What are effective descent morphisms of Priestley spaces?

George Janelidze[*] and Manuela Sobral[†]

Dedicated to the memory of Sergio Salbany

Abstract

We discuss the problem formulated in the title. We solve it only in two very special cases: for maps with finite codomains and for maps that are open and order-open, or, equivalently, open and order-closed.

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0 Introduction

A morphism $p\colon E \to B$ in a category $\mathbb{C}$ with pullbacks is said to be an effective descent morphism if the pullback functor $p^*\colon (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ is monadic. This definition was used many times since the early nineties by various authors, who also explained where it comes from and how to deal with it. Nevertheless let us briefly recall:

- Intuitively, when $p\colon E \to B$ is a *good surjection*, one can think of $E = (E, p)$ as an extension of B. If so, then given a problem on a certain category $\mathbb{A}^B$ associated with B, one can try first to solve it for $\mathbb{A}^E$ and then to use descent from E to B. This requires to have an induced functor $p^*\colon \mathbb{A}^B \to \mathbb{A}^E$, and to be able to describe the category $\mathbb{A}^B$ as the category $Des(p)$, called the *category of descent data* for p, and constructed as the category of objects in $\mathbb{A}^E$ equipped with a certain additional structure defined using p^*. Accordingly, the morphism p is said to be an *effective descent morphism* if a certain comparison functor $\mathbb{A}^B \to Des(p)$ is a category equivalence. This general idea of descent theory is due to A. Grothendieck (see e.g. [3] and [4]).

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- There are several ways, later proposed by several authors, to describe the category $Des(p)$ at various levels of generality recalled in the survey papers [8] and [7]. In the 'basic' case of *global descent*, which we are considering in the present paper: $\mathbb{A}^B = (\mathbb{C} \downarrow B)$ is the category of pairs (A, f), where $f\colon A \to B$ is a morphism in $\mathbb{C}$; the functor $p^*\colon (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ is defined by $p^*(A, f) = (E \times_B A, \pi_1)$ using the pullback $E \times_B A$ of p and f; p^* has a left adjoint $p_!$, which is defined by $p_!(D, g) = (D, pg)$, and $Des(p)$ is defined as the category $(\mathbb{C} \downarrow E)^{\mathbb{T}^p}$ of algebras over the corresponding monad $\mathbb{T}^p$ on $(\mathbb{C} \downarrow E)$; the monadicity of p^* means that the standard comparison functor $(\mathbb{C} \downarrow B) \to Des(p)$ is a category equivalence.

- The expression "good surjection" used above is suggested by the fact that when $\mathbb{C}$ is 'good' (e.g. Barr exact), p is an effective descent morphism if and only if it is a regular epimorphism. A general characterization of effective descent morphisms is given in [7], but there are many concrete examples, including $\mathbb{C} = \mathsf{Top}$ (the category of topological spaces), where a lot of further work is needed to understand its meaning. Some of them are mentioned in Example 0.1 below.

Example 0.1. (a) For $\mathbb{C} = \mathsf{Top}$, the effective descent morphisms are characterized by J. Reiterman and W. Tholen ([13]) in terms of ultrafilter convergence.

(b) Let $\mathbb{C}$ be either the category of preorders (that is, sets equipped with a reflexive and transitive relation) or the category of finite preorders. Then $p\colon E \to B$ is an effective descent morphism if and only if for every $b_2 \leq b_1 \leq b_0$ in B there exists $e_2 \leq e_1 \leq e_0$ in E with $p(e_i) = b_i$ $(i = 0, 1, 2)$. This was shown in [5] (published as a preprint in 1999), as a simplified version of the above-mentioned Reiterman-Tholen result. Note that this result on preorders easily implies similar results for equivalence relations and for order relations (finite or not).

(c) Since the category of compact Hausdorff spaces is Barr exact and its regular epimorphisms are nothing but (continuous) surjections, its effective descent morphisms also are nothing but surjections. However, the same is true for Stone spaces, whose category is only regular; this was first observed by M. Makkai (unpublished).

(d) As explained in [2], using (b) and (c) one can easily describe effective descent morphisms of preordered Stone spaces and of ordered Stone spaces: they are the same as continuous maps that are effective descent morphisms of underlying preorders.

(e) For $\mathbb{C}$ being the category of compact (not necessarily Hausdorff) 0-dimensional spaces the effective descent morphisms are characterized in [6], although that category does not admit arbitrary pullbacks, and so the existence of relevant pullbacks becomes a part of the definition of an effective descent morphism there.

