrak{S}^{\mathrm{op}} \to \mathsf{Cat}, \mathscr{D} : \mathfrak{S} \to \mathfrak{A}$ are pseudofunctors, the weighted bicolimit $\mathscr{W}' *_{\mathrm{bi}} \mathscr{D}$ is the weighted bilimit $\{\mathscr{W}', \mathscr{D}^{\mathrm{op}}\}_{\mathrm{bi}}$ in $\mathfrak{A}^{\mathrm{op}}$. That is to say, it is an object $\mathscr{W}' *_{\mathrm{bi}} \mathscr{D}$ of $\mathfrak{A}$ endowed with a pseudonatural equivalence (in X) $\mathfrak{A}(\mathscr{W}' *_{\mathrm{bi}} \mathscr{D}, X) \simeq [\mathfrak{S}^{\mathrm{op}}, \mathsf{Cat}]_{PS}(\mathscr{W}', \mathfrak{A}(\mathscr{D}-, X))$. By the bicategorical Yoneda Lemma, $\{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}, \mathscr{W}' *_{\mathrm{bi}} \mathscr{D}$ are unique up to equivalence, if they exist.

Remark 5.1.2. If $\mathscr{W}$ and $\mathscr{D}$ are 2-functors, $\{\mathscr{W}, \mathscr{D}\}_{\mathrm{bi}}$ and $\{\mathscr{W}, \mathscr{D}\}$ may exist, without being equivalent to each other. This problem is related to the notion of flexible presheaves/weights (see [8]): whenever $\mathscr{W}$ is *flexible*, these two types of limits are equivalent, if they exist.

Definition 5.1.3. Let $R : \mathfrak{A} \to \mathfrak{B}, E : \mathfrak{B} \to \mathfrak{A}$ be pseudofunctors. E is *left biadjoint* to R (or R is *right biadjoint* to E) if there exist

1. pseudonatural transformations $\rho : \mathrm{Id}_{\mathfrak{B}} \longrightarrow RE$ and $\varepsilon : ER \longrightarrow \mathrm{Id}_{\mathfrak{A}}$
2. invertible modifications $v : \mathrm{id}_E \Longrightarrow (\varepsilon E)(E\rho)$ and $w : (R\varepsilon)(\rho R) \Longrightarrow \mathrm{id}_R$

satisfying coherence axioms [77].

Remark 5.1.4. Recall that a biadjunction $(E \dashv R, \rho, \varepsilon, v, w)$ has an associated pseudonatural equivalence $\chi : \mathfrak{B}(-, R-) \simeq \mathfrak{A}(E-, -)$, in which

$$\begin{aligned} \chi_{(X,Z)} : \mathfrak{B}(X, RA) \quad & \to \mathfrak{A}(EX, A) \\ f \quad & \mapsto \varepsilon_A E(f) \\ \mathfrak{m} \quad & \mapsto \mathrm{id}_{\varepsilon_A} * E(\mathfrak{m}) \end{aligned}$$

$$\left(\chi_{(h,g)}\right)_f := \left(\mathrm{id}_{\varepsilon_A} * \mathfrak{e}_{(hf)R(g)}\right) \cdot \left(\varepsilon_g * \mathfrak{e}_{hf}\right).$$

If L, U are 2-functors, we say that L is *left* 2-*adjoint* to U whenever there is a biadjunction $(L \dashv U, \eta, \varepsilon, s, t)$ in which s, t are identities and η, ε are 2-natural transformations. In this case, we say that $(L \dashv U, \eta, \varepsilon)$ is a 2-*adjunction*.

5.1.5 On computads

We employ the concept of computad, introduced in [103], to define the 2-categories $\dot{\Delta}_\ell, \dot{\Delta}, \Delta_\ell$ in Section 5.3. For this reason, we give a short introduction to computads in this subsection.

Herein a *graph* $\mathrm{G} = (d_1, d_0)$ is a pair of functors $d_0, d_1 : \mathrm{G}_1 \to \mathrm{G}_0$ between discrete categories of CAT. In this case, G_0 is called the collection of objects and, for each pair of objects (a, b) of G_0, $d_0^{-1}(a) \cap d_1^{-1}(b) = \mathrm{G}(a, b)$ is the collection of arrows between a and b. A *graph morphism* T between G, G' is a function $\mathrm{T} : \mathrm{G}_0 \to \mathrm{G}'_0$ endowed with a function $\mathrm{T}_{(a,b)} : \mathrm{G}(a, b) \to \mathrm{G}'(\mathrm{T}a, \mathrm{T}b)$ for each pair (a, b) of objects in G_0. That is to say, a graph morphism $\mathrm{T} = (\mathrm{T}_1, \mathrm{T}_0)$ is a natural transformation between graphs. The category of graphs is denoted by GRPH.

We also define the full subcategories of GRPH, denoted by Grph and grph: the objects of Grph are graphs in the subcategory of discrete categories of Cat and the objects of grph, called *small graphs*, are graphs in the subcategory of discrete categories of cat. The forgetful functors $\mathsf{CAT} \to \mathsf{GRPH}$, $\mathsf{Cat} \to \mathsf{Grph}$ and $\mathsf{cat} \to \mathsf{grph}$ have left adjoints.

We denote by $\mathscr{F} : \mathsf{GRPH} \to \mathsf{CAT}$ the functor left adjoint to $\mathsf{CAT} \to \mathsf{GRPH}$ and $\mathfrak{F}$ the monad on GRPH induced by this adjunction. If $\mathrm{G} = (d_1, d_0)$ is an object of GRPH, $\mathscr{F}\mathrm{G}$ is the coinserter of this diagram (d_1, d_0).

Recall that $\mathscr{F}\mathrm{G}$, called the category freely generated by G, can be seen as the category with the same objects of G but the arrows between two objects a, b are the paths between a, b (including the empty path): composition is defined by juxtaposition of paths.

Definition 5.1.6. [Computad] A *computad* $\mathfrak{c}$ is a graph $\mathfrak{c}^{\mathrm{G}}$ endowed with a graph $\mathfrak{c}(a, b)$ such that $\mathfrak{c}(a, b)_0 = \left(\mathscr{F}\mathfrak{c}^{\mathrm{G}}\right)(a, b)$ for each pair (a, b) of objects of $\mathfrak{c}^{\mathrm{G}}$.

Remark 5.1.7. A *small computad* is a computad $\mathfrak{c}$ such that the graphs $\mathfrak{c}^{\mathrm{G}}$ and $\mathfrak{c}(a,b)$ are small for every pair (a,b) of objects of $\mathfrak{c}^{\mathrm{G}}$. Such a computad can be entirely described by a diagram

$$\begin{tikzcd}
\mathfrak{c}_2 \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & \left(\mathfrak{F}\mathfrak{c}^{\mathrm{G}}\right)_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathfrak{c}^{\mathrm{G}}_0
\end{tikzcd}$$

in Set such that:

- (d_1, d_0) is the graph $\mathfrak{F}\mathfrak{c}^{\mathrm{G}}$;
- $\mathfrak{c}_2 := \bigcup\limits_{(a,b)\in\mathfrak{c}^{\mathrm{G}}_0\times\mathfrak{c}^{\mathrm{G}}_0} \mathfrak{c}(a,b)_1$;
- $d_1\partial_1 = d_1\partial_0$ and $d_0\partial_1 = d_0\partial_0$.

A *morphism T between computads* $\mathfrak{c}, \mathfrak{c}'$ is a graph morphism $\mathrm{T}^{\mathrm{G}} : \mathfrak{c}^{\mathrm{G}} \to \mathfrak{c}'^{\mathrm{G}}$ endowed with a graph morphism $\mathrm{T}^{(a,b)} : \mathfrak{c}(a,b) \to \mathfrak{c}(\mathrm{T}^{\mathrm{G}}a, \mathrm{T}^{\mathrm{G}}b)$ for each pair of objects (a,b) in $\mathfrak{c}^{\mathrm{G}}$ such that $\mathrm{T}^{(a,b)}_0$ coincides with $\mathscr{F}(\mathrm{T}^{\mathrm{G}})_{(a,b)}$. The *category of computads* is denoted by CMP.

We can define a forgetful functor $\mathscr{U} : \mathsf{Cat}\text{-}\mathsf{CAT} \to \mathsf{CMP}$ in which $(\mathscr{U}\mathfrak{A})^{\mathrm{G}}$ is the underlying graph of the underlying category of $\mathfrak{A}$. Recall that, for each pair of objects (a,b) of $(\mathscr{U}\mathfrak{A})^{\mathrm{G}}$, an object f of $(\mathscr{U}\mathfrak{A})(a,b)$ is a path between a and b. Then the composition defines a map $\circ : (\mathscr{U}\mathfrak{A})(a,b) \to \mathfrak{A}(a,b)$ and we can define the arrows of the graphs $(\mathscr{U}\mathfrak{A})(a,b)$ as follows: $(\mathscr{U}\mathfrak{A})(a,b)(f,g) := \mathfrak{A}(a,b)(\circ(f), \circ(g))$.

The left reflection of a small computad $\mathfrak{c}$ along $\mathscr{U}$ is denoted by $\mathscr{L}\mathfrak{c}$ and called the 2*-category freely generated by* $\mathfrak{c}$. The underlying category of $\mathscr{L}\mathfrak{c}$ is $\mathscr{F}\mathfrak{c}^{\mathrm{G}}$ and its 2-dimensional structure is constructed below.

$$\begin{tikzcd}
\mathfrak{c}_2 \amalg \left(\mathfrak{F}\mathfrak{c}^{\mathrm{G}}\right)_1 \arrow[rr, shift left, "d_0\partial_0,\ d_0"] \arrow[rr, shift right, "d_1\partial_0,\ d_1"'] \arrow[d, shift right, "\partial_1{,}\mathrm{id}"'] \arrow[d, shift left, "\partial_0{,}\mathrm{id}"] & & \mathfrak{c}^{\mathrm{G}}_0 \arrow[d, "\mathrm{id}"] \\
\left(\mathfrak{F}\mathfrak{c}^{\mathrm{G}}\right)_1 \arrow[rr, shift left, "d_0"] \arrow[rr, shift right, "d_1"'] & & \mathfrak{c}^{\mathrm{G}}_0
\end{tikzcd}$$

The diagram of Remark 5.1.7 induces the graph morphisms $((\partial_0, \mathrm{id}), \mathrm{id})$ and $((\partial_1, \mathrm{id}), \mathrm{id})$ above between a graph denoted by $\mathfrak{c}^-$ and $\mathfrak{F}\mathfrak{c}^{\mathrm{G}}$. Using the multiplication of the monad $\mathfrak{F}$, these morphisms induce two morphisms $\mathfrak{F}\mathfrak{c}^- \to \mathfrak{F}\mathfrak{c}^{\mathrm{G}}$. These two morphisms define in particular the graph $\mathfrak{c}_-$ below and $\mathscr{F}\mathfrak{c}_-$ defines the 2-dimensional structure of $\mathscr{L}\mathfrak{c}$.

$$\begin{tikzcd}
\left(\mathfrak{F}\mathfrak{c}^-\right)_1 - \left(\mathfrak{F}^2\mathfrak{c}^{\mathrm{G}}\right)_1 \arrow[r, shift left] \arrow[r, shift right] & \left(\mathfrak{F}\mathfrak{c}^{\mathrm{G}}\right)_1
\end{tikzcd}$$

Defining all compositions by juxtaposition, we have a sesquicategory (see [106]). We define $\mathscr{L}\mathfrak{c}$ to be the 2-category obtained from the quotient of this sesquicategory, forcing the interchange laws.

Remark 5.1.8. Let Preord the category of preordered sets. We have an inclusion $\mathsf{Preord} \to \mathsf{Cat}$ which is right adjoint. This adjunction induces a 2-adjunction between $\mathsf{Preord}\text{-}\mathsf{CAT}$ and $\mathsf{Cat}\text{-}\mathsf{CAT}$.

If $\mathfrak{c}$ is a computad, *the locally preordered* 2*-category freely generated by* $\mathfrak{c}$ is the image of $\mathscr{L}\mathfrak{c}$ by the left 2-adjoint functor $\mathsf{Cat}\text{-}\mathsf{CAT} \to \mathsf{Preord}\text{-}\mathsf{CAT}$.

5.2 Lifting of biadjoints

In this section, we assume that a small weight $\mathscr{W} : \mathfrak{S} \to \mathsf{Cat}$, a right biadjoint pseudofunctor $R : \mathfrak{A} \to \mathfrak{B}$ and a pseudofunctor $J : \mathfrak{A} \to \mathfrak{C}$ are given. We investigate whether J is right biadjoint.

We establish Theorem 5.2.3 and its immediate corollary on biadjoint triangles. We omit the proof of Lemma 5.2.2, since it is analogous to the proof of Lemma 5.2.1.

