Pullback attractors of the Jeffreys–Oldroyd equations*

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Abstract

We study the dynamics of the Jeffreys–Oldroyd equation using the theory of trajectory pullback attractors. We prove an existence theorem for weak solutions and use it to construct a family of trajectory spaces and to specify the class of attracted families of sets, which includes families bounded in the past. Finally, we prove the existence of the trajectory and global pullback attractors of the model.

Keywords: pullback attractors, trajectory attractors, Jeffreys–Oldroyd equations, weak solutions, existence theorems.

1. Introduction

When mathematically describing the dynamics of a system, we identify possible states of the system with points of an abstract mathematical space called the phase space, and we prescribe an evolution law. In the simplest case of a deterministic autonomous system without memory the evolution law can be given in terms of a semigroup of evolutionary operators acting in the phase space. More sophisticated cases require the notion of a process (a biparametric family of operators) or trajectory spaces and families thereof. In this context the trajectory is a function $\mathbb{R}_+ \to E$, where $\mathbb{R}_+ = [0, +\infty)$ and E is the phase space. Each trajectory corresponds to a particular scenario of the evolution. If the dynamics of the system is described in terms of a semigroup of operators or a process, the trajectories can be defined by means of the evolutionary equations. If trajectory spaces are used, trajectories are postulated. Approaches

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involving trajectory spaces are the most general ones. In particular, trajectories make it possible to consider indeterministic dynamics, i. e. such that the initial state of the system may not uniquely determine the evolutions. For this reason trajectory and global attractors of trajectory spaces are very important in mathematical fluid mechanics, as in this domain the lack of uniqueness results is a commonplace.

There exists a class of systems such that a major part of potential states are unobservable in the sense that their existence is limited in time. The theory of dynamical systems introduces the notion of the attractor to deal with such systems. If a system has an attractor, initial data may become 'forgotten' under the evolution, and the limit regimes are due to intrinsic properties of the system. Mathematically, an attractor is characterised by the attraction property, which is usually accompanied by such requirements as minimality and compactness.

The notion of a trajectory attractor is the most important one in the theory of attractors of trajectory spaces. Trajectory attractors consist of functions $\mathbb{R}_+ \to E$ representing prototypic behaviour of trajectories. This means that the behaviour of any trajectory eventually resembles that of the functions belonging to the trajectory attractor.

The theory of trajectory attractors goes back to [1, 2]. These papers feature the attractors of three-dimensional Navier–Stokes equations, which are notorious for its lack of uniqueness of weak solutions, and became a breakthrough in the analysis of indeterministic dynamics. The theory was further developed in [3, 4, 5, 6, 7], see also the reviews [8, 9, 10] and monographs [11, 12, 13].

Given an autonomous system with an attractor, it is only a matter of elapsed time, when the initial data gets 'forgotten'. In nonautonomous systems the absolute times of both start and check are to be taken into account. As a consequence, there is more than one way to generalise the notion of attractor to nonautonomous systems.

One well-established approach is to consider uniform attractors [3, 5]. The uniformity of attraction is understood with respect to the initial time. Thus, given a bounded set D in the phase space, the trajectories starting in D are expected to land in a given neighbourhood of the attractor in a fixed time h_D no matter *when* they start. This resembles the attraction in autonomous system, as the absolute time of start and check is essentially irrelevant. For this reason uniform attractors are rather strong, i. e. they only exist in a rather narrow class of systems. In particular, when dealing with specific equation, one has to impose rather restricting assumptions on the time-dependent terms.

Another option is to consider pullback attractors, which are less exigent. They were first considered in [14, 15]. Initially, the theory of pullback attractors was naturally developed in the framework of processes (biparametric families of operators describing the evolution of nonautonomous systems). The infinite-dimensional setting of this theory has become quite rich both in abstract results and in applications, see e. g. the excellent monograph [16]. In particular, there are a number of results concerning pullback attractors of Newtonian fluids as well as certain non-Newtonian ones [17, 18, 19, 20, 21, 22, 23]. However, typical lack of uniqueness impedes the use of processes in fluid mechanics.

The notion of pullback attractor has recently been ported to trajectory spaces [7]. Even though the definitions of trajectory and global attractors are rather involved, the concept of pullback attraction is fairly intuitive. We sketch it here.

Considering a nonautonomous system, we start by defining a family of trajectory spaces. Given initial time $\xi \in \mathbb{R}$, consider the set $\mathcal{H}_\xi^+$ of trajectories starting at this time. The trajectories are parametrised not with absolute time, but with relative time elapsed from time ξ. Fix the time of check $\theta \in \mathbb{R}$. We are interested in the behaviour exhibited at time θ by trajectories belonging to $\mathcal{H}_\xi^+$ under the assumption that $\theta - \xi$ is large, i. e. $\xi \to -\infty$. To be specific, take a bounded set D in the phase space and consider the trajectories starting within D at time ξ, which we do not fix (the set of such trajectories is denoted $\mathcal{H}_\xi^+(D)$). The question is which is the set of values these trajectories take at time θ, i. e. the image of D under the evolution. Obviously, the image depends on the initial moment ξ, which we send to $-\infty$. Then it may happen that the image of D at time θ is situated in the vicinity of a set $\mathcal{A}_\theta \subset E$ independent of D. Formally, this means that the semidistance of the set $\{u(\theta-\xi)\colon u \in \mathcal{H}_\xi^+(D)\}$ from the set $\mathcal{A}_\theta$ tends to 0 as $\xi \to -\infty$ for any bounded D (we write $u(\theta - \xi)$ rather than $u(\theta)$ because the trajectories are parametrised with relative time). In this case it is natural to say that $\mathcal{A}_\theta$ pullback attracts bounded sets. Assuming that at any time θ there exists a pullback attracting set, we obtain a family of sets $\mathcal{A} = \{\mathcal{A}_\theta \colon \theta \in \mathbb{R}\}$ pullback attracting bounded sets. If this family is bounded, compact and minimal in the senses to be specified below, it is the global pullback attractor (of bounded sets). We stress that the global pullback attractor is a family of sets in the phase space, which can be thought of as a time-dependent set in the phase space.

Given a pullback attractor, the attraction holds not only for sets of trajectories starting within the same bounded set D. Specifically, if the family $\mathbf{D} = \{D_\xi \colon \xi \in \mathbb{R}\}$ is bounded in the past, i. e. the union $\bigcup_{\xi \le \xi_0} D_\xi$ is bounded for some ξ_0, then it can be shown that pullback attraction holds for the sets $\mathcal{H}^+(\mathbf{D}) = \{\mathcal{H}_\xi^+(D_\xi)\colon \xi \in \mathbb{R}\}$. More generally, the definition of a pullback attractor can be extended to include a class $\mathfrak{D}$ of families of sets subject to attraction. In applications, $\mathfrak{D}$ usually contains not only families bounded in the past, but allows for certain growth as ξ goes to $-\infty$.

The definition of the trajectory pullback attractor is more involved, but employs similar ideas.

Up to now there are scarce results about trajectory pullback attractors in fluid mechanics: [7, 24, 25]. In this paper we consider one more application and prove the existence of trajectory and global pullback attractors of the Jeffreys–Oldroyd equations with substantial derivative.