(f) Generalizing (c), all categories monadic over the category Set of sets (which includes all varieties of universal algebras) are Barr exact and their effective descent morphisms are exactly those morphisms that are mapped to surjections by the forgetful functor to Set. However, this is not the case for some quasi-varieties; the first simple counter-examples were given in the first part of [8], and much more information, also about relational structures was obtained by A. H. Roque (see [14], [15], [16]).

(g) When $\mathbb{C}$ is the opposite category of commutative rings (with 1), $p : E \to B$ is an effective descent morphism if and only if, considered as a B-module homomorphism, it is a pure monomorphism. We refer to the third part of [8] for the proof; however, that proof essentially follows the first published proof, due to B. Mesablishvili ([10]). In general, describing effective descent morphisms in the opposite categories of varieties of universal algebras is often a hard problem; some results in this direction, but very different from the commutative ring case, were obtained by D. Zangurashvili (see [18] and [20]; see also [19] for effective descent morphisms in some opposite topological categories).

As its title shows, this paper is about effective descent morphisms of Priestley spaces, and we shall make some general remarks about them before describing the content of the paper. Let us begin with a well-known result that goes back to G. Birkhoff [1], but is formulated categorically-precisely:

Theorem 0.2. *("Birkhoff Duality") The dual category* $(\mathsf{FDLat})^{op}$ *of finite distributive lattices is equivalent to the category* FOrd *of finite ordered sets. Both functors making the equivalence can be defined as hom*$(-, \mathbf{2})$*, where:*

(a) when L is a distributive lattice, hom$(L, \mathbf{2})$ is defined as the ordered set of lattice homomorphisms from L to the two-element lattice $\mathbf{2} = \{0, 1\}$;

(b) when X is an ordered set, hom$(X, \mathbf{2})$ is defined as the lattice of order preserving maps from X to the two-element ordered set $\mathbf{2} = \{0, 1\}$.

Next, as observed e.g. by P. T. Johnstone in [9], the duality above extends from finite to all distributive lattices simply by observing that:

- Since every finitely generated distributive lattice is finite, the category DLat of distributive lattices is equivalent to the filtered colimit completion of FDLat.
- Therefore the category $(\mathsf{DLat})^{op}$ is equivalent to what Johnstone (and some other authors, but not us) calls the category of ordered Stone spaces, since that category is equivalent to the filtered limit completion of FOrd.

However:

- The more complicated topological (or almost topological) approach to the extended duality was developed long before the categorical one, independently by several authors, but this again goes back to [1], and to further ideas of M. H. Stone (see the details in the Introduction of [9]; but see also [17] and its Zentralblatt review by G. Birkhoff).

- The more recent work of H. Priestley [11], partly independent and proving a clearer picture in a sense, has influenced many authors interested in (also in) universal algebra, and especially in various concrete algebraic dualities. We formulate Theorem 0.3, as these authors would expect, replacing the term "ordered Stone space" with "Priestley space". Another reason for this replacement is that we also need to reserve the term "ordered Stone space" for merely Stone spaces equipped with an order relation, as it is done in [2].

Furthermore, it will be convenient for us to use three other related terms, namely "Priestley covering family" (Problem 1.1), "Priestley-separated" (proof of Proposition 1.3), and "Priestley extension" (Problem 2.1). In particular, according to this terminology, an ordered Stone space A is a Priestley space if and only if every pair (a, a') of elements in A with $a \nleq a'$ in A, can be Priestley-separated; this means that there exists an up-closed clopen subset U of A with $a \in U$ and $a' \notin U$. The Priestley form of the above mentioned extended duality theorem formulates as:

Theorem 0.3. *The dual category* $(\mathsf{DLat})^{op}$ *of distributive lattices is equivalent to the category of Priestley spaces. Both functors making the equivalence can be defined as* $hom(-, \mathbf{2})$*, where:*

(a) when L *is a distributive lattice,* $hom(L, \mathbf{2})$ *is defined as the Priestley space of lattice homomorphisms from* L *to the two-element lattice* $\mathbf{2} = \{0, 1\}$*;*

(b) when X *is a Priestley space,* $hom(X, \mathbf{2})$ *is defined as the lattice of continuous order preserving maps from* X *to the two-element ordered set* $\mathbf{2} = \{0, 1\}$ *equipped with the discrete topology.*

The morphisms of Priestley spaces are, of course, the order preserving continuous maps; since all finite Priestley spaces are discrete, they become just order preserving maps in the finite case. Working with Priestley spaces we shall freely use their simple well-known properties, such as, e.g. separation of closed subsets instead of separation of points, or the fact that the up-closure of a (topologically) closed subset is closed, or the fact that the category of Priestley spaces is closed under pullbacks in the category of ordered topological spaces.

