On the exponential decay of waves with memory

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Abstract

In this paper we consider general linear damped wave equations with memory. We establish energy estimates that under the assumption of exponentially bounded kernels, induce exponential decaying solutions. Numerical waves that mimic their continuous counterpart are also introduced using a finite element approach.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. This paper is concerned with the study of the decay of the solutions of the following damped wave equation with memory

$$\rho u''(t) + cu'(t) + \mathcal{A}u(t) = \int_0^t Ker\,(t-s;\tau)\mathcal{B}u(s)ds + f(t),\ t \in \mathbb{R}^+. \tag{1}$$

In (1) $u : \overline{\Omega} \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ and, for $t \in \mathbb{R}_0^+$, $u(\cdot, t)$ can be seen as a function defined from $\overline{\Omega}$ into $\mathbb{R}$ that is denoted by $u(t)$, c is a function depending only on spatial variables and accounts for the damping of the wave, Ker denotes a function, called the *memory kernel*, that depends on a parameter $\tau > 0$, f denotes a source term and $\mathcal{A}$ and $\mathcal{B}$ are second order differential operators. Equation (1) is completed with homogeneous Dirichlet boundary conditions and the following initial conditions

$$\begin{cases} u(0) = u_0, \\ u'(0) = u_1. \end{cases} \tag{2}$$

This type of differential problem arises in many contexts, such as modelling the displacement of materials with viscoelastic properties. Indeed, let u denote the displacement of the material, f an external force being applied to the material and $\boldsymbol{\sigma}$ the stress tensor associated. Newton's second law states that

$$\rho u''(t) = \nabla \cdot \boldsymbol{\sigma}(t) + f(t), \tag{3}$$

where ρ is the density of the material. Usually, the relation considered between the stress tensor $\boldsymbol{\tau}$ and the strain tensor $\boldsymbol{\epsilon}$ is

$$\boldsymbol{\sigma}(t) = \mathbf{D}\boldsymbol{\epsilon}(t) \tag{4}$$

where $\mathbf{D}$ is an elastic tensor. Assuming that the components of the strain and the displacement satisfy

$$\boldsymbol{\epsilon}(t) = \frac{1}{2}\left(\nabla u(t) + \nabla u(t)^t\right),$$

relation (4) accounts for a fickian type effect. However, if we assume that the material has viscoelastic properties modelled by a Maxwell-Wiechert model and assume the following constitutive equation

$$\boldsymbol{\sigma}(t) = E(0)\mathbf{D}\epsilon(t) - \int_0^t \frac{\partial}{\partial s} E(t-s)\mathbf{D}\boldsymbol{\epsilon}(s)\, ds,$$

where

$$E(t) = E_0 + \sum_{i=1}^{N} E_i e^{-\alpha_i t}$$

and E_0 is the Young modulus of the spring arm, E_i, $i = 1, \ldots, N$, are the Young modulus of the Maxwell arms and $\alpha_i = \frac{E_i}{\mu_i}$, $i = 1, \ldots, N$, being $\mu_i, i = 1, \ldots, N$, their associated viscosities, then from (3) we obtain for the displacement the following second order integro-differential equation

$$\rho u''(t) - \nabla \cdot \left(\frac{1}{2} E(0)\mathbf{D}(\nabla u(t) + \nabla u(t)^t)\right) = - \int_0^t Ker\,(t-s)\nabla \cdot \big(\mathbf{D}(\nabla u(s) + \nabla u(s)^t)\big)\, ds + f(t),$$

with $Ker\,(t) = \frac{1}{2}\sum_{i=1}^{N} E_i \alpha_i e^{-\alpha_i t}$, $t \geqslant 0$.

Equations of type (1) have already been introduced in the literature, see [5, 13, 15], to model viscoelastic physical phenomena.

Let us consider the classical wave equation with homogeneous Dirichlet boundary conditions. It is well known that the energy of its solution (which is the sum of kinetics and potential energies) is conserved in time. If a damping effect is added then it can be shown that such energy decreases exponentially in time (see Section 3). In certain scenarios, the wave equation with a memory term can be seen as a singular perturbation of the diffusion equation with memory. The solution of this last equation has, in several cases, an energy that goes to zero exponentially (see Section 4).

A question that naturally arises is which conditions on the memory kernels lead to the same energy behaviour for the solution of the wave equation or its generalization in presence of a memory effect. This problem has been object of research in recent years and will be addressed in the present paper.

The study of qualitative properties of partial differential problems defined by equations of type (1) was presented for instance in [1, 4, 3, 8, 12, 14, 16, 18]. However, these works deal essentially with energy estimates for the case when $\mathcal{A}$ and $\mathcal{B}$ represent the Laplace operator, combined with exponential or polynomial decaying kernels. For example, in [3], the authors studied the energy decay for a wave equation with nonlinear boundary damping. Also, in [14], acoustic boundary conditions were considered and the authors established energy decrease results when the kernel function does not necessarily decay exponentially. Similar results were obtained in [18] considering homogeneous Dirichlet boundary conditions but imposing weak assumptions on the memory kernel. The study of the decay of the solution of systems of wave equations has also been addressed in [1, 16]. In [1] the authors established energy decreasing results for systems of two linear wave equations with memory with homogeneous Dirichlet boundary conditions with kernels exponentially dominated.

Energy decreasing results for quasilinear wave equations with memory were considered in [4, 12]. In the first paper the authors consider a nonlinear reaction term and a wave equation where the coefficient of the second derivative depends on the solution was introduced in the second paper. Wave equations with memory as singular perturbations of nonfickian diffusion equations with memory have also been studied. Without being exhaustive we mention [2, 9, 10, 11].

This work aims at establishing energy estimates (and show their exponential decay) for several variants of equation (1). This shall be accomplished in the case $\mathcal{A}$ and $\mathcal{B}$ represent the Laplace operator, but also in the more general setting as presented by (1), always under the assumption that the memory kernel decays exponentially.

The paper is organized as follows: we start in section 2 by introducing the functional context necessary for the development of the energy estimates, as well as some properties of the kernels. In section 3 we start by considering the wave equation with no memory ($\mathcal{A} = \mathcal{B} = -\Delta$) and review classical estimates for this case. In section 4, we explore the case where the coefficient of the second time derivative vanishes, that is, the wave equation with memory is replaced by the diffusion equation with memory that is usually used to model diffusion phenomena (characterized by a nonfickian behaviour). We show that under suitable assumptions on the parameters of the equation, exponential decay of the waves is obtained. Damped wave equations with memory is the object of study of section 5. In this section we introduce a new energy functional that is obtained from the classical one adding a new term induced by its memory character. We establish conditions that lead to the exponential decreasing of such energy functional. To measure the deviation of the gradient of the solution and its evolution in time, a new term is added to the energy functional under analysis. For this new energy functional we prove also its exponential decreasing. The techniques used to obtain these estimates (as well as the estimates themselves) are the motivation of a new energy functional definition for the first equation to be explored in the coming section. Indeed, similar results are established in section 6 for more general problems. Numerical wave equations that mimic their continuous counterpart are introduced in section 7 and their behaviour is explored in section 8. Finally we summarize some conclusions in section 9.

2 Notations and preliminaries

We introduce now the functional context needed in the following sections. Let $L^2(\Omega), L^\infty(\Omega)$ and $H_0^1(\Omega)$ be the usual Sobolev spaces. In $L^2(\Omega)$ we consider the usual inner product $(\cdot,\cdot)$ and the norm induced by this inner product is denoted by $\|\cdot\|$. In $H_0^1(\Omega)$ we consider the usual norm $\|\cdot\|_1$. Let $L^2(\mathbb{R}^+, H_0^1(\Omega))$ be the space of functions $v : \mathbb{R}^+ \longrightarrow H_0^1(\Omega)$ such that

$$\int_0^T \|v(t)\|_1^2 \, dt < \infty, \, \forall T > 0.$$

Let $H^1(\mathbb{R}^+, H_0^1(\Omega))$ be the subspace of $L^2(\mathbb{R}^+, H_0^1(\Omega))$ of all functions v such that its weak derivative $v' : \mathbb{R}^+ \longrightarrow H_0^1(\Omega)$ belongs to $L^2(\mathbb{R}^+, H_0^1(\Omega))$. By $H^2(\mathbb{R}^+, L^2(\Omega))$ we represent the subspace of $L^2(\mathbb{R}^+, L^2(\Omega))$ of all functions v such that its weak derivatives $v^{(j)} : \mathbb{R}^+ \longrightarrow L^2(\Omega)$, $j = 1, 2$, belong to $L^2(\mathbb{R}^+, L^2(\Omega))$.

We start by proving the following auxiliar lemmas for the kernel function.

Lemma 1. *Let $Ker \in L^2(\mathbb{R}^+)$. If there exist constants $K, \alpha > 0$, such that*

$$|Ker\,(s)| \leq K e^{-\alpha s}, \; s \in \mathbb{R}_0^+, \tag{5}$$

then

$$\|Ker\|_{L^1} \leq \frac{K}{\alpha} \quad \textit{and} \quad \|Ker\|_{L^2} \leq \frac{K}{\sqrt{2\alpha}}.$$

Let γ be a nonnegative real and let us denote Ker_γ the function defined by

$$Ker_\gamma\,(s) = e^{\gamma s} Ker\,(s), \; s \in \mathbb{R}_0^+.$$

For this function, the following result holds, which generalizes Lemma 1.

Lemma 2. *If Ker satisfies the hypothesis of Lemma 1 and $\gamma < \alpha$ then*

$$\left\|Ker_\gamma\right\|_{L^1} \leq \frac{K}{\alpha-\gamma} \quad \textit{and} \quad \left\|Ker_\gamma\right\|_{L^2} \leq \frac{K}{\sqrt{2(\alpha-\gamma)}}.$$

Moreover, if $Ker \in H^1(\mathbb{R}^+)$ and Ker' satisfies (5) *then*

$$\left\|Ker'_\gamma\right\|_{L^1} \leq \frac{K(1+\gamma)}{\alpha-\gamma} \quad \textit{and} \quad \left\|Ker'_\gamma\right\|_{L^2} \leq \frac{K(1+\gamma)}{\sqrt{2(\alpha-\gamma)}}.$$

3 Damped wave equation with no memory

We consider in this section the following (simpler) version of equation (1),

$$\rho u''(t) + cu'(t) - D_1\Delta u(t) = -D_2\int_0^t Ker\,(t-s;\tau)\Delta u(s)ds,\ t\in\mathbb{R}^+, \tag{6}$$

where D_1, D_2, ρ denote positive constants and $c \in L^\infty(\Omega)$ is such that there exists a constant $c_0 > 0$ such that $c_0 \leqslant c$. The variational formulation for (6) reads as: let $u \in L^2(\mathbb{R}^+, H^1_0(\Omega)) \cap H^2(\mathbb{R}^+, L^2(\Omega))$ and, for all $T > 0$, holds the following

$$\begin{cases} (\rho u''(t)+cu'(t),w)+D_1(\nabla u(t),\nabla w) = D_2\int_0^t Ker\,(t-s)(\nabla u(s),\nabla w)\,ds, \\ \qquad\qquad\qquad\qquad \text{a. e. in } (0,T),\ \forall w\in H^1_0(\Omega), \\ u'(0)=u_1, \\ u(0)=u_0, \end{cases}$$

where τ is a parameter. If we assume that the kernel function Ker, when $\tau \to 0$, is such that the integral term in equation (6) formally reduces to $-D_2\Delta u(t)$, then equation (6) is replaced by the wave equation

$$\rho u''(t) + cu'(t) - (D_1-D_2)\Delta u(t) = 0,\ t\in\mathbb{R}^+. \tag{7}$$

We remark that this is the case for exponential kernels of the type $Ker\,(s) = \frac{1}{\tau}e^{-\frac{s}{\tau}}$. Indeed, the wave equation with memory is reduced to the classical wave equation.