In this paper we establish the existence of pullback attractors for the Jeffreys–Oldroyd equations with the substantial derivative in a bounded domain. The weak solvability of this problem is proved in [26]. Its trajectory attractors are studied in [6, 5, 13] under different assumptions. In particular, in [5] the existence of uniform attractors is proved in the nonautonomous case under rather restricting assumptions on the body force. Here we revisit the nonautonomous in a different framework and with less rigid assumptions. Our main tool is the topological approximation method (see [13, 27]).

Consider the initial and boundary value problem for the Jeffreys–Oldroyd equations in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with piecewise Lipschitz boundary:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} + \operatorname{grad} p = \operatorname{Div} \sigma + f, \tag{1.1}$$

$$\operatorname{div} u = 0. \tag{1.2}$$

$$\sigma + \lambda_1 \left(\frac{\partial \sigma}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \sigma}{\partial x_i} \right) = 2\eta \left(\mathcal{E} + \lambda_2 \left(\frac{\partial \mathcal{E}}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \mathcal{E}}{\partial x_i} \right) \right), \tag{1.3}$$

$$u\Big|_{\partial\Omega} = 0 \tag{1.4}$$

$$u\Big|_{t=0} = a, \tau\Big|_{t=0} = \tau_0. \tag{1.5}$$

Here $u(x,t)$ is the velocity vector of the particle occupying the point x at time t; $p(x,t)$ is the pressure at x at time t, which is a scalar function; $f(x,t)$ is the body force vector; $\sigma = (\sigma_{ij}(x))$ is the stress deviator, which is a symmetric matrix of order n;

$$\operatorname{Div} \sigma = \left(\frac{\partial \sigma_{11}}{\partial x_1} + \cdots + \frac{\partial \sigma_{1n}}{\partial x_n}, \ldots, \frac{\partial \sigma_{n1}}{\partial x_1} + \cdots + \frac{\partial \sigma_{nn}}{\partial x_n} \right),$$

$\mathcal{E} = (\mathcal{E}_{ij})$ is the strain velocity tensor, which is a symmetrical matrix with entries given by

$$\mathcal{E}_{ij} = \mathcal{E}_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The parameter $\eta > 0$ is called the viscosity of the Jeffreys body, $\lambda_1 > 0$ is the relaxation time, and $\lambda_2 > 0$ is the retardation time, where $\lambda_2 < \lambda_1$.

The unknowns in (1.1), (1.2) are v, p, and σ.

Equations (1.1)–(1.3) are adequate for such viscoelastic media as polymer solutions, bitumen, concrete, or earth crust (see [28]).

When dealing with the Jeffreys–Oldroyd equations (in particular, when considering their solvability and attractors), it is convenient to change variables. Put $\mu_1 = \eta\lambda_2/\lambda_1$, $\mu_2 = (\eta - \mu_1)/\lambda_1$ (observe that $\mu_1 > 0$ and $\mu_2 > 0$) and consider the unknown

$$\tau = \sigma - 2\mu_1 \mathcal{E}(u). \tag{1.6}$$

instead of σ. Thus, we have the following problem equivalent to (1.1)–(1.5):

$$\frac{\partial u}{\partial t} - \mu_1 \Delta u + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} - \operatorname{Div} \tau + \operatorname{grad} p = f, \tag{1.7}$$

$$\frac{\partial \tau}{\partial t} + \frac{\tau}{\lambda_1} + \sum_{i=1}^{n} u_i \frac{\partial \tau}{\partial x_i} = 2\mu_2 \mathcal{E}(u), \tag{1.8}$$

$$\operatorname{div} u = 0, \tag{1.9}$$

$$u\Big|_{\partial\Omega} = 0, \ u\Big|_{t=0} = a, \ \tau\Big|_{t=0} = \tau_0. \tag{1.10}$$

In problem (1.7)–(1.10) we fix the body force $f \in L_2^{\mathrm{loc}}(\mathbb{R}; (L_2(\Omega))^n)$ and the coefficients. The initial data a and τ_0 can be chosen arbitrarily in correspondent function spaces.

In Section 2 we define the weak solution and prove the main result of the paper, Theorem 2.5, establishing the existence of the trajectory and global pullback attractors under certain assumptions on f. In Section 3 we prove the existence of weak solutions satisfying a special estimate.

2. Pullback attractors of the Jeffreys–Oldroyd equations

We start with basic definitions and results of the abstract theory of trajectory pullback attractors [7].

Let $E \subset E_0$ be Banach spaces such that E is reflexive and the embedding is continuous. Consider the class of functions

$$\mathcal{T} := C(\mathbb{R}_+; E_0) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; E).$$

By a familiar embedding result (see e. g. [29, Chapter 3, Lemma 8.1] we have

$$\mathcal{T} \subset C_w(\mathbb{R}_+, E).$$

In particular, this implies that for any $u \in \mathcal{T}$ we have $u(t) \in E$ for all $t \geq 0$ and that

$$\|u\|_{L_\infty(t_1,t_2;E)} = \sup_{t\in[t_1,t_2]} \|u(t)\|_E \tag{2.1}$$

for any segment $[t_1, t_2] \subset \mathbb{R}_+$.

To each $\theta \in \mathbb{R}$ assign a nonempty set

$$\mathcal{H}_\theta^+ \subset \mathcal{T} := C(\mathbb{R}_+; E_0) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; E). \tag{2.2}$$

We refer to the sets $\mathcal{H}_\theta^+$ as *trajectory spaces*, elements thereof as *trajectories*, and the family $\mathbf{H}^+ = \{\mathcal{H}_\theta^+\}_{\theta\in\mathbb{R}}$ as the *family of trajectory spaces*.

We shall also consider other families of sets depending on a parameter varying over $\mathbb{R}$. We always assume that the sets belonging to such families are nonempty. If all the sets in such a family are subsets of E, we say that it is a family *over E*.

Fix a class of families of sets $\mathfrak{D}$ over E assuming that for any family $\mathbf{D} = \{D_\theta\} \in \mathfrak{D}$ we have $D_\theta \neq \varnothing$ for all $\theta \in \mathbb{R}$. Given $\mathbf{D} = \{D_\theta\} \in \mathfrak{D}$, by $\mathbf{H}^+(\mathbf{D}) = \{\mathcal{H}^+_\theta(\mathbf{D})\}$ denote the family of sets defined by

$$\mathcal{H}^+_\theta(\mathbf{D}) = \{u \in \mathcal{H}^+_\theta \colon u(0) \in D_\theta\}. \tag{2.3}$$

By $\mathrm{T}(h)$ ($h \in \mathbb{R}$) denote the translation operator, which takes a function g to the function $\mathrm{T}(h)g$ given by

$$(\mathrm{T}(h)g)(s) = g(s+h).$$

Given a family $\mathbf{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) and $h \in \mathbb{R}$, by $\mathrm{T}(h)\mathbf{P}$ denote the family of sets

$$(\mathrm{T}(h)P)_\theta = \mathrm{T}(h)P_{\theta-h} \quad (\theta \in \mathbb{R}). \tag{2.4}$$

By definition, the inclusion of families $\mathbf{P} \subset \mathbf{P}'$, where $\mathbf{P} = \{P_\theta\}$, $\mathbf{P}' = \{P'_\theta\}$ ($P_\theta, P'_\theta \subset \mathcal{T}$) means that $P_\theta \subset P'_\theta$ for any $\theta \in \mathbb{R}$.

