ON 2-STAR-PERMUTABILITY IN REGULAR MULTI-POINTED CATEGORIES

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Abstract. 2-star-permutable categories were introduced in a joint work with Z. Janelidze and A. Ursini as a common generalisation of regular Mal'tsev categories and of normal subtractive categories. In the present article we first characterise these categories in terms of what we call star-regular pushouts. We then show that the 3×3 Lemma characterising normal subtractive categories and the Cuboid Lemma characterising regular Mal'tsev categories are special instances of a more general homological lemma for star-exact sequences. We prove that 2-star-permutability is equivalent to the validity of this lemma for a star-regular category.

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Introduction

The theory of *Mal'tsev categories* in the sense of A. Carboni, J. Lambek and M.C. Pedicchio [6] provides a beautiful example of the way how categorical algebra leads to a structural understanding of algebraic varieties (in the sense of universal algebra). Among regular categories, Mal'tsev categories are characterised by the property of 2-permutability of equivalence relations: given two equivalence relations R and S on the same object A, the two relational composites RS and SR are equal:

$$RS = SR.$$

In the case of a variety of universal algebras this property is actually equivalent to the existence of a ternary term $p(x, y, z)$ satisfying the identities $p(x, y, y) = x$ and $p(x, x, y) = y$ [20]. In the pointed context, that is when the category has a zero object, there is also a suitable notion of 2-permutability, called "2-permutability at 0" [21]. In a variety this property can be expressed by requiring that, whenever for a given element x in an algebra A there is an element y with $xRyS0$ (here 0 is the unique constant in A), then there is also an element z in A with $xSzR0$. The validity of this property is equivalent to the existence of a binary term $s(x, y)$ such that the identities $s(x, 0) = x$ and $s(x, x) = 0$ hold true [21]. Among regular categories, the ones where the property of 2-permutability at 0 holds true are precisely the *subtractive categories* introduced in [16].

The aim of this paper is to look at regular Mal'tsev and at subtractive categories as special instances of the general notion of 2-star-permutable categories introduced in collaboration with Z. Janelidze and A. Ursini in [9]. This generalisation is achieved by working in the context of a *regular multi-pointed category*, i.e. a regular category equipped with an ideal $\mathcal{N}$ of distinguished morphisms [7]. When

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$\mathcal{N}$ is the class of all morphisms, a situation which we refer to as the *total context*, regular multi-pointed categories are just regular categories, and 2-star-permutable categories are precisely the regular Mal'tsev categories. When $\mathcal{N}$ is the class of all zero morphisms in a pointed category, we call this the *pointed context*, regular multi-pointed categories are regular pointed categories, and 2-star-permutable categories are the regular subtractive categories.

This paper follows the same line of research as in [9] which was mainly focused on the property of 3-star-permutability, a generalised notion which captures Goursat categories in the total context and, again, subtractive categories in the pointed context.

In this work we study two remarkable aspects of the property of 2-star-permutability. First we provide a characterisation of 2-star-permutable categories in terms of a special kind of pushouts (Proposition 2.4), that we call *star-regular pushouts* (Definition 2.2). Then we examine a homological diagram lemma of star-exact sequences, which can be seen as a generalisation of the 3×3 Lemma, whose validity is equivalent to 2-star-permutability. We call this lemma the Star-Upper Cuboid Lemma (Theorem 3.3). The validity of this lemma turns out to give at once a characterisation of regular Mal'tsev categories (extending a result in [11]) and, in the pointed context, a characterisation of those normal categories which are subtractive (this was first discovered in [17]).

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1. Star-regular categories

1.1. **Regular categories and relations.** A finitely complete category $\mathbb{C}$ is said to be a *regular* category [1] when any kernel pair has a coequaliser and, moreover, regular epimorphisms are stable under pullbacks. In a regular category any morphism $f \colon X \to Y$ has a factorisation $f = m \cdot p$, where p is a regular epimorphism and m is a monomorphism. The corresponding (regular epimorphism, monomorphism) factorisation system is then stable under pullbacks.

A relation ϱ from X to Y is a subobject $\langle \varrho_1, \varrho_2 \rangle \colon R \rightarrowtail X \times Y$. The opposite relation, denoted ϱ°, is given by the subobject $\langle \varrho_2, \varrho_1 \rangle \colon R \rightarrowtail Y \times X$. We identify a morphism $f : X \to Y$ with the relation $\langle 1_X, f \rangle \colon X \rightarrowtail X \times Y$ and write f° for the opposite relation. Given another relation σ from Y to Z, the composite relation of ϱ and σ is a relation $\sigma\varrho$ from X to Z. With this notation, we can write the above relation as $\varrho = \varrho_2 \varrho_1^\circ$. The following properties are well known (see [5], for instance); we collect them in a lemma for future references.