Lemma 5.2.1. *Assume that, for each object* y *of* $\mathfrak{C}$*, there are pseudofunctors* $\mathfrak{D}_\mathrm{y} : \mathfrak{S} \times \mathfrak{A} \to \mathsf{Cat}, \mathscr{A}_\mathrm{y} : \mathfrak{S}^{\mathrm{op}} \to \mathfrak{A}$ *such that* $\mathfrak{D}_\mathrm{y} \simeq \mathfrak{A}(\mathscr{A}_\mathrm{y}-,-)$ *and* $\left\{\mathscr{W}, \mathfrak{D}_\mathrm{y}(-,A)\right\}_{\mathrm{bi}} \simeq \mathfrak{C}(\mathrm{y}, JA)$ *for each object* A *of* $\mathfrak{A}$*. The pseudofunctor* J *is right biadjoint if and only if, for every object* y *of* $\mathfrak{C}$*, the weighted bicolimit* $\mathscr{W} *_{\mathrm{bi}} \mathscr{A}_\mathrm{y}$ *exists in* $\mathfrak{A}$*. In this case,* J *is right biadjoint to* G*, defined by* $G\mathrm{y} = \mathscr{W} *_{\mathrm{bi}} \mathscr{A}_\mathrm{y}$*.*

Proof. There is a pseudonatural equivalence (in A)

$$\left\{\mathscr{W}, \mathfrak{A}(\mathscr{A}_\mathrm{y}-,A)\right\}_{\mathrm{bi}} \simeq \left\{\mathscr{W}, \mathfrak{D}_\mathrm{y}(-,A)\right\}_{\mathrm{bi}} \simeq \mathfrak{C}(\mathrm{y}, JA).$$

Thereby, an object $G\mathrm{y}$ of $\mathfrak{A}$ is the weighted bicolimit $\mathscr{W} *_{\mathrm{bi}} \mathscr{A}_\mathrm{y}$ if and only if there is a pseudonatural equivalence (in A) $\mathfrak{A}(G\mathrm{y}, A) \simeq \left\{\mathscr{W}, \mathfrak{A}(\mathscr{A}_\mathrm{y}-,A)\right\}_{\mathrm{bi}} \simeq \mathfrak{C}(\mathrm{y}, JA)$. That is to say, an object $G\mathrm{y}$ of $\mathfrak{A}$ is the weighted bicolimit $\mathscr{W} *_{\mathrm{bi}} \mathscr{A}_\mathrm{y}$ if and only if $G\mathrm{y}$ is a birepresentation of $\mathfrak{C}(\mathrm{y}, J-)$. □

Lemma 5.2.2. *Assume that* $J, \mathscr{W}$ *are* 2*-functors and, for each object* y *of* $\mathfrak{C}$*, there are* 2*-functors* $\mathfrak{D}_\mathrm{y} : \mathfrak{S} \times \mathfrak{A} \to \mathsf{Cat}, \mathscr{A}_\mathrm{y} : \mathfrak{S}^{\mathrm{op}} \to \mathfrak{A}$ *such that there is a* 2*-natural isomorphism* $\mathfrak{D}_\mathrm{y} \cong \mathfrak{A}(\mathscr{A}_\mathrm{y}-,-)$ *and* $\left\{\mathscr{W}, \mathfrak{D}_\mathrm{y}(-,A)\right\} \cong \mathfrak{C}(\mathrm{y}, JA)$ *for every object* A *of* $\mathfrak{A}$*. The* 2*-functor* J *is right* 2*-adjoint if and only if, for every object* y *of* $\mathfrak{C}$*, the weighted colimit* $\mathscr{W} * \mathscr{A}_\mathrm{y}$ *exists in* $\mathfrak{A}$*. In this case,* J *is right* 2*-adjoint to* G*, defined by* $G\mathrm{y} = \mathscr{W} * \mathscr{A}_\mathrm{y}$*.*

Let $\mathscr{D} : \mathfrak{S} \times \mathfrak{A} \to \mathsf{Cat}$ be a pseudofunctor. We denote by $|\mathscr{D}| : \mathfrak{S}_0 \times \mathfrak{A} \to \mathsf{Cat}$ the restriction of $\mathscr{D}$ in which $\mathfrak{S}_0$ is the discrete 2-category of the objects of $\mathfrak{S}$. Also, herein we say that $|\mathscr{D}|$ can be factorized through $R^* := \mathfrak{B}(-, R-)$ if there are a pseudofunctor $\mathscr{D}' : \mathfrak{S}_0 \to \mathfrak{B}^{\mathrm{op}}$ and a pseudonatural equivalence $|\mathscr{D}| \simeq R^* \circ (\mathscr{D}' \times \mathrm{Id}_{\mathfrak{A}})$.

Theorem 5.2.3. *Assume that, for each object* y *of* $\mathfrak{C}$*, there is a pseudofunctor* $\mathfrak{D}_\mathrm{y} : \mathfrak{S} \times \mathfrak{A} \to \mathsf{Cat}$ *such that* $\left|\mathfrak{D}_\mathrm{y}\right|$ *can be factorized through* R^* *and* $\left\{\mathscr{W}, \mathfrak{D}_\mathrm{y}(-,A)\right\}_{\mathrm{bi}} \simeq \mathfrak{C}(\mathrm{y}, JA)$ *for every object* A *of* $\mathfrak{A}$*. In this setting, for each object* y *of* $\mathfrak{C}$ *there are a pseudofunctor* $\mathscr{A}_\mathrm{y} : \mathfrak{S}^{\mathrm{op}} \to \mathfrak{A}$ *and a pseudonatural equivalence* $\mathfrak{D}_\mathrm{y} \simeq \mathfrak{A}(\mathscr{A}_\mathrm{y}-,-)$*.*

As a consequence, the pseudofunctor J *is right biadjoint if and only if, for every object* y *of* $\mathfrak{C}$*, the weighted bicolimit* $\mathscr{W} *_{\mathrm{bi}} \mathscr{A}_\mathrm{y}$ *exists in* $\mathfrak{A}$*. In this case,* J *is right biadjoint to* G*, defined by* $G\mathrm{y} = \mathscr{W} *_{\mathrm{bi}} \mathscr{A}_\mathrm{y}$*.*

Proof. Indeed, if $E : \mathfrak{B} \to \mathfrak{A}$ is left biadjoint to R, then there is a pseudonatural equivalence $R^* \simeq \mathfrak{A}(E-,-)$. Therefore, by the hypotheses, for each object Y of $\mathfrak{C}$, there is a pseudofunctor $\mathfrak{D}'_\mathrm{y} : \mathfrak{S}_0 \to \mathfrak{B}^{\mathrm{op}}$ such that $\left|\mathfrak{D}_\mathrm{y}\right| \simeq R^* \circ (\mathfrak{D}'_\mathrm{y} \times \mathrm{Id}_{\mathfrak{A}}) \simeq \mathfrak{A}(E\mathfrak{D}'_\mathrm{y}-,-)$. From the bicategorical Yoneda lemma, it follows that we can choose a pseudofunctor $\mathscr{A}_\mathrm{y} : \mathfrak{S}^{\mathrm{op}} \to \mathfrak{A}$ which is an extension of $E\mathfrak{D}'_\mathrm{y}$ such that $\mathfrak{A}(\mathscr{A}_\mathrm{y}-,-) \simeq \mathfrak{D}_\mathrm{y}$. The consequence follows from Lemma 5.2.1. □

Corollary 5.2.4 (Biadjoint Triangle)**.** *Assume that $\mathcal{V} : \mathfrak{C}' \to \mathfrak{C}$ is a pseudofunctor and*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[rd, "R"'] & & \mathfrak{C}' \arrow[ld, "U"] \\
& \mathfrak{B} &
\end{tikzcd}$$

is a commutative triangle of pseudofunctors satisfying the following: for each object y *of* $\mathfrak{C}$*, there is a pseudofunctor* $\mathscr{D}_{\mathrm{y}} : \mathfrak{S} \times \mathfrak{C}' \to \mathsf{Cat}$ *such that* $\left|\mathscr{D}_{\mathrm{y}}\right|$ *can be factorized through* U^* *and* $\left\{\mathscr{W}, \mathscr{D}_{\mathrm{y}}(-,\mathrm{x})\right\}_{\mathrm{bi}} \simeq \mathfrak{C}(\mathrm{y}, \mathcal{V}\mathrm{x})$ *for each object* x *of* $\mathfrak{C}'$*. In this setting, for each object* y *of* $\mathfrak{C}$*, there is a pseudofunctor* $\mathscr{A}_{\mathrm{y}} : \mathfrak{S}^{\mathrm{op}} \to \mathfrak{A}$ *such that* $\mathscr{D}_{\mathrm{y}}(-,J-) \simeq \mathfrak{A}(\mathscr{A}_{\mathrm{y}}-,-)$*.*

As a consequence, the pseudofunctor $\mathcal{V} \circ J$ *is right biadjoint if and only if, for every object* y *of* $\mathfrak{C}$*, the weighted bicolimit* $\mathscr{W} *_{\mathrm{bi}} \mathscr{A}_{\mathrm{y}}$ *exists in* $\mathfrak{A}$*. In this case,* $\mathcal{V} \circ J$ *is right biadjoint to* G*, defined by* $G\mathrm{y} = \mathscr{W} *_{\mathrm{bi}} \mathscr{A}_{\mathrm{y}}$*.*

Proof. We prove that $\mathfrak{D}_{\mathrm{y}} := \mathscr{D}_{\mathrm{y}}(-,J-)$ satisfies the hypotheses of Theorem 5.2.3. We have that, for each object y of $\mathfrak{C}$ and each object A of $\mathfrak{A}$,

$$\left\{\mathscr{W}, \mathfrak{D}_{\mathrm{y}}(-,A)\right\}_{\mathrm{bi}} \simeq \mathfrak{C}(\mathrm{y}, \mathcal{V}JA).$$

Also, for each object y of $\mathfrak{C}$, there is a pseudofunctor $\mathscr{D}'_{\mathrm{y}} : \mathfrak{S}_0 \to \mathfrak{B}^{\mathrm{op}}$ such that $U^* \circ (\mathscr{D}'_{\mathrm{y}} \times \mathrm{Id}_{\mathfrak{C}}) \simeq \left|\mathscr{D}_{\mathrm{y}}\right|$. Therefore

$$R^* \circ (\mathscr{D}'_{\mathrm{y}} \times \mathrm{Id}_{\mathfrak{A}}) \simeq U^* \circ (\mathscr{D}'_{\mathrm{y}} \times J) \simeq \left|\mathscr{D}_{\mathrm{y}}\right| \circ (\mathrm{Id}_{\mathfrak{S}_0} \times J) \simeq \left|\mathfrak{D}_{\mathrm{y}}\right|.$$

□

Corollary 5.10 of [77] is a direct consequence of the last corollary and Proposition 5.7 of [77]. In particular, if $\mathcal{T}$ is a pseudomonad on $\mathfrak{B}$ and $U : \mathsf{Ps}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \to \mathfrak{B}$ is the forgetful 2-functor, Proposition 5.5 of [77] shows that the category of pseudomorphisms between two pseudoalgebras is given by a descent object (which is a type of weighted bilimit) of a diagram satisfying the hypotheses of Corollary 5.2.4. Therefore, assuming the existence of codescent objects in $\mathfrak{A}$, J has a left biadjoint.

In Section 5.4, we define the 2-category of lax algebras of a pseudomonad $\mathcal{T}$. There, we also show Proposition 5.4.5 which is precisely the analogue and a generalization of Proposition 5.5 of [77]: the category of lax morphisms and the category of pseudomorphisms between lax algebras are given by appropriate types of weighted bilimits. Then, we can apply Corollary 5.2.4 to get our desired result on lifting of biadjoints to the 2-category of lax algebras: Theorem 5.5.2. Next section, we define and study the weighted bilimits appropriate to our problem, called lax descent objects and descent objects.

To finish this section, we get a trivial consequence of Corollary 5.2.4:

Corollary 5.2.5. *If* $RJ = U$ *are pseudofunctors in which* R *is right biadjoint and* U *is locally an equivalence, then* J *is right biadjoint as well. Actually, if* E *is left biadjoint to* R*,* $G\mathrm{y} := EU\mathrm{y}$ *defines the pseudofunctor left biadjoint to* J*.*

5.3 Lax descent objects

In this section we describe the 2-categorical limits called lax descent objects and descent objects [67, 79, 103–105, 107].

In page 177 of [103], without establishing the name "lax descent objects", it is shown that given a 2-monad $\mathscr{T}$, for each pair y, z of strict $\mathscr{T}$-algebras, there is a diagram of categories for which its lax descent category (object) is the category of lax morphisms between y and z. We establish a generalization of this result for lax algebras: Proposition 5.4.5.

In order to establish such result, our approach in defining the lax descent objects is different from [103], commencing with the definition of our "domain 2-category", denoted by Δ_ℓ.

Definition 5.3.1. [$\mathsf{t} : \Delta_\ell \to \dot{\Delta}_\ell$ and $\mathsf{j} : \Delta_\ell \to \dot{\Delta}$] We denote by $\dot{\mathfrak{d}}_\ell$ the computad defined by the diagram

$$\begin{tikzcd}
0 \arrow[r, "d"] & 1 \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & 2 \arrow[l, "s^0" description] \arrow[r, shift left=2, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=2, "\partial^2"'] & 3
\end{tikzcd}$$

with the 2-cells:

$$\begin{aligned}
\sigma_{00} &: \partial^0 d^0 \Rightarrow \partial^1 d^0, & n_0 &: \mathrm{id}_1 \Rightarrow s^0 d^0,\\
\sigma_{20} &: \partial^2 d^0 \Rightarrow \partial^0 d^1, & n_1 &: \mathrm{id}_1 \Rightarrow s^0 d^1,\\
\sigma_{21} &: \partial^2 d^1 \Rightarrow \partial^1 d^1, & \vartheta &: d^1 d \Rightarrow d^0 d.
\end{aligned}$$

The 2-*category* $\dot{\Delta}_\ell$ is, herein, the locally preordered 2-category freely generated by $\dot{\mathfrak{d}}_\ell$. The full sub-2-category of $\dot{\Delta}_\ell$ with objects $1, 2, 3$ is *denoted by* Δ_ℓ and the full inclusion by $\mathsf{t} : \Delta_\ell \to \dot{\Delta}_\ell$.