Figure 1 illustrates the behaviour of the solution of the IBVP defined by (6) with $\Omega = (-1,1)^2$, with homogeneous Dirichlet boundary conditions, a gaussian profile $u(x,y,0) = e^{-10(x^2+y^2)}$, $(x,y) \in \overline{\Omega}$, as initial data and $Ker\,(s;\tau) = \tau^{-1}e^{-\frac{s}{\tau}}$, $\rho = c = 1$ for different values of τ at $t = 4$. When the memory parameter τ decreases, we observe that the corresponding solution approximates the case with no memory and wave coefficient $(D_1 - D_2)$.

We recall that for the solution of the IBVP involving equation (7), the energy

$$\mathbb{E}_u\,(t) = \frac{\rho}{2}\left\|u'(t)\right\|^2 + \frac{D}{2}\left\|\nabla u(t)\right\|^2,\ t\in\mathbb{R}^+_0,$$

where $D = D_1 - D_2$, satisfies the following:

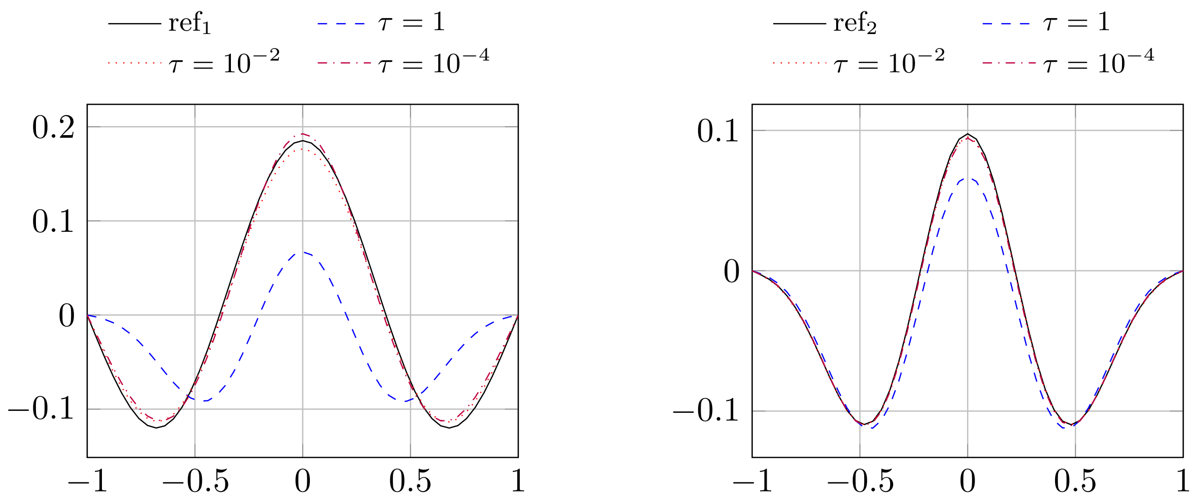
1. when the damping effect is zero ($c = 0$),

$$\mathbb{E}_u\,(t) = \mathbb{E}_u\,(0)\,,\ t\in\mathbb{R}^+_0; \tag{8}$$

2. if $c \neq 0$, then

$$\mathbb{E}_u\,(t) + c_0\left\|u'(t)\right\|^2 ds \leq \mathbb{E}_u\,(0) \tag{9}$$

and the energy has an upper bound.



(a) $D_2 = 0.1$ (b) $D_2 = 0.01$

Figure 1: Restriction of solution at $[-1, 1] \times \{0\}$ for $c = 0.25$ and two choices of D_2 at $t = 4$. The solid line solutions ref_1 and ref_2 correspond, respectively, to the choice of parameters $D_1 = 0.9$ and $D_2 = 0$ and $D_1 = 0.99$ and $D_2 = 0$. These figures were obtained with the fully discrete method presented in section 8, using a fine mesh and small timestep.

The proofs of (8) and (9) are classical and omitted here. However, the decay of $\mathbb{E}_u$, which is not guaranteed in the previous cases, can be established in the presence of an additional term $m^2 u$, that is, if we consider instead

$$\rho u''(t) + cu'(t) - (D_1 - D_2)\Delta u(t) + m^2 u(t) = 0, \; t \in \mathbb{R}^+. \tag{10}$$

In this case it can be even shown a stronger result that states that

$$\left\|u'(t)\right\|^2 + \|u(t)\|_1^2 \longrightarrow 0, \, t \to \infty,$$

exponentially (see [7] and the references cited in this paper). Indeed, considering the new variable $u_\gamma(t) = e^{\gamma t} u(t)$, $t \in \mathbb{R}_0^+$, the wave equation for u_γ and the energy method, it can be shown that there exists a class of wave problems (10) and a corresponding constant $\gamma > 0$ such that

$$\|u'(t)\|^2 + \|u(t)\|_1^2 \leq Ce^{-2\gamma t} \left(\|u(0)\|_1^2 + \|u'(0)\|^2\right), \, t \in \mathbb{R}_0^+, \tag{11}$$

where $C > 0$ denotes a constant that depends on the coefficients of the wave equation (10) and on γ. Estimate (11) leads to the exponential decay of $\mathbb{E}_u$ when $t \to 0$.

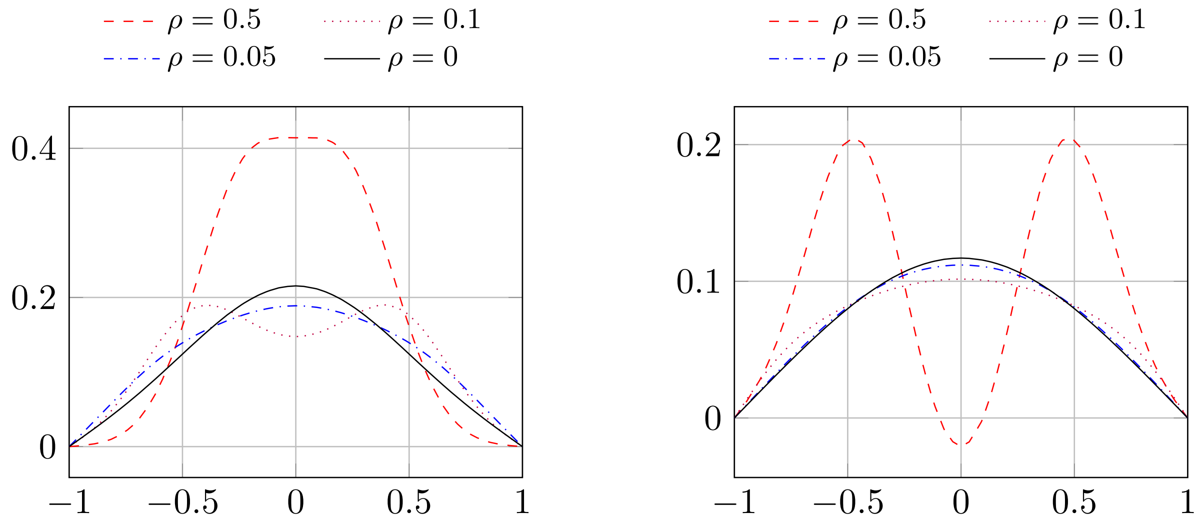
4 Damped wave and diffusion equations with memory

Let us consider now in equation (6) the damping factor $c = 1$ and $\rho \to 0$. Formally, we obtain the following diffusion equation with memory

$$u'(t) - D_1 \Delta u(t) = -D_2 \int_0^t Ker\,(t - s)\Delta u(s)ds, \; t \in \mathbb{R}^+. \tag{12}$$

Figure 2 illustrates the behaviour of the solution of the IBVP defined by (6) with $\Omega = (-1, 1)^2$, with homogeneous Dirichlet boundary conditions, the same gaussian profile used for obtaining

Figure 1 as initial data and $Ker\,(s) = \tau^{-1}e^{-\frac{s}{\tau}}$, $s \in \mathbb{R}_0^+$, $c = 1$, $D_1 = 0.1$, $D_2 = 0.01$, $\tau = 0.001$ for different values of ρ. When ρ decreases we observe that for two different time instances, that solution of the wave problem approaches the one of the diffusion equation.



(a) $t = 0.5$ (b) $t = 1$

Figure 2: Restriction of solution at $[-1, 1] \times \{0\}$ for $c = 1$ and two values of t. These figures were obtained with the fully discrete method presented in section 8, using a fine mesh and small timestep.

We present now two different estimates for the energy

$$\mathbb{E}_u\,(t) = \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2\,ds,\ t \in \mathbb{R}_0^+,$$

of the solution of the IBVP defined by (12) with homogeneous Dirichlet boundary conditions. The first one is obtained using the energy method and the second one is similar to the one established for instance in [6].