Definition 2.1. A family $\mathbf{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) is called *pullback $\mathfrak{D}$-attracting* for $\mathcal{H}^+$, if for any family $\mathbf{D} \in \mathfrak{D}$ and any $\theta \in \mathbb{R}$ we have

$$\sup_{u \in \mathcal{H}^+_\xi(\mathbf{D})} \inf_{v \in P_\theta} \|\mathrm{T}(\theta - \xi)u - v\|_{C(\mathbb{R}_+; E_0)} \to 0 \quad (\xi \to -\infty). \tag{2.5}$$

Definition 2.2. A family $\mathbf{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) is called pullback $\mathfrak{D}$-*absorbing* for $\mathbf{H}^+$, if for any family $\mathbf{D} \in \mathfrak{D}$ and any $\theta \in \mathbb{R}$ there exists $\vartheta_{\mathbf{D}}(\theta) \leq \theta$ such that for any $\xi \leq \vartheta_{\mathbf{D}}(\theta)$ the inclusion

$$\mathrm{T}(\theta - \xi)\mathcal{H}^+_\tau(\mathbf{D}) \subset P_\theta \tag{2.6}$$

holds, and the function $\vartheta_{\mathbf{D}} \colon \mathbb{R} \to \mathbb{R}$ is nondecreasing.

Definition 2.3. A family $\mathbf{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) is called $\mathcal{T}$*-precompact*, if

(i) P_θ is precompact in $C(\mathbb{R}_+; E_0)$ for any $\theta \in \mathbb{R}$;

(ii) for any $\theta \in \mathbb{R}$ there exists a continuous function $\varphi_\theta \colon \mathbb{R}_+ \to \mathbb{R}$ such that for any trajectory $u \in P_\theta$ the inequality $\|u(t)\|_E \leq \varphi_\theta(t)$ holds for all $t \in \mathbb{R}$.

This family is called $\mathcal{T}$*-compact*, if in addition P_θ is closed (and thus, compact) in $C(\mathbb{R}_+; E_0)$ for any $\theta \in \mathbb{R}$.

Remark 2.1. The requirement (ii) in Definition 2.3 can be replaced by the following:

(ii$'$) given $\theta \in \mathbb{R}$, there exists a continuous function $\tilde{\varphi}_\theta \colon \mathbb{R}_+ \to \mathbb{R}$ such that for any trajectory $u \in P_\theta$ the inequality $\|u\|_{L_\infty(t,t+1;E)} \leq \varphi_\theta(t)$ holds for all $t \in \mathbb{R}$.

Indeed, if (ii) holds, then (ii$'$) holds with the function $\tilde{\varphi}(\theta) = \max_{\xi \in [\theta, \theta+1]} \varphi(\xi)$. Conversely, if (ii$'$) holds, then (ii) holds with the function $\varphi(\theta) = \tilde{\varphi}(\theta)$, which follows from (2.1).

Definition 2.4. A family $\mathbf{P}$ consisting of nonempty subsets of $\mathcal{T}$ is called a *trajectory pullback $\mathfrak{D}$-semiattractor* for $\mathbf{H}^+$, if

(i) $\mathbf{P}$ is $\mathcal{T}$-compact;

(ii) $\mathrm{T}(h)\mathbf{P} \subset \mathbf{P}$ for any $h \geq 0$;

(iii) $\mathbf{P}$ is pullback $\mathfrak{D}$-attracting.

Definition 2.5. A trajectory pullback $\mathfrak{D}$-semiattractor $\mathbf{P}$ for $\mathbf{H}^+$ is called a *trajectory pullback $\mathfrak{D}$-attractor* for $\mathbf{H}^+$, if $\mathrm{T}(h)\mathbf{P} = \mathbf{P}$ for any $h \geq 0$.

Definition 2.6. A trajectory pullback $\mathfrak{D}$-attractor $\mathbf{U} = \{\mathcal{U}_\theta\}$ ($\mathcal{U}_\theta \subset \mathcal{T}$) for $\mathbf{H}^+$ is called *minimal*, if it is contained in any trajectory pullback $\mathfrak{D}$-attractor $\mathbf{P} = \{P_\theta\}$.

Definition 2.7. A family $\mathbf{A} = \{\mathcal{A}_\theta\}$ over E is called a *global pullback $\mathfrak{D}$-attractor* for $\mathbf{H}^+$, if

(i) $\mathcal{A}_\theta$ is compact in E_0 and bounded in E for all $\theta \in \mathbb{R}$;

(ii) for any $\mathbf{D} \in \mathfrak{D}$ and $\theta \in \mathbb{R}$ the pullback attraction

$$\sup_{u \in \mathcal{H}^+_\tau(\mathbf{D})} \inf_{a \in \mathcal{A}_\theta} \|u(\theta - \tau) - a\|_{E_0} \to 0 \quad (\tau \to -\infty) \tag{2.7}$$

holds;

(iii) $\mathbf{A}$ is contained in any family $\mathbf{A}' = \{\mathcal{A}'_\theta\}$ ($\mathcal{A}'_\theta \subset E$) satisfying (i) and (ii).

Remark 2.2. The minimal trajectory pullback attractor is unique, and so is the global pullback attractor.

The following theorems are proved in [7].

Theorem 2.1. *Suppose that $\mathbf{H}^+$ admits a $\mathcal{T}$-precompact pullback $\mathfrak{D}$-absorbing family $\mathbf{P}$, and let $\overline{\mathbf{P}}$ denote the closure of $\mathbf{P}$ with respect to the topology of $C(\mathbb{R}_+; E_0)$. Then there exists a minimal trajectory pullback $\mathfrak{D}$-attractor $\mathbf{U} \subset \overline{\mathbf{P}}$.*

Theorem 2.2. *Suppose that $\mathbf{H}^+$ has a trajectory pullback $\mathfrak{D}$-semiattractor $\mathbf{P}$. Then it also has the minimal trajectory pullback $\mathfrak{D}$-attractor $\mathcal{U} \subset \mathbf{P}$.*

Theorem 2.3. *Let $\mathbf{U} = \{\mathcal{U}_\theta\}$ be the minimal trajectory pullback $\mathfrak{D}$-attractor for $\mathbf{H}^+$. Then the family $\mathbf{A} = \{\mathcal{A}_\theta\}$, where $\mathcal{A}_\theta = \{u(0) \colon u \in \mathcal{U}_\theta\} \subset E$, is the global pullback $\mathfrak{D}$-attractor for $\mathbf{H}^+$.*

Trajectory attractors are associated with differential equations in the following way. Consider a nonautonomous differential equation

$$u'(\theta) = A(\theta, u(\theta)). \tag{2.8}$$

where $A \colon D(A) \to R(A)$, $D(A) = \mathbb{R} \times E_A$, $E_A \subset E$. Also consider an auxiliary equation

$$v'(t) = A(t + \tau, v(t)), \tag{2.9}$$

so that $u(\theta)$ is a solution of equation (2.8) on $[\tau, +\infty)$ if and only if $v(t) = u(\tau + t)$. Given $\tau \in \mathbb{R}$, the trajectory space $\mathcal{H}^+_\tau$ is chosen as a set of solutions of (2.9). The solutions can be understood in any appropriate sense, e. g. weak or strong, but it is essential that the trajectory space should be nonempty and contained in the class $\mathcal{T}$. Thus, the trajectories belonging to $\mathcal{H}_\tau^+$ can be viewed as evolution scenarios starting at absolute time τ, but parametrised with the relative time elapsed since the start.