Lemma 1.1. *Let $f : X \to Y$ be any morphism in a regular category $\mathbb{C}$. Then:*

(a) $ff^\circ f = f$ *and* $f^\circ f f^\circ = f^\circ$*;*
(b) $ff^\circ = 1_Y$ *if and only if f is a regular epimorphism.*

A kernel pair of a morphism $f : X \to Y$, denoted by

$$(\pi_1, \pi_2) \colon \mathsf{Eq}(f) \rightrightarrows X,$$

is called an *effective equivalence relation*; we write it either as $\mathsf{Eq}(f) = f^\circ f$, or as $\mathsf{Eq}(f) = \pi_2 \pi_1^\circ$, as mentioned above. When f is a regular epimorphism, then f is the coequaliser of π_1 and π_2 and the diagram

$$\mathsf{Eq}(f) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

is called an *exact fork*. In a regular category any effective equivalence relation is the kernel pair of a regular epimorphism.

1.2. **Star relations.** We now recall some notions introduced in [10], which are useful to develop a unified treatment of pointed and non-pointed categorical algebra. Let $\mathbb{C}$ denote a category with finite limits, and $\mathcal{N}$ a distinguished class of morphisms that forms an *ideal*, i.e. for any composable pair of morphisms g, f, if either g or f belongs to $\mathcal{N}$, then the composite $g \cdot f$ belongs to $\mathcal{N}$. An $\mathcal{N}$*-kernel* of a morphism $f : X \to Y$ is defined as a morphism $\mathrm{n}_f : \mathrm{N}_f \to X$ such that $f \cdot \mathrm{n}_f \in \mathcal{N}$ and n_f is universal with this property (note that such n_f is automatically a monomorphism). A pair of parallel morphisms, denoted by $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with $\sigma_1 \in \mathcal{N}$, is called a *star*; it is called a monic star, or a *star relation*, when the pair (σ_1, σ_2) is jointly monomorphic.

Given a relation $\varrho = (\varrho_1, \varrho_2) : R \rightrightarrows X$ on an object X, we denote by $\varrho^* : R^* \rightrightarrows X$ the biggest subrelation of ϱ which is a (monic) star. When $\mathbb{C}$ has $\mathcal{N}$-kernels, it can be constructed by setting $\varrho^* = (\,\varrho_1 \cdot \mathrm{n}_{\rho_1}, \varrho_2 \cdot \mathrm{n}_{\rho_1}\,)$, where n_{ρ_1} is the $\mathcal{N}$-kernel of ϱ_1. In particular, if we denote the discrete equivalence relation on an object X by $\Delta_X = (1_X, 1_X) : X \rightrightarrows X$, then $\Delta_X^* = (\,\mathrm{n}_{1_X}, \mathrm{n}_{1_X}\,)$, where n_{1_X} is the $\mathcal{N}$-kernel of 1_X.

The *star-kernel* of a morphism $f : X \to Y$ is a universal star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with the property $f \cdot \sigma_1 = f \cdot \sigma_2$; it is easy to see that the star-kernel of f coincides with $\mathsf{Eq}(f)^* \rightrightarrows X$ whenever $\mathcal{N}$-kernels exist.

A category $\mathbb{C}$ equipped with an ideal $\mathcal{N}$ of morphisms is called a *multi-pointed* category [10]. If, moreover, every morphism admits an $\mathcal{N}$-kernel, then $\mathbb{C}$ will be called a *multi-pointed category with kernels*.

Definition 1.2. [10] A regular multi-pointed category $\mathbb{C}$ with kernels is called a *star-regular category* when every regular epimorphism in $\mathbb{C}$ is a coequaliser of a star.

In the total context stars are pairs of parallel morphisms, $\mathcal{N}$-kernels are isomorphisms, star-kernels are kernel pairs and a star-regular category is precisely a regular category. In the pointed context, the first morphism σ_1 in a star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ is the unique null morphism $S \to X$ and hence a star σ can be identified with a morphism (its second component σ_2). Then, $\mathcal{N}$-kernels and star-kernels become the usual kernels, and a star-regular category is the same as a normal category [18], i.e. a pointed regular category in which any regular epimorphism is a normal epimorphism.

1.3. **Calculus of star relations.** The calculus of star relations [9] can be seen as an extension of the usual calculus of relations (in a regular category) to the regular multi-pointed context. First of all note that for any relation $\varrho : R \rightrightarrows X$ we have

$$\varrho^* = \varrho\Delta_X^*.$$

Inspired by this formula, for any relation ϱ from X to an object Y, we define

$$\varrho^* = \varrho\Delta_X^* \quad \text{and} \quad {}^*\varrho = \Delta_Y^*\varrho.$$

Note that associativity of composition yields

$$^*(\varrho^*) = (^*\varrho)^*$$

and so we can write $^*\varrho^*$ for the above.