We consider also the computad $\dot{\mathfrak{d}}$ which is defined as the computad $\dot{\mathfrak{d}}_\ell$ with one extra 2-cell $d^0 d \Rightarrow d^1 d$. We denote by $\dot{\Delta}$ the locally preordered 2-category freely generated by $\dot{\mathfrak{d}}$. Of course, there is also a full inclusion $\mathsf{j} : \Delta_\ell \to \dot{\Delta}$.

We define, also, the computad $\mathfrak{d}_\ell$ which is the full subcomputad of $\dot{\mathfrak{d}}_\ell$ with objects $1, 2, 3$.

Proposition 5.3.2. *Let $\mathfrak{A}$ be a 2-category. There is a bijection between the 2-functors $\Delta_\ell \to \mathfrak{A}$ and the maps of computads $\mathfrak{d}_\ell \to \mathscr{U}\mathfrak{A}$. In other words, Δ_ℓ is the 2-category freely generated by the computad $\mathfrak{d}_\ell$.*

Also, there is a bijection between 2-functors $\overline{\mathscr{D}} : \dot{\Delta}_\ell \to \mathfrak{A}$ and the maps of computads $\mathscr{D} : \mathfrak{d}_\ell \to \mathscr{U}\mathfrak{A}$ which satisfy the following equations:

- *Associativity:*

$$\begin{tikzcd}
\mathscr{D}0 \arrow[r, "\mathscr{D}(d)"] \arrow[d, "\mathscr{D}(d)"'] \arrow[dr, phantom, "\overset{\mathscr{D}(\vartheta)}{\Longrightarrow}"] & \mathscr{D}1 \arrow[d, "\mathscr{D}(d^0)"] \\
\mathscr{D}1 \arrow[r, "\mathscr{D}(d^1)"] \arrow[d, "\mathscr{D}(d^1)"'] \arrow[dr, phantom, "\overset{\mathscr{D}(\sigma_{21})}{\Longrightarrow}"] & \mathscr{D}2 \arrow[d, "\mathscr{D}(\partial^1)"] \\
\mathscr{D}2 \arrow[r, "\mathscr{D}(\partial^2)"'] & \mathscr{D}3
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{D}3 & & \mathscr{D}2 \arrow[ll, "\mathscr{D}(\partial^1)"'] \\
& \mathscr{D}2 \arrow[ul, "\mathscr{D}(\partial^0)"] \arrow[ur, phantom, "\overset{\mathscr{D}(\sigma_{00})}{\Longrightarrow}"] & \mathscr{D}1 \arrow[u, "\mathscr{D}(d^0)"'] \arrow[l, "\mathscr{D}(d^0)"'] \\
\mathscr{D}2 \arrow[uu, "\mathscr{D}(\partial^2)"] \arrow[ur, phantom, "\overset{\mathscr{D}(\sigma_{20})}{\Longrightarrow}"] & \mathscr{D}1 \arrow[l, "\mathscr{D}(d^0)"'] \arrow[u, "\mathscr{D}(d^1)"'] \arrow[r, phantom, "\overset{\mathscr{D}(\vartheta)}{\Longrightarrow}"] & \\
\mathscr{D}1 \arrow[u, "\mathscr{D}(d^1)"] \arrow[ur, phantom, "\overset{\mathscr{D}(\vartheta)}{\Longrightarrow}"] & \mathscr{D}0 \arrow[l, "\mathscr{D}(d)"] \arrow[u, "\mathscr{D}(d)"'] \arrow[uur, bend right, "\mathscr{D}(d)"'] &
\end{tikzcd}$$

– *Identity:*

$$\begin{tikzcd}
\mathscr{D}0 \arrow[r, "\mathscr{D}(d)"] \arrow[d, "\mathscr{D}(d)"'] \arrow[rd, phantom, "\overset{\mathscr{D}(\vartheta)}{\Longrightarrow}"] & \mathscr{D}1 \arrow[d, "\mathscr{D}(d^0)"] \\
\mathscr{D}1 \arrow[r, "\mathscr{D}(d^1)"'] \arrow[rd, equal, bend right] \arrow[rd, phantom, "\overset{\mathscr{D}(n_1)}{\Longrightarrow}"] & \mathscr{D}2 \arrow[d, "\mathscr{D}(s^0)"] \\
& \mathscr{D}1
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{D}0 \arrow[d, "\mathscr{D}(d)"'] & \\
\mathscr{D}1 \arrow[r, "\mathscr{D}(d^0)"'] \arrow[rd, equal, bend right] \arrow[rd, phantom, "\overset{\mathscr{D}(n_0)}{\Longrightarrow}"] & \mathscr{D}2 \arrow[d, "\mathscr{D}(s^0)"] \\
& \mathscr{D}1
\end{tikzcd}$$

Moreover, there is a bijection between 2*-functors* $\dot{\Delta} \to \mathfrak{A}$ *and* 2*-functors* $\overline{\mathscr{D}} : \dot{\Delta}_\ell \to \mathfrak{A}$ *such that* $\overline{\mathscr{D}}(\vartheta)$ *is an invertible* 2*-cell.*

Let $\mathfrak{A}$ be a 2-category and $\mathscr{D} : \Delta_\ell \to \mathfrak{A}$ be a pseudofunctor. If the weighted bilimit $\left\{\dot{\Delta}(0,\mathsf{j}-), \mathscr{D}\right\}_{\mathrm{bi}}$ exists, we say that $\left\{\dot{\Delta}(0,\mathsf{j}-), \mathscr{D}\right\}_{\mathrm{bi}}$ is the *descent object* of $\mathscr{D}$. Moreover, if the weighted bilimit $\left\{\dot{\Delta}_\ell(0,\mathsf{t}-), \mathscr{D}\right\}_{\mathrm{bi}}$ exists, it is called the *lax descent object* of $\mathscr{D}$.

Analogously, if such $\mathscr{D}$ is a 2-functor and the (strict) weighted 2-limit $\left\{\dot{\Delta}(0,\mathsf{j}-), \mathscr{D}\right\}$ exists, we call it the *strict descent object* of $\mathscr{D}$. Finally, the (strict) weighted 2-limit $\left\{\dot{\Delta}_\ell(0,\mathsf{t}-), \mathscr{D}\right\}$ is called the *strict lax descent object* of $\mathscr{D}$, if it exists.

Lemma 5.3.3. *Strict lax descent objects are lax descent objects and strict descent objects are descent objects. That is to say, the weights* $\dot{\Delta}_\ell(0,\mathsf{t}-) : \Delta_\ell \to \mathsf{Cat}, \dot{\Delta}(0,\mathsf{j}-) : \Delta_\ell \to \mathsf{Cat}$ *are flexible.*

The dual notions of lax descent object and descent object are called the codescent object and the lax codescent object. If $\mathscr{A} : \Delta_\ell^{\mathrm{op}} \to \mathfrak{A}$ is a 2-functor, the *codescent object* of $\mathscr{A}$ is, if it exists, $\dot{\Delta}(0,\mathsf{j}-) *_{\mathrm{bi}} \mathscr{A}$ and the *lax codescent object* of $\mathscr{A}$ is $\dot{\Delta}_\ell(0,\mathsf{t}-) *_{\mathrm{bi}} \mathscr{A}$ if it exists.

Also, the weighted colimits $\dot{\Delta}(0,\mathsf{j}-) * \mathscr{A}, \dot{\Delta}_\ell(0,\mathsf{t}-) * \mathscr{A}$ are called, respectively, the *strict codescent object* and the *strict lax codescent object* of $\mathscr{A}$.

Remark 5.3.4. If $\mathscr{D} : \Delta_\ell \to \mathsf{Cat}$ is a 2-functor, then

$$\left\{\dot{\Delta}_\ell(0,\mathsf{t}-), \mathscr{D}\right\} \cong [\Delta_\ell, \mathsf{Cat}]\left(\dot{\Delta}_\ell(0,\mathsf{t}-), \mathscr{D}\right).$$

Thereby, we can describe the strict lax descent object of $\mathscr{D} : \Delta_\ell \to \mathsf{Cat}$ explicitly as follows:

1. Objects are 2-natural transformations $\mathtt{f} : \dot{\Delta}_\ell(0,\mathsf{t}-) \longrightarrow \mathscr{D}$. We have a bijective correspondence between such 2-natural transformations and pairs $(f, \langle\bar{\mathsf{f}}\rangle)$ in which f is an object of $\mathscr{D}1$ and $\langle\bar{\mathsf{f}}\rangle : \mathscr{D}(d^1)f \to \mathscr{D}(d^0)f$ is a morphism in $\mathscr{D}2$ satisfying the following equations:

   – Associativity:

   $$\left(\mathscr{D}(\sigma_{00})_f\right)\left(\mathscr{D}(\partial^0)(\langle\bar{\mathsf{f}}\rangle)\right)\left(\mathscr{D}(\sigma_{20})_f\right)\left(\mathscr{D}(\partial^2)(\langle\bar{\mathsf{f}}\rangle)\right) = \left(\mathscr{D}(\partial^1)(\langle\bar{\mathsf{f}}\rangle)\right)\left(\mathscr{D}(\sigma_{21})_f\right)$$

   – Identity:

   $$\left(\mathscr{D}(s^0)(\langle\bar{\mathsf{f}}\rangle)\right)\left(\mathscr{D}(n_1)_f\right) = \left(\mathscr{D}(n_0)_f\right)$$

   If $\mathtt{f} : \dot{\Delta}(0,-) \longrightarrow \mathscr{D}$ is a 2-natural transformation, we get such pair by the correspondence $\mathtt{f} \mapsto (\mathtt{f}_1(d), \mathtt{f}_2(\vartheta))$.

2. The morphisms are modifications. In other words, a morphism $\mathtt{m} : \mathtt{f} \to \mathtt{h}$ is determined by a morphism $\mathfrak{m} : f \to g$ in $\mathscr{D}1$ such that $\mathscr{D}(d^0)(\mathfrak{m})\left\langle\bar{\mathsf{f}}\right\rangle = \left\langle\bar{\mathsf{h}}\right\rangle\mathscr{D}(d^1)(\mathfrak{m})$.

Furthermore, there is a full inclusion $\left\{\dot{\Delta}(0,\mathsf{j}-),\mathscr{D}\right\} \to \left\{\dot{\Delta}_\ell(0,\mathsf{t}-),\mathscr{D}\right\}$ such that the objects of $\left\{\dot{\Delta}(0,\mathsf{j}-),\mathscr{D}\right\}$ are precisely the pairs $(f,\left\langle\bar{\mathsf{f}}\right\rangle)$ (described above) with one further property: $\left\langle\bar{\mathsf{f}}\right\rangle$ is actually an isomorphism in $\mathscr{D}2$.

5.4 Pseudomonads and lax algebras

Pseudomonads in 2-Cat are defined in [77, 79] (that is to say, Chapter 4 and Chapter 3). The definition agrees with the theory of pseudomonads for Gray-categories [66, 83, 84, 86] and with the definition of *doctrines* of [104].

For each pseudomonad $\mathscr{T}$ on a 2-category $\mathfrak{B}$, there is an associated (right biadjoint) forgetful 2-functor Ps-$\mathscr{T}$-Alg $\to \mathfrak{B}$, in which Ps-$\mathscr{T}$-Alg is the 2-category of pseudoalgebras. In this section, we give the definitions of the 2-category of lax algebras Lax-$\mathscr{T}$-Alg_ℓ and its associated forgetful 2-functor Lax-$\mathscr{T}$-$\mathsf{Alg}_\ell \to \mathfrak{B}$, which are slight generalizations of the definitions given in [67, 102].