If pullback attractors attract bounded sets, the class of families $\mathfrak{D}$ can consist at least of all the families bounded in the past (see Introduction). Depending on the estimates available for the solutions, it can be broader.

We use this approach to construct pullback attractors of problem (1.7)–(1.10). Accordingly, we need time-shifted solutions $v(x,t) = u(x,t+h)$, which solve the following problem:

$$\frac{\partial v}{\partial t} - \mu_1 \Delta v + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \operatorname{Div} \tau + \operatorname{grad} p = F, \tag{2.10}$$

$$\frac{\partial \tau}{\partial t} + \frac{\tau}{\lambda_1} + \sum_{i=1}^{n} v_i \frac{\partial \tau}{\partial x_i} = 2\mu_2 \mathcal{E}(v), \tag{2.11}$$

$$\operatorname{div} v = 0, \tag{2.12}$$

$$v\Big|_{\partial\Omega} = 0, \ v\Big|_{t=0} = v_0, \ \tau\Big|_{t=0} = \tau_0, \tag{2.13}$$

where $F(x,t) = f(x,t+h)$ for some $h \in \mathbb{R}$.

Now we specify the functional spaces required for the weak formulation of problem (2.10)–(2.13).

We use standard notation for the Lebesgue and Sobolev spaces. The Hilbert space of symmetric matrices of order n is denoted $M_S(n)$. We often use simplified notations for matrix-valued functions, e. g. writing L_2 instead of $L_2(\Omega; M_S(n))$. We use parentheses to denote the L_2 inner product and angular brackets to denote the pairing between a Banach space and its dual.

We use standard hydrodynamic spaces of vector functions (see [30]). Let $\mathcal{V}$ be the set of smooth nondivergent functions $\Omega \to \mathbb{R}^n$ with compact support contained in Ω, and let H, V, and V_α be the closed subspaces of $(L_2(\Omega))^n$, $(H^1(\Omega))^n$, and $(H^\alpha(\Omega))^n$ respectively spanned by $\mathcal{V}$. Here $\alpha \in (0,1]$ is fixed. The space V is equipped with the norm

$$\|u\|_V = \|\nabla u\|_{L_2}, \tag{2.14}$$

equivalent to the standard norm of $(H^1(\Omega))^n$. By H^*, V^*, and V_α^* denote the dual spaces. Identifying the Hilbert space H with H^* by means of the Riesz isomorphism, we have the chain of embeddings

$$V \subset H \equiv H^* \subset V_\alpha^* \subset V^*. \tag{2.15}$$

We use the Poincaré inequality

$$\|u\|_H \le K_0 \|u\|_V \quad (u \in V), \tag{2.16}$$

where K_0 is the first eigenvalue of the Stokes operator.

We denote by $C_0^\infty = C_0^\infty(\Omega, M_S(n))$ the space of smooth matrix functions having compact support in Ω. The closed subspace of H^s spanned by C_0^∞, $s > 0$ is standardly denoted by H_0^s, and its dual, H^{-s}. We need the following spaces of matrix functions:

$$H_0^2 \subset H_0^1 \subset L_2 \equiv L_2^* \subset H^{-\alpha} \subset H^{-1} \subset H^{-2}. \tag{2.17}$$

Definition 2.8. A *weak solution* of problem (2.10)–(2.13) on $\mathbb{R}_+$ is a pair of functions (v, τ) belonging to the classes

$$\left.\begin{array}{c} v \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H), \\ v' \in L_1^{\mathrm{loc}}(\mathbb{R}_+; V^*); \end{array}\right\} \tag{2.18}$$

$$\left.\begin{array}{c} \tau \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; L_2), \\ \tau' \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{-2}); \end{array}\right\} \tag{2.19}$$

which satisfy the identities

$$\frac{d}{dt}(v, \varphi) - \sum_{i=1}^n \left(v_i v, \frac{\partial \varphi}{\partial x_i}\right) + \mu_1(\nabla v, \nabla \varphi) + (\tau, \nabla \varphi) = (F, \varphi) \qquad (\varphi \in \mathcal{V}), \tag{2.20}$$

$$\frac{d}{dt}(\tau, \Phi) + \frac{1}{\lambda_1}(\tau, \Phi) - \sum_{i=1}^n \left(v_i \tau, \frac{\partial \Phi}{\partial x_i}\right) + 2\mu_2(v, \operatorname{Div} \Phi) = 0 \qquad (\Phi \in C_0^\infty) \tag{2.21}$$

almost everywhere on $\mathbb{R}_+$, as well as the initial conditions

$$v(0) = v_0, \quad \tau(0) = \tau_0. \tag{2.22}$$

Remark 2.3. Given v and τ satisfying (2.18) and (2.19), we at least have $v \in C(\mathbb{R}_+; V^*)$ and $\tau \in C(\mathbb{R}_+; H^{-2})$, so initial conditions (2.22) make sense.

We have the following existence theorem for weak solutions:

Theorem 2.4. *Suppose that $F \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V^*)$. Then for any $v_0 \in H$ and $\tau_0 \in L_2$ there exists a weak solution (v, τ) of problem* (2.10)–(2.13) *satisfying the estimate*

$$\begin{aligned} &\frac{1}{4}\|v\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|v\|^2_{L_2(t,t+1;V)} \\ \le{}& \frac{1}{2\mu_1}\|F\|^2_{L_2(t,t+1;V^*)} + \frac{1}{\mu_1}\int_0^{t+1} e^{-2\gamma(t-s)}\|F(s)\|^2_{V^*}\,ds + \left(\|v_0\|_H^2 + \frac{1}{2\mu_2}\|\tau_0\|^2_{L_2}\right)e^{-2\gamma t} \end{aligned} \tag{2.23}$$

for $t \ge 0$, where

$$\gamma = \min\left\{\frac{1}{\lambda_1}, \frac{\mu_1}{2K_0^2}\right\}, \tag{2.24}$$

and K_0 is the constant from inequality (2.16).

Theorem 2.4 is proved in Section 3.

The weak solutions of (2.20)–(2.22) provided by Theorem 2.4 allow the following estimate for the time derivative (cf. [13, Lemma 6.8.1]):

Lemma 2.1. *Suppose that $F \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V^*)$ and the pair (v, τ) satisfies (2.20) and (2.21) almost everywhere on $\mathbb{R}_+$. Then*

$$\begin{aligned} &\|v'\|_{L_{4/3}(t,t+1,V^*)} + \|\tau'\|_{L_2(t,t+1;H^{-2})} \\ &\le K_1(\|v\|_{L_\infty(t,t+1;H)}, \|v\|_{L_2(t,t+1;V)}, \|\tau\|_{L_\infty(t,t+1;L_2)}, \|F\|_{L_2(t,t+1;V^*)}), \end{aligned} \tag{2.25}$$

where K_1 continuously depends on its argument and is nondecreasing with respect to each argument.

Now we consider pullback attractors of problem (1.7)–(1.10).