Proposition 1. *Let* $Ker \in L^1(\mathbb{R}^+)$ *be a kernel satisfying* (5). *If the weak solution* $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^1(\mathbb{R}^+, L^2(\Omega))$ *then there exist* $\epsilon \neq 0$, $\gamma < \alpha$ *and* $C_1, C_2 > 0$ *such that*

$$\|u(t)\|^2 + C_1 \int_0^t \|\nabla u(s)\|^2\,ds \leq \|u(0)\|^2\,, \tag{13}$$

and

$$\|u(t)\|^2 + C_2 \int_0^t e^{-2\gamma(t-s)}\,\|\nabla u(t)\|^2 \leq e^{-2\gamma t}\,\|u(0)\|^2\,, \tag{14}$$

where

$$C_1 = 2\left(D_1 - \epsilon^2 - \frac{D_2^2K^2}{4\epsilon^2\alpha^2}\right),\quad C_2 = C_1 + \frac{D_2^2K^2}{2\epsilon^2\alpha^2}\left(\frac{1+(\alpha-\gamma)^2}{(\alpha-\gamma)^2}\right) - \gamma C_\Omega,$$

and C_Ω *is the constant from the Friedrichs-Poincaré inequality.*

Proof. We start by proving (13). Using the energy method, it can be shown that, for $t > 0$,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + D_1\|\nabla u(t)\|^2 = \left(D_2\int_0^t Ker\,(t-s)\nabla u(s)\,ds, \nabla u(t)\right),$$

which leads to

$$\frac{d}{dt}\left\|u(t)\right\|^2 + 2(D_1-\epsilon^2)\left\|\nabla u(t)\right\|^2 \leq \frac{D_2^2}{2\epsilon^2}\left(\int_0^t |Ker\,(t-s)|\left\|\nabla u(s)\right\| ds\right)^2. \tag{15}$$

As

$$\left(\int_0^t |Ker\,(t-s)|\left\|\nabla u(s)\right\| ds\right)^2 \leq \left\|Ker\right\|_{L^1}\int_0^t |Ker\,(t-s)|\left\|\nabla u(s)\right\|^2 ds,$$

using Lemma 1, from (15) we obtain

$$\left\|u(t)\right\|^2 + 2(D_1-\epsilon^2)\int_0^t \left\|\nabla u(t)\right\|^2 ds$$
$$\leq \frac{D_2^2}{2\epsilon^2}\frac{K}{\alpha}\int_0^t\int_0^s |Ker\,(s-\mu)|\left\|\nabla u(\mu)\right\|^2 d\mu ds + \left\|u(0)\right\|^2.$$

Moreover, as

$$\int_0^t\int_0^s |Ker\,(s-\mu)|\left\|\nabla u(\mu)\right\|^2 d\mu ds \leq \left\|Ker\right\|_{L^1}\int_0^t \left\|\nabla u(s)\right\|^2 ds,$$

we conclude (13).

To prove estimate (14), we use the technique presented in [17]. Let $\gamma > 0$ be a fixed constant and let $u_\gamma(t) = e^{\gamma t}u(t)$, $t \in \mathbb{R}^+$ for $\gamma < \alpha$. Then, we have

$$\frac{1}{2}\frac{d}{dt}\left\|u_\gamma(t)\right\|^2 - \gamma\left\|u_\gamma(t)\right\|^2 + D_1\left\|\nabla u_\gamma(t)\right\|^2$$
$$= \left(D_2\int_0^t Ker_\gamma\,(t-s)\nabla u_\gamma(s)\,ds, \nabla u_\gamma(t)\right).$$

Considering that $\left\|u_\gamma(t)\right\|^2 \leq C_\Omega\left\|\nabla u_\gamma(t)\right\|^2$ and using Lemma 2, the analysis presented before allow to conclude that

$$\left\|u_\gamma(t)\right\|^2 + C_2\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds \leq \left\|u(0)\right\|^2, \tag{16}$$

where $C_2 = 2\left(D_1 - \epsilon^2 - C_\Omega\gamma - \frac{D_2^2}{4\epsilon^2}\frac{K^2}{(\alpha-\gamma)^2}\right)$. Inequality (16) leads to (14). □

For the energy $\mathbb{E}_u$ we conclude its boundedness in bounded time intervals. Moreover, from (14) we can establish conditions on the coefficients such that $\left\|u(t)\right\|$ decreases exponentially.

Corollary 1. *Under the assumptions of Proposition 1, if*

$$D_1 - \frac{D_2 K}{\alpha} > 0, \tag{17}$$

then there exist constants $C >$ *and* $0 < \gamma < \alpha$ *such that*

$$\left\|u(t)\right\|^2 + \int_0^t e^{-2\gamma(t-s)}\left\|\nabla u(s)\right\|^2 ds \leq Ce^{-2\gamma t}\left\|u(0)\right\|^2,\ t \in \mathbb{R}_0^+. \tag{18}$$

Proof. From Proposition 1 we have (14), that is,

$$\|u(t)\|^2 + g(\gamma)\int_0^t e^{-2\gamma(t-s)}\|\nabla u(s)\|^2\,ds \le e^{-2\gamma t}\|u(0)\|^2,$$

with $g(\gamma) = 2\left(D_1 - \epsilon^2 - C_\Omega\gamma - \frac{D_2^2}{4\epsilon^2}\frac{K^2}{(\alpha-\gamma)^2}\right)$. Taking $\epsilon^2 = \frac{D_2 K}{2\alpha}$, as $g(0) = D_1 - \epsilon^2 - \frac{D_2^2}{4\epsilon^2}\frac{K^2}{\alpha^2}$, it follows from (17) that $g(0) > 0$. Therefore, there exists $0 < \gamma < \alpha$ such that (18) holds.

□

Corollary 1 establishes sufficient conditions that lead to the exponential decreasing of

$$\|u(t)\|^2 + \int_0^t e^{-2\gamma(t-s)}\|\nabla u(s)\|^2\,ds.$$

We remark that condition (17) means that the fickian character of the diffusion process dominates the nonfickian counterpart.

5 Wave equation with memory

We consider in what follows the IBVP defined by the following damped wave equation with memory

$$\rho u''(t) + cu'(t) - D_1\Delta u(t) + m^2 u(t) = -D_2\int_0^t Ker\,(t-s)\Delta u(s)\,ds,\ t\in\mathbb{R}^+, \tag{19}$$

with homogeneous Dirichlet boundary conditions and D_1, D_2, ρ and c satisfy the assumptions made in section 3.

We start by establishing a stability result, under a general assumption on the kernel function Ker, for the energy

$$\mathbb{E}_{u,\gamma}(t) = \|u'(t)\|^2 + \|u(t)\|_1^2 + \int_0^t e^{-2\gamma(t-s)}\|\nabla u(s)\|^2\,ds,\ t\in\mathbb{R}_0^+, \tag{20}$$

where $\gamma > 0$ is a constant. The last term in the definition of $\mathbb{E}_{u,\gamma}$ is motivated by the energy functional for the diffusion equation with memory. If the kernel Ker satisfies (5) then we show that $\mathbb{E}_{u,\gamma}$ decreases to zero exponentially. We observe that the energy functional introduced here incorporates more terms that those considered in the literature. It should be pointed out that other versions of the last energy functional were also studied in the literature. For instance, in [3] and [18], the authors considered the classical energy functional

$$\mathbb{E}_u(t) = \|u(t)\|_1^2,\ t\in\mathbb{R}^+,$$

while in [14] a term induced by the boundary conditions was added to the last energy functional. In [4], for a quasilinear problem, a term related with the reaction term was also taken into account. The energy functional

$$\mathbb{E}_u(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\left(1 - \int_0^t Ker\,(t-s)ds\right)\|\nabla u(t)\|^2 + \int_0^t Ker\,(t-s)\|\nabla u(t) - \nabla u(s)\|^2\,ds,$$

was studied in [16]. A similar definition was analyzed in [18].

A modification of the functional energy (20) will be introduced in this work adding the term

$$\left\| \int_0^t Ker\,(t-s)\nabla u(s)ds - \nabla u(t) \right\|^2 .$$

For this new functional energy we also prove its exponential decay.

Proposition 2. *Let $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^2(\mathbb{R}^+, L^2(\Omega))$ be the weak solution of the IBVP defined by* (19) *with homogeneous Dirichlet boundary conditions. If $Ker \in H^1(\mathbb{R}^+)$ is a kernel such that Ker and Ker' satisfies* (5)*, then the following estimate holds*

$$\begin{aligned} \rho \left\|u'(t)\right\|^2 + 2c_0 \int_0^t \left\|u'(s)\right\|^2 ds + (D_1 - \epsilon^2)\left\|\nabla u(t)\right\|^2 + m^2 \left\|u(t)\right\|^2 \\ + 2D_2 \left(Ker\,(0) - \left(\frac{D_2 K}{4\epsilon^2} + 1 \right) \frac{K}{\alpha} \right) \int_0^t \left\|\nabla u(s)\right\|^2 ds \\ \leq \rho \left\|u'(0)\right\|^2 + D_1 \left\|\nabla u(0)\right\|^2 + m^2 \left\|u(t)\right\|^2 . \end{aligned} \tag{21}$$

where $\epsilon \neq 0$.

Proof. Let

$$I(t) = \int_0^t Ker\,(t-s)\nabla u(s)ds \quad \text{and} \quad I_d(t) = \int_0^t Ker'\,(t-s)\nabla u(s)ds,$$

for $t > 0$. Using the energy method it can be shown that

$$\frac{\rho}{2}\frac{d}{dt}\left\|u'(t)\right\|^2 + c_0 \left\|u'(t)\right\|^2 + \frac{D_1}{2}\frac{d}{dt}\left\|\nabla u(t)\right\|^2 + \frac{m^2}{2}\left\|u(t)\right\| \leq D_2 \left(I(t), \nabla u'(t)\right).$$

As

$$\frac{d}{dt}\left(I(t), \nabla u(t)\right) = Ker\,(0)\left\|\nabla u(t)\right\|^2 + \left(I_d(t), \nabla u(t)\right) + \left(I(t), \nabla u'(t)\right),$$

we obtain

$$\begin{aligned} \frac{\rho}{2}\left\|u'(t)\right\|^2 + c_0 \int_0^t \left\|u'(s)\right\|^2 ds + \frac{D_1}{2}\left\|\nabla u(t)\right\|^2 \\ + D_2 Ker\,(0) \int_0^t \left\|\nabla u(s)\right\|^2 ds + \frac{m^2}{2}\left\|u(t)\right\|^2 \\ \leq D_2\left(I(t), \nabla u(t)\right) - D_2 \int_0^t \left(I_d(s), \nabla u(s)\right) ds \\ + \frac{\rho}{2}\left\|u'(0)\right\|^2 + \frac{D_1}{2}\left\|\nabla u(0)\right\|^2 + \frac{m^2}{2}\left\|u(0)\right\|^2 . \end{aligned} \tag{22}$$

We remark that, for $\epsilon \neq 0$, holds the following

$$2D_2\left(I(t), \nabla u(t)\right) \leq \frac{D_2^2}{\epsilon^2}\left\|Ker\right\|_{L^2}^2 \int_0^t \left\|\nabla u(s)\right\|^2 + \epsilon^2 \left\|\nabla u(t)\right\|^2 ,$$

and

$$-2\int_0^t \left(I_d(s), \nabla u(s)\right) ds \leq 2\left\|Ker'\right\|_{L^1} \int_0^t \left\|\nabla u(s)\right\|^2 ds.$$

Taking the last estimates in (22) we obtain

$$\rho \left\|u'(t)\right\|^2 + 2c_0 \int_0^t \left\|u'(s)\right\|^2 ds + (D_1 - \epsilon^2) \left\|\nabla u(t)\right\|^2$$
$$+ 2D_2 \left(Ker\,(0) - \frac{D_2}{2\epsilon^2} \left\|Ker\right\|_{L^2}^2 - \left\|Ker'\right\|_{L^1}\right) \int_0^t \left\|\nabla u(s)\right\|^2 ds + m^2 \left\|u(t)\right\|^2$$
$$\leq \rho \left\|u'(0)\right\|^2 + D_1 \left\|\nabla u(0)\right\|^2 + m^2 \left\|u(t)\right\|^2. \quad (23)$$

Using Lemma 2 (with $\gamma = 0$), from (23) we obtain (21).