Recall that a *pseudomonad* $\mathscr{T}$ on a 2-category $\mathfrak{B}$ consists of a sextuple $(\mathscr{T}, m, \eta, \mu, \iota, \tau)$, in which $\mathscr{T} : \mathfrak{B} \to \mathfrak{B}$ is a pseudofunctor, $m : \mathscr{T}^2 \longrightarrow \mathscr{T}, \eta : \mathrm{Id}_{\mathfrak{B}} \longrightarrow \mathscr{T}$ are pseudonatural transformations and $\tau : \mathrm{Id}_{\mathscr{T}} \Longrightarrow (m)(\mathscr{T}\eta)$, $\iota : (m)(\eta\mathscr{T}) \Longrightarrow \mathrm{Id}_{\mathscr{T}}$, $\mu : m(\mathscr{T}m) \Rightarrow m(m\mathscr{T})$ are invertible modifications satisfying the following coherence equations:

– Associativity:

$$\begin{tikzcd}
\mathscr{T}^4 \arrow[r, "\mathscr{T}^2 m"] \arrow[d, "m\mathscr{T}^2"'] \arrow[dr, "\mathscr{T}m\mathscr{T}" description] & \mathscr{T}^3 \arrow[dr, "\mathscr{T}m"] \arrow[r, phantom, ""{name=A}] & {} \\
\mathscr{T}^3 \arrow[dr, "m\mathscr{T}"'] \arrow[r, phantom, "\overset{\mu\mathscr{T}}{\Longleftarrow}"] & \mathscr{T}^3 \arrow[r, "\mathscr{T}m"] \arrow[d, "m\mathscr{T}" description] \arrow[ur, phantom, "\overset{\widehat{\mathscr{T}\mu}}{\Longleftarrow}"] \arrow[dr, phantom, "\overset{\mu}{\Longleftarrow}"] & \mathscr{T}^2 \arrow[d, "m"] \\
{} & \mathscr{T}^2 \arrow[r, "m"'] & \mathscr{T}
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
\mathscr{T}^4 \arrow[r, "\mathscr{T}^2 m"] \arrow[d, "m\mathscr{T}^2"'] \arrow[dr, phantom, "\overset{m_m^{-1}}{\Longleftarrow}"] & \mathscr{T}^3 \arrow[d, "m\mathscr{T}" description] \arrow[dr, "\mathscr{T}m"] & {} \\
\mathscr{T}^3 \arrow[r, "\mathscr{T}m"] \arrow[dr, "m\mathscr{T}"'] & \mathscr{T}^2 \arrow[r, phantom, "\overset{\mu}{\Longleftarrow}"] \arrow[dr, "m" description] \arrow[d, phantom, "\overset{\mu}{\Longleftarrow}"] & \mathscr{T}^2 \arrow[d, "m"] \\
{} & \mathscr{T}^2 \arrow[r, "m"'] & \mathscr{T}
\end{tikzcd}$$

– Identity:

$$\begin{tikzcd}
{} & \mathscr{T}^2 \arrow[dl, "\mathscr{T}\eta\mathscr{T}"'] \arrow[dr, "\mathscr{T}\eta\mathscr{T}"] \arrow[dd, "\mathrm{Id}_{\mathscr{T}^2}" description] & {} \\
\mathscr{T}^3 \arrow[dr, "m\mathscr{T}"'] \arrow[r, phantom, "\overset{\tau\mathscr{T}}{\Longleftarrow}"] & {} \arrow[r, phantom, "\overset{\widehat{\mathscr{T}\iota}}{\Longleftarrow}"] & \mathscr{T}^3 \arrow[dl, "\mathscr{T}m"] \\
{} & \mathscr{T}^2 \arrow[d, "m" description] & {} \\
{} & \mathscr{T} & {}
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
{} & \mathscr{T}^2 \arrow[d, "\mathscr{T}\eta\mathscr{T}" description] & {} \\
{} & \mathscr{T}^3 \arrow[dl, "m\mathscr{T}" description] \arrow[dr, "\mathscr{T}m" description] & {} \\
\mathscr{T}^2 \arrow[dr, "m" description] \arrow[rr, phantom, "\overset{\mu}{\Longleftarrow}"] & {} & \mathscr{T}^2 \arrow[dl, "m" description] \\
{} & \mathscr{T} & {}
\end{tikzcd}$$

in which

$$\widehat{\mathscr{T}\iota} := (\mathsf{t}_{\mathscr{T}})^{-1}(\mathscr{T}\iota)\left(\mathsf{t}_{(m)(\eta\mathscr{T})}\right) \qquad \widehat{\mathscr{T}\mu} := \left(\mathsf{t}_{(m)(m\mathscr{T})}\right)^{-1}(\mathscr{T}\mu)\left(\mathsf{t}_{(m)(\mathscr{T}m)}\right).$$

Recall that the Cat-enriched notion of monad is a pseudomonad $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ such that the invertible modifications μ, ι, τ are identities and m, η are 2-natural transformations. In this case, we say that $\mathscr{T} = (\mathscr{T}, m, \eta)$ is a 2-monad, omitting the identities.

Definition 5.4.1. [Lax algebras] Let $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ be a pseudomonad on $\mathfrak{B}$. We define the 2-category Lax-$\mathscr{T}$-Alg_ℓ as follows:

1. Objects: *lax $\mathscr{T}$-algebras* are defined by $\mathtt{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \overline{\mathsf{z}}, \overline{\mathsf{z}}_0)$ in which $\mathfrak{alg}_{\mathsf{z}} : \mathscr{T}Z \to Z$ is a morphism of $\mathfrak{B}$ and $\overline{\mathsf{z}} : \mathfrak{alg}_{\mathsf{z}}\mathscr{T}(\mathfrak{alg}_{\mathsf{z}}) \Rightarrow \mathfrak{alg}_{\mathsf{z}} m_Z, \overline{\mathsf{z}}_0 : \mathrm{Id}_Z \Rightarrow \mathfrak{alg}_{\mathsf{z}}\eta_Z$ are 2-cells of $\mathfrak{B}$ satisfying the coherence axioms:

$$\begin{tikzcd}
\mathscr{T}^3Z \arrow[r, "\mathscr{T}^2(\mathfrak{alg}_{\mathsf{z}})"] \arrow[d, "m_{\mathscr{T}Z}"'] \arrow[dr, "\mathscr{T}(m_Z)"] & \mathscr{T}^2Z \arrow[dr, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"] \arrow[d, phantom, "\overset{\widehat{\mathscr{T}(\overline{\mathsf{z}})}}{\Longleftarrow}"] & & & \mathscr{T}^3Z \arrow[r, "\mathscr{T}^2(\mathfrak{alg}_{\mathsf{z}})"] \arrow[d, "m_{\mathscr{T}Z}"'] \arrow[dr, phantom, "\overset{m^{-1}_{\mathfrak{alg}_{\mathsf{z}}}}{\Longleftarrow}"] & \mathscr{T}^2Z \arrow[d, "m_Z"] \arrow[dr, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"] & \\
\mathscr{T}^2Z \arrow[r, phantom, "\overset{\mu_Z}{\Longleftarrow}"] \arrow[dr, "m_Z"'] & \mathscr{T}^2Z \arrow[r, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"] \arrow[d, "m_Z"] \arrow[dr, phantom, "\overset{\overline{\mathsf{z}}}{\Longleftarrow}"] & \mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"] & = & \mathscr{T}^2Z \arrow[r, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"'] \arrow[dr, "m_Z"'] & \mathscr{T}Z \arrow[r, phantom, "\overset{\overline{\mathsf{z}}}{\Longleftarrow}"] \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[d, phantom, "\overset{\overline{\mathsf{z}}}{\Longleftarrow}"] & \mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"] \\
& \mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"'] & Z & & & \mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"'] & Z
\end{tikzcd}$$

in which $\widehat{\mathscr{T}(\overline{\mathsf{z}})} := \left(\mathfrak{t}_{(\mathfrak{alg}_{\mathsf{z}})(m_Z)}\right)^{-1} (\mathscr{T}(\overline{\mathsf{z}})) \left(\mathfrak{t}_{(\mathfrak{alg}_{\mathsf{z}})(\mathscr{T}(\mathfrak{alg}_{\mathsf{z}}))}\right)$ and the 2-cells

$$\begin{tikzcd}
\mathscr{T}Z \arrow[rrr, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[dd, equal] \arrow[dr, "\eta_{\mathscr{T}Z}"] & & & Z \arrow[dd, equal] \arrow[dl, "\eta_Z"'] & & \mathscr{T}Z \arrow[rr, equal] \arrow[dd, equal] \arrow[dr, "\mathscr{T}(\eta_Z)"] & & \mathscr{T}Z \arrow[d, equal] \\
\overset{\iota_Z}{\Longleftarrow} & \mathscr{T}^2Z \arrow[r, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"'] \arrow[dl, "m_Z"] & \mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[u, phantom, "\overset{\eta^{-1}_{\mathfrak{alg}_{\mathsf{z}}}}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\overline{\mathsf{z}}}{\Longleftarrow}"] & \overset{\overline{\mathsf{z}}_0}{\Longleftarrow} & & \overset{\tau^{-1}_Z}{\Longleftarrow} & \mathscr{T}^2Z \arrow[r, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"'] \arrow[dl, "m_Z"] \arrow[ur, phantom, "\overset{\widehat{\mathscr{T}(\overline{\mathsf{z}}_0)}}{\Longleftarrow}"] \arrow[dr, phantom, "\overset{\overline{\mathsf{z}}}{\Longleftarrow}"] & \mathscr{T}Z \arrow[d, "\mathfrak{alg}_{\mathsf{z}}"] \\
\mathscr{T}Z \arrow[rrr, "\mathfrak{alg}_{\mathsf{z}}"'] & & & Z & & \mathscr{T}Z \arrow[rr, "\mathfrak{alg}_{\mathsf{z}}"'] & & Z
\end{tikzcd}$$

are identities in which $\widehat{\mathscr{T}(\overline{\mathsf{z}}_0)} := \left(\mathfrak{t}_{(\mathfrak{alg}_{\mathsf{z}})(\eta_Z)}\right)^{-1} (\mathscr{T}(\overline{\mathsf{z}}_0))(\mathfrak{t}_{\mathscr{T}Z})$. Recall that, if a lax algebra $\mathtt{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \overline{\mathsf{z}}, \overline{\mathsf{z}}_0)$ is such that $\overline{\mathsf{z}}, \overline{\mathsf{z}}_0$ are invertible 2-cells, then $\mathtt{z}$ is called a pseudoalgebra.

2. Morphisms: *lax $\mathscr{T}$-morphisms* $\mathtt{f} : \mathtt{y} \to \mathtt{z}$ between lax $\mathscr{T}$-algebras $\mathtt{y} = (Y, \mathfrak{alg}_{\mathsf{y}}, \overline{\mathsf{y}}, \overline{\mathsf{y}}_0)$, $\mathtt{z} = (Z, \mathfrak{alg}_{\mathsf{z}}, \overline{\mathsf{z}}, \overline{\mathsf{z}}_0)$ are pairs $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$ in which $f : Y \to Z$ is a morphism in $\mathfrak{B}$ and $\langle\overline{\mathsf{f}}\rangle : \mathfrak{alg}_{\mathsf{z}}\mathscr{T}(f) \Rightarrow f\mathfrak{alg}_{\mathsf{y}}$ is a 2-cell of $\mathfrak{B}$ such that, defining $\widehat{\mathscr{T}(\langle\overline{\mathsf{f}}\rangle)} := \mathfrak{t}^{-1}_{(f)(\mathfrak{alg}_{\mathsf{y}})}\mathscr{T}(\langle\overline{\mathsf{f}}\rangle)\mathfrak{t}_{(\mathfrak{alg}_{\mathsf{z}})(\mathscr{T}(f))}$, the equations

$$\begin{tikzcd}
& \mathscr{T}^2Y \arrow[r, "\mathscr{T}^2(f)"] \arrow[dl, "m_Y"'] \arrow[d, phantom, "\overset{m^{-1}_f}{\Longleftarrow}"] & \mathscr{T}^2Z \arrow[dl, "m_Z"] \arrow[dr, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"] & & & & \mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"] & Z & \\
\mathscr{T}Y \arrow[r, "\mathscr{T}(f)"'] \arrow[dr, "\mathfrak{alg}_{\mathsf{y}}"'] & \mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[rr, phantom, "\overset{\overline{\mathsf{z}}}{\Longleftarrow}"] \arrow[d, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longleftarrow}"] & & \mathscr{T}Z \arrow[dl, "\mathfrak{alg}_{\mathsf{z}}"] & = & \mathscr{T}^2Z \arrow[ur, "\mathscr{T}(\mathfrak{alg}_{\mathsf{z}})"] \arrow[rr, phantom, "\overset{\widehat{\mathscr{T}(\langle\overline{\mathsf{f}}\rangle)}}{\Longrightarrow}"] & & \mathscr{T}Y \arrow[ul, "\mathscr{T}(f)"'] \arrow[r, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[u, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longrightarrow}"] \arrow[d, phantom, "\overset{\overline{\mathsf{y}}}{\Longrightarrow}"] & Y \arrow[ul, "f"'] \\
& Y \arrow[r, "f"'] & Z & & & & \mathscr{T}^2Y \arrow[ull, "\mathscr{T}^2(f)"] \arrow[ur, "\mathscr{T}(\mathfrak{alg}_{\mathsf{y}})"] \arrow[r, "m_Y"'] & \mathscr{T}Y \arrow[ur, "\mathfrak{alg}_{\mathsf{y}}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[d, "\eta_Y"'] \arrow[dr, phantom, "\overset{\eta_f^{-1}}{\Longleftarrow}"] & & Z \arrow[dl, "\eta_Z"'] \arrow[dd, equals] \\
\mathscr{T}Y \arrow[r, "\mathscr{T}(f)"'] \arrow[d, "\mathfrak{alg}_{\mathsf{y}}"'] \arrow[dr, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longleftarrow}"] & \mathscr{T}Z \arrow[dr, "\mathfrak{alg}_{\mathsf{z}}"] \arrow[r, phantom, "\overset{\overline{\mathsf{z}}_0}{\Longleftarrow}"] & \\
Y \arrow[rr, "f"'] & & Z
\end{tikzcd}
=
\begin{tikzcd}
& Y \arrow[dl, "\eta_Y"'] \arrow[dd, equals] & \\
\mathscr{T}Y \arrow[dr, "\mathfrak{alg}_{\mathsf{y}}"'] \arrow[r, phantom, "\overset{\overline{\mathsf{y}}_0}{\Longleftarrow}"] & & \\
& Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

hold. Recall that a lax $\mathscr{T}$-morphism $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$ is called a $\mathscr{T}$-pseudomorphism if $\langle\overline{\mathsf{f}}\rangle$ is an invertible 2-cell. Moreover, if $\langle\overline{\mathsf{f}}\rangle$ is an identity, $\mathtt{f}$ is called a (strict) $\mathscr{T}$-morphism.

