SUMS OF SQUARES ON THE HYPERCUBE

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Abstract. Let X be a finite set of points in $\mathbb{R}^n$. A polynomial p nonnegative on X can be written as a sum of squares of rational functions modulo the vanishing ideal $I(X)$. From the point of view of applications, such as polynomial optimization, we are interested in rational function representations *of small degree*. We derive a general upper bound in terms of the Hilbert function of X, and we show that this upper bound is tight for the case of quadratic functions on the hypercube $C = \{0,1\}^n$, a very well studied case in combinatorial optimization. Using the lower bounds for C we construct a family of globally nonnegative quartic polynomials, which are not sums of squares of rational functions of small degree. To our knowledge this is the first construction for Hilbert's 17th problem of a family of polynomials of bounded degree which need increasing degrees in rational function representations as the number of variables n goes to infinity. We note that representation theory of the symmetric group S_n plays a crucial role in our proofs of the lower bounds.

1. Introduction

Certifying that a polynomial p is nonnegative on a finite set X in $\mathbb{R}^n$ is an important problem in optimization, as certificates of nonnegativity can often be leveraged into optimization algorithms. One frequently used certificate is writing p as a sum of squares of polynomials modulo the vanishing ideal $I(X)$ of X. These certificates lead to semidefinite relaxations for the problem of optimizing a polynomial on X [8, 15]. For instance, when X is the hypercube $\{0,1\}^n$, maximizing a quadratic polynomial on X specializes to many famous combinatorial optimization problems such as MAXCUT. Sums of squares certificates provide a way of automatically constructing semidefinite relaxations for these problems. The celebrated Goemans-Williamson relaxation algorithm, for instance, can be seen as such a sum of squares relaxation [1, Chapter 2 and 3], [4, 14].

In general, one might be required to use polynomials of high degree to certify that p is nonnegative on X. Since Hilbert's 17th problem, it is classical in real algebraic geometry to certify nonnegativity of a polynomial by writing it as a sum of squares of rational functions, instead of polynomials. This can be reformulated as follows:

Given p find a sum of squares h, such that ph is a sum of squares modulo $I(X)$.

When $X = \mathbb{R}^n$, the existence of such certificates for any nonnegative polynomial corresponds to Hilbert's 17th problem, and was answered affirmatively by Artin. For a general semialgebraic set the existence of such certificates is guaranteed by Stengle's Positivstllensatz, which was later refined by Schmüdgen, Putinar and Jacobi. See for example [14, 16] for an in-depth discussion of these topics. We are interested in showing *degree bounds* on the

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degree of the multiplier h. There are known general upper bounds coming from real algebraic geometry for rational function certificates on any real semialgebraic set X [12, 13, 18]. However, they result in bounds which are multiple towers of exponentials. We are not aware of any general lower degree bounds, even for Hilbert's 17th problem. For some specific small cases see [7].

For the case when X is a finite set of points, one of our main results is an elementary uniform upper bound on the degree of the multiplier h, in terms of the Hilbert function of X and the degree of p. Our second main result is showing this bound is tight for the case of quadratic functions on the hypercube $C = \{0,1\}^n$. We leverage the tightness of the bound on C into a construction of a globally nonnegative polynomial p of degree 4 in n variables such that ph is not a sum of squares for all sums of squares h of degree at most $2\lfloor n/2 \rfloor - 4$. While this bound can very likely be improved, to our knowledge this is the first construction for Hilbert's 17th problem of a polynomial of bounded degree, which needs multipliers h of increasing degree as the number of variables n goes to infinity.

1.1. **Background, Discussion and Main Results.** Let $X \subset \mathbb{R}^n$ be a real variety and let $I = I(X)$ be its vanishing ideal. Let $\mathbb{R}[X] = \mathbb{R}[x_1, \dots, x_n]/I$ be the coordinate ring of X. Given $f \in \mathbb{R}[X]$ we define *degree* of f as the lowest degree of any polynomial in the equivalence class $f + I$. Let $\mathbb{R}[X]_{\leq d}$ be the real vector space of polynomials of degree at most d in $\mathbb{R}[X]$. The Hilbert function $H_X(t)$ of X is defined by:

$$H_X(t) = \dim \mathbb{R}[X]_{\leq t},$$

i.e. the value of the Hilbert function at t simply records the dimension of the vector space of polynomial functions of degree at most t on X. It is known that for t large enough the Hilbert function becomes a polynomial, this polynomial is called the Hilbert polynomial of X. See [] for more properties and the background on the Hilbert function.