□

Corollary 2. *Under the assumptions of Proposition 2 and if*

$$Ker\,(0) - \frac{K}{\alpha} > 0 \quad (24)$$

and

$$\frac{D_2}{D_1} - 4\frac{\alpha}{K^2}\left(Ker\,(0) - \frac{K}{\alpha}\right) > 0, \quad (25)$$

then there exists a constant $C > 0$ such that

$$\mathbb{E}_{u,0}\,(t) + \int_0^t \left\|u'(s)\right\|^2 ds \leq C\left(\left\|u'(0)\right\|^2 + \left\|u(0)\right\|_1^2\right),\ t \in \mathbb{R}^+.$$

Corollary 2 establishes that $\mathbb{E}_{u,0}$ is bounded. In what follows we establish conditions that allow us to conclude that the energy decreases to zero exponentially.

Theorem 1. *Let $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^2(\mathbb{R}^+, L^2(\Omega))$ be the weak solution of the IBVP defined by (19) with homogeneous Dirichlet boundary conditions. If $Ker \in H^1(\mathbb{R}^+)$ is a kernel such that Ker and Ker' satisfies (5), then for $0 < \gamma < \alpha$, there exists a constant $C >$ such that*

$$\frac{\rho}{2} \left\|u'(t)\right\|^2 + (D_1 - \epsilon^2) \left\|\nabla u(t)\right\|^2 + 2(c_0 - 2\rho\gamma)e^{-2\gamma t} \int_0^t \left\|u_\gamma'(s)\right\|^2 ds$$
$$+ (m^2 - \gamma \left\|c\right\|_\infty) \left\|u(t)\right\|^2 + 2D_2 g(\gamma) e^{-2\gamma t} \int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds$$
$$\leq Ce^{-2\gamma t}\left(\left\|u_\gamma'(0)\right\|^2 + \left\|\nabla u(0)\right\|^2 + \left\|u(0)\right\|^2\right),\ t \in \mathbb{R}_0^+, \quad (26)$$

where $u_\gamma(t) = e^{\gamma t} u(t)$,

$$g(\gamma) = Ker\,(0) - \left(\frac{D_2 K(1+\gamma)}{4\epsilon^2} + 1\right)\frac{K}{\alpha - \gamma}, \quad (27)$$

and $\epsilon \neq 0$.

Proof. Lets consider $\gamma > 0$ and define $u_\gamma(t) = e^{\gamma t} u(t)$, $t \geq 0$. It can be shown that, for u_γ, we have

$$\rho u_\gamma'' + (c - 2\gamma\rho)u_\gamma' - D_1 \Delta u_\gamma(t) + (\rho\gamma^2 - c\gamma + m^2)u_\gamma = -D_2 \int_0^t Ker_\gamma\,(t-s)\Delta u_\gamma(s).$$

Following the analysis presented in Proposition 2, it can be shown there exist ϵ and η, arbitrary nonzero constants, and $\gamma < \alpha$ such that

$$\rho \left\|u_\gamma'(t)\right\|^2 + 2(c_0 - 2\gamma\rho)\int_0^t \left\|u_\gamma'(s)\right\|^2 ds$$
$$+ (D_1 - \epsilon^2)\left\|\nabla u_\gamma(t)\right\|^2 + (\rho\gamma^2 - \|c\|_\infty \gamma + m^2)\left\|u_\gamma(t)\right\|^2$$
$$+ 2D_2\left(Ker\,(0) - \left(\frac{D_2 K(1+\gamma)}{4\epsilon^2} + 1\right)\frac{K}{\alpha-\gamma}\right)\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds$$
$$\leq \rho\left\|u_\gamma'(0)\right\|^2 + D_1\left\|\nabla u_\gamma(0)\right\|^2 + (\rho\gamma^2 - \gamma\|c\|_\infty + m^2)\left\|u_\gamma(t)\right\|^2. \quad (28)$$

We compute now a lower bound for $\left\|u_\gamma'(t)\right\|^2$. Since, for $\theta \neq 0$, we have

$$\left\|u_\gamma'(t)\right\|^2 \geq (1-\theta^2)\left\|e^{\gamma t}u'(t)\right\|^2 + \gamma^2\left(1 - \frac{1}{\theta^2}\right)\left\|u_\gamma(t)\right\|^2,$$

we deduce, for $\theta^2 = \dfrac{1}{2}$,

$$\left\|u_\gamma'(t)\right\|^2 \geq \frac{1}{2}\left\|e^{\gamma t}u'(t)\right\|^2 - \gamma^2\left\|u_\gamma(t)\right\|^2. \quad (29)$$

Considering (29) in (28) we obtain

$$\frac{\rho}{2}\left\|e^{\gamma t}u'(t)\right\|^2 + (D_1 - \epsilon^2)\left\|\nabla u_\gamma\right\|^2 + 2(c_0 - 2\rho\gamma)\int_0^t \left\|u_\gamma'(s)\right\|^2 ds$$
$$+ (m^2 - \gamma\|c\|_\infty)\left\|u_\gamma(t)\right\|^2 + D_2 g(\gamma)\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds$$
$$\leq \rho\left\|u_\gamma'(0)\right\|^2 + D_1\left\|\nabla u_\gamma(0)\right\|^2 + (\rho\gamma^2 - \gamma\|c\|_\infty + m^2)\left\|u_\gamma(t)\right\|^2,$$

where $g(\gamma)$ is defined by (27). This inequality leads immediately to (26). □

Corollary 3. *Under the assumptions of Theorem 1 and if* (2) *and* (25) *hold, then there exist constants* $C, \gamma > 0$ *such that, for all* $t \in \mathbb{R}_0^+$,

$$\mathbb{E}_{u,\gamma}(t) + e^{-2\gamma t}\int_0^t \left\|u_\gamma'(s)\right\|^2 ds \leq Ce^{-2\gamma t}\left(\left\|u'(0)\right\|^2 + \|u(0)\|_1^2\right). \quad (30)$$

Proof. By (24) and (25) and choosing a suitable value for ϵ,

$$g(0) = \left(Ker\,(0) - \left(\frac{D_2 K}{4\epsilon^2} + 1\right)\frac{K}{\alpha}\right)$$

is positive. Then there exists $\gamma \in \left(0, \min\left\{\alpha, \frac{c_0}{2\rho}, \frac{m^2}{\|c\|_\infty}\right\}\right)$ such that, from (26), we obtain (30). □

From Corollary 3 we conclude that

$$\lim_{t\to\infty}\left(\mathbb{E}_{u,\gamma}(t) + e^{-2\gamma t}\int_0^t \left\|u_\gamma'(s)\right\|^2 ds\right) = 0$$

exponentially. We observe that condition (25) imposed to guarantee the boundedness of $\mathbb{E}_{u,0}$ is sufficient to prove the existence of $\gamma > 0$ such that $\mathbb{E}_{u,\gamma}$ decreases to zero exponentially.

Remark 1. *If we consider a wave propagation in viscoelastic material following by a Maxwell-Wiechert model, then* $Ker\,(s) = \dfrac{1}{2}\sum_{i=1}^{n} E_i e^{-\alpha_i s}$, $s \in \mathbb{R}_0^+$ *where* $\alpha_i = \frac{E_i}{\mu_i}$. *In this case we can take*

$$\alpha = \frac{\min_{i=1,\ldots,n} E_i}{\max_{i=1,\ldots,n} \mu_i}.$$

To guarantee condition (25) *we need to assume that the Young models* $E_i, i = 1, \ldots, n$, *are significantly larger than the viscosities* $\mu_i, i = 1, \ldots, n$.

We establish in what follows an estimate for the energy functional

$$\mathbb{E}_{u,\nabla,\gamma}(t) = \mathbb{E}_{u,\gamma}\,(t) + \left\| \int_0^t Ker\,(t-s)\nabla u(s)\,ds - \nabla u(t) \right\|^2, \tag{31}$$

for $t \in \mathbb{R}^+$, where u is a solution of (42). Under suitable regularity conditions and using the energy method, it is straightforward to show the following result.

Theorem 2. *Let* $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^2(\mathbb{R}^+, L^2(\Omega))$ *be the weak solution of the IBVP defined by* (19) *with homogeneous Dirichlet boundary conditions. If* $Ker \in H^1(\mathbb{R}^+)$ *is a kernel such that* Ker *and* Ker' *satisfies* (5), *then for* $0 < \gamma < \alpha$, *there exists* $C > 0$ *such that*

$$\begin{aligned}
\frac{\rho}{2}\left\|u'(t)\right\|^2 &+ (m^2 - \gamma\left\|c\right\|_\infty)\left\|u(t)\right\|^2 + (D_1 - D_2)\left\|\nabla u(t)\right\|^2 \\
&+ 2(c_0 - 2\rho\gamma)e^{-2\gamma t}\int_0^t \left\|u'_\gamma(s)\right\|^2 ds \\
&+ D_2\left\|\int_0^t Ker\,(t-s)\nabla u(s)ds - \nabla u(t)\right\|^2 \\
&+ D_2 g(\gamma)\int_0^t e^{-2\gamma(t-s)}\left\|\nabla u(s)\right\|^2 ds \le Ce^{-2\gamma t}\left(\left\|u'(0)\right\|^2 + \left\|u(0)\right\|_1^2\right),\ t \in \mathbb{R}_0^+,
\end{aligned} \tag{32}$$

where $\epsilon \neq 0$, $u_\gamma(t) = e^{\gamma t}u(t)$ *and* $g(\gamma)$ *is defined by*

$$g(\gamma) = Ker\,(0) - \frac{K}{\alpha-\gamma}\left(1 + \gamma + Ker\,(0) + \frac{K(1+\gamma)}{\alpha-\gamma}\right). \tag{33}$$

Proof. Let $\gamma > 0$ be a real such that $\gamma < \alpha$ and let

$$I(t) = \int_0^t Ker_\gamma\,(t-s)\nabla u_\gamma(s)ds \quad \text{and} \quad I_d(t) = \int_0^t Ker'_\gamma\,(t-s)\nabla u_\gamma(s)ds,$$

for $t > 0$. It can be shown that $u_\gamma(t)$ satisfies the following relation

$$\begin{aligned}
\rho\frac{d}{dt}\left\|u'_\gamma(t)\right\|^2 + (\rho\gamma^2 - \gamma\left\|c\right\|_\infty + m^2)\frac{d}{dt}\left\|u_\gamma(t)\right\|^2 + 2(c_0 - 2\rho\gamma)\left\|u'_\gamma(t)\right\|^2 \\
+ D_1\frac{d}{dt}\left\|\nabla u_\gamma(t)\right\|^2 \le 2D_2\left(I_d(t), \nabla u'_\gamma(t)\right).
\end{aligned} \tag{34}$$