3. 2-cells: a *$\mathscr{T}$-transformation* $\mathfrak{m} : \mathtt{f} \Rightarrow \mathtt{h}$ between lax $\mathscr{T}$-morphisms $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$, $\mathtt{h} = (h, \langle\overline{\mathsf{h}}\rangle)$ is a 2-cell $\mathfrak{m} : f \Rightarrow h$ in $\mathfrak{B}$ such that the equation below holds.

$$\begin{tikzcd}
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[d, bend right=60, "\mathscr{T}(f)"'] \arrow[d, bend left=60, "\mathscr{T}(h)"] \arrow[d, phantom, "\overset{\mathscr{T}(\mathfrak{m})}{\Longrightarrow}"] \arrow[dr, phantom, "\overset{\langle\overline{\mathsf{h}}\rangle}{\Longrightarrow}"] & Y \arrow[d, "h"] \\
\mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{T}Y \arrow[r, "\mathfrak{alg}_{\mathsf{y}}"] \arrow[d, "\mathscr{T}(f)"'] \arrow[dr, phantom, "\overset{\langle\overline{\mathsf{f}}\rangle}{\Longrightarrow}"] & Y \arrow[d, bend right=60, "f"'] \arrow[d, bend left=60, "h"] \arrow[d, phantom, "\overset{\mathfrak{m}}{\Rightarrow}"] \\
\mathscr{T}Z \arrow[r, "\mathfrak{alg}_{\mathsf{z}}"'] & Z
\end{tikzcd}$$

The compositions are defined in the obvious way and these definitions make Lax-$\mathscr{T}$-Alg_ℓ a 2-category. The full sub-2-category of the pseudoalgebras of Lax-$\mathscr{T}$-Alg_ℓ is denoted by Ps-$\mathscr{T}$-Alg_ℓ. Also, the locally full sub-2-category Lax-$\mathscr{T}$-Alg_ℓ consisting of lax algebras and pseudomorphisms between them is denoted by Lax-$\mathscr{T}$-Alg. Finally, the full sub-2-category of the pseudoalgebras of Lax-$\mathscr{T}$-Alg is denoted by Ps-$\mathscr{T}$-Alg. In short, we have locally full inclusions:

$$\begin{tikzcd}
\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[r] \arrow[d] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \arrow[d] \\
\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[r, "\ell" description] & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell
\end{tikzcd}$$

Remark 5.4.2. If $\mathscr{T} = (\mathscr{T}, m, \eta)$ is a 2-monad, we denote by $\mathscr{T}$-Alg_ℓ the full sub-2-category of *strict algebras* of Lax-$\mathscr{T}$-Alg_ℓ. That is to say, the objects of $\mathscr{T}$-Alg_ℓ are the lax $\mathscr{T}$-algebras $\mathrm{y} = (Y, \mathfrak{alg}_{\mathsf{y}}, \underline{\overline{\mathrm{y}}}, \underline{\overline{\mathrm{y}}}_0)$ such that its 2-cells $\underline{\overline{\mathrm{y}}}, \underline{\overline{\mathrm{y}}}_0$ are identities.

Also, we denote by $\mathscr{T}$-Alg the locally full sub-2-category of $\mathscr{T}$-Alg_ℓ consisting of strict algebras and pseudomorphisms between them. Finally, $\mathscr{T}$-$\mathsf{Alg}_{\mathsf{s}}$ is the locally full sub-2-category $\mathscr{T}$-Alg_ℓ consisting of strict algebras and *strict morphisms* between them. That is to say, the 1-cells of $\mathscr{T}$-$\mathsf{Alg}_{\mathsf{s}}$ are the pseudomorphisms $\mathtt{f} = (f, \langle\overline{\mathsf{f}}\rangle)$ such that $\langle\overline{\mathsf{f}}\rangle$ is the identity. In this case, we have locally full inclusions

$$\begin{tikzcd}
\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[r] \arrow[d] & \mathscr{T}\text{-}\mathsf{Alg} \arrow[r] \arrow[d] & \mathscr{T}\text{-}\mathsf{Alg}_{\ell} \arrow[d] \\
\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[r] \arrow[d] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[r] \arrow[d] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell} \arrow[d] \\
\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \arrow[r] & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[r, "\ell"] & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell}
\end{tikzcd}$$

in which the vertical arrows are full.

Remark 5.4.3. There is a vast literature of examples of pseudomonads, 2-monads and their respective algebras, pseudoalgebras and lax algebras [9, 46, 65, 95]. The reader can keep in mind three very simple examples:

- The "free 2-monad" $\mathscr{T}$ on Cat whose pseudoalgebras are unbiased monoidal categories. This is defined by $\mathscr{T}X := \coprod_{n=0}^{\infty} X^n$, in which $X^{n+1} := X^n \times X$ and $X^0 := I$ is the terminal category, with the obvious pseudomonad structure. In this case, the $\mathscr{T}$-pseudomorphisms are the so called strong monoidal functors, while the lax $\mathscr{T}$-morphisms are the lax monoidal functors [75].

- The most simple example is the pseudomonad rising from a monoidal category. A monoidal category M is just a pseudomonoid [25] of Cat and, therefore, it gives rise to a pseudomonad $\mathscr{T} : \mathsf{Cat} \to \mathsf{Cat}$ defined by $\mathscr{T}X = M \times X$ with obvious unit and multiplication (and invertible modifications) coming from the monoidal structure of M. The pseudoalgebras and lax algebras of this pseudomonad are called, respectively, the pseudoactions and lax actions of M. Lax actions of a monoidal category M are also called *graded monads* (see [37]).

  The inclusion $\mathsf{Set} \to \mathsf{Cat}$ is a strong monoidal functor w.r.t. the cartesian structures, since this functor preserves products. In particular, it takes monoids of Set to monoids of Cat. In short, this means that we can see a monoid M as a (discrete) strict monoidal category. Therefore, a monoid M gives rise to a 2-monad $\mathscr{T}X = M \times X$ as defined above. In this case, the 2-categories $\mathscr{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$, $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ and $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_{\ell}$ are, respectively, the 2-categories of (strict) actions, pseudoactions (as defined in [26]) and lax actions of this monoid M on categories. A lax action of the trivial monoid on a category is the same as a monad.

- Let $\mathfrak{S}$ be a small 2-category and $\mathfrak{A}$ a 2-category. We denote by $\mathfrak{S}_0$ the discrete 2-category of the objects of $\mathfrak{S}$ and by $[\mathfrak{S},\mathfrak{A}]$ the 2-category of 2-functors, 2-natural transformations and modifications. If the restriction $[\mathfrak{S},\mathfrak{A}] \to [\mathfrak{S}_0,\mathfrak{A}]$ has a left 2-adjoint (called the global left Kan extension), then the restriction is 2-monadic and $[\mathfrak{S},\mathfrak{A}]_{PS}$ is the 2-category of $\mathscr{T}$-pseudoalgebras (in which $\mathscr{T}$ is the 2-monad induced by the 2-adjunction). Also, the 2-category of lax algebras is the 2-category $[\mathfrak{S},\mathsf{Cat}]_{Lax}$ of lax functors $\mathfrak{S} \to \mathfrak{A}$, lax natural transformations and modifications [9].

  Again, if M is a monoid (of Set), M can be seen as a category with only one object [76], usually denoted by $\sum M$. That is to say, the locally discrete 2-category $\sum M$ has only one object $*$ and

$\sum M(*,*) := M$ is the discrete category with the composition of 1-cells given by the product of the monoid. In this case, the restriction

$$\left[\sum M, \mathsf{Cat}\right] \to \left[\left(\sum M\right)_0, \mathsf{Cat}\right] \cong \mathsf{Cat}$$

has a left 2-adjoint (and, as explained, it is 2-monadic). The left 2-adjoint is given by

$$X \mapsto \mathscr{L}an_{(\Sigma M)_0 \to \Sigma M} X$$

in which

$$\begin{aligned} \mathscr{L}an_{(\Sigma M)_0 \to \Sigma M} X : \sum M &\to \mathsf{Cat} \\ * &\to M \times X \\ M \ni g &\mapsto \overline{g} : (h,x) \mapsto (gh,x). \end{aligned}$$

This 2-adjunction is precisely the same 2-adjunction between strict $\mathscr{T}$-algebras and the base 2-category Cat, if $\mathscr{T}$ is the 2-monad $\mathscr{T}X = M \times X$ described above. Hence the 2-category of pseudoalgebras $[\sum M, \mathsf{Cat}]_{PS}$ and the 2-category $[\sum M, \mathsf{Cat}]_{Lax}$ are, respectively, isomorphic to the 2-category of pseudoactions and the 2-category of lax actions of M on categories. Moreover, $\mathscr{T}\text{-}\mathsf{Alg}_{\mathrm{s}} \to \mathsf{Cat}$ is 2-comonadic.

More generally, if M is a monoidal category, M can be seen as a bicategory with only one object (see [5, 75]), also denoted by $\sum M$. The restriction 2-functor $[\sum M, \mathsf{Cat}]_{PS} \to [(\sum M)_0, \mathsf{Cat}]_{PS} \cong \mathsf{Cat}$ is pseudomonadic and pseudocomonadic. Furthermore, it coincides with the forgetful pseudofunctor $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Cat}$ in which $\mathscr{T}X = M \times X$ is given by the structure of the monoidal category (as above).

Remark 5.4.4. Let $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ be a pseudomonad on a 2-category $\mathfrak{B}$. If $\mathfrak{C}$ is any sub-2-category of $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$, we have a forgetful 2-functor

$$\begin{aligned} U : \mathfrak{C} &\to \mathfrak{B} \\ \mathtt{z} = (Z, \mathfrak{alg}_{\mathtt{z}}, \overline{\underline{\mathtt{z}}}, \overline{\underline{\mathtt{z}}}_0) &\mapsto Z \\ \mathtt{f} = (f, \langle \overline{\mathsf{f}} \rangle) &\mapsto f \\ \mathfrak{m} &\mapsto \mathfrak{m} \end{aligned}$$

Proposition 5.4.5. *Let $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ be a pseudomonad on a 2-category $\mathfrak{B}$. Given lax $\mathscr{T}$-algebras $\mathtt{y} = (Y, \mathfrak{alg}_{\mathtt{y}}, \overline{\underline{\mathtt{y}}}, \overline{\underline{\mathtt{y}}}_0)$, $\mathtt{z} = (Z, \mathfrak{alg}_{\mathtt{z}}, \overline{\underline{\mathtt{z}}}, \overline{\underline{\mathtt{z}}}_0)$ the category $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell(\mathtt{y}, \mathtt{z})$ is the strict lax descent object of the diagram $\mathbb{T}^{\mathtt{y}}_{\mathtt{z}} : \Delta_\ell \to \mathsf{Cat}$*

$$\begin{tikzcd}[column sep=huge] \mathfrak{B}(U\mathtt{y}, U\mathtt{z}) \arrow[r, shift left=3, "\mathfrak{B}(\mathscr{T}U\mathtt{y}{,}\mathfrak{alg}_{\mathtt{z}}) \circ \mathscr{T}_{(U\mathtt{y}{,}U\mathtt{z})}"] \arrow[r, shift right=3, "\mathfrak{B}(\mathfrak{alg}_{\mathtt{y}}{,}U\mathtt{z})"'] & \mathfrak{B}(\mathscr{T}U\mathtt{y}, U\mathtt{z}) \arrow[l, "\mathfrak{B}(\eta_{U\mathtt{y}}{,}U\mathtt{z})" description] \arrow[r, shift left=3, "\mathfrak{B}(\mathscr{T}^2U\mathtt{y}{,}\mathfrak{alg}_{\mathtt{z}}) \circ \mathscr{T}_{(\mathscr{T}U\mathtt{y}{,}U\mathtt{z})}"] \arrow[r, "\mathfrak{B}(m_{U\mathtt{y}}{,}U\mathtt{z})" description] \arrow[r, shift right=3, "\mathfrak{B}(\mathscr{T}(\mathfrak{alg}_{\mathtt{y}}){,}U\mathtt{z})"'] & \mathfrak{B}(\mathscr{T}^2U\mathtt{y}, U\mathtt{z}) \end{tikzcd} \qquad (\mathbb{T}^{\mathtt{y}}_{\mathtt{z}})$$

such that

$$\begin{aligned}
\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(\sigma_{20})_f &:= \left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * \mathsf{t}_{(f)(\mathfrak{alg}_{\mathsf{y}})}\right) & \mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(\sigma_{00})_f &:= \left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * m_f^{-1}\right)\cdot\left(\overline{\underline{\mathsf{z}}} * \mathrm{id}_{\mathcal{T}^2(f)}\right)\cdot\left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * \mathsf{t}^{-1}_{(\mathfrak{alg}_{\mathsf{z}})(\mathcal{T}(f))}\right)\\
\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(\sigma_{21})_f &:= \left(\mathrm{id}_f * \overline{\underline{\mathsf{y}}}\right) & \mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(n_0)_f &:= \left(\mathrm{id}_{\mathfrak{alg}_{\mathsf{z}}} * \eta_f^{-1}\right)\cdot\left(\overline{\underline{\mathsf{z}}}_0 * \mathrm{id}_f\right)\\
\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}(n_1)_f &:= \left(\mathrm{id}_f * \overline{\underline{\mathsf{y}}}_0\right) & &
\end{aligned}$$

Furthermore, the strict descent object of $\mathbb{T}^{\mathsf{y}}_{\mathsf{z}}$ *is* $\mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}(\mathsf{y},\mathsf{z})$.