As explained in [2], using general results of descent theory and (b) and (c) in Example 0.1, the problem of describing effective descent morphisms of Priestley spaces reduces to a problem that can be formulated in simple terms not involving any categorical constructions except a single pullback. The reduction theorem can be formulated as:

Theorem 0.4. *A morphism $p\colon E \to B$ of Priestley spaces is an effective descent morphism if and only if it satisfies the following conditions:*

(a) for every $b_2 \leq b_1 \leq b_0$ in B there exists $e_2 \leq e_1 \leq e_0$ in E with $p(e_i) = b_i$ $(i = 0, 1, 2)$;

(b) for every morphism $f\colon A \to B$ of ordered Stone spaces, A is a Priestley space whenever so is the pullback $E \times_B A$.

However, condition 0.4(b) needs a further clarification, and, moreover, we do not even know whether it follows from condition 0.4(a). In fact we are formulating this problem in Section 3, while in Sections 1 and 2 we establish preliminary results and formulate related problems. In Section 3 we also present our two main results, namely Theorem 3.1 that describes effective descent morphisms with finite codomain, and Theorem 3.3 that gives another wide class of effective descent morphisms. In Section 4 we briefly consider the passage from descent for Priestley spaces to codescent for distributive lattices via Theorem 0.3.

We hope our problems and rather simple results will be of interest for categorical and point-set topologists working with Stone and Priestley spaces.

1 Remarks on Priestley spaces mapped to finite ordered sets

The example mentioned by H. A. Priestley in her review [12] (which is, as she says, due to W. G. Bowen) of an ordered Stone space S that is not a Priestley space seems to be indeed 'the' simplest one: it is the topological coproduct $\{x_n |\ n \leq \omega\} + \{y_n |\ n \leq \omega\}$ of two copies of the ordinal $\omega + 1$, with the order that has $u < v$ if and only if $u = x_n, v = y_n$ for some $n \neq \omega$. This space S admits, however, an order-preserving continuous map to the ordered set $\{0, 1\}$, whose fibres are (order discrete) Priestley spaces, suggesting the following:

Problem 1.1. Let B a Priestley space, and let us define a Priestley covering family of B as a covering family $(U_i)_{i \in I}$ of B with the following property: If A is an ordered Stone space and $f : A \to B$ a continuous order preserving map, such that each $f^{-1}(U_i)$ $(i \in I)$ is a Priestley space, then A is a Priestley space.

(a) Is it possible to characterize Priestley covering families?

(b) If not, what are reasonably wide classes of such coverings?

According to the above-mentioned example, $\{\{0\},\{1\}\}$ (considered as a two-member family) is not a Priestley covering of $\{0,1\}$. Nevertheless Proposition 1.3 below (see also Remark 1.4) gives a satisfactory answer to Problem 1.1(b) for finite B. In order to prove it, we need the following very simple lemma:

Lemma 1.2. *Let $f\colon A \to B$ be an order preserving map of ordered sets, X and Y subsets in A and B, respectively, with $f(X) \subseteq Y$, and b an element in B. Then*

$$(X\uparrow) \cap f^{-1}(b) = \cup_{y\in Y} A_y,$$

where A_y is the intersection of $f^{-1}(b)$ and the up-closure of $X \cap (f^{-1}(y) \cup f^{-1}(b))$ in $f^{-1}(y) \cup f^{-1}(b)$.

Proof. If a is an element in $(X\uparrow) \cap f^{-1}(b)$, then a is in $f^{-1}(b)$ and there exists $x \in X$ with $x \leq a$. For $y = f(x)$ we then have

$$x \in X \cap f^{-1}(y) \subseteq X \cap (f^{-1}(y) \cup f^{-1}(b)),$$

$x \leq a$, and $a \in f^{-1}(b) \subseteq f^{-1}(y) \cup f^{-1}(b)$. Therefore a is in A_y. That is $(X\uparrow) \cap f^{-1}(b) \subseteq \cup_{y\in Y} A_y$. The opposite inclusion is trivial. □

Proposition 1.3. *If B is a finite Priestley space that has more than one element, then its covering family formed by all two-element subsets of B is a Priestley covering family.*

Proof. Let A be an ordered Stone space and $f\colon A \to B$ a continuous order preserving map, such that $f^{-1}(b) \cup f^{-1}(b')$ is a Priestley space for each pair (b, b') of elements in B. We have to prove that A is a Priestley space. That is, we have to prove that every pair (a, a') of elements in A, with $a \nleq a'$ in A, can be Priestley-separated, in the sense that there exists an up-closed clopen subset U of A with $a \in U$ and $a' \notin U$.