For any relation σ (from some object Y to Z), the associativity of composition also gives

$$(\sigma^*)\varrho = \sigma(^*\varrho),$$

and

$$(\sigma\varrho)^* = \sigma\varrho^*.$$

It is easy to verify that for any morphism $f : X \to Y$ we have

$$f^* = {}^*f^* \quad \text{and} \quad {}^*f^\circ = {}^*f^{\circ *}.$$

2. 2-STAR-PERMUTABILITY AND STAR-REGULAR PUSHOUTS

Recall that a finitely complete category $\mathbb{C}$ is called a *Mal'tsev category* when any reflexive relation in $\mathbb{C}$ is an equivalence relation [6, 5]. We recall the following well known characterisation of the regular categories which are Mal'tsev categories:

Proposition 2.1. *A regular category $\mathbb{C}$ is a Mal'tsev category if and only if the composition of effective equivalence relations in $\mathbb{C}$ is commutative:*

$$\mathsf{Eq}(f)\mathsf{Eq}(g) = \mathsf{Eq}(g)\mathsf{Eq}(f)$$

for any pair of regular epimorphisms f and g in $\mathbb{C}$ with the same domain.

There are many known characterisations of regular Mal'tsev categories (see Section 2.5 in [2], for instance, and references therein). The one that will play a central role in the present work is expressed in terms of commutative diagrams of the form

$$\begin{tikzcd} C \arrow[r, two heads, "c"] \arrow[d, shift right, "g"'] & A \arrow[d, shift right, "f"'] \\ D \arrow[r, two heads, "d"'] \arrow[u, shift right, "t"'] & B, \arrow[u, shift right, "s"'] \end{tikzcd} \tag{1}$$

where f and g are split epimorphisms ($f \cdot s = 1_B$, $g \cdot t = 1_D$), $f \cdot c = d \cdot g$, $s \cdot d = c \cdot t$, and c and d are regular epimorphisms. A diagram of type (1) is always a pushout; it is called a *regular pushout* [4] (alternatively, a *double extension* [15, 13]) when, moreover, the canonical morphism $\langle g, c \rangle \colon C \twoheadrightarrow D \times_B A$ to the pullback $D \times_B A$ of d and f is a regular epimorphism. Among regular categories, Mal'tsev categories can be characterized as those ones where any square (1) is a regular pushout: this easily follows from the results in [4], and a simple proof of this fact is given in [12].

Observe that a commutative diagram of type (1) is a regular pushout if and only if $cg^\circ = f^\circ d$ or, equivalently, $gc^\circ = d^\circ f$. This suggests to introduce the following notion:

Definition 2.2. A commutative diagram (1) is a *star-regular pushout* if it satisfies the identity $cg^{\circ *} = f^\circ d^*$ (or, equivalently, $gc^{\circ *} = d^\circ f^*$).

Diagrammatically, the property of being a star-regular pushout can be expressed as follows. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{N}_g \arrow[dd, tail, "\mathrm{n}_g"'] \arrow[dr, tail] & & & \\
& \mathrm{N}_a \arrow[r, dotted] \arrow[dd, tail, "\mathrm{n}_a"] & \mathrm{N}_x \arrow[ddd, tail, "\mathrm{n}_x"] & \\
C \arrow[rrr, two heads, "c"] \arrow[dr, two heads, "p"] \arrow[ddd, shift right, "g"'] & & & A \arrow[ddd, shift right, "f"'] \\
& M \arrow[urr, "b"] \arrow[dr, "m"'] \arrow[ddl, "a"'] & & \\
& & D \times_B A \arrow[uur, "y"'] \arrow[dll, "x"] & \\
D \arrow[rrr, two heads, "d"'] \arrow[uuu, shift right] & & & B, \arrow[uuu, shift right]
\end{tikzcd} \tag{2}$$

where $(D \times_B A, x, y)$ is the pullback of (f, d), $m \cdot p$ is the (regular epimorphism, monomorphism) factorisation of the induced morphism $\langle g, c\rangle \colon C \to D \times_B A$. Then the identity $cg^\circ = ba^\circ$ allows one to identify $cg^{\circ *}$ with the relation $(a \cdot \mathrm{n}_a, b \cdot \mathrm{n}_a)$, while $f^\circ d = yx^\circ$ says that $f^\circ d^*$ can be identified with the relation $(x \cdot \mathrm{n}_x, y \cdot \mathrm{n}_x)$. Accordingly, diagram (1) is a star-regular pushout precisely when the dotted arrow from N_a to N_x is an isomorphism. Notice that in the total context the $\mathcal{N}$-kernels are isomorphisms, so that m is an isomorphism if and only if (1) is a regular pushout, as expected.

The "star-version" of the notion of Mal'tsev category can be defined as follows:

Definition 2.3. [9] A regular multi-pointed category with kernels $\mathbb{C}$ is said to be a 2-*star-permutable category* if

$$\mathsf{Eq}(f)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(f)^*$$

for any pair of regular epimorphisms f and g in $\mathbb{C}$ with the same domain.

One can check that the equality $\mathsf{Eq}(f)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(f)^*$ in the definition above can be actually replaced by $\mathsf{Eq}(f)\mathsf{Eq}(g)^* \leq \mathsf{Eq}(g)\mathsf{Eq}(f)^*$.

In the total context the property of 2-star-permutability characterises the regular categories which are Mal'tsev. In the pointed context this same property characterises the regular categories which are subtractive [16] (this follows from the characterisation of subtractivity given in Theorem 6.9 in [17]).