Without loss of generality we assume that the body force f in (1.7) belongs to $L_2^{\mathrm{loc}}(\mathbb{R}; H)$ and verifies

$$\int_{-\infty}^{t} e^{-2\gamma s} \|f(s)\|_{V^*}^2 \, ds < \infty \tag{2.26}$$

for all $t \in \mathbb{R}$, where γ is defined by (2.24).

Fix $\alpha \in (0, 1]$. To introduce the trajectory spaces for the Jeffreys–Oldroyd equations, we use the Banach spaces $E = H \times L_2(\Omega, M_s(n))$ and $E_0 = V_\alpha^* \times H^{-\alpha}(\Omega, M_s(n))$.

Take $\theta \in \mathbb{R}$. We define the trajectory space $\mathcal{H}_\theta^+$ of problem (1.7)–(1.10) to be the set of all pairs (v, τ) being weak solutions of (2.10)–(2.13) on $\mathbb{R}_+$ with the right-hand side $F = \mathrm{T}(\theta) f$ and an initial condition $(v(0), \tau(0)) \in H \times L_2$, which can be chosen individually for each solutions, satisfying the estimate

$$\begin{aligned} &\frac{1}{4}\|v\|_{L_\infty(t,t+1;H)}^2 + \frac{1}{8\mu_2}\|\tau\|_{L_\infty(t,t+1;L_2)}^2 + \frac{\mu_1}{2}\|v\|_{L_2(t,t+1;V)}^2 \\ &\qquad \le \frac{1}{2\mu_1}\|f\|_{L_2(t+\theta,t+\theta+1;V^*)}^2 + \frac{1}{\mu_1}\int_{-\infty}^{t+1} e^{-2\gamma(t-s)} \|f(s+\theta)\|_{V^*}^2 \, ds \\ &\qquad\qquad + \left(\|v(0)\|_H^2 + \frac{1}{2\mu_2}\|\tau(0)\|_{L_2}^2\right) e^{-2\gamma t} \quad (t \in \mathbb{R}), \end{aligned} \tag{2.27}$$

where γ is the constant from Theorem 2.4. Thus, we obtain the family of trajectory spaces $\mathbf{H}^+ = \{\mathcal{H}_\theta^+\}$.

Remark 2.4. The definition of the trajectory spaces imply the inequality

$$\begin{aligned} &\|v\|_{L_\infty(t,t+1;H)}^2 + \|\tau\|_{L_\infty(t,t+1;L_2)}^2 + \|v\|_{L_2(t,t+1;V)}^2 \\ &\qquad \le C\Big\{\|f\|_{L_2(t+\theta,t+\theta+1;V^*)}^2 + \int_{\infty}^{t+1} e^{-2\gamma(t-s)} \|f(s+\theta)\|_{V^*}^2 \, ds \\ &\qquad\qquad + \left(\|v(0)\|_H^2 + \|\tau(0)\|_{L_2}^2\right) e^{-2\gamma t}\Big\} \quad (t \in \mathbb{R}) \end{aligned} \tag{2.28}$$

for all trajectories $(v, \tau) \in \mathcal{H}_\theta^+$ with a constant C independent of θ and the trajectory. By Lemma 2.1 we have $v' \in L_{4/3}^{\mathrm{loc}}(\mathbb{R}_+; V^*)$ and $\tau' \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{-2})$.

It follows from Theorem 2.4 that given $b \in E$ and $\theta \in \mathbb{R}$, there exists a trajectory $(v, \tau) \in \mathcal{H}_\theta^+$ satisfying the initial condition $(v(0), \tau(0)) = b$.

It is easy to check the inclusion

$$\mathcal{H}_\theta^+ \subset \mathcal{T} \quad (\theta \in \mathbb{R}).$$

Firstly, the inclusion $\mathcal{H}_\theta^+ \subset L_\infty^{\mathrm{loc}}(\mathbb{R}_+; E)$ follows from inequality (2.27). Now we check the continuity. Let (v, τ) be a trajectory. Take $T > 0$. According to Remark 2.4, we see that $v \in L_\infty(0, T; H)$ and $v' \in L_{4/3}(0, T; V^*)$, so by the Aubin–Lions lemma applied to the spaces $H \subset V_\alpha^* \subset V^*$ we have $v \in C([0, T]; V_\alpha^*)$. This is true for any T, so $v \in C(\mathbb{R}_+; V_\alpha^*)$ as claimed. The continuity of τ is proved in the same way.

Now we specify the class of families of sets $\mathfrak{D}$ which is attracted by our attractor. By $\mathcal{R}$ denote the set of functions $r \colon \mathbb{R} \to \mathbb{R}_+$ such that the function $\theta \mapsto e^{2\gamma\theta}(r(\theta))^2$ increases and

$$\lim_{\theta \to -\infty} e^{2\gamma\theta}(r(\theta))^2 = 0. \tag{2.29}$$

By definition, a family of sets $\mathbf{D} = \{D_\theta\}$ ($\varnothing \neq D_\theta \subset V^1$) belongs to $\mathfrak{D}$ if there exists a function $r_{\mathbf{D}} \in \mathcal{R}$ such that $\|v_0\|_H^2 + \|\tau_0\|_{L_2}^2 \leq r_{\mathbf{D}}(\theta)^2$ for all $\theta \in \mathbb{R}$ and $(v_0, \tau_0) \in D_\theta$.

Thus, a family of sets is attracted to the pullback attractors if it grows at most exponentially as θ decreases, with the exponent being less than γ. In particular, this is true for all families of sets bounded in the past.

Theorem 2.5. *Suppose that $f \in L_2^{\mathrm{loc}}(\mathbb{R}, V^*)$ satisfies (2.26). Then the family of trajectory spaces $\mathbf{H}^+$ has a minimal trajectory pullback $\mathfrak{D}$-attractor $\mathbf{U}$ and a global pullback $\mathfrak{D}$-attractor $\mathbf{A} = \mathbf{U}(0)$.*

Proof. We construct a family of sets $\mathbf{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) that is $\mathcal{T}$-precompact and pullback absorbing and use Theorems 2.1 and 2.3.

Define P_θ ($\theta \in \mathbb{R}$) to be the set of pairs of functions $(v, \tau) \in \mathcal{T}$ satisfying

$$\begin{aligned} &\|v\|^2_{L_\infty(t,t+1;H)} + \|\tau\|^2_{L_\infty(t,t+1;L_2)} + \|v\|^2_{L_2(t,t+1;V)} \\ &\leq C\left\{\|f\|^2_{L_2(t+\theta,t+\theta+1;V^*)} + \int_\infty^{t+1} e^{-2\gamma(t-s)}\|f(s+\theta)\|^2_{V^*}\,ds + 1\right\} \quad (t \in \mathbb{R}) \end{aligned} \tag{2.30}$$

$$\begin{aligned} &\|v'\|_{L_{4/3}(t,t+1,V^*)} + \|\tau'\|_{L_2(t,t+1;H^{-2})} \\ &\leq K_1(\|v\|_{L_\infty(t,t+1;H)}, \|v\|_{L_2(t,t+1;V)}, \|\tau\|_{L_\infty(t,t+1;L_2)}, \|f\|_{L_2(t+\theta,t+\theta+1;V^*)},) \end{aligned} \tag{2.31}$$

for all $t \geq 0$, where C is the constant from (2.28) and K_1 is the function from (2.25).