It follows that

$$\begin{aligned}
\left(I(t), \nabla u'_\gamma(t)\right) = &-\frac{1}{2}\frac{d}{dt}\left\|I(t) - \nabla u_\gamma(t)\right\|^2 - Ker\,(0)\left\|\nabla u_\gamma(t)\right\|^2 \\
&+ \frac{1}{2}\frac{d}{dt}\left\|\nabla u_\gamma(t)\right\|^2 + Ker\,(0)\left(I(t), \nabla u_\gamma(t)\right) \\
&- \left(I_d(t), \nabla u_\gamma(t)\right) + \left(I(t), I_d(t)\right),
\end{aligned}$$

and, together with (34), we obtain

$$\begin{aligned}\rho\left\|u_\gamma'(t)\right\|^2 + (\rho\gamma^2 - \gamma\left\|c\right\|_\infty + m^2)\left\|u_\gamma(t)\right\|^2 + 2(c_0 - 2\rho\gamma)\int_0^t \left\|u_\gamma'(s)\right\|^2 ds \\ + (D_1 - D_2)\left\|\nabla u_\gamma(t)\right\|^2 + D_2\left\|I(t) - \nabla u_\gamma(t)\right\|^2 \\ + 2D_2 Ker\,(0)\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds \le 2D_2 Ker\,(0)\int_0^t (I(s), \nabla u_\gamma(s))\,ds \\ - 2D_2\int_0^t (I_d(s), \nabla u_\gamma(s))\,ds + 2D_2\int_0^t (I(s), I_d(s))\,ds \\ + \rho\left\|u_\gamma'(0)\right\|^2 + (\rho\gamma^2 - \gamma\left\|c\right\|_\infty + m^2)\left\|u_\gamma(0)\right\|^2 + D_1\left\|\nabla u_\gamma(0)\right\|^2. \quad (35)\end{aligned}$$

It can be shown the following

$$\int_0^t (I(s), \nabla u_\gamma(s))\,ds \le \left\|Ker_\gamma\right\|_{L^1}\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds, \quad (36)$$

$$-\int_0^t (I_d(s), \nabla u_\gamma(s))\,ds \le \left\|Ker_\gamma'\right\|_{L^1}\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds$$

and

$$\int_0^t (I(s), I_d(s))\,ds \le \left\|Ker_\gamma\right\|_{L^1}\left\|Ker_\gamma'\right\|_{L^1}\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds. \quad (37)$$

Using Lemma 2 and (36)-(37), from (35) we get

$$\begin{aligned}\rho\left\|u_\gamma'(t)\right\|^2 + (\rho\gamma^2 - \gamma\left\|c\right\|_\infty + m^2)\left\|u_\gamma(t)\right\|^2 + 2(c_0 - 2\rho\gamma)\int_0^t \left\|u_\gamma'(s)\right\|^2 ds \\ + (D_1 - D_2)\left\|\nabla u_\gamma(t)\right\|^2 + D_2\left\|I(t) - \nabla u_\gamma(t)\right\|^2 \\ + 2D_2\left(Ker\,(0) - \frac{K}{\alpha-\gamma}\left(1 + \gamma + Ker\,(0) + \frac{K(1+\gamma)}{\alpha-\gamma}\right)\right)\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds \\ \le \rho\left\|u_\gamma'(0)\right\|^2 + (\rho\gamma^2 - \gamma\left\|c\right\|_\infty + m^2)\left\|u_\gamma(0)\right\|^2 + (D_1 - D_2)\left\|\nabla u_\gamma(0)\right\|^2.\end{aligned}$$

Considering that

$$\left\|u_\gamma'(t)\right\|^2 \ge \frac{1}{2}\left\|e^{\gamma t}u'(t)\right\|^2 - \gamma^2\left\|u_\gamma(t)\right\|^2,$$

we obtain

$$\begin{aligned}\frac{\rho}{2}\left\|e^{\gamma t}u'(t)\right\|^2 + (m^2 - \gamma\left\|c\right\|_\infty)\left\|u_\gamma(t)\right\|^2 + 2(c_0 - 2\rho\gamma)\int_0^t \left\|u_\gamma'(s)\right\|^2 ds \\ + (D_1 - D_2)\left\|\nabla u_\gamma(t)\right\|^2 + D_2\left\|\int_0^t Ker_\gamma\,(t-s)\nabla u_\gamma(s)ds - \nabla u_\gamma(t)\right\|^2 \\ + 2D_2 g(\gamma)\int_0^t \left\|\nabla u_\gamma(s)\right\|^2 ds \\ \le \rho\left\|u_\gamma'(0)\right\|^2 + (\rho\gamma^2 - \gamma\left\|c\right\|_\infty + m^2)\left\|u_\gamma(0)\right\|^2 + (D_1 - D_2)\left\|\nabla u_\gamma(0)\right\|^2, \quad (38)\end{aligned}$$

where $g(\gamma)$ is given by (33). Inequality (32) is easily obtained from (38).

□

Corollary 4. *Under the assumption of Theorem 2, if*

$$0 < \frac{K}{\alpha}\frac{K+\alpha}{K-\alpha} < Ker\,(0) \tag{39}$$

and

$$D_1 - D_2 > 0, \tag{40}$$

then there exist constants $C, \gamma > 0$ such that

$$\mathbb{E}_{u,\nabla,\gamma}(t) + e^{-2\gamma t}\int_0^t \left\|u'_\gamma(s)\right\|^2 ds \leq Ce^{-2\gamma t}\left(\left\|u'(0)\right\|^2 + \|u(0)\|_1^2\right),\ t \in \mathbb{R}_0^+, \tag{41}$$

where $u_\gamma(t) = e^{\gamma t}u(t)$.

Proof. From (39) it follows that $g(0) > 0$. Then there exists

$$\gamma \in \left(0, \min\{\alpha, \frac{c_0}{2\rho}, \frac{m^2}{\|c\|_\infty}\}\right)$$

and $C > 0$ such that

$$\begin{aligned}\left\|u'(t)\right\|^2 + \|u(t)\|^2 + e^{-2\gamma t}\int_0^t \left\|u'_\gamma(s)\right\|^2 ds \\ + \|\nabla u(t)\|^2 + \left\|\int_0^t Ker\,(t-s)\nabla u(s)ds - \nabla u(t)\right\|^2 \\ \int_0^t \|\nabla u(s)\|^2 ds \leq Ce^{-2\gamma t}\left(\left\|u'(0)\right\|^2 + \|u(0)\|_1^2\right),\end{aligned}$$

and this inequality leads to (41) .

□

From the last result we conclude

$$\lim_{t\to\infty}\left(\mathbb{E}_{u,\nabla,\gamma}(t) + e^{-2\gamma t}\int_0^t \left\|u'_\gamma(s)\right\|^2 ds\right) = 0,$$

exponentially, and consequently $\mathbb{E}_{u,\nabla,\gamma}$ decreases exponentially.

6 Continuous energy estimates for general operators

In this section we extend the results presented before for the wave equation to general integro-differential equation (1) where the operators $\mathcal{A}$ and $\mathcal{B}$ are defined by

$$\mathcal{A}v = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) + \sum_{i=1}^n \frac{\partial}{\partial x_i}(a_i v) + a_0 v,$$

$$\mathcal{B}v = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial v}{\partial x_j}\right) + \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i v) + b_0 v,$$

where $a_{ij}, b_{ij},\ i,j = 1,\dots,n$ and $a_i, b_i,\ i = 0,\dots,n$ are functions whose regularity shall be specified later and $v \in C^2(\Omega)$.

Let us introduce the weak form of the IBVP (1)-(2): let $u \in L^2(\mathbb{R}^+, H_0^1(\Omega)) \cap H^2(\mathbb{R}^+, L^2(\Omega))$ and, for all $T > 0$, holds the following

$$\begin{cases} \left(\rho u''(t) + cu'(t), w\right) + a(u(t), w) = \displaystyle\int_0^t Ker\,(t-s) b(u(s), w)\, ds + (f(t), w), \\ \qquad\qquad\qquad\qquad \text{a. e. in } (0,T),\ \forall w \in H_0^1(\Omega), \\ u'(0) = u_1, \\ u(0) = u_0, \end{cases} \tag{42}$$

where, for $v, w \in H_0^1(\Omega)$,

$$a(v,w) = \sum_{i,j=1}^n \left(a_{ij}\frac{\partial v}{\partial x_j}, \frac{\partial w}{\partial x_i}\right) - \sum_{i=1}^n \left(a_i v, \frac{\partial w}{\partial x_i}\right) + (a_0 v, w),$$

$$b(v,w) = \sum_{i,j=1}^n \left(b_{ij}\frac{\partial v}{\partial x_j}, \frac{\partial w}{\partial x_i}\right) - \sum_{i=1}^n \left(b_i v, \frac{\partial w}{\partial x_i}\right) + (b_0 v, w).$$

We assume that $a_{ij}, b_{ij} \in L^\infty(\Omega)$, $i,j = 1,\ldots,n$, $a_i, b_i \in L^\infty(\Omega)$, $i = 1\ldots,n$ and $a_0, b_0, c \in L^\infty(\Omega)$ and these coefficients satisfy the following assumptions:

H1. There exists $c_0 > 0$ such that $c \geq c_0$ in $\overline{\Omega}$.

H2. $a(\cdot,\cdot)$ is symmetric, continuous and elliptic, ie,

$$a(u,v) = a(v,u),\ \forall u,v \in H_0^1(\Omega),$$

and there exist $a_e, a_c > 0$ such that

$$|a(u,v)| \leq a_c \|u\|_1 \|v\|_1,\ \forall u,v \in H_0^1(\Omega),$$

and

$$a(u,u) \geq a_e \|u\|_1^2,\ \forall u \in H_0^1(\Omega)$$

H3. $b(\cdot,\cdot)$ is continuous and elliptic, ie, there exist $b_e, b_c > 0$ such that

$$|b(u,v)| \leq b_c \|u\|_1 \|v\|_1,\ \forall u,v \in H_0^1(\Omega),$$

and

$$b(u,u) \geq b_e \|u\|_1^2,\ \forall u \in H_0^1(\Omega)$$

Let $\mathbb{E}_{u,\gamma}$ be defined by (20). In the first result we establish an estimate for the usual energy for the wave equation

$$\mathbb{E}_{u,0}(t) + \int_0^t \left\|u'(s)\right\|^2 ds,\ t \in \mathbb{R}_0^+,$$

where u is a solution of (42), that leads to the boundedness of $\mathbb{E}_{u,0}$ in bounded time intervals.