Since B is a Priestley space, this is trivial when $f(a) \nleq f(a')$ in B, and we can therefore assume $f(a) \leq f(a')$. The case $f(a) = f(a')$ can also be excluded, since $f^{-1}f(a)$ is a clopen Priestley subspace of A, and if $x \leq y \leq z$ in A with x and z in $f^{-1}f(a)$, then y is in $f^{-1}f(a)$. We shall therefore assume $f(a) < f(a')$. On the other hand, using induction by the number of elements in B, we can assume that if B' is a proper subset of B, then $f^{-1}(B')$ is a Priestley space.

We shall construct the desired set U in several steps as follows:

- We choose a clopen subset V of the Priestley space $f^{-1}f(a) \cup f^{-1}f(a')$, up-closed in it, that contains a but not a'.

- We put $B' = \{b \in B|\ f(a) < b\}$ and $C = \cup_{b \in B'}((V \uparrow) \cap f^{-1}(b))$; obviously C has no element less or equal to a'. We claim that each $(V \uparrow) \cap f^{-1}(b)$ is closed, and since B' is finite, this implies that C is closed. Our claim follows from Lemma 1.2 applied to $X = V$ and $Y = \{f(a), f(a')\}$. Indeed, $(V \uparrow) \cap f^{-1}(b)$ becomes the union of the following two closed sets: the intersection of $f^{-1}(b)$ and the up-closure of $V \cap (f^{-1}f(a) \cup f^{-1}(b))$ in the Priestley space $f^{-1}f(a) \cup f^{-1}(b)$, and the intersection of $f^{-1}(b)$ and the up-closure of $V \cap (f^{-1}f(a') \cup f^{-1}(b))$ in the Priestley space $f^{-1}f(a') \cup f^{-1}(b)$.

- Since $B' = \{b \in B|\ f(a) < b\}$ has strictly less elements than B, $f^{-1}(B')$ is a Priestley space. Since C is a closed subset in it, with no element less or equal to a', we can choose a clopen W, up-closed in $f^{-1}(B')$ and not containing a'.

- Finally, we take $U = (V \cap f^{-1}f(a)) \cup W$.

All we need to show now is that U is up-closed in A. Moreover, since W is up-closed in $f^{-1}(B')$ and B' is up-closed in B, it suffices to show that if u is in $V \cap f^{-1}f(a)$ and $u \leq t$ in A, then t is in U. Since $u \leq t$ and $f(u) = f(a)$, we either have $f(a) = f(t)$, or $f(a) < f(t)$. If $f(a) = f(t)$, then t is in $V \cap f^{-1}f(a)$ since V is up-closed in $f^{-1}f(a) \cup f^{-1}f(a')$. If $f(a) < f(t)$, then $f(t)$ is in B', and t is in $(V \uparrow) \cap f^{-1}(f(t)) \subseteq C \subseteq W \subseteq U$. □

Remark 1.4. Let A be an ordered Stone space, B a Priestley space, $f\colon A \to B$ a continuous order preserving map, and a and a' elements in A. Since B is a Priestley space, whenever $f(a) \not\leq f(a')$ in B, the pair (a, a') can be Priestley-separated. Therefore, formulating Proposition 1.3 we can replace all two-element subsets of B with all subsets $\{b, b'\}$ with (at most two elements and) either $b \leq b'$ or $b' \leq b$. Moreover, since every closed subspace of a Priestley space is itself a Priestley space, we can make a further reformulation: If B is a finite Priestley space and $(U_i)_{i \in I}$ a covering family of B such that, for every $b \leq b'$ in B, there exists $i \in I$ with b and b' in U_i, then $(U_i)_{i \in I}$ is a Priestley covering family.

2 Involving pullbacks

Problem 2.1. Let us define a Priestley extension as a morphism $p\colon E \to B$ of Priestley spaces such that for every pullback diagram of the form

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

in the category of ordered Stone spaces, where $E \times_B A$ is a Priestley space, the space A also is a Priestley space.