We claim that the family $\mathbf{P} = \{P_\theta\}$ is $\mathcal{T}$-precompact.

Fix $\theta \in \mathbb{R}$. It follows from (2.30) and (2.31) that for any $t \geq 0$ the set P_θ is bounded in $L_\infty(t, t+1; H \times L_2)$ and the set $P'_\theta = \{(v', \tau') \colon (v, \tau) \in P_\theta\}$ is bounded in $L_{4/3}(t, t+1; V^* \times H^{-2})$. By the Aubin–Lions lemma for the spaces $H \times L_2 \subset E_0 \subset V^* \times H^{-2}$ we have that P_θ is precompact in $C([t, t+1]; E_0)$. As t was arbitrary, we see that P_θ is precompact in $C(\mathbb{R}_+; E_0)$.

Condition (ii′) of Remark 2.1 holds with the function $\tilde{\varphi}_\theta$ defined by

$$(\tilde{\varphi}_\theta(t))^2 = C\left\{\|f\|^2_{L_2(t+\theta, t+\theta+1; V^*)} + \int_\infty^{t+1} e^{-2\gamma(t-s)}\|f(s+\theta)\|^2_{V^*}\,ds + 1\right\}.$$

Indeed, (2.30) implies

$$\|(v, \tau)\|^2_{L_\infty(t, t+1; E)} \leq \|v\|^2_{L_\infty(t, t+1; H)} + \|\tau\|^2_{L_\infty(t, t+1; L_2)} \leq (\tilde{\varphi}_\theta(t))^2.$$

for any pair $(v, \tau) \in P_\theta$ for all $t \geq 0$. It follows from (2.26) that $\tilde{\varphi}_\theta$ is finite, as

$$\int_\infty^{t+1} e^{-2\gamma(t-s)}\|f(s+\theta)\|^2_{V^*}\,ds \leq e^{2\gamma\xi}\int_{-\infty}^{\theta+t+1} e^{-2\gamma(t-\xi)}\|f(\xi)\|^2_{V^*}\,d\xi.$$

It is obvious that $\tilde{\varphi}_\theta$ is continuous in t.

Thus, $\mathbf{P}$ is $\mathcal{T}$-precompact.

Now we check that the conditions of Definition 2.2 hold for $\mathbf{P}$. Take $\mathbf{D} = \{D_\theta\} \in \mathfrak{D}$ and choose $\theta \in \mathbb{R}$. We must prove that there exists $\vartheta_{\mathbf{D}}(\theta) \leq \theta$ such that

$$\mathrm{T}(\theta - \xi)\mathcal{H}^+_\xi(\mathbf{D}) \subset P_\theta \tag{2.32}$$

whenever $\xi \leq \vartheta_{\mathbf{D}}(\theta)$, and the function $\vartheta_{\mathbf{D}}$ increases.

According to the way we defined $\mathfrak{D}$, there exists a function $r_{\mathbf{D}} \colon \mathbb{R} \to \mathbb{R}_+$ such that any $(v_0, \tau_0) \in D_\theta$ satisfies $\|v_0\|^2_H + \|\tau_0\|^2_{L_2} \leq (r_{\mathbf{D}}(\theta))^2$ and the function $\chi_{\mathbf{D}}(\theta) = e^{2\gamma\theta}(r_{\mathbf{D}}(\theta))^2$ is increasing as θ increases and tends to 0 as $\theta \to -\infty$. By monotonicity, χ has the inverse χ^{-1}, which also increases.

If $\sup_{\xi \in \mathbb{R}} \chi(\xi) \leq e^{2\gamma\theta}$, set $\vartheta_{\mathbf{D}}(\theta) = \theta$, otherwise set $\vartheta_{\mathbf{D}}(\theta) = \min\{\chi^{-1}(e^{2\gamma\theta}), \theta\}$. In both cases $\vartheta_{\mathbf{D}}(\theta) \leq \theta$. Also, for $\xi \leq \vartheta_{\mathbf{D}}(\theta)$ we have $\chi(\xi) \leq \chi(\vartheta_{\mathbf{D}}(\theta)) \leq e^{2\gamma\theta}$ by the monotonicity of χ or, equivalently,

$$e^{-2\gamma(\theta-\xi)}(r_{\mathbf{D}}(\xi))^2 \leq 1 \quad (\xi \leq \vartheta_{\mathbf{D}}(\theta)). \tag{2.33}$$

It only remains to show the inclusion (2.32) for $\xi \leq \vartheta_{\mathbf{D}}(\theta)$. Take $(\tilde{v}, \tilde{\tau}) \in \mathrm{T}(\theta-\xi)\mathcal{H}^+_\xi(\mathbf{D})$, then $\tilde{v} = \mathrm{T}(\theta-\xi)v$, $\tilde{\tau} = \mathrm{T}(\theta-\xi)\tau$ for the trajectory $(v, \tau) \in \mathcal{H}^+_\xi$, and $(v(0), \tau(0)) \in D_\xi$. We estimate the initial data of the trajectory by means of (2.33) and obtain

$$e^{-2\gamma(\theta-\xi)}(\|v(0)\|^2_H + \|\tau(0)\|^2_{L_2}) \leq e^{-2\gamma(\theta-\xi)}(r_{\mathbf{D}}(\xi))^2 \leq 1. \tag{2.34}$$

Let us check that inequality (2.30) holds with $v = \tilde{v}$ and $\tau = \tilde{\tau}$. Using the definition of the trajectory space $\mathcal{H}_\xi^+$ and inequality (2.34), we obtain:

$$
\begin{aligned}
&\|\tilde v\|^2_{L_\infty(t,t+1;H)} + \|\tilde\tau\|^2_{L_\infty(t,t+1;L_2)} + \|\tilde v\|^2_{L_2(t,t+1;V)} \\
&= \|v\|^2_{L_\infty(\theta-\xi+t,\theta-\xi+t+1;H)} + \|\tau\|^2_{L_\infty(\theta-\xi+t,\theta-\xi+t+1;L_2)} + \|v\|^2_{L_2(\theta-\xi+t,\theta-\xi+t+1;V)} \\
&\quad \le C\Bigg\{ \|f\|^2_{L_2(t+\theta,t+\theta+1;V^*)} + \int_{-\infty}^{\theta-\xi+t+1} e^{-2\gamma(\theta-\xi+t-s)}\|f(s+\xi)\|^2_{V^*}\,ds \\
&\qquad\qquad + e^{-2\gamma(\theta-\xi+t)}(\|v(0)\|^2_H + \|\tau(0)\|^2_{L_2})\Bigg\} \\
&\qquad \le C\Bigg\{ \|f\|^2_{L_2(t+\theta,t+\theta+1;V^*)} + \int_{-\infty}^{t+1} e^{-2\gamma(t-s)}\|f(s+\theta)\|^2_{V^*}\,ds + 1\Bigg\},
\end{aligned}
$$

as claimed.

Inequality (2.31) for $(\tilde v,\tilde\tau)$ on the segment $[t,t+1]$ is equivalent for the same inequality for (v,τ) on the segment $[\theta-\xi+t,\theta-\xi+t+1]$.

Thus, $(\tilde v,\tilde\tau)\in P_\theta$, and inclusion (2.32) is proved.