The next result gives a useful characterisation of 2-star-permutable categories. Given a commutative diagram of type (1), we write $g\langle \mathsf{Eq}(c)\rangle$ and $g\langle \mathsf{Eq}(c)^*\rangle$ for the direct images of the relations $\mathsf{Eq}(c)$ and $\mathsf{Eq}(c)^*$ along the split epimorphism g. The vertical split epimorphisms are such that both the equalities $g\langle \mathsf{Eq}(c)\rangle = \mathsf{Eq}(d)$ and $g\langle \mathsf{Eq}(c)^*\rangle = \mathsf{Eq}(d)^*$ hold true in $\mathbb{C}$.

Proposition 2.4. *For a regular multi-pointed category with kernels $\mathbb{C}$ the following statements are equivalent:*

(a) $\mathbb{C}$ *is a 2-star-permutable category;*
(b) *any commutative diagram of the form (1) is a star-regular pushout.*

Proof. (a) $\Rightarrow$ (b) Given a pushout (1) we have

$$\begin{array}{lll} f^\circ d^* & = cc^\circ f^\circ d^* & \text{(Lemma 1.1(2))} \\ & = cg^\circ d^\circ d^* & (f \cdot c = d \cdot g) \\ & = cg^\circ g c^\circ c^* g^\circ & (\mathsf{Eq}(d)^* = g\langle \mathsf{Eq}(c)^* \rangle) \\ & = cc^\circ c g^\circ g^* g^\circ & (\mathsf{Eq}(g)\mathsf{Eq}(c)^* = \mathsf{Eq}(c)\mathsf{Eq}(g)^* \text{ by Definition 2.3}) \\ & \leq cc^\circ c g^\circ g g^\circ & (g^* \leq g) \\ & = cg^\circ. & \text{(Lemma 1.1(1))} \end{array}$$

Since $cg^{\circ *}$ is the largest star contained in cg°, it follows that $f^\circ d^* \leq cg^{\circ *}$. The inclusion $cg^{\circ *} \leq f^\circ d^*$ always holds, so that $cg^{\circ *} = f^\circ d^*$.
(b) $\Rightarrow$ (a) Let us consider regular epimorphisms $f\colon X \twoheadrightarrow Y$ and $g\colon X \twoheadrightarrow Z$. We want to prove that $\mathsf{Eq}(f)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(f)^*$. For this we build the following diagram

$$\begin{tikzcd} \mathsf{Eq}(f) \arrow[r, two heads, "c"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & g\langle \mathsf{Eq}(f) \rangle \arrow[d, shift right, "\rho_1"'] \arrow[d, shift left, "\rho_2"] \\ X \arrow[r, two heads, "g"'] \arrow[d, "f"'] & Z \\ Y & \end{tikzcd}$$

that represents the regular image of $\mathsf{Eq}(f)$ along g. The relation $g\langle \mathsf{Eq}(f) \rangle = (\rho_1, \rho_2)$ is reflexive and, consequently, ρ_1 is a split epimorphism. By assumption, we then know that the equality

$$(A) \qquad \rho_1^\circ g^* = c\pi_1^{\circ *}$$

holds true. This implies that

$$\begin{array}{lll} \mathsf{Eq}(f)\mathsf{Eq}(g)^* & = \pi_2 \pi_1^\circ g^\circ g^* & \\ & = \pi_2 c^\circ \rho_1^\circ g^* & (g \cdot \pi_1 = \rho_1 \cdot c) \\ & = \pi_2 c^\circ c \pi_1^{\circ *} & \text{(A)} \\ & \leq \pi_2 c^\circ c \pi_2^\circ \pi_2 \pi_1^{\circ *} & (\Delta_{\mathsf{Eq}(f)} \leq \pi_2^\circ \pi_2) \\ & = \mathsf{Eq}(g) \pi_2 \pi_1^{\circ *} & (\pi_2 \langle \mathsf{Eq}(c) \rangle = \mathsf{Eq}(g)) \\ & = \mathsf{Eq}(g)\mathsf{Eq}(f)^*, & \end{array}$$

where the equality $\pi_2\langle \mathsf{Eq}(c) \rangle = \mathsf{Eq}(g)$ follows from the fact that the split epimorphisms π_2 and ρ_2 induce a split epimorphism from $\mathsf{Eq}(c)$ to $\mathsf{Eq}(g)$. $\square$

In the total context, Proposition 2.4 gives the characterisation of regular Mal'tsev categories through regular pushouts (see [4] and Proposition 3.4 of [12]), as expected. In the pointed context, condition (b) of Proposition 2.4 translates into the pointed version of the *right saturation* property [9] for any commutative diagram of type (1): the induced morphism $\bar{c} : \mathsf{Ker}(g) \to \mathsf{Ker}(f)$, from the kernel of g to the kernel of f is also a regular epimorphism. This can be seen by looking at diagram (2), where the $\mathcal{N}$-kernels now represent actual kernels, so that $\mathsf{Ker}(a) = \mathsf{Ker}(x) = \mathsf{Ker}(f)$.