(a) Is the necessary condition for p to be a Priestley extension, given in Proposition 2.3 below, also sufficient? (Proposition 2.4 gives the affirmative answer in the case of finite B.)

(b) If not, how can one characterize the class of Priestley extensions?

(c) What are reasonably wide classes of such extensions?

Remark 2.2. The Problems 1.1 and 2.1 are closely related of course. In particular, if $(U_i)_{i\in I}$ is a finite covering family of a Priestley space B, in which each U_i is a closed subspace of B, then the coproduct $\coprod_{i\in I} U_i$ is a Priestley space and the canonical map $\coprod_{i\in I} U_i \to B$ is a Priestley extension if and only if $(U_i)_{i\in I}$ is a Priestley covering.

Proposition 2.3. *If $p\colon E \to B$ is a Priestley extension, then for every $b \leq b'$ in B there exist $e \leq e'$ in E with $p(e) = b$ and $p(e') = b'$.*

Proof. We shall exclude the obvious case $b = b'$. After that, given $b < b'$ in B, we take A to be the space S considered at the beginning of Section 1, and define $f\colon A \to B$ by $f(x_n) = b$ and $f(y_n) = b'$. Since A is not a Priestley space, in order to prove the existence of $e \leq e'$ in E with $p(e) = b$ and $p(e') = b'$, it suffices to prove that if there is no such $e \leq e'$, then $E \times_B A$ is a Priestley space.

The ordered Stone space $E \times_B A$ is a topological coproduct of the Priestley spaces $p^{-1}(b) \times f^{-1}(b)$ and $p^{-1}(b') \times f^{-1}(b')$, such that:

- No element of $f^{-1}(b')$ is smaller than any element of $f^{-1}(b)$, and so no element of $p^{-1}(b') \times f^{-1}(b')$ is smaller than any element of $p^{-1}(b) \times f^{-1}(b)$.

- To say that there is no $e \leq e'$ in E with $p(e) = b$ and $p(e') = b'$, is to say that there is no element of $p^{-1}(b)$ smaller than any element of $p^{-1}(b')$, and so no element of $p^{-1}(b) \times f^{-1}(b)$ is smaller than any element of $p^{-1}(b') \times f^{-1}(b')$.

That is, if there is no $e \leq e'$ in E with $p(e) = b$ and $p(e') = b'$, then $E \times_B A$ is the coproduct of the Priestley spaces $p^{-1}(b) \times f^{-1}(b)$ and $p^{-1}(b') \times f^{-1}(b')$ in the category of ordered Stone spaces, and so it is a Priestley space. $\square$

Proposition 2.4. *Let $p\colon E \to B$ be a morphism of Priestley spaces, in which B is finite and for every $b \leq b'$ in B there exist $e \leq e'$ in E with $p(e) = b$ and $p(e') = b'$. Then p is a Priestley extension.*

Proof. Let F be the coproduct of all subsets in B with at most two elements in the category of finite ordered sets (or, equivalently, in the category of Priestley spaces), let $q\colon F \to B$ be the map induced by the inclusion map, and let $u\colon F \to E$ be any order preserving map with $pu = q$. Such a u does exist by our assumption on p. For an arbitrary order preserving map $f\colon A \to B$, in which A is an ordered Stone space, consider the diagram

$$\begin{array}{ccccc} F \times_B A & \xrightarrow{u\times 1} & E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1'}\downarrow & & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ F & \xrightarrow[u]{} & E & \xrightarrow[p]{} & B \end{array}$$

in obvious notation, whose both squares are pullbacks in the category of ordered Stone spaces. If $E \times_B A$ is a Priestley space, then:

- $F \times_B A$ is a Priestley space, since the left-hand square is a pullback.
- Since $F \times_B A$ is a Priestley space, then so is A by Proposition 1.3 (see also Remarks 1.4 and 2.2).

$\square$

3 Descent theorems

This section is devoted to our two main results on effective descent morphisms of Priestley spaces. The first of them is an immediate consequence of (Theorem 0.4 and) Proposition 2.4:

Theorem 3.1. *Let $p\colon E \to B$ be a morphism of Priestley spaces with finite B. Then p is an effective descent morphism if and only if for every $b_2 \leq b_1 \leq b_0$ in B there exists $e_2 \leq e_1 \leq e_0$ in E with $p(e_i) = b_i$ $(i = 0, 1, 2)$.*

Before formulating the second one, let us state:

Problem 3.2. Let $p\colon E \to B$ be a morphism of Priestley spaces. According to Theorem 0.4, and our definition of a Priestley extension in Problem 2.1, it is an effective descent morphism if it satisfies condition 0.4(a) (repeated in Theorem 3.1) and is a Priestley extension. Therefore Problem 2.1 suggests asking:

(a) Can one drop the finiteness assumption in Theorem 3.1?