We have proved that the family $\mathbf{P}$ is $\mathcal{T}$-precompact and pullback $\mathfrak{D}$-absorbing, so the theorem follows from Theorems 2.2 and 2.3. $\square$

3. Existence of weak solutions

In this section we prove Theorem 2.4.

Proof of Theorem 2.4. We start with establishing the inequality (2.23) for the solutions of an approximating problem.

Take $\varepsilon,\delta,T>0$ and approximate (2.20)–(2.22) on $[0,T]$ by the following problem:

$$
\begin{aligned}
&\frac{d}{dt}(\tau,\Phi) + \frac{1}{\lambda_1}(\tau,\Phi) - \sum_{i=1}^n \left(\frac{v_i\tau}{1+\delta\left(\frac{|\tau|^2}{2\mu_2}+|v|^2\right)}, \frac{\partial\Phi}{\partial x_i}\right) \\
&\quad + 2\mu_2(v,\operatorname{Div}\Phi) + \frac{\varepsilon}{\lambda_1}(\nabla\tau,\nabla\Phi) = 0 \quad (\Phi\in H^1),
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
&\frac{d}{dt}(v,\varphi) - \sum_{i=1}^n \left(\frac{v_i v}{1+\delta\left(\frac{|\tau|^2}{2\mu_2}+|v|^2\right)}, \frac{\partial\varphi}{\partial x_i}\right) + \mu_1(\nabla v,\nabla\varphi) \\
&\quad + (\tau,\nabla\varphi) = \langle F,\varphi\rangle \quad (\varphi\in V).
\end{aligned} \tag{3.2}
$$

$$
v(0)=v_0, \quad \tau(0)=\tau_0. \tag{3.3}
$$

It is shown in [26, 13] that problem (3.1)–(3.3) has a solution (v,τ) in the classes

$$
\left.\begin{aligned}
v\in L_2(0,T;V),\ v'\in L_2(0,T;V^*),\\
\tau\in L_2(0,T;H_0^1(\Omega,M_S(n))),\ \tau'\in L_2(0,T;H^{-1}(\Omega,M_S(n))).
\end{aligned}\right\} \tag{3.4}
$$

Substitute $\varphi = v$ in (3.2) and $\Phi = \tau/(2\mu_2)$ in (3.1) and add the equalities thus obtaining, then

$$\frac{1}{2}\frac{d}{dt}\|v\|_H^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\tau\|_{L_2}^2 + \mu_1\|v\|_V^2 + \frac{1}{2\lambda_1\mu_2}\|\tau\|_{L_2}^2 + \frac{\varepsilon}{2\lambda_1\mu_2}\|\nabla\tau\|_{L_2}$$
$$-\left\{\frac{1}{2\mu_2}\sum_{i=1}^{n}\left(\frac{v_i\tau}{1+\delta\left(\frac{|\tau|^2}{2\mu_2}+|v|^2\right)}, \frac{\partial\tau}{\partial x_i}\right) + \left(\sum_{i=1}^{n}\frac{v_i v}{1+\delta\left(\frac{|\tau|^2}{2\mu_2}+|v|^2\right)}, \frac{\partial v}{\partial x_i}\right)\right\}$$
$$+\Big\{(v, \operatorname{Div}\tau) + (\tau, \nabla v)\Big\} = \langle F, v\rangle. \quad (3.5)$$

In (3.5) the terms in braces annihilate (see e. g. [13, Subsection 6.1.2]). Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|_H^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\tau\|_{L_2}^2 + \mu_1\|v\|_V^2 + \frac{1}{2\lambda_1\mu_2}\|\tau\|_{L_2}^2 + \frac{\varepsilon}{2\lambda_1\mu_2}\|\nabla\tau\|_{L_2}^2 = \langle F, v\rangle, \quad (3.6)$$

which holds a. e. on $(0, T)$.

In (3.6) substitute $v(t) = e^{-\gamma t}\bar{v}(t)$, $\tau(t) = e^{-\gamma t}\bar{\tau}(t)$. We get

$$\frac{1}{2}\frac{d}{dt}\left(e^{-2\gamma t}\|\bar{v}\|_H^2\right) + \frac{1}{4\mu_2}\frac{d}{dt}\left(e^{-2\gamma t}\|\bar{\tau}\|_{L_2}^2\right) + \mu_1\left(e^{-2\gamma t}\|\bar{v}\|_V^2\right)$$
$$+\frac{1}{2\lambda_1\mu_2}\left(e^{-2\gamma t}\|\bar{\tau}\|_{L_2}^2\right) + \frac{\varepsilon}{2\lambda_1\mu_2}\left(e^{-2\gamma t}\|\nabla\bar{\tau}\|_{L_2}^2\right) = \langle F, e^{-\gamma t}\bar{v}\rangle.$$

Expanding the derivative of the product and multiplying both sides by $e^{2\gamma t}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\bar{v}\|_H^2 - \gamma\|\bar{v}\|_H^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}\|_{L_2}^2 - \frac{\gamma}{2\mu_2}\|\bar{\tau}\|_{L_2}^2 + \mu_1\|\bar{v}\|_V^2 + \frac{1}{2\lambda_1\mu_2}\|\bar{\tau}\|_{L_2}^2$$
$$+\frac{\varepsilon}{2\lambda_1\mu_2}\|\nabla\bar{\tau}\|_{L_2}^2 = e^{\gamma t}\langle F, \bar{v}\rangle. \quad (3.7)$$

By the definition of γ we have

$$-\gamma\|\bar{v}\|_H^2 + \mu_1\|\bar{v}\|_V^2 \geq -\frac{\mu_1}{2K_0^2}\cdot K_0^2\|\bar{v}\|_V^2 + \mu_1\|\bar{v}\|_V^2 = \frac{\mu_1}{2}\|\bar{v}\|_V^2;$$
$$-\frac{\gamma}{2\mu_2}\|\bar{\tau}\|_{L_2}^2 + \frac{1}{2\lambda_1\mu_2}\|\bar{\tau}\|_{L_2}^2 \geq -\frac{1}{2\lambda_1\mu_2}\|\bar{\tau}\|_{L_2}^2 + \frac{1}{2\lambda_1\mu_2}\|\bar{\tau}\|_{L_2}^2 = 0.$$

Using (3.7) and the last two estimates, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\bar{v}\|_H^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}\|_{L_2}^2 + \frac{\mu_1}{2}\|\bar{v}\|_V^2 \leq e^{\gamma t}\|F\|_{V^*}\|\bar{v}\|_V.$$

Using the Cauchy inequality to estimate the right-hand side, we have

$$\frac{1}{2}\frac{d}{dt}\|\bar{v}\|_H^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}\|_{L_2}^2 + \frac{\mu_1}{2}\|\bar{v}\|_V^2 \leq \frac{e^{2\gamma t}}{2\mu_1}\|F\|_{V^*}^2 + \frac{\mu_1}{2}\|\bar{v}\|_V^2,$$

whence

$$\frac{1}{2}\frac{d}{dt}\|\bar{v}\|_H^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\bar{\tau}\|_{L_2}^2 \le \frac{e^{2\gamma t}}{2\mu_1}\|F\|_{V^*}^2. \tag{3.8}$$