2.1. **The star of a pullback relation.** Consider the pullback relation $\pi = (\pi_1, \pi_2)$ of a pair (g, δ) of morphisms as in the diagram

$$\begin{tikzcd} W \times_D C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ W \arrow[r, "\delta"'] & D. \end{tikzcd}$$

The *star of the pullback relation* π is defined as $\pi^* = \pi\Delta_W^*$. It can be described as the universal relation $\nu = (\nu_1, \nu_2)$ from W to C such that $\nu_1 \in \mathcal{N}$ and $\delta \cdot \nu_1 = g \cdot \nu_2$ as in the diagram

$$\begin{tikzcd} (W \times_D C)^* \arrow[dr, "\mathrm{n}_{\pi_1}"] \arrow[drr, bend left, "\nu_2"] \arrow[ddr, bend right, "\nu_1"'] & & \\ & W \times_D C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ & W \arrow[r, "\delta"'] & D, \end{tikzcd}$$

where n_{π_1} is the $\mathcal{N}$-kernel of π_1, $\nu_1 = \pi_1 \cdot \mathrm{n}_{\pi_1}$ and $\nu_2 = \pi_2 \cdot \mathrm{n}_{\pi_1}$.

By using the composition of relations one has the equalities $\pi = \pi_2\pi_1^\circ = g^\circ\delta$, so that

$$\pi^* = \pi_2\pi_1^{\circ *} = g^\circ\delta^*.$$

In the total context, the star of a pullback relation is precisely that pullback relation. In the pointed context, the star of the pullback (relation) of (g, δ) is given by $\pi^* = (0, \ker(g))$.

A morphism $f\colon X \to Y$ in a multi-pointed category with kernels is said to be *saturating* [9] when the induced dotted morphism from the $\mathcal{N}$-kernel of 1_X to the $\mathcal{N}$-kernel of 1_Y making the diagram

$$\begin{tikzcd} N_{1_X} \arrow[r, dotted, two heads] \arrow[d, tail, "\mathrm{n}_{1_X}"'] & N_{1_Y} \arrow[d, tail, "\mathrm{n}_{1_Y}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commute is a regular epimorphism. All morphisms are saturating in the pointed context. This is also the case for any *quasi-pointed category* [3], namely a finitely complete category with an initial object 0 and a terminal object 1 such that the arrow $0 \to 1$ is a monomorphism. As in the pointed case, it suffices to choose for $\mathcal{N}$ the class of morphisms which factor through the initial object 0. In this case we shall speak of the *quasi-pointed context.* In the total context, any regular epimorphism is saturating. The proof of the following result is straightforward:

Lemma 2.5. [9] *Let $\mathbb{C}$ be a regular multi-pointed category with kernels. For a morphism $f : X \to Y$ the following conditions are equivalent:*

(a) *f is saturating;*
(b) *$\Delta_Y^* = f^* f^\circ$.*

The next result gives a characterisation of 2-star-permutable categories which will be useful in the following section.

Proposition 2.6. *For a regular multi-pointed category $\mathbb{C}$ with kernels and saturating regular epimorphisms the following statements are equivalent:*

(a) $\mathbb{C}$ *is a 2-star-permutable category;*
(b) *for any commutative diagram*

$$\begin{tikzcd}
(W \times_D C)^* \arrow[rr, dotted, "\lambda"] \arrow[dd, "\nu_1"'] \arrow[dr, "\nu_2"] & & (Y \times_B A)^* \arrow[dd, dashed, "\chi_1"'] \arrow[dr, "\chi_2"] & \\
& C \arrow[rr, two heads, "c"] \arrow[dd, shift right, "g"'] & & A \arrow[dd, shift right, "f"'] \\
W \arrow[rr, dashed, two heads, "w"] \arrow[dr, "\delta"'] & & Y \arrow[dr, dashed, "\beta"] & \\
& D \arrow[rr, two heads, "d"'] \arrow[uu, shift right, "t"'] & & B, \arrow[uu, shift right, "s"']
\end{tikzcd} \tag{3}$$

where the front square is of the form (1), $\beta \cdot w = d \cdot \delta$, w is a regular epimorphism, $((W \times_D C)^*, \nu_1, \nu_2)$ and $((Y \times_B A)^*, \chi_1, \chi_2)$ are stars of the corresponding pullback relations, then the comparison morphism $\lambda \colon (W \times_D C)^* \to (Y \times_B A)^*$ is also a regular epimorphism.

Proof. (a) $\Rightarrow$ (b) To prove that the arrow λ in the cube above is a regular epimorphism, we must show that $\langle \chi_1, \chi_2 \rangle \lambda$ in the commutative diagram

$$\begin{tikzcd}
(W \times_D C)^* \arrow[r, "\lambda"] \arrow[d, tail, "\langle \nu_1, \nu_2 \rangle"'] & (Y \times_B A)^* \arrow[d, tail, "\langle \chi_1, \chi_2 \rangle"] \\
W \times C \arrow[r, two heads, "w \times c"'] & Y \times A
\end{tikzcd}$$

is the (regular epimorphism, monomorphism) factorisation of the morphism $\langle w \cdot \nu_1, c \cdot \nu_2 \rangle \colon (W \times_D C)^* \to Y \times A$. That is, we must have $c\nu_2\nu_1^\circ w^\circ = \chi_2 \chi_1^\circ$ or, equivalently, $cg^\circ \delta^* w^\circ = f^\circ \beta^*$, since $\nu_2 \nu_1^\circ = \nu^* = g^\circ \delta^*$ and $\chi_2 \chi_1^\circ = \chi^* = f^\circ \beta^*$ (see Section 2.1).