Theorem 3. *Let $u \in H^2(\mathbb{R}^+, L^2(\Omega)) \cap L^2(\mathbb{R}^+, H_0^1(\Omega))$ be a solution of (42). If hypothesis* ***H1-H3*** *hold, $Ker \in H^1(\mathbb{R})$ and Ker and Ker' satisfy (5), then, for $\eta, \epsilon \neq 0$, we have*

$$\begin{aligned} \rho\left\|u'(t)\right\|^2 + \left(a_e - \epsilon^2\right)\|u(t)\|_1^2 + \left(2c_0 - \eta^2\right)\int_0^t \left\|u'(s)\right\|^2 ds \\ + \left(2Ker\,(0) b_e - \left(\frac{b_c K}{2\epsilon^2} - 2\right)\frac{K b_c}{\alpha}\right)\int_0^t \|u(s)\|_1^2\, ds \\ \leq \rho\left\|u'(0)\right\|^2 + a_c\|u(0)\|_1^2 + \frac{1}{\eta^2}\int_0^t \|f(s)\|^2\, ds,\ t \in \mathbb{R}_0^+. \end{aligned} \tag{43}$$

Proof. Considering in (42) $w = u'(t)$ we obtain

$$\rho\left(u''(t), u'(t)\right) + \left(cu'(t), u'(t)\right) + a\left(u(t), u'(t)\right) = \int_0^t Ker\,(t-s) b\left(u(s), u'(t)\right)\,ds + \left(f(t), u'(t)\right),$$

which can be rewritten in the following equivalent form

$$\rho \frac{d}{dt}\left\|u'(t)\right\|^2 + 2c_0\left\|u'(t)\right\|^2 + \frac{d}{dt}a(u(t), u(t)) \leq 2\int_0^t Ker\,(t-s) b(u(s), u'(t))\,ds + 2\left(f(t), u'(t)\right). \quad (44)$$

It can be shown that holds the following

$$\int_0^t Ker\,(t-s) b\left(u(s), u'(t)\right)\,ds = \frac{d}{dt}\int_0^t Ker\,(t-s) b(u(s), u(t))\,ds - Ker\,(0) b(u(t), u(t)) - \int_0^t Ker'\,(t-s) b(u(s), u(t)) ds. \quad (45)$$

Considering the representation (45) in (44) we further deduce that, for all $\eta \neq 0$,

$$\rho \frac{d}{dt}\left\|u'(t)\right\|^2 + 2c_0\left\|u'(t)\right\|^2 + \frac{d}{dt}a(u(t), u(t)) + 2Ker\,(0) b(u(t), u(t)) \leq 2\frac{d}{dt}\int_0^t Ker\,(t-s) b(u(s), u(t))\,ds - 2\int_0^t Ker'\,(t-s) b(u(s), u(t))\,ds + \frac{1}{\eta^2}\left\|f(t)\right\|^2 + \eta^2\left\|u'(t)\right\|^2.$$

Integrating over $[0, t]$ and using **H3** leads to

$$\rho\left\|u'(t)\right\|^2 + a(u(t), u(t)) + \left(2c_0 - \eta^2\right)\int_0^t \left\|u'(s)\right\|^2 ds + 2b_e Ker\,(0)\int_0^t \left\|u(s)\right\|_1^2\,ds \leq 2\int_0^t Ker\,(t-s) b(u(s), u(t))\,ds + 2\int_0^t\int_0^s Ker'\,(s-\mu) b(u(\mu), u(s))\,d\mu ds + \frac{1}{\eta^2}\int_0^t \left\|f(s)\right\|^2 ds + \rho\left\|u'(0)\right\|^2 + a(u(0), u(0)). \quad (46)$$

Using Lemma 1 it can be shown that

$$2\int_0^t Ker\,(t-s) b(u(s), u(t))\,ds \leq \frac{b_c^2}{2\epsilon^2}\frac{K^2}{\alpha}\int_0^t \left\|u(s)\right\|_1^2\,ds + \epsilon^2\left\|u(t)\right\|_1^2$$

and

$$\int_0^t\int_0^s Ker'\,(s-\mu) b(u(\mu), u(s))\,d\mu ds \leq b_c\left\|Ker'\right\|_{L^1}\int_0^t \left\|u(s)\right\|_1^2\,ds.$$

Considering the last two inequalities in (46) we obtain (43). □

Corollary 5. *Under the assumptions of Theorem 3, if $\epsilon, \xi, \eta \neq 0$, are such that*

$$a_e - \epsilon^2 > 0, \tag{47}$$

$$2c_0 - \eta^2 > 0, \tag{48}$$

$$2Ker\,(0)b_e - \left(\frac{b_c K}{2\epsilon^2} - 2\right)\frac{K b_c}{\alpha} > 0 \tag{49}$$

there exists a constant $C > 0$ such that

$$\mathbb{E}_{u,0}\,(t) + \int_0^t \left\|u'(s)\right\|_1^2 \, ds \leq C\left(\left\|u'(0)\right\|^2 + \|u(0)\|_1^2 + \int_0^t \|f(s)\|^2 \, ds\right), \; t \in \mathbb{R}_0^+. \tag{50}$$

From the upper bound (50) we conclude that, for an isolated system ($f = 0$), $\mathbb{E}_{u,0}$ is bounded by the the energy of the system at $t = 0$.

In what follows we prove, for a class of differential operators $\mathcal{A}, \mathcal{B}$ and kernels Ker, that there exists a constant $\gamma > 0$ such that

$$\mathbb{E}_{u,\gamma}\,(t) + e^{-2\gamma t}\int_0^t \left\|u_\gamma'(s)\right\|_1^2 \, ds$$

decays to zero when $t \to \infty$. We start by establishing and upper bound for $\mathbb{E}_{u,\gamma}$.

Theorem 4. *Under the assumptions of Theorem 3, for $\epsilon, \eta \neq 0$ and $0 < \gamma < \alpha$, we have*

$$\begin{aligned}
&\frac{\rho}{2}\left\|u'(t)\right\|^2 + (a_e - \gamma\|c\|_\infty)\,\|u(t)\|^2 + (a_e - \epsilon^2)\,\|\nabla u(t)\|^2 \\
&\quad + (2(c_0 - 2\gamma\rho) - \eta^2)e^{-2\gamma t}\int_0^t \left\|u_\gamma'(s)\right\|^2 ds + g(\gamma)\int_0^t e^{-2\gamma(t-s)}\|u(s)\|_1^2 ds \\
&\quad \leq \frac{1}{\eta^2}\int_0^t e^{2\gamma(t-s)}\,\|f(s)\|^2 \, ds + c_p\left(\left\|u'(0)\right\|^2 + \|u(0)\|^2\right), \; t \in \mathbb{R}_0^+,
\end{aligned} \tag{51}$$

where $c_p = \max\{\rho, \gamma, \rho\gamma^2 - \gamma\|c\|_\infty, a_e\}$, $u_\gamma = e^{\gamma t}u(t)$, $t \in \mathbb{R}_0^+$ and

$$g(\gamma) = 2Ker\,(0)b_e - \left(\frac{b_c K(1+\gamma)}{2\epsilon^2} - 2\right)\frac{K(1+\gamma)b_c}{\alpha - \gamma} \tag{52}$$

Proof. Let $u_\gamma(t) = e^{\gamma t}u(t)$. This function satisfies

$$\begin{aligned}
&\rho\left(u_\gamma''(t), w\right) + (c_0 - 2\gamma\rho)\left(\frac{du_\gamma}{dt}(t), w\right) + (\gamma^2\rho - \gamma\|c\|_\infty)(u(t), w) + a(u_\gamma(t), w) \\
&\quad \leq \int_0^t Ker_\gamma\,(t-s)b(u_\gamma(s), w)\, ds + (f_\gamma(t), w),
\end{aligned}$$

for $w \in H_0^1(\Omega)$, where $Ker\,(s;\gamma) = Ker\,(s)e^{\gamma s}$. Following the proof of Theorem 3, it can be shown the following

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\left\|u_\gamma'(t)\right\|^2 + \frac{1}{2}(\gamma^2\rho - \gamma\|c\|_\infty)\frac{d}{dt}\|u_\gamma(t)\|^2 + \frac{1}{2}\frac{d}{dt}a(u_\gamma(t), u_\gamma(t)) \\
&\quad + Ker\,(0)b(u_\gamma(t), u_\gamma(t)) + (c_0 - 2\rho\gamma)\left\|u_\gamma'(t)\right\|^2 \\
&\quad = \int_0^t Ker_\gamma'\,(t-s)b(u_\gamma(s), u_\gamma(t))ds \\
&\quad\quad + \frac{d}{dt}\int_0^t Ker_\gamma\,(t-s)b(u_\gamma(s), u_\gamma(t))\, ds + \left(f_\gamma(t), u'(t)\right),
\end{aligned}$$

that leads to

$$\rho\left\|u_\gamma'(t)\right\|^2+(\rho\gamma^2-\gamma\|c\|_\infty)\|u_\gamma(t)\|^2+a_e\|u_\gamma(t)\|_1^2$$
$$+2Ker\,(0)b_e\int_0^t\|u_\gamma(s)\|_1^2\,ds+2(c_0-2\gamma\rho-\eta^2)\int_0^t\left\|u_\gamma'(s)\right\|^2ds$$
$$\leq 2b_c\int_0^t\int_0^s Ker'\,(s-\mu;\gamma)\,\|u_\gamma(\mu)\|_1\,\|u_\gamma(s)\|_1\,d\mu ds$$
$$+2b_c\int_0^t Ker_\gamma\,(t-s)\,\|u_\gamma(s)\|_1\,\|u_\gamma(t)\|_1\,ds+\frac{1}{2\eta^2}\int_0^t\|f_\gamma(s)\|^2\,ds$$
$$+\rho\left\|u_\gamma'(0)\right\|^2+(\rho\gamma^2-\gamma c)\|u_\gamma(0)\|^2+a_c\|u_\gamma(0)\|_1^2,$$

where $\eta\neq 0$.
As before, for $0<\gamma<\alpha$, we also have

$$\rho\left\|u_\gamma'(t)\right\|^2+(\rho\gamma^2-\gamma\|c\|_\infty+a_e)\|u_\gamma(t)\|^2+(a_e-\epsilon^2)\|\nabla u_\gamma(t)\|^2$$
$$+(2(c_0-2\gamma\rho)-\eta^2)\int_0^t\left\|u_\gamma'(s)\right\|^2ds+g(\gamma)\int_0^t\|u_\gamma(s)\|_1^2\,ds$$
$$\leq\frac{1}{\eta^2}\int_0^t\|f_\gamma(s)\|^2\,ds+\rho\left\|u_\gamma'(0)\right\|^2+(\rho\gamma^2-\gamma\|c\|_\infty)\|u_\gamma(0)\|^2+a_e\|u_\gamma(0)\|_1^2,$$

where $\epsilon\neq 0$, that implies

$$\frac{\rho}{2}\left\|e^{\gamma t}u'(t)\right\|^2+(a_e-\gamma\|c\|_\infty)\|u_\gamma(t)\|^2+(a_e-\epsilon^2)\|\nabla u_\gamma(t)\|^2$$
$$+(2(c_0-2\gamma\rho)-\eta^2)\int_0^t\left\|u_\gamma'(s)\right\|^2ds+g(\gamma)\int_0^t\|u_\gamma(s)\|_1^2\,ds$$
$$\leq\frac{1}{\eta^2}\int_0^t\|f_\gamma(s)\|^2\,ds+\rho\left\|u_\gamma'(0)\right\|^2$$
$$+(\rho\gamma^2-\gamma\|c\|_\infty)\|u_\gamma(0)\|^2+a_e\|u_\gamma(0)\|_1^2,\quad (53)$$

where $g(\gamma)$ is defined by (52). Finally, (51) follows immediately from (53).