(b) If not, how can one characterize the class of effective descent morphisms of Priestley spaces?

(c) What are reasonably wide classes of such morphisms?

Of course Problem 3.2(c) might have many answers, including the following:

Theorem 3.3. *A morphism $p\colon E \to B$ of Priestley spaces is an effective descent morphism whenever it is:*

(a) surjective;

(b) open and order-closed, that is, for every $b_1 \leq b_0$ in B and e_1 in E with $p(e_1) = b_1$, there exists e_0 in E with $p(e_0) = b_0$ and $e_1 \leq e_0$.

Proof. Condition 0.4(a) obviously follows from conditions 3.3(a) and 3.3(b). Therefore we only need to prove that these conditions imply 0.4(b). First, considering the pullback diagram displayed in the formulation of Problem 2.1, we observe that the projection $\pi_2\colon E \times_B A \to A$ is open since so is $p\colon E \to B$. On the other hand $\pi_2\colon E \times_B A \to A$ is closed since it is a continuous map between compact Hausdorff spaces. Furthermore, it preserves up-closed subsets, as easily follows from the fact that p is order-closed. Therefore it preserves up-closed clopen subsets. Now we can show that A is a Priestley space whenever so is $E \times_B A$. Suppose $a \nleq a'$ in A. We choose $e \in E$ with $p(e) = a$ and observe that $(e, a) \nleq (e', a')$ for every $e' \in p^{-1}(f(a'))$. Since $E \times_B A$ is a Priestley space, this allows us to choose an up-closed clopen subset U of $E \times_B A$ with $(e, a) \in U$ and $p^{-1}(f(a')) \times U = \emptyset$. It follows that $\pi_2(U)$ is an up-closed clopen subset of A with $a \in \pi_2(U)$ and $a' \notin \pi_2(U)$. □

Reversing the orders we obtain:

Corollary 3.4. *A morphism $p\colon E \to B$ of Priestley spaces is an effective descent morphism whenever it is:*

(a) surjective;

(b) open and order-open, that is, for every $b_1 \leq b_0$ in B and e_0 in E with $p(e_0) = b_0$, there exists e_1 in E with $p(e_1) = b_1$ and $e_1 \leq e_0$.

4 From Priestley spaces to distributive lattices

Let L be a distributive lattice. A homomorphism $l\colon L \to \mathbf{2}$ is completely determined by the inverse image $l^{-1}(0)$ of $0 \in \mathbf{2}$, which is a prime ideal of L, and, conversely, every prime ideal of L is of this form for some homomorphism $l\colon L \to \mathbf{2}$. This gives a more traditional (from the viewpoint of lattice theory) formulation of Theorems 0.2 and 0.3, where $hom(L, \mathbf{2})$ is replaced with the ordered set $PI(L)$ of prime ideals of L (one also replaces, for a Priestley space X, the lattice $hom(X, \mathbf{2})$ either with the lattice of up-closed clopen subsets of X, or with the lattice of down-closed clopen subsets of X there, but we shall not need that). Theorem 3.1, translated literally, becomes:

Theorem 4.1. *Let $p\colon B \to E$ be a homomorphism of distributive lattices with finite B. Then p is an effective codescent morphism, that is, it makes the induced pushout functor $(B \downarrow \mathsf{DLat}) \to (E \downarrow \mathsf{DLat})$ comonadic, if and only if for every triple (b_2, b_1, b_0) of prime ideals of B with $b_2 \subseteq b_1 \subseteq b_0$, there exist prime ideals $e_2 \subseteq e_1 \subseteq e_0$ in E with $p^{-1}(e_i) = b_i$ $(i = 0, 1, 2)$.*

Everything else we say in the previous sections can also be reformulated for distributive lattices. Let us omit that and only mention that it would be interesting to compare the results on distributive lattices with those for commutative rings (see the third part of [8]), and try to unify them in the context of commutative semirings.

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George Janelidze
Dept. Math. and Appl. Math.
University of Cape Town
Rondebosh 7700, Cape Town
South Africa
george.janelidze@uct.ac.za

Manuela Sobral
Departamento de Matemática
Universidade de Coimbra
Ap. 3008, 3001-454 Coimbra
Portugal
sobral@mat.uc.pt