Proof. It follows from Definition 5.4.1 and Remark 5.3.4. ☐

Remark 5.4.6. In the context of the proposition above, we can define a pseudofunctor $\mathbb{T}^{\mathsf{y}} : \Delta_\ell \times \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \to \mathsf{Cat}$ in which $\mathbb{T}^{\mathsf{y}}(-,\mathsf{z}) := \mathbb{T}^{\mathsf{y}}_{\mathsf{z}}$, since the morphisms defined above are actually pseudonatural in z w.r.t. $\mathcal{T}$-pseudomorphisms and $\mathcal{T}$-transformations.

Assume that the triangles below are commutative, R is a right biadjoint pseudofunctor and the arrows without labels are the forgetful 2-functors of Remark 5.4.4. By Corollary 5.2.4, it follows from Proposition 5.4.5 (and last remark) that, whenever $\mathfrak{A}$ has lax codescent objects, $\ell \circ J$ is right biadjoint to a pseudofunctor G. Also, for each lax algebra y, there is a diagram $\mathscr{A}_{\mathsf{y}}$ such that $G\mathsf{y} \simeq \dot{\Delta}_\ell(0,\mathsf{t}-) *_{\mathrm{bi}} \mathscr{A}_{\mathsf{y}}$ defines the left biadjoint to $\ell \circ J$. Moreover, J is right biadjoint as well if $\mathfrak{A}$ has codescent objects of these diagrams $\mathscr{A}_{\mathsf{y}}$. Next section, we give precisely the diagrams $\mathscr{A}_{\mathsf{y}}$ and prove a strict version of our theorem as a consequence of Lemma 5.2.2.

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "J"] \arrow[dr, "R"'] & \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \arrow[r, "\ell"] \arrow[d] & \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg}_\ell \arrow[dl] \\
 & \mathfrak{B} &
\end{tikzcd}$$

5.5 Lifting of biadjoints to lax algebras

In this section, we give our results on lifting right biadjoints to the 2-category of lax algebras of a given pseudomonad. As explained above, we already have such results by Corollary 5.2.4 and Proposition 5.4.5. But, in this section, we present an explicit calculation of the diagrams $\mathscr{A}_{\mathsf{y}}$ whose lax codescent objects are needed in the construction of our left biadjoint.

Definition 5.5.1. Let $(E \dashv R, \rho, \varepsilon, v, w)$ be a biadjunction and $\mathcal{T} = (\mathcal{T}, m, \eta, \mu, \iota, \tau)$ a pseudomonad on $\mathfrak{B}$ such that

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "R"'] & & \mathsf{Lax}\text{-}\mathcal{T}\text{-}\mathsf{Alg} \arrow[dl, "U"] \\
 & \mathfrak{B} &
\end{tikzcd}$$

is commutative, in which U is the forgetful 2-functor defined in Remark 5.4.4. In this setting, for each lax $\mathscr{T}$-algebra $\mathrm{y} = (Y, \mathfrak{alg}_{\mathrm{y}}, \underline{\overline{\mathrm{y}}}, \underline{\overline{\mathrm{y}}}_0)$, we define the 2-functor $\mathscr{A}_{\mathrm{y}} : \Delta_\ell^{\mathrm{op}} \to \mathfrak{A}$

$$\begin{tikzcd}[column sep=huge]
EU\mathrm{y} \arrow[r, "E(\eta_{U\mathrm{y}})" description] \arrow[r, leftarrow, shift left=4, "\varepsilon_{EU\mathrm{y}} E(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}}))"] \arrow[r, leftarrow, shift right=4, "E(\mathfrak{alg}_{\mathrm{y}})"'] & E\mathscr{T}U\mathrm{y} \arrow[r, leftarrow, "E(m_{U\mathrm{y}})" description] \arrow[r, leftarrow, shift left=4, "\varepsilon_{E\mathscr{T}U\mathrm{y}} E(\mathfrak{alg}_{JE\mathscr{T}U\mathrm{y}} \mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}}))"] \arrow[r, leftarrow, shift right=4, "E\mathscr{T}(\mathfrak{alg}_{\mathrm{y}})"'] & E\mathscr{T}^2U\mathrm{y}
\end{tikzcd} \qquad (\mathscr{A}_{\mathrm{y}})$$

in which

$$\mathscr{A}_{\mathrm{y}}(\sigma_{21}) := \mathfrak{e}^{-1}_{(\mathfrak{alg}_{\mathrm{y}})(m_{U\mathrm{y}})} \cdot E(\underline{\overline{\mathrm{y}}}) \cdot \mathfrak{e}_{(\mathfrak{alg}_{\mathrm{y}})(\mathscr{T}(\mathfrak{alg}_{\mathrm{y}}))} \qquad \mathscr{A}_{\mathrm{y}}(n_0) := \mathfrak{e}^{-1}_{(\mathfrak{alg}_{\mathrm{y}})(\eta_{U\mathrm{y}})} \cdot E(\underline{\overline{\mathrm{y}}}_0) \cdot \mathfrak{e}_{U\mathrm{y}}$$

$$\begin{aligned}
\mathscr{A}_{\mathrm{y}}(n_1) := {} & \left(\left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} \right) * \left(\mathfrak{e}^{-1}_{(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}}))(\eta_{U\mathrm{y}})} \cdot E(\mathrm{id}_{\mathfrak{alg}_{JEU\mathrm{y}}} * \eta^{-1}_{\rho_{U\mathrm{y}}}) \cdot E(\underline{\overline{JEU\mathrm{y}}}_0 * \mathrm{id}_{\rho_{U\mathrm{y}}}) \right) \right) \cdot v_{U\mathrm{y}} \\
\mathscr{A}_{\mathrm{y}}(\sigma_{20}) := {} & \left(\varepsilon^{-1}_{E(\mathfrak{alg}_{\mathrm{y}})} * \mathrm{id}_{E(\mathfrak{alg}_{JE\mathscr{T}\mathrm{y}} \mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}}))} \right) \cdot \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * \mathfrak{e}^{-1}_{(RE(\mathfrak{alg}_{\mathrm{y}}))(\mathfrak{alg}_{JE\mathscr{T}U\mathrm{y}} \mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}}))} \right) \cdot \\
& \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * \left(E(\left\langle \overline{JE(\mathfrak{alg}_{\mathrm{y}})} \right\rangle * \mathrm{id}_{\mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}})}) \cdot E(\mathrm{id}_{\mathfrak{alg}_{JEU\mathrm{y}}} * (\mathscr{T}\rho)^{-1}_{\mathfrak{alg}_{\mathrm{y}}}) \right) \right) \cdot \\
& \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * \left(\mathfrak{e}_{(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}}))(\mathscr{T}(\mathfrak{alg}_{\mathrm{y}}))} \right) \right)
\end{aligned}$$

$$\begin{aligned}
\mathscr{A}_{\mathrm{y}}(\sigma_{00}) := {} & \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * \left(\mathfrak{e}^{-1}_{(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}}))(m_{U\mathrm{y}})} \cdot E(\mathrm{id}_{\mathfrak{alg}_{JEU\mathrm{y}}} * m^{-1}_{\rho_{U\mathrm{y}}}) \cdot E(\underline{\overline{JEU\mathrm{y}}} * \mathrm{id}_{\mathscr{T}^2(\rho_{U\mathrm{y}})}) \right) \right) \cdot \\
& \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * E\left(\mathrm{id}_{\mathfrak{alg}_{JE\mathscr{T}U\mathrm{y}}} * \left(\mathscr{T}(w_{EU\mathrm{y}}) \cdot \mathfrak{t}^{-1}_{(R(\varepsilon_{EU\mathrm{y}}))(\rho_{REU\mathrm{y}})} \right) * \mathrm{id}_{\mathscr{T}(\mathfrak{alg}_{JEU\mathrm{y}})\mathscr{T}^2(\rho_{U\mathrm{y}})} \right) \right) \cdot \\
& \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * E\left(\left\langle \overline{J(\varepsilon_{EU\mathrm{y}})} \right\rangle^{-1} * \mathrm{id}_{\mathscr{T}(\rho_{REU\mathrm{y}})} * \mathfrak{t}^{-1}_{(\mathfrak{alg}_{JEU\mathrm{y}})(\mathscr{T}(\rho_{U\mathrm{y}}))} \right) \right) \cdot \\
& \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * \mathfrak{e}_{(R(\varepsilon_{EU\mathrm{y}}))(\mathfrak{alg}_{JEU\mathscr{T}U\mathrm{y}} \mathscr{T}(\rho_{REU\mathrm{y}})\mathscr{T}(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}})))} \right) \cdot \\
& \left(\varepsilon_{\varepsilon_{EU\mathrm{y}}} * E\left(\left(\mathrm{id}_{\mathfrak{alg}_{JEU\mathscr{T}U\mathrm{y}}} * (\mathscr{T}\rho)_{\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}})} \right) \cdot \left(\left\langle \overline{JE(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}}))} \right\rangle^{-1} \right) \right) \right) \cdot \\
& \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}} \varepsilon_{EREU\mathrm{y}}} * \mathfrak{e}_{(RE(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}})))(\mathfrak{alg}_{JE\mathscr{T}U\mathrm{y}} \mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}}))} \right) \cdot \\
& \left(\mathrm{id}_{\varepsilon_{EU\mathrm{y}}} * \varepsilon_{E(\mathfrak{alg}_{JEU\mathrm{y}} \mathscr{T}(\rho_{U\mathrm{y}}))} * \mathrm{id}_{E(\mathfrak{alg}_{JE\mathscr{T}U\mathrm{y}} \mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}}))} \right)
\end{aligned}$$

Theorem 5.5.2 (Biadjoint Triangle Theorem)**.** *Let* $(E \dashv R, \rho, \varepsilon, v, w)$ *be a biadjunction,* $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ *a pseudomonad on* $\mathfrak{B}$ *and* $\ell : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$ *the inclusion. Assume that*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "R"'] & & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[dl, "U"] \\
& \mathfrak{B} &
\end{tikzcd}$$

is commutative. The pseudofunctor $\ell \circ J$ *is right biadjoint if and only if* $\mathfrak{A}$ *has the lax codescent object of the diagram* $\mathscr{A}_{\mathrm{y}} : \Delta_\ell^{\mathrm{op}} \to \mathfrak{A}$ *for every lax* $\mathscr{T}$*-algebra* y*. In this case, the left biadjoint* G *is defined by* $G\mathrm{y} = \dot{\Delta}_\ell(0, \mathsf{t}-) *_{\mathrm{bi}} \mathscr{A}_{\mathrm{y}}$

Furthermore, J is right biadjoint if and only if $\mathfrak{A}$ has the codescent object of the diagram $\mathscr{A}_{\mathrm{y}} : \Delta_{\ell}^{\mathrm{op}} \to \mathfrak{A}$ for every lax $\mathscr{T}$-algebra y*. In this case, the left biadjoint G' is defined by $G'\mathrm{y} = \dot{\Delta}(0,\mathsf{j}-) \ast_{\mathsf{bi}} \mathscr{A}_{\mathrm{y}}$*

Proof. By Lemma 5.2.1, Proposition 5.4.5 and Remark 5.4.6, it is enough to observe that, for each lax $\mathscr{T}$-algebra y, there is a pseudonatural equivalence

$$\psi^{\mathrm{y}} : \mathbb{T}^{\mathrm{y}}(-,J-) \longrightarrow \mathfrak{A}(\mathscr{A}_{\mathrm{y}}-,-)$$

defined by

$$\begin{aligned}
\psi^{\mathrm{y}}_{(1,A)} &:= \chi_{(U\mathrm{y},A)} : \mathfrak{B}(U\mathrm{y},RA) \to \mathfrak{A}(EU\mathrm{y},A)\\
\psi^{\mathrm{y}}_{(2,A)} &:= \chi_{(\mathscr{T}U\mathrm{y},A)} : \mathfrak{B}(\mathscr{T}U\mathrm{y},RA) \to \mathfrak{A}(E\mathscr{T}U\mathrm{y},A)\\
\psi^{\mathrm{y}}_{(3,A)} &:= \chi_{(\mathscr{T}^2U\mathrm{y},A)} : \mathfrak{B}(\mathscr{T}^2U\mathrm{y},RA) \to \mathfrak{A}(E\mathscr{T}^2U\mathrm{y},A)
\end{aligned}$$

in which $\chi : \mathfrak{B}(-,R-) \simeq \mathfrak{A}(E-,-)$ is the pseudonatural equivalence corresponding to the biadjunction $(E \dashv R, \rho, \varepsilon, v, w)$ (see Remark 5.1.4). Also,

$$\begin{aligned}
(\psi^{\mathrm{y}}_{s^0})_f &:= \mathrm{id}_{\varepsilon_A} \ast \mathfrak{e}_{(f)(\eta_{U\mathrm{y}})} & \qquad (\psi^{\mathrm{y}}_{\partial^1})_f &:= \mathrm{id}_{\varepsilon_A} \ast \mathfrak{e}_{(f)(m_{U\mathrm{y}})}\\
(\psi^{\mathrm{y}}_{d^1})_f &:= \mathrm{id}_{\varepsilon_A} \ast \mathfrak{e}_{(f)(\mathfrak{alg}_{\mathrm{y}})} & \qquad (\psi^{\mathrm{y}}_{\partial^2})_f &:= \mathrm{id}_{\varepsilon_A} \ast \mathfrak{e}_{(f)(\mathscr{T}(\mathfrak{alg}_{\mathrm{y}}))}
\end{aligned}$$