□

Corollary 6. *Under the assumptions of Theorem 4, if the parameters $\epsilon,\eta\neq 0$ satisfy the inequalities (47), (48) and (49), then there exist constants $C,\gamma>0$ such that*

$$\mathbb{E}_{u,\gamma}\,(t)+e^{-2\gamma t}\int_0^t\left\|u_\gamma'(s)\right\|^2ds$$
$$\leq Ce^{-2\gamma t}\left(\int_0^t e^{2\gamma s}\|f(s)\|^2\,ds+\left\|u'(0)\right\|^2+\|u(0)\|_1^2\right),\ t\in\mathbb{R}_0^+,$$

where $u_\gamma(t)=e^{\gamma t}u(t)$.

Corollary 6 allows to conclude that in a isolated system, that is, with $f=0$, we have

$$\lim_{t\to\infty}\left(\mathbb{E}_u\,(t)+e^{-2\gamma t}\int_0^t\left\|u_\gamma'(s)\right\|^2ds\right)=0,$$

exponentially and consequently $\mathbb{E}_u$ decreases to zero with the same rate.

7 Decay of numerical waves with memory

In this section we establish that numerical approximations for the solution of the IBVP (1)-(2) with homogeneous Dirichlet boundary conditions present the same qualitative behaviour of the solution of this problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and let $h > 0$ be a fixed parameter and let $\mathcal{T}_h$ be an admissible triangulation of Ω with diameter h, that is,

$$h = \max_{\triangle \in \mathcal{T}_h} \operatorname{diam}(\triangle),$$

where $\operatorname{diam}(\triangle)$ denotes the diameter of the element $\triangle$ Let $\mathcal{V}_h$ be the space of piecewise polynomials of degree m defined in $\mathcal{T}_h$, that is

$$\mathcal{V}_h = \{v \in C^0(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega,\, v = p_m \text{ in } \triangle,\, \triangle \in \mathcal{T}_h\},$$

where p_m denotes a polynomial of degree at most m. By $\mathcal{P}_{\partial\Omega}$ and $\mathcal{P}_\Omega$ we represent the set of nodes of $\mathcal{T}_h$ on $\partial\Omega$ and Ω, respectively. Let $\{\phi_P, P \in \mathcal{P}_\Omega\}$ be a basis of $\mathcal{V}_h$. The finite element approximation for the solution of the IBVP (1), (2) with homogeneous Dirichlet boundary conditions is $u_h(x,t) = \sum_{P \in \mathcal{P}_\Omega} \alpha_P(t)\phi_P(x)$ that satisfies the following

$$\begin{cases} (u_h''(t), w_h) + (cu_h'(t), w_h) + a(u_h(t), w_h) \\ \qquad = \displaystyle\int_0^t Ker\,(t-s) b(u_h(s), w_h)\, ds + (f(t), w_h), \\ \qquad\qquad\qquad\qquad \text{a. e. in } \mathbb{R}^+,\ \forall w_h \in \mathcal{V}_h, \\ u_h'(0) = u_{1,h}, \\ u_h(0) = u_{0,h}. \end{cases} \tag{54}$$

In (54) $u_{1,h}$ and $u_{0,h}$ are approximations of u_1 and u_0 in $\mathcal{V}_h$. To compute $u_h(t)$ we need to solve the following system of second order integro-differential equations

$$\begin{cases} M_h\alpha''(t) + C_h\alpha'(t) + A_h\alpha(t) = \displaystyle\int_0^t Ker\,(t-s) B_h\alpha(s)\, ds + F_h(t),\ t \in \mathbb{R}^+, \\ \alpha'(0) = \mathcal{U}_{1,h}, \\ \alpha(0) = \mathcal{U}_{0,h}, \end{cases} \tag{55}$$

where $\alpha(t) = [(\alpha_P(t))_{P \in \mathcal{P}_\Omega}]$, $\mathcal{U}_{i,h}, i = 0,1$, are the vectors whose components are the coordinates of $u_{i,h}, i = 0,1$, with respect to the basis $\{\phi_P, P \in \mathcal{P}_\Omega\}$, and

$$\begin{aligned} M_h &= [((\phi_P, \phi_Q))_{P,Q \in \mathcal{P}_\Omega}], \\ C_h &= [((c\phi_P, \phi_Q))_{P,Q \in \mathcal{P}_\Omega}], \\ A_h &= [(a(\phi_P, \phi_Q))_{P,Q \in \mathcal{P}_\Omega}], \\ B_h &= [(b(\phi_P, \phi_Q))_{P,Q \in \mathcal{P}_\Omega}], \\ F_h(t) &= [((f(t), \phi_Q))_{Q \in \mathcal{P}_\Omega}]. \end{aligned}$$

Introducing the new variable $Z(t) = (z_1(t), z_2(t))$ where $z_1(t) = \alpha(t), z_2(t) = \alpha'(t)$, then the initial value problem (55) of second order is equivalent to

$$\begin{cases} Z'(t) = \mathcal{A}_h Z(t) + \displaystyle\int_0^t Ker\,(t-s)\mathcal{B}_h Z(s)\, ds + \mathcal{F}_h(t),\ t \in \mathbb{R}^+, \\ Z(0) = \mathcal{U}_h, \end{cases} \tag{56}$$

where

$$\mathcal{A}_h = \begin{bmatrix} 0 & I \\ -M_h^{-1}A_h & -M_h^{-1}C_h \end{bmatrix}, \mathcal{B}_h = \begin{bmatrix} M_h^{-1}B_h & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{F}_h(t) = \begin{bmatrix} 0 \\ M_h^{-1}F_h \end{bmatrix}, \mathcal{U}_h = \begin{bmatrix} U_{0,h} \\ U_{1,h} \end{bmatrix}.$$

As the unique solution of the IVP (56) is smooth enough, then for the unique solution $u_h(t) \in \mathcal{V}_h$ of (54) it can be shown the following result.

Proposition 3. *Let us suppose that the assumptions of Theorem 3 hold for the finite element solution u_h. If (47), (48) and (49) also hold then there exist constants $C, \gamma > 0$ such that*

$$\mathbb{E}_{u_h,\gamma}(t) + e^{-2\gamma t}\int_0^t \left\|u'_{h,\gamma}(s)\right\|^2 ds \\ \leq Ce^{-\gamma t}\left(\int_0^t \|f(s)\|^2\, ds + \left\|u'_h(0)\right\|^2 + \|u_h(0)\|_1^2\right),\ t \in \mathbb{R}_0^+,$$

where $u_{h,\gamma}(t) = e^{\gamma t}u_h(t)$.

For $f = 0$ we have

$$\lim_{t\to\infty}\left(\mathbb{E}_{u_h,\gamma}(t) + e^{-\gamma t}\int_0^t \left\|u'_{h,\gamma}(s)\right\|^2 ds\right) = 0,$$

exponentially.

For the particular case $\mathcal{A} = \mathcal{B} = -\Delta$ the previous result can be improved. In fact the following result can be stated for the energy $\mathbb{E}_{u_h,\nabla,\gamma}$.

Proposition 4. *Let us suppose that the assumptions of Theorem 1 are valid for the finite element solution u_h. If the conditions (39) and (40) also hold then there exist constants $C, \gamma > 0$ such that*

$$\mathbb{E}_{u_h,\nabla,\gamma}(t) + e^{-2\gamma t}\int_0^t \left\|u'_{h,\gamma}(s)\right\|^2 ds \\ \leq Ce^{-2\gamma t}\left(\int_0^t \|f(s)\|^2\, ds + \left\|u'_h(0)\right\|^2 + \|u_h(0)\|_1^2\right),\ t \in \mathbb{R}_0^+.$$

For $f = 0$ we have

$$\lim_{t\to\infty}\left(\mathbb{E}_{u_h,\nabla,\gamma}(t) + e^{-2\gamma t}\int_0^t \|u_{h,\gamma}(s)\|^2\, ds\right) = 0,$$

exponentially.

8 Numerical results

In this section we illustrate the qualitative behaviour of numerical solutions of (55) for the equation studied in section 6, a particular choice of kernel and a set of associated parameters. The choice of kernel is motivated by the example given and its frequent reference in literature.

Let us introduce the specifics of our test problem. Let $\Omega = (-1,1)^2$. In this setup, consider the following differential problem

$$\begin{cases} u''(t) + cu'(t) - D_1 \Delta u(t) = -D_2 \displaystyle\int_0^t Ker\,(t-s)\Delta u(s)ds, \ t \in (0,T), \\ u(x,y,0) = e^{-10(x^2+y^2)}, (x,y) \in \Omega, \\ u'(x,y,0) = 0, \ (x,y) \in \Omega, \\ u(x,y,t) = 0, \ (x,y) \in \partial\Omega, t \in [0,T], \end{cases}$$

where $T > 0$ and $c, D_1, D_2, \tau > 0$ are constants. In this equation we take exponential kernels of the form $Ker\,(t) = \tau^{-1} e^{-\frac{t}{\tau}}$, $t \in \mathbb{R}_0^+$.

Following the spatial discretisation in (54), we introduce the time step Δt and a uniform partition $t_j = j\Delta t, j = 0,1,2,\ldots,N = [\frac{T}{\Delta t}]$. Applying standard centered finite differences schemes in time and the composite trapezoidal rule to the formulation (55), the following second order in time method is obtained:

$$\left(\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}, v\right) + c\left(\frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, v\right) + D_1\left(\nabla u_h^{n+1}, \nabla v\right)$$
$$= \frac{D_2 \Delta t}{2\tau} \sum_{j=0}^{n} \left(e^{-\frac{t_{n+1}-t_{j+1}}{\tau}} \nabla u_h^{j+1} + e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla u_h^j, \nabla v\right), \quad (57)$$

where u_h^j is an approximation for $u(t_j)$, $j = 0,1,\ldots,N$.