The front square of diagram (3) is a star-regular pushout by Proposition 2.4, which means that the equality

$$(B) \qquad cg^{\circ *} = f^\circ d^*$$

holds true. Now, we always have

$$\begin{aligned}
cg^\circ \delta^* w^\circ &\leqslant f^\circ d \delta^* w^\circ && \text{(commutativity of the front face of (3))} \\
&= f^\circ \beta w^* w^\circ && (d \cdot \delta = \beta \cdot w) \\
&= f^\circ \beta \Delta_Y^* && \text{(Lemma 2.5)} \\
&= f^\circ \beta^*.
\end{aligned}$$

The other inequality follows from

$$\begin{aligned}
cg^\circ \delta^* w^\circ &\geqslant cg^{\circ *} \delta^* w^\circ && (g^\circ \geqslant g^{\circ *}) \\
&= f^\circ d^* \delta^* w^\circ && \text{(B)} \\
&= f^\circ d \delta^* w^\circ && ({}^*\delta^* = \delta^*;\ \text{Section 1.3}) \\
&= f^\circ \beta^*. && \text{(as in the inequality above)}
\end{aligned}$$

(b) ⇒ (a) A commutative diagram of type (1) induces a commutative cube

$$\begin{tikzcd}
\mathrm{N}_g \arrow[rr, dotted, two heads, "\lambda"] \arrow[dr, "\mathrm{n}_g"] \arrow[dd, "g\cdot\mathrm{n}_g"'] & & (D\times_B A)^* \arrow[dr, "\chi_2"] \arrow[dd, dashed, "\chi_1"'] & \\
& C \arrow[rr, two heads, "c"] \arrow[dd, shift right, "g"'] & & A \arrow[dd, shift right, "f"'] \\
D \arrow[rr, equal] \arrow[dr, equal] & & D \arrow[dr, "d"] & \\
& D \arrow[uu, shift right, "t"'] \arrow[rr, two heads, "d"'] & & B, \arrow[uu, shift right, "s"']
\end{tikzcd}$$

where $\nu = (g\cdot\mathrm{n}_g, \mathrm{n}_g)$ is the star of the pullback (relation) of $(g, 1_D)$. By assumption, λ is a regular epimorphism which translates into the equality $cg^\circ 1_D^* 1_D = f^\circ d^*$, as observed in the first part of the proof. We get the equality $cg^{\circ *} = f^\circ d^*$, and this proves that diagram (1) is a star-regular pushout and, consequently, that $\mathbb{C}$ is a 2-star-permutable category by Proposition 2.4.

□

In the total context, Proposition 2.6 is the "star version" of Proposition 3.6 in [12] (see also Proposition 4.1 in [4]). In the pointed context condition (b) of Proposition 2.6 also reduces to the pointed version of the right saturation property (in the sense of [9]). Indeed, in this context that condition says that, in the following commutative diagram

$$\begin{tikzcd}
\mathsf{Ker}(g) \arrow[rr, dotted, "\bar{c}"] \arrow[dr, "\ker(g)"] \arrow[dd, "0"'] & & \mathsf{Ker}(f) \arrow[dr, "\ker(f)"] \arrow[dd, dashed, "0"'] & \\
& C \arrow[rr, two heads, "c"] \arrow[dd, shift right, "g"'] & & A \arrow[dd, shift right, "f"'] \\
W \arrow[rr, dashed, two heads, "w"] \arrow[dr, "\delta"'] & & Y \arrow[dr, dashed, "\beta"'] & \\
& D \arrow[uu, shift right, "t"'] \arrow[rr, two heads, "d"'] & & B, \arrow[uu, shift right, "s"']
\end{tikzcd} \tag{4}$$

the induced arrow $\bar{c}\colon \mathsf{Ker}(g) \to \mathsf{Ker}(f)$ is a regular epimorphism.

We conclude this section with the pointed version of Propositions 2.4 and 2.6:

Corollary 2.7. (see Theorem 2.12 in [9]) *For a pointed regular category $\mathbb{C}$ the following statements are equivalent:*

(a) $\mathbb{C}$ *is a subtractive category;*
(b) *any commutative diagram of the form* (1) *is right saturated, i.e. the comparison morphism* $\bar{c}\colon \mathsf{Ker}(g) \to \mathsf{Ker}(f)$ *is a regular epimorphism.*

3. The Star-Cuboid Lemma

In [12] it was shown that regular Mal'tsev categories can be characterised through the validity of a homological lemma called the Upper Cuboid Lemma, a strong form of the denormalised 3×3 Lemma [4, 19, 11]. We are now going to extend this result to the star-regular context. We shall then observe that, in the pointed context, it gives back the classical Upper 3×3 Lemma characterising subtractive normal categories.