We say that $f \in \mathbb{R}[X]$ is k-sos if there exist $g_1, \dots, g_m \in \mathbb{R}[X]_{\leq k}$ such that $f = g_1^2 + \cdots + g_m^2$. The set of all k-sos polynomials will be denoted by $\Sigma(X)_{\leq 2k}$. This set of polynomials has attracted strong attention from the optimization community in recent years, as a relaxation for the cone of polynomials nonnegative on X [5, 6, 9, 11]. The reason for this is that checking whether a polynomial is k-sos is a semidefinite feasibility problem and, even better, one can use semidefinite programming to optimize a linear functional over the cone of k-sos polynomials [1, Chapters 2 and 6].

For a compact variety X, Schmüdgen's Positivstellensatz implies that any polynomial that is strictly positive on X is k-sos for large enough k. However there may be no uniform bounds on this k for all polynomials of fixed degree. This situation improves considerably if we allow sums of squares of rational functions. We say that $p \in \mathbb{R}[X]_{\leq 2s}$ is (d,k)-rsos (rational sum of squares) if there exists non-zero $h \in \Sigma(X)_{\leq 2d}$ such that $ph \in \Sigma(X)_{\leq 2k}$. We will omit d and write simply that p is k-rsos for the case $d = k - s$. It follows from Stengle's Positivstellensatz that for any polynomial p nonnegative on X there is a $k \in \mathbb{N}$ for which p is k-rsos. Moreover, there is a bound on k that depends only on the degree of p and the variety X. The trade-off is that, computationally, this certificate has worse properties: while checking if a polynomial is k-rsos is still a semidefinite feasibility problem, the set of all such polynomials has no direct semidefinite description, and tools other than semidefinite programming have to be used to optimize over it. Moreover, when X is a *reducible* variety, a non-zero sum of squares multiplier h such that ph is a sum of squares is not necessarily a certificate of nonnegativity of p. This happens since h may vanish identically on a component of X, and on this component nonnegativity of p is not certified. Therefore, we will also be interested in the existence of *strictly positive* sum of squares multipliers h.

In the case X is a finite set of points in $\mathbb{R}^n$, there exist uniform degree bounds for k-sos representations. The *Hilbert regularity* $h(X)$ of X is the smallest degree d for which $H_X(d) = |X|$ and, consequently, $H_X(t) = |X|$ for all $t \geq h(X)$. A polynomial $f \in \mathbb{R}[X]$ is uniquely determined by its values on X, so we may identify elements of $\mathbb{R}[X]$ with functions on X, an $|X|$-dimensional space. The Hilbert regularity $h(X)$ of X is the smallest degree d for which $H_X(d) = |X|$, i.e. $\mathbb{R}[X] = \mathbb{R}[X]_{\leq d}$ and, consequently, $H_X(t) = |X|$ for all $t \geq h(X)$.

For a point $v \in X$ let $\delta_v : X \to \mathbb{R}$ be the interpolator of v: $\delta_v(v) = 1$ and $\delta_v(x) = 0, \quad x \neq v$. We note that $h(X)$ is the smallest degree d such that $\delta_v \in \mathbb{R}[X]_{\leq d}$ for all $v \in X$. Furthermore, using interpolators we can write any $p \in \mathbb{R}[X]$ as:

$$p = \sum_{v \in X} p(v)\delta_v^2.$$

It follows that any nonnegative polynomial $p \in \mathbb{R}[X]$ is $h(X)$-sos. It is not difficult to construct examples of finite sets X and nonnegative polynomials $p \in \mathbb{R}[X]$ of any degree, such that p is not $(h(X)-1)$-sos, i.e. we may need to go all the way up to Hilbert regularity to certify nonnegativity of p.