Integrating from 0 to t, we obtain

$$\frac{1}{2}\|\bar{v}\|_H^2 + \frac{1}{4\mu_2}\|\bar{\tau}\|_{L_2}^2 \le \frac{1}{2}\|v_0\|_H^2 + \frac{1}{4\mu_2}\|\tau_0\|_{L_2}^2 + \frac{1}{2\mu_1}\int_0^t e^{2\gamma s}\|F(s)\|_{V^*}^2\, ds.$$

Now reverting to the functions v and τ and multiplying both sides by $2e^{-2\gamma t}$, we get

$$\|v(t)\|_H^2 + \frac{1}{2\mu_2}\|\tau(t)\|_{L_2}^2 \le \left(\|v_0\|_H^2 + \frac{1}{2\mu_2}\|\tau_0\|_{L_2}^2\right)e^{-2\gamma t} + \frac{1}{\mu_1}\int_0^t e^{-2\gamma(t-s)}\|F(s)\|_{V^*}^2\, ds.$$

By (3.4), we have $v \in C([0,T];H)$ and $\tau \in C([0,T];L_2)$, so the last inequality holds everywhere on $[0,T]$ and we can take the maximum over $[t,t+1] \subset [0,T]$. We obtain the estimate

$$\max_{s\in[t,t+1]}\left(\|v(s)\|_H^2 + \frac{1}{2\mu_2}\|\tau(s)\|_{L_2}^2\right) \le \left(\|v_0\|_H^2 + \frac{1}{2\mu_2}\|\tau_0\|_{L_2}^2\right)e^{-2\gamma t} + \frac{1}{\mu_1}\int_0^{t+1} e^{-2\gamma(t-s)}\|F(s)\|_{V^*}^2\, ds. \tag{3.9}$$

Now we go back to (3.6). Estimate its right-hand side by means of the Cauchy inequality:

$$|\langle F, v\rangle| \le \|F\|_{V^*}\|v\|_V \le \frac{1}{2\mu_1}\|F\|_{V^*}^2 + \frac{\mu_1}{2}\|v\|_V^2,$$

and we have

$$\frac{1}{2}\frac{d}{dt}\|v\|_H^2 + \frac{1}{4\mu_2}\frac{d}{dt}\|\tau\|_{L_2}^2 + \frac{\mu_1}{2}\|v\|_V^2 + \frac{\varepsilon}{2\lambda_1\mu_2}\|\nabla\tau\|_{L_2}^2 \le \frac{1}{2\mu_1}\|F\|_{V^*}^2.$$

Integrating over $[t,t+1] \subset [0,T]$, we obtain

$$\frac{1}{2}\|v(t+1)\|_H^2 - \frac{1}{2}\|v(t)\|_H^2 + \frac{1}{4\mu_2}\|\tau(t+1)\|_{L_2}^2 - \frac{1}{4\mu_2}\|\tau(t)\|_{L_2}^2 + \frac{\mu_1}{2}\int_t^{t+1}\|v(s)\|_V^2 ds + \frac{\varepsilon}{2\lambda_1\mu_2}\int_t^{t+1}\|\nabla\tau(s)\|_{L_2}^2 ds \le \frac{1}{2\mu_1}\int_t^{t+1}\|F(s)\|_{V^*}^2\, ds.$$

Adding the last inequality to (3.9), we get

$$\begin{aligned}
&\frac{1}{2}\|v(t+1)\|_H^2 + \frac{1}{4\mu_2}\|\tau(t+1)\|_{L_2}^2 \\
&\qquad + \max_{s\in[t,t+1]}\left(\frac{1}{2}\|v(s)\|_H^2 + \frac{1}{4\mu_2}\|\tau(s)\|_{L_2}^2\right) \\
&\qquad + \frac{\mu_1}{2}\int_t^{t+1}\|v(s)\|_V^2\, ds + \frac{\varepsilon}{2\lambda_1\mu_2}\int_t^{t+1}\|\nabla\tau(s)\|_{L_2}^2\, ds \\
&\le \left(\|v_0\|_H^2 + \frac{1}{2\mu_2}\|\tau_0\|_{L_2}^2\right)e^{-2\gamma t} + \frac{1}{2\mu_1}\int_t^{t+1}\|F(s)\|_{V^*}^2\, ds + \frac{1}{\mu_1}\int_0^t e^{-2\gamma(t-s)}\|F(s)\|_{V^*}^2\, ds
\end{aligned}$$

This inequality implies (2.23) for the solutions of the approximating problem (3.1)–(3.3). Indeed, it suffices to use the obvious inequality $\max(A+B) \geq (\max A + \max B)/2$ valid for nonnegative A and B.

Now we proceed to estimating the solutions of the original problem (2.20)–(2.22). Take sequences $T_m \to \infty$, $\varepsilon_m \to 0$, and $\delta_m \to 0$ and consider pairs of functions (v_m, τ_m) solving (3.1)–(3.3) on $[0, T_m]$ and extended by constant to a continuous function on $\mathbb{R}_+$. Given $T \geq 1$ and large enough m, the function (v_m, τ_m) solve the approximating problem on $[0, T]$ and satisfy (2.23) whenever $t+1 \leq T$, whence it is clear that the sequence $\{v_m\}$ is bounded in $L_\infty(0, T; H)$ and $L_2(0, T, V)$ and the sequence $\{\tau_m\}$ is bounded in $L_\infty(0, T; L_2)$. Refining the sequence if necessary, we can assume that there exist functions

$$\begin{gathered} v \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H), \\ \tau \in L_\infty(\mathbb{R}_+; L_2) \end{gathered}$$

such that for any $T \geq 1$ we have $v_m \to v$ weakly in $L_2(0, T; V)$ and weakly* in $L_\infty(0, T; H)$ and $\tau_m \to \tau$ weakly* in $L_\infty(0, T; L_2)$. It is shown in [26, 13] that (v, τ) solves (2.20)–(2.22) on $[0, T]$ for any $T \geq 1$, i. e. (v, τ) is a weak solution of (2.10)–(2.13) on $\mathbb{R}_+$. Finally, by the lower semicontinuity of the norm, we have

$$\begin{aligned} &\frac{1}{4}\|v\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|v\|^2_{L_2(t,t+1;V)} \\ &\leq \frac{1}{4}\varliminf_{m\to\infty}\|v_m\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\varliminf_{m\to\infty}\|\tau_m\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\varliminf_{m\to\infty}\|v_m\|^2_{L_2(t,t+1;V)} \\ &\quad \leq \varliminf_{m\to\infty}\left(\frac{1}{4}\|v_m\|^2_{L_\infty(t,t+1;H)} + \frac{1}{8\mu_2}\|\tau_m\|^2_{L_\infty(t,t+1;L_2)} + \frac{\mu_1}{2}\|v_m\|^2_{L_2(t,t+1;V)}\right) \\ &\leq \frac{1}{2\mu_1}\|F\|^2_{L_2(t,t+1;V^*)} + \frac{1}{\mu_1}\int_0^{t+1} e^{-2\gamma(t-s)}\|F(s)\|^2_{V^*}\,ds + \left(\|v_0\|^2_H + \frac{1}{2\mu_2}\|\tau_0\|^2_{L_2}\right)e^{-2\gamma t}. \end{aligned}$$

Thus the weak solution we have constructed satisfies (2.23), which concludes the proof. □

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