3.1. **$\mathcal{N}$-trivial objects.** An object X in a multi-pointed category is said to be *$\mathcal{N}$-trivial* when $1_X \in \mathcal{N}$. If a composite $f \cdot g$ belongs to $\mathcal{N}$ and g is a strong epimorphism, then also f belongs to $\mathcal{N}$. This implies that $\mathcal{N}$-trivial objects are closed under strong quotients. One says that a multi-pointed category $\mathbb{C}$ *has enough trivial objects* [8] when $\mathcal{N}$ is a closed ideal [14], i.e. any morphism in $\mathcal{N}$ factors through an $\mathcal{N}$-trivial object and, moreover, the class of $\mathcal{N}$-trivial objects is closed under subobjects and squares, where the latter property means that, for any $\mathcal{N}$-trivial object X, the object $X^2 = X \times X$ is $\mathcal{N}$-trivial. An equivalent way of expressing the existence of enough trivial objects is recalled in the following:

Proposition 3.1. [8] *Let $\mathbb{C}$ be a regular multi-pointed category with kernels. The following conditions are equivalent:*

(a) *if $(\sigma_1, \sigma_2) : S \rightrightarrows X$ is a relation on X such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$;*
(b) *$\mathbb{C}$ has enough trivial objects.*

In the following we shall also assume that $\mathcal{N}$-trivial objects are closed under binary products. Remark that in the total and in the (quasi-)pointed contexts there are enough trivial objects, and $\mathcal{N}$-trivial objects are closed under binary products.

Under the presence of enough trivial objects the assumption that $\mathcal{N}$-trivial objects are closed under binary products is equivalent to the following condition:

(a') if $(\sigma_1, \sigma_2) : S \rightarrowtail X \times Y$ is a relation from X to Y such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$.

Whenever the category has enough trivial objects, condition (a') implies that star-kernels "commute" with stars of pullback relations:

Lemma 3.2. *Let $\mathbb{C}$ be a multi-pointed category with kernels, enough trivial objects, and assume that $\mathcal{N}$-trivial objects are closed under binary products. Given a commutative cube*

$$\begin{tikzcd}
(W \times_D C)^* \arrow[rr, "\lambda"] \arrow[dr, "\nu_2"] \arrow[dd, "\nu_1"'] & & (Y \times_B A)^* \arrow[dr, "\chi_2"] \arrow[dd, dashed, "\chi_1"' near start] & \\
& C \arrow[rr, two heads, "c" near start] \arrow[dd, "g" near end] & & A \arrow[dd, "f"] \\
W \arrow[rr, dashed, two heads, "w" near end] \arrow[dr, "\delta"'] & & Y \arrow[dr, dashed, "\beta"'] & \\
& D \arrow[rr, two heads, "d"'] & & B
\end{tikzcd}$$

in $\mathbb{C}$, consider the star-kernels of c, d and w, and the induced morphisms $\overline{\delta} \colon \mathsf{Eq}(w)^* \to \mathsf{Eq}(d)^*$ and $\overline{g} \colon \mathsf{Eq}(c)^* \to \mathsf{Eq}(d)^*$. Then the following constructions are equivalent (up to isomorphism):

- *taking the horizontal star-kernel of λ and then the induced morphisms $\mathsf{Eq}(\lambda)^* \to \mathsf{Eq}(w)^*$ and $\mathsf{Eq}(\lambda)^* \to \mathsf{Eq}(c)^*$;*

- *taking the star of the pullback (relation) of $\overline{g}$ and $\overline{\delta}$ and then the induced morphisms* $(\mathsf{Eq}(w)^* \times_{\mathsf{Eq}(d)^*} \mathsf{Eq}(c)^*)^* \rightrightarrows (W \times_D C)^*$.

Proof. This follows easily by the usual commutation of kernel pairs with pullbacks and condition (a'). $\square$

In a star-regular category, a *(short) star-exact sequence* is a diagram

$$\begin{tikzcd} \mathsf{Eq}(f)^* \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \arrow[r, two heads, "f"] & Y \end{tikzcd}$$

where $\mathsf{Eq}(f)^*$ is a star-kernel of f and f is a coequaliser of f_1 and f_2 (which, by star-regularity, is the same as to say that f is a regular epimorphism). In the total context, a star-exact sequence is just an exact fork, while in the (quasi-)pointed context it is a short exact sequence in the usual sense.