$$\begin{aligned}
(\psi^{\mathrm{y}}_{d^0})_f := {} & \left(\mathrm{id}_{\varepsilon_A} \ast \left(E(\mathrm{id}_{\mathfrak{alg}_{JA}} \ast \mathscr{T}(w_A) \ast \mathrm{id}_{\mathscr{T}(f)}) \cdot E(\mathrm{id}_{\mathfrak{alg}_{JA}} \ast \mathfrak{t}_{(R(\varepsilon_A))(\rho_{RA})} \ast \mathrm{id}_{\mathscr{T}(f)})\right)\right) \cdot\\
& \left(\mathrm{id}_{\varepsilon_A} \ast \left(E(\left\langle\overline{J(\varepsilon_A)}\right\rangle^{-1} \ast \mathrm{id}_{\mathscr{T}(\rho_{RA})\mathscr{T}(f)}) \cdot \mathfrak{e}_{(R(\varepsilon_A))(\mathfrak{alg}_{JERA}\mathscr{T}(\rho_{RA})\mathscr{T}(f))}\right)\right) \cdot\\
& \left(\mathrm{id}_{\varepsilon_A ER(\varepsilon_A)} \ast \left(E(\mathrm{id}_{\mathfrak{alg}_{JERA}} \ast (\mathscr{T}\rho)_f) \cdot E(\left\langle\overline{JE(f)}\right\rangle^{-1} \ast \mathrm{id}_{\mathscr{T}(\rho_{U\mathrm{y}})})\right)\right) \cdot\\
& \left(\varepsilon_{\varepsilon_A} \ast \mathfrak{e}_{(RE(f))(\mathfrak{alg}_{JEU\mathrm{y}}\mathscr{T}(\rho_{U\mathrm{y}}))}\right) \cdot \left(\mathrm{id}_{\varepsilon_A} \ast \varepsilon_f \ast \mathrm{id}_{E(\mathfrak{alg}_{JEU\mathrm{y}}\mathscr{T}(\rho_{U\mathrm{y}}))}\right)\\
(\psi^{\mathrm{y}}_{\partial^0})_f := {} & \left(\mathrm{id}_{\varepsilon_A} \ast \left(E(\mathrm{id}_{\mathfrak{alg}_{JA}} \ast \mathscr{T}(w_A) \ast \mathrm{id}_{\mathscr{T}(f)}) \cdot E(\mathrm{id}_{\mathfrak{alg}_{JA}} \ast \mathfrak{t}_{(R(\varepsilon_A))(\rho_{RA})} \ast \mathrm{id}_{\mathscr{T}(f)})\right)\right) \cdot\\
& \left(\mathrm{id}_{\varepsilon_A} \ast \left(E(\left\langle\overline{J(\varepsilon_A)}\right\rangle^{-1} \ast \mathrm{id}_{\mathscr{T}(\rho_{RA})\mathscr{T}(f)}) \cdot \mathfrak{e}_{(R(\varepsilon_A))(\mathfrak{alg}_{JERA}\mathscr{T}(\rho_{RA})\mathscr{T}(f))}\right)\right) \cdot\\
& \left(\mathrm{id}_{\varepsilon_A ER(\varepsilon_A)} \ast \left(E(\mathrm{id}_{\mathfrak{alg}_{JERA}} \ast (\mathscr{T}\rho)_f) \cdot E(\left\langle\overline{JE(f)}\right\rangle^{-1} \ast \mathrm{id}_{\mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}})})\right)\right) \cdot\\
& \left(\varepsilon_{\varepsilon_A} \ast \mathfrak{e}_{(RE(f))(\mathfrak{alg}_{JE\mathscr{T}U\mathrm{y}}\mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}}))}\right) \cdot \left(\mathrm{id}_{\varepsilon_A} \ast \varepsilon_f \ast \mathrm{id}_{E(\mathfrak{alg}_{JE\mathscr{T}U\mathrm{y}}\mathscr{T}(\rho_{\mathscr{T}U\mathrm{y}}))}\right)
\end{aligned}$$

This defines a pseudonatural transformation which is a a pseudonatural equivalence, since it is objectwise an equivalence. □

Theorem 5.5.3 (Strict Biadjoint Triangle)**.** *Let* $(E \dashv R, \rho, \varepsilon)$ *be a* 2*-adjunction,* $(\mathscr{T}, m, \eta)$ *a* 2*-monad on* $\mathfrak{B}$ *and* $\ell : \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$ *the inclusion. Assume that*

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "J"] \arrow[dr, "R"'] & & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[dl, "U"] \\
& \mathfrak{B} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A} \arrow[rr, "\tilde{J}"] \arrow[dr, "J"'] & & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_s \arrow[dl] \\
& \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg} &
\end{tikzcd}$$

are commutative triangles, in which $\mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_s \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ *is the locally full inclusion of the* 2*-category of lax algebras and strict* $\mathscr{T}$*-morphisms into the* 2*-category of lax algebras and* $\mathscr{T}$*-pseudomorphisms. The pseudofunctor* $\ell \circ J$ *is right biadjoint if and only if* $\mathfrak{A}$ *has the strict lax codescent object of the diagram* $\mathscr{A}_{\mathtt{y}} : \Delta_\ell^{\mathrm{op}} \to \mathfrak{A}$ *for every lax* $\mathscr{T}$*-algebra* $\mathtt{y}$*. In this case, the left* 2*-adjoint G is defined by* $G\mathtt{y} = \dot{\Delta}_\ell(0, \mathsf{t}-) \ast \mathscr{A}_{\mathtt{y}}$

Furthermore, J is right 2*-adjoint if and only if* $\mathfrak{A}$ *has the strict codescent object of the diagram* $\mathscr{A}_{\mathtt{y}} : \Delta_\ell^{\mathrm{op}} \to \mathfrak{A}$ *for every lax* $\mathscr{T}$*-algebra* $\mathtt{y}$*.*

Proof. We have, in particular, the setting of Theorem 5.5.2. Therefore, we can define ψ as it is done in the last proof. However, in our setting, we get a 2-natural transformation which is an objectwise isomorphism. Therefore ψ is a 2-natural isomorphism.

By Lemma 5.2.2, Proposition 5.4.5 and Remark 5.4.6, this completes our proof. □

5.6 Coherence

As mentioned in the introduction, *the* 2*-monadic approach to coherence* consists of studying the inclusions induced by a 2-monad $\mathscr{T}$ of Remark 5.4.2 to get general coherence results [9, 67, 93].

Given a 2-monad $(\mathscr{T}, m, \eta)$ on a 2-category $\mathfrak{B}$, the inclusions of Remark 5.4.2 and the forgetful functors of Remark 5.4.4 give in particular the commutative diagram below, in which $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathfrak{B}$ is right biadjoint and $\mathscr{T}\text{-}\mathsf{Alg}_{\mathrm{s}} \to \mathfrak{B}$ is right 2-adjoint.

$$\begin{tikzcd}
\mathscr{T}\text{-}\mathsf{Alg}_{\mathrm{s}} \arrow[r] \arrow[dr] & \mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \arrow[r] \arrow[d] & \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell \arrow[dl] \\
& \mathfrak{B} &
\end{tikzcd}$$

In this section, we are mainly concerned with the triangles involving the 2-category of lax algebras. We refer to [77] for the remaining triangles involving the 2-category of pseudoalgebras. The inclusion $\mathscr{T}\text{-}\mathsf{Alg}_{\mathrm{s}} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$ is also studied in [67]. Therein, it is proved that it has a left 2-adjoint whenever the 2-category $\mathscr{T}\text{-}\mathsf{Alg}_{\mathrm{s}}$ has the lax codescent objects of some diagrams called therein *lax coherence data*. This is of course the immediate consequence of Theorem 5.5.3 applied to the large triangle above.

Actually, we can study other inclusions of Remark 5.4.2 with the techniques of this paper. For instance, by Theorem 5.5.3 and Corollary 5.2.5, the inclusion of $\mathscr{T}\text{-}\mathsf{Alg}$ into any 2-category of $\mathscr{T}$-algebras and lax $\mathscr{T}$-morphisms of Remark 5.4.2 has a left biadjoint provided that $\mathscr{T}\text{-}\mathsf{Alg}$ has lax codescent objects. Also, the inclusion of this 2-category into any 2-category of $\mathscr{T}$-algebras and

$\mathscr{T}$-pseudomorphisms (*i.e.* vertical arrows with domain in $\mathscr{T}$-Alg of Remark 5.4.2) has a left biadjoint provided that $\mathscr{T}$-Alg has codescent objects.

In the more general context of pseudomonads, we can apply Theorem 5.5.2 and Theorem 5.5.3 to understand precisely when the inclusions $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}_\ell$ and $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ have left biadjoints. In particular, we have:

Theorem 5.6.1. *Let* $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ *be a pseudomonad on a* 2*-category* $\mathfrak{B}$. *If* $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ *has lax codescent objects, then the inclusion* $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ *has a left biadjoint. Furthermore, if* $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ *has codescent objects,* $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg} \to \mathsf{Lax}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ *has a left biadjoint.*

In particular, if $\mathscr{T} = (\mathscr{T}, m, \eta, \mu, \iota, \tau)$ is a pseudomonad that preserves lax codescent objects, then $\mathsf{Ps}\text{-}\mathscr{T}\text{-}\mathsf{Alg}$ has lax codescent objects and, therefore, satisfies the hypothesis of the first part of the result above. Similarly, if $\mathscr{T}$ preserves codescent objects, it satisfies the hypothesis of the second part.

References

[1] Barr, M. (1972). The point of the empty set. *Cahiers Topologie Géom. Différentielle*, 13:357–368, 442.

[2] Barr, M. and Wells, C. (2005). Toposes, triples and theories. *Repr. Theory Appl. Categ.*, TAC(12):x+288. Corrected reprint of the 1985 original [MR0771116].

[3] Batanin, M. A. (1998). Computads for finitary monads on globular sets. In *Higher category theory (Evanston, IL, 1997)*, volume 230 of *Contemp. Math.*, pages 37–57. Amer. Math. Soc., Providence, RI.

[4] Beck, J. M. (2003). Triples, algebras and cohomology. *Repr. Theory Appl. Categ.*, TAC(2):1–59.

[5] Bénabou, J. (1967). Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin.

[6] Bénabou, J. and Roubaud, J. (1970). Monades et descente. *C. R. Acad. Sci. Paris Sér. A-B*, 270:A96–A98.

[7] Betti, R., Carboni, A., Street, R., and Walters, R. (1983). Variation through enrichment. *J. Pure Appl. Algebra*, 29(2):109–127.

[8] Bird, G. J., Kelly, G. M., Power, A. J., and Street, R. H. (1989). Flexible limits for 2-categories. *J. Pure Appl. Algebra*, 61(1):1–27.

[9] Blackwell, R., Kelly, G. M., and Power, A. J. (1989). Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41.

[10] Börger, R., Tholen, W., Wischnewsky, M. B., and Wolff, H. (1981). Compact and hypercomplete categories. *J. Pure Appl. Algebra*, 21(2):129–144.

[11] Bourke, J. (2014). Two-dimensional monadicity. *Adv. Math.*, 252:708–747.

[12] Brown, R. (2006). *Topology and groupoids*. BookSurge, LLC, Charleston, SC. Third edition of Elements of modern topology [McGraw-Hill, New York, 1968; MR0227979], With 1 CD-ROM (Windows, Macintosh and UNIX).

[13] Brown, R., Higgins, P. J., and Sivera, R. (2011). *Nonabelian algebraic topology*, volume 15 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich. Filtered spaces, crossed complexes, cubical homotopy groupoids, With contributions by Christopher D. Wensley and Sergei V. Soloviev.

[14] Burroni, A. (1971). T-catégories (catégories dans un triple). *Cahiers Topologie Géom. Différentielle*, 12:215–321.

[15] Burroni, A. (1993). Higher-dimensional word problems with applications to equational logic. *Theoret. Comput. Sci.*, 115(1):43–62. 4th Summer Conference on Category Theory and Computer Science (Paris, 1991).

[16] Chikhladze, D. (2015). Lax formal theory of monads, monoidal approach to bicategorical structures and generalized operads. *Theory Appl. Categ.*, 30:No. 10, 332–386.

[17] Clementino, M. M. and Hofmann, D. (2002). Triquotient maps via ultrafilter convergence. *Proc. Amer. Math. Soc.*, 130(11):3423–3431.

[18] Clementino, M. M. and Hofmann, D. (2004). Effective descent morphisms in categories of lax algebras. *Appl. Categ. Structures*, 12(5-6):413–425.

[19] Clementino, M. M. and Hofmann, D. (2006). Exponentiation in V-categories. *Topology Appl.*, 153(16):3113–3128.

[20] Clementino, M. M., Hofmann, D., and Stubbe, I. (2009). Exponentiable functors between quantaloid-enriched categories. *Appl. Categ. Structures*, 17(1):91–101.

[21] Clementino, M. M. and Janelidze, G. (2011). A note on effective descent morphisms of topological spaces and relational algebras. *Topology Appl.*, 158(17):2431–2436.

[22] Clementino, M. M. and Tholen, W. (2003). Metric, topology and multicategory—a common approach. *J. Pure Appl. Algebra*, 179(1-2):13–47.