Let

$$I_{n+1} = \frac{D_2 \Delta t}{2\tau} \sum_{j=0}^{n} \left(e^{-\frac{t_{n+1}-t_{j+1}}{\tau}} \nabla u_h^{j+1} + e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla u_h^j\right).$$

It is easy to show that I_n satisfies

$$\begin{cases} I_{n+1} = e^{-\frac{\Delta t}{\tau}} I_n + \dfrac{D_2 \Delta t}{2\tau}\left(e^{-\frac{\Delta t}{\tau}} \nabla u_h^n + \nabla u_h^{n+1}\right), \quad n > 1, \\ I_1 = \dfrac{D_2 \Delta t}{2\tau}\left(e^{-\frac{\Delta t}{\tau}} \nabla u_h^0 + \nabla u_h^1\right). \end{cases}$$

With this new notation, method (57) can be rewritten as

$$\left(\left(\frac{1}{\Delta t^2} + \frac{c}{2\Delta t}\right) u_h^{n+1}, v\right) + \left(D_1 - \frac{D_2 \Delta t}{2\tau}\right)\left(\nabla u_h^{n+1}, \nabla v\right)$$
$$= \left(\frac{2}{\Delta t^2} u_h^n + \left(\frac{c}{2\Delta t} - \frac{1}{\Delta t^2}\right) u_h^{n-1}, v\right) + e^{-\frac{\Delta t}{\tau}}\left(I_n, \nabla v\right). \quad (58)$$

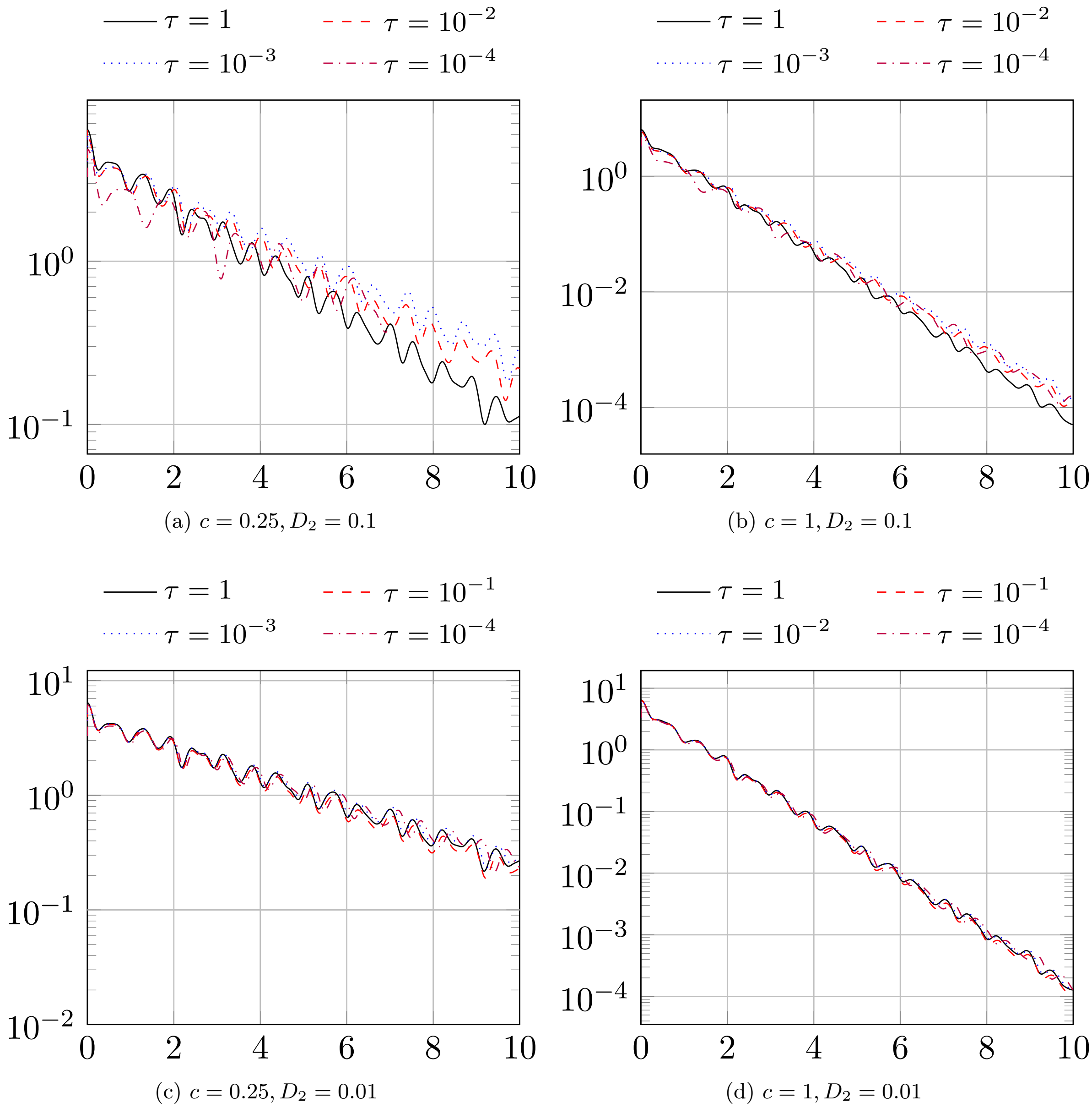
Remark 2. *The integral term in* (42)*, discretized in* (57)*, should be implemented following* (58)*.*

Let the fully discretisation of $\mathbb{E}_{u_h,\nabla,\gamma}$ (31) be defined by

$$\mathbb{E}_{h,n} = \left\| \frac{u_h^n - u_h^{n-2}}{2\Delta t} \right\|^2 + \|u_h^n\|_1^2 + \|I_n - \nabla u_n^n\|^2, \ n \geqslant 2.$$

The behaviour of $\mathbb{E}_{h,n}$ is clearly illustrated in Figure 3, for different values of D_2 and τ. It can be observed that the larger the damping factor c is, the faster the discrete energy approximates zero.

A similar result is observed when analysing the numerical solution at the central point $(0,0)$ of the square $[-1,1]^2$. As expected from the previous results, the solution at this point approximates zero. In Figure 4 we plot the numerical solution at this point, for the same profiles as in Figure 3.



(a) $c = 0.25, D_2 = 0.1$

(b) $c = 1, D_2 = 0.1$

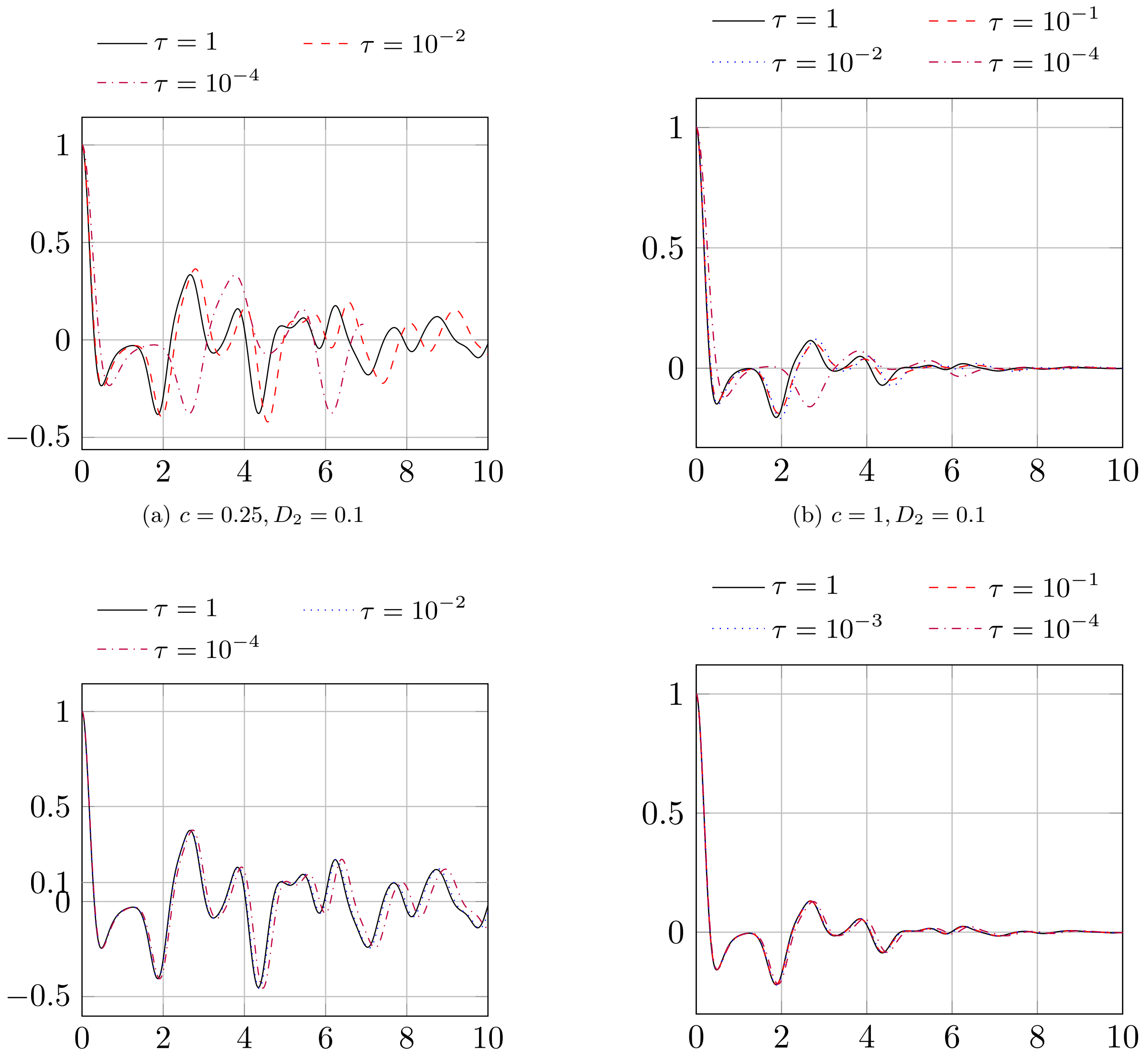
(c) $c = 0.25, D_2 = 0.01$

(d) $c = 1, D_2 = 0.01$

Figure 3: Plot of discrete energy for different damping factors and coefficients D_2, τ $(D_1 = 1)$.

9 Conclusions

Wave equations with memory, can be reduced in certain scenarios, to a classical wave equation or to the diffusion equation with memory that is often used to model diffusion processes characterized by fickian and nonfickian mass fluxes. Based in these two facts, a new energy functional for the wave equation with memory is introduced in this paper. Using the energy method, upper bounds for this new energy functional are established. Such upper bounds are then used to establish sufficient conditions for its exponential decay. We remark that exponential decay of other energy functionals were proved in the literature and some of them can be obtained from the results presented here. The results obtained for the wave equation were generalized for a more general class of problems.



(a) $c = 0.25, D_2 = 0.1$

(b) $c = 1, D_2 = 0.1$

(c) $c = 0.25, D_2 = 0.01$

(d) $c = 1, D_2 = 0.01$

Figure 4: Discrete solution at point (0,0) for different damping factors and coefficients D_2, τ.

To simulate the energy behaviour we introduce a fully discrete model based on finite element approach. We showed that the semi-discrete counterpart of the equation (obtained by discretisation in space with finite elements) inherits the same property. The numerical waves defined using the exponential kernel $Ker\,(s) = \tau^{-1} e^{-\frac{s}{\tau}}$, $s \in \mathbb{R}_0^+$ also exhibit the same qualitative behaviour.

10 Acknowledgements

This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

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