The Star-Upper Cuboid Lemma
Let $\mathbb{C}$ be a star-regular category. Consider a commutative diagram of morphisms and stars in $\mathbb{C}$

$$\begin{tikzcd}[column sep=small]
 & P \arrow[rr, shift left, "\pi"] \arrow[rr, shift right] \arrow[dl, "\tau_1"'] \arrow[dd, "\tau_2"] & & (W \times_D C)^* \arrow[rr, "\lambda"] \arrow[dl, dashed, "\nu_1"'] \arrow[dd, "\nu_2"] & & (Y \times_B A)^* \arrow[dl, dashed, "\chi_1"'] \arrow[dd, "\chi_2"] \\
\mathsf{Eq}(w)^* \arrow[rr, dashed, shift left] \arrow[rr, dashed, shift right] \arrow[dd, "\bar{\delta}"'] & & W \arrow[rr, dashed, two heads, "w"] \arrow[dd, dashed, "\delta"'] & & Y \arrow[dd, dashed, "\beta"'] & \\
 & \mathsf{Eq}(c)^* \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dl, "\bar{g}"'] & & C \arrow[rr, two heads, "c"'] \arrow[dl, "g"] & & A \arrow[dl, "f"] \\
S \arrow[rr, shift left] \arrow[rr, shift right, "\sigma"'] & & D \arrow[rr, "d"'] & & B, &
\end{tikzcd} \tag{5}$$

where the three diamonds are stars of pullback (relations) of regular epimorphisms along arbitrary morphisms (so that $P = (\mathsf{Eq}(w)^* \times_S \mathsf{Eq}(c)^*)^*$) and the two middle rows are star-exact sequences. Then *the upper row is a star-exact sequence whenever the lower row is.*

Note that, in the diagram (5) above, d is necessarily a regular epimorphism, $d \cdot \sigma_1 = d \cdot \sigma_2$ since $\bar{g}$ is an epimorphism, and $\lambda \cdot \pi_1 = \lambda \cdot \pi_2$, because the pair of morphisms (χ_1, χ_2) is jointly monomorphic.

Theorem 3.3. *Let $\mathbb{C}$ be a star-regular category with saturating regular epimorphisms, enough trivial objects, and assume that $\mathcal{N}$-trivial objects are closed under binary products. The following conditions are equivalent:*

(a) $\mathbb{C}$ *is a 2-star-permutable category;*
(b) *the Star-Upper Cuboid Lemma holds true in* $\mathbb{C}$.

Proof. (a) $\Rightarrow$ (b) Suppose that the lower row is a star-exact sequence. The fact that $\pi = \mathsf{Eq}(\lambda)^*$ follows from Lemma 3.2. As explained in Proposition 2.6, λ is a regular epimorphism if and only if $cg^\circ\delta^* w^\circ \geqslant f^\circ\beta^*$. In fact we have

$$\begin{array}{lll}
cg^\circ\delta^* w^\circ & = cc^\circ cg^\circ gg^\circ\delta^* w^\circ & \text{(Lemma 1.1(1))} \\
& \geqslant cc^\circ cg^\circ g^* g^\circ\delta^* w^\circ & (\mathsf{Eq}(g) \geqslant \mathsf{Eq}(g)^*) \\
& = cg^\circ gc^\circ c^* g^\circ\delta^* w^\circ & (\mathsf{Eq}(c)\mathsf{Eq}(g)^* = \mathsf{Eq}(g)\mathsf{Eq}(c)^*;\ \text{Definition 2.3}) \\
& = cg^\circ d^\circ d^*\delta^* w^\circ & (g\langle\mathsf{Eq}(c)^*\rangle = \mathsf{Eq}(d)^* \text{ by assumption}) \\
& = cg^\circ d^\circ d\delta^* w^\circ & ({}^*\delta^* = \delta^*;\ \text{Section 1.3}) \\
& = cc^\circ f^\circ\beta w^* w^\circ & (d\cdot g = f\cdot c,\ d\cdot\delta = \beta\cdot w) \\
& = f^\circ\beta w^* w^\circ & \text{(Lemma 1.1(2))} \\
& = f^\circ\beta\Delta_Y^* & \text{(Lemma 2.5)} \\
& = f^\circ\beta^*. & \text{(Section 1.3)}
\end{array}$$

(b) $\Rightarrow$ (a) Consider a commutative cube of the form (3). We construct a commutative diagram of type (5) by taking the star-kernels of c, w, d and λ, so that $\bar{g}$, $\bar{\delta}$, τ_1 and τ_2 are the induced arrows between the star-kernels. By Lemma 3.2 we know that (τ_1, τ_2) is the star above the pullback (relation) of $(\bar{g}, \bar{\delta})$. By applying the Star-Upper Cuboid Lemma to this diagram we conclude that the upper row is a star-exact sequence and, consequently, λ is a regular epimorphism. By Proposition 2.6, $\mathbb{C}$ is a 2-star-permutable category. $\square$

In the total context, Theorem 3.3 is precisely Theorem 4.3 in [12], which gives a characterisation of regular Mal'tsev categories through the Upper Cuboid Lemma, as expected. In the pointed context, the Star-Upper Cuboid Lemma gives the classical Upper 3×3 Lemma: in the pointed version of diagram (5), the back part is irrelevant (like in diagram (4)). Then the front part is a 3×3 diagram where all columns and the middle row are short exact sequences. The Star-Upper Cuboid Lemma claims that the upper row is a short exact sequence whenever the lower row is, i.e. the same as the Upper 3×3 Lemma. The pointed version of Theorem 3.3 is Theorem 5.4 of [18] which characterises normal subtractive categories. Note that in the pointed context, the Upper 3×3 Lemma is also equivalent to the Lower 3×3 Lemma as shown in [18].

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