[23] Conduché, F. (1972). Au sujet de l'existence d'adjoints à droite aux foncteurs "image réciproque" dans la catégorie des catégories. *C. R. Acad. Sci. Paris Sér. A-B*, 275:A891–A894.

[24] Cruttwell, G. S. H. and Shulman, M. A. (2010). A unified framework for generalized multicategories. *Theory Appl. Categ.*, 24:No. 21, 580–655.

[25] Day, B. and Street, R. (1997). Monoidal bicategories and Hopf algebroids. *Adv. Math.*, 129(1):99–157.

[26] Deligne, P. (1997). Action du groupe des tresses sur une catégorie. *Invent. Math.*, 128(1):159–175.

[27] Descotte, M. E., Dubuc, E. J., and Szyld, M. (2016). On the notion of flat 2-functors. *ArXiv e-prints*. arXiv:1610.09429.

[28] Dostál, M. (2017). Bénabou's theorem for pseudoadjunctions. *ArXiv e-prints*. arXiv:1707.04074.

[29] Dror Farjoun, E. (2004). Fundamental group of homotopy colimits. *Adv. Math.*, 182(1):1–27.

[30] Dubuc, E. (1968). Adjoint triangles. In *Reports of the Midwest Category Seminar, II*, pages 69–91. Springer, Berlin.

[31] Dubuc, E. J. (1970). *Kan extensions in enriched category theory*. Lecture Notes in Mathematics, Vol. 145. Springer-Verlag, Berlin-New York.

[32] Duskin, J. (1975). Simplicial methods and the interpretation of "triple" cohomology. *Mem. Amer. Math. Soc.*, 3(issue 2, 163):v+135.

[33] Dyckhoff, R. and Tholen, W. (1987). Exponentiable morphisms, partial products and pullback complements. *J. Pure Appl. Algebra*, 49(1-2):103–116.

[34] Ehresmann, C. (1965). *Catégories et structures*. Dunod, Paris.

[35] Eilenberg, S. and Kelly, G. M. (1966). Closed categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 421–562. Springer, New York.

[36] Eilenberg, S. and Moore, J. C. (1965). Adjoint functors and triples. *Illinois J. Math.*, 9:381–398.

[37] Fujii, S., Katsumata, S.-y., and Melliès, P.-A. (2016). Towards a formal theory of graded monads. In *Foundations of software science and computation structures*, volume 9634 of *Lecture Notes in Comput. Sci.*, pages 513–530. Springer, Berlin.

[38] Garner, R. and Gurski, N. (2009). The low-dimensional structures formed by tricategories. *Math. Proc. Cambridge Philos. Soc.*, 146(3):551–589.

[39] Giraud, J. (1964). Méthode de la descente. *Bull. Soc. Math. France Mém.*, 2:viii+150.

[40] Gordon, R., Power, A. J., and Street, R. (1995). Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81.

[41] Gray, J. W. (1974a). *Formal category theory: adjointness for* 2*-categories*. Lecture Notes in Mathematics, Vol. 391. Springer-Verlag, Berlin-New York.

[42] Gray, J. W. (1974b). Quasi-Kan extensions for 2-categories. *Bull. Amer. Math. Soc.*, 80:142–147.

[43] Grothendieck, A. (1995). Technique de descente et théorèmes d'existence en géometrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 190, 299–327. Soc. Math. France, Paris.

[44] Guiraud, Y. and Malbos, P. (2012). Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3-4):2294–2351.

[45] Hermida, C. (2000). Representable multicategories. *Adv. Math.*, 151(2):164–225.

[46] Hermida, C. (2001). From coherent structures to universal properties. *J. Pure Appl. Algebra*, 165(1):7–61.

[47] Hermida, C. (2004). Descent on 2-fibrations and strongly 2-regular 2-categories. *Appl. Categ. Structures*, 12(5-6):427–459.

[48] Huber, P. J. (1961). Homotopy theory in general categories. *Math. Ann.*, 144:361–385.

[49] Janelidze, G., Schumacher, D., and Street, R. (1993). Galois theory in variable categories. *Appl. Categ. Structures*, 1(1):103–110.

[50] Janelidze, G., Sobral, M., and Tholen, W. (2004). Beyond Barr exactness: effective descent morphisms. In *Categorical foundations*, volume 97 of *Encyclopedia Math. Appl.*, pages 359–405. Cambridge Univ. Press, Cambridge.

[51] Janelidze, G. and Tholen, W. (1994). Facets of descent. I. *Appl. Categ. Structures*, 2(3):245–281.

[52] Janelidze, G. and Tholen, W. (1997). Facets of descent. II. *Appl. Categ. Structures*, 5(3):229–248.

[53] Janelidze, G. and Tholen, W. (2004). Facets of descent. III. Monadic descent for rings and algebras. *Appl. Categ. Structures*, 12(5-6):461–477.

[54] Johnson, D. L. (1990). *Presentations of groups*, volume 15 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge.

[55] Johnstone, P. T. (1975). Adjoint lifting theorems for categories of algebras. *Bull. London Math. Soc.*, 7(3):294–297.

[56] Kampen, E. R. V. (1933). On Some Lemmas in the Theory of Groups. *Amer. J. Math.*, 55(1-4):268–273.

[57] Kelly, G. M. (1974). Doctrinal adjunction. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 257–280. Lecture Notes in Math., Vol. 420. Springer, Berlin.

[58] Kelly, G. M. (1982). *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York.

[59] Kelly, G. M. (1989). Elementary observations on 2-categorical limits. *Bull. Austral. Math. Soc.*, 39(2):301–317.

[60] Kelly, G. M. and Lack, S. (1997). On property-like structures. *Theory Appl. Categ.*, 3:No. 9, 213–250.

[61] Kelly, G. M., Lack, S., and Walters, R. F. C. (1993). Coinverters and categories of fractions for categories with structure. *Appl. Categ. Structures*, 1(1):95–102.

[62] Kelly, G. M. and Schmitt, V. (2005). Notes on enriched categories with colimits of some class. *Theory Appl. Categ.*, 14:no. 17, 399–423.

[63] Kelly, G. M. and Street, R. (1974). Review of the elements of 2-categories. *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 75–103. Lecture Notes in Math., Vol. 420.

[64] Kleisli, H. (1965). Every standard construction is induced by a pair of adjoint functors. *Proc. Amer. Math. Soc.*, 16:544–546.

[65] Kock, A. (1995). Monads for which structures are adjoint to units. *J. Pure Appl. Algebra*, 104(1):41–59.

[66] Lack, S. (2000). A coherent approach to pseudomonads. *Adv. Math.*, 152(2):179–202.

[67] Lack, S. (2002). Codescent objects and coherence. *J. Pure Appl. Algebra*, 175(1-3):223–241. Special volume celebrating the 70th birthday of Professor Max Kelly.

[68] Lack, S. (2010a). A 2-categories companion. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 105–191. Springer, New York.

[69] Lack, S. (2010b). Icons. *Appl. Categ. Structures*, 18(3):289–307.

[70] Lack, S. and Paoli, S. (2008). 2-nerves for bicategories. *K-Theory*, 38(2):153–175.

[71] Lawvere, F. W. (1969). Ordinal sums and equational doctrines. In *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 141–155. Springer, Berlin.

[72] Le Creurer, I. J. (1999). *Descent of internal categories*. PhD thesis, Université catholique de Louvain, Louvain-la-Neuve.

[73] Le Creurer, I. J., Marmolejo, F., and Vitale, E. M. (2002). Beck's theorem for pseudo-monads. *J. Pure Appl. Algebra*, 173(3):293–313.

[74] Lee, J. M. (2011). *Introduction to topological manifolds*, volume 202 of *Graduate Texts in Mathematics*. Springer, New York, second edition.

[75] Leinster, T. (2004). *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge.

[76] Leinster, T. (2014). *Basic category theory*, volume 143 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

[77] Lucatelli Nunes, F. (2016). On biadjoint triangles. *Theory Appl. Categ.*, 31:Paper No. 9, 217–256.

[78] Lucatelli Nunes, F. (2016a). On lifting of biadjoints and lax algebras. *ArXiv e-prints*. arXiv:1607.03087, to appear in *Categories and General Algebraic Structures with Applications*.

[79] Lucatelli Nunes, F. (2016b). Pseudo-Kan Extensions and Descent Theory. *ArXiv e-prints*. arXiv:1606.04999, under review.

[80] Lucatelli Nunes, F. (2017). Freely generated n-categories, coinserters and presentations of low dimensional categories. *ArXiv e-prints*. arXiv:1704.04474.

[81] Mac Lane, S. (1998). *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.

[82] Makkai, M. and Zawadowski, M. (2008). The category of 3-computads is not Cartesian closed. *J. Pure Appl. Algebra*, 212(11):2543–2546.

[83] Marmolejo, F. (1997). Doctrines whose structure forms a fully faithful adjoint string. *Theory Appl. Categ.*, 3:No. 2, 24–44.

[84] Marmolejo, F. (1999). Distributive laws for pseudomonads. *Theory Appl. Categ.*, 5:No. 5, 91–147.

[85] Marmolejo, F. (2004). Distributive laws for pseudomonads. II. *J. Pure Appl. Algebra*, 194(1-2):169–182.

[86] Marmolejo, F. and Wood, R. J. (2008). Coherence for pseudodistributive laws revisited. *Theory Appl. Categ.*, 20:No. 5, 74–84.

[87] May, J. P. (1999). *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL.

[88] Métayer, F. (2003). Resolutions by polygraphs. *Theory Appl. Categ.*, 11:No. 7, 148–184.

[89] Métayer, F. (2016). Strict ω-categories are monadic over polygraphs. *Theory Appl. Categ.*, 31:Paper No. 27, 799–806.

[90] Niefield, S. (2001). Exponentiable morphisms: posets, spaces, locales, and Grothendieck toposes. *Theory Appl. Categ.*, 8:16–32.

[91] Niefield, S. B. (1982). Cartesianness: topological spaces, uniform spaces, and affine schemes. *J. Pure Appl. Algebra*, 23(2):147–167.

[92] Power, A. J. (1988). A unified approach to the lifting of adjoints. *Cahiers Topologie Géom. Différentielle Catég.*, 29(1):67–77.

[93] Power, A. J. (1989). A general coherence result. *J. Pure Appl. Algebra*, 57(2):165–173.

[94] Power, A. J. (1991). An n-categorical pasting theorem. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 326–358. Springer, Berlin.

[95] Power, A. J., Cattani, G. L., and Winskel, G. (2000). A representation result for free cocompletions. *J. Pure Appl. Algebra*, 151(3):273–286.

[96] Reiterman, J. and Tholen, W. (1994). Effective descent maps of topological spaces. *Topology Appl.*, 57(1):53–69.

[97] Schanuel, S. and Street, R. (1986). The free adjunction. *Cahiers Topologie Géom. Différentielle Catég.*, 27(1):81–83.

[98] Shulman, M. (2011). Comparing composites of left and right derived functors. *New York J. Math.*, 17:75–125.

[99] Shulman, M. A. (2012). Not every pseudoalgebra is equivalent to a strict one. *Adv. Math.*, 229(3):2024–2041.

[100] Street, R. (1972a). The formal theory of monads. *J. Pure Appl. Algebra*, 2(2):149–168.

[101] Street, R. (1972b). Two constructions on lax functors. *Cahiers Topologie Géom. Différentielle*, 13:217–264.

[102] Street, R. (1974). Fibrations and Yoneda's lemma in a 2-category. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 104–133. Lecture Notes in Math., Vol. 420. Springer, Berlin.

[103] Street, R. (1976). Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8(2):149–181.

[104] Street, R. (1980). Fibrations in bicategories. *Cahiers Topologie Géom. Différentielle*, 21(2):111–160.

[105] Street, R. (1987). Correction to: "Fibrations in bicategories" [Cahiers Topologie Géom. Différentielle **21** (1980), no. 2, 111–160; MR0574662 (81f:18028)]. *Cahiers Topologie Géom. Différentielle Catég.*, 28(1):53–56.

[106] Street, R. (1996). Categorical structures. In *Handbook of algebra, Vol. 1*, volume 1 of *Handb. Algebr.*, pages 529–577. Elsevier/North-Holland, Amsterdam.

[107] Street, R. (2004). Categorical and combinatorial aspects of descent theory. *Appl. Categ. Structures*, 12(5-6):537–576.

[108] Street, R., Tholen, W., Wischnewsky, M., and Wolff, H. (1980). Semitopological functors. III. Lifting of monads and adjoint functors. *J. Pure Appl. Algebra*, 16(3):299–314.

[109] Street, R. and Verity, D. (2010). The comprehensive factorization and torsors. *Theory Appl. Categ.*, 23:No. 3, 42–75.

[110] Street, R. and Walters, R. (1978). Yoneda structures on 2-categories. *J. Algebra*, 50(2):350–379.

[111] Tholen, W. (1975). Adjungierte Dreiecke, Colimites und Kan-Erweiterungen. *Math. Ann.*, 217(2):121–129.

[112] Wood, R. J. (1982). Abstract proarrows. I. *Cahiers Topologie Géom. Différentielle*, 23(3):279–290.

[113] Wood, R. J. (1985). Proarrows. II. *Cahiers Topologie Géom. Différentielle Catég.*, 26(2):135–168.

[114] Zöberlein, V. (1973). Doktrinen auf 2-kategorien. Manuscript, Math. Inst. Univ. Zürich, Zürich.