For the rational function representations we provide better upper bounds by using the following result.

Theorem 1.1. *Let X be a finite set of points in $\mathbb{R}^n$. Let $p \in \mathbb{R}[X]_{\leq 2s}$ be a polynomial of degree at most $2s$ nonnegative on X. Suppose that for some $k \in \mathbb{N}$ we have*

$$H_X(k+s) + H_X(k) > H_X(2k+2s).$$

Then p is $(k+s)$-rsos on X, i.e. there exists $h \in \Sigma(X)_{\leq 2k}$ such that $ph \in \Sigma(X)_{\leq 2s+2k}$.

The key to proving Theorem 1.1 is Lemma 2.2, which gives a way of bounding degree of multipliers via an elementary sign counting argument. An important application of the above theorem is to quadratic polynomials on the hypercube $C = \{0,1\}^n$. It is easy to show that $H_C(t) = \sum_{i=0}^{t} \binom{n}{i}$ and therefore $H_C(n) = 2^n = |C|$, while $H_C(\lfloor \frac{n}{2} \rfloor + 1) + H_C(\lfloor \frac{n}{2} \rfloor) > 2^n$. This implies that all nonnegative quadratic polynomials on the hypercube are $(\lfloor \frac{n}{2} \rfloor + 1)$-rsos. In fact this result is tight since we also show the following:

Theorem 1.2. *Let $k = \lfloor \frac{n}{2} \rfloor$ and let $f \in \mathbb{R}[C]$ be given by*

$$f = (x_1 + \cdots + x_n - k)(x_1 + \cdots + x_n - k - 1).$$

Then f is nonnegative on C but f is not k-rsos.

Our proof relies on symmetries of the polynomial $(x_1 + \cdots + x_n - k)(x_1 + \cdots + x_n - k - 1)$ and we use representation theory of the symmetric group S_n in an essential way. More general lower bounds for rational function representations of symmetric polynomials on the hypercube are given in Theorem 3.4 and Theorem 1.2 is a direct corollary.

From Theorem 1.2 we can derive two interesting results. First, it immediately recovers a result by Laurent [10] concerning the power of k-sos representations for relaxations of the MAXCUT problem. In fact, we significantly strengthen that result by proving that it remains true even for rational sums of squares representations, and by proving that in this case, the bounds are optimal.

If we demand the certificates to be strictly positive, the case most pertinent to optimization, we prove in Theorem 2.5 that for the case of quadratic functions on the hypercube C the bound of Theorem 1.1 needs to be increased by at most 1 degree, and thus it is still almost optimal. If a quadratic polynomial p on C remains invariant under the substitution $x \to 1 - x_i$, i.e. $p(x_1, \ldots, x_n) = p(1 - x_1, \ldots, 1 - x_n)$ then it was recently shown by Fawzi, Parrilo and Saunderson in[3] that p is actually $\lceil \frac{n}{2} \rceil$-sos, thus proving a conjecture of Laurent from [10].

We also use Theorem 1.2 to provide lower bounds for the degree of the denominators in Hilbert's 17th problem. More precisely, we use the quadratic polynomial nonnegative on the hypercube to construct a family of globally nonnegative quartic polynomials in n variables which are not $\lfloor \frac{n}{2} \rfloor$-rsos. This is, to our knowledge, the first example of a family of polynomials of bounded degree which needs denominators of increasing degree in their representations as sums of squares of rational functions.

2. Upper Bound on Multipliers

Let $X = \{v_1, \ldots, v_m\}$ be a finite set of points in $\mathbb{R}^n$. We first show that the set of (d_1, d_2)-rsos polynomials is always closed.

Lemma 2.1. *Fix $d_1, d_2 \in \mathbb{N}$. The set of polynomials in $\mathbb{R}[X]_{\leq 2d}$ which are (d_1, d_2)-rsos is closed for all d_1, d_2, and d.*

Proof. One can check that $\Sigma(X)_{\leq 2d}$ is a closed pointed convex cone in $\mathbb{R}[X]_{\leq 2d}$ [1, Chapter 4]. Suppose that $f_i \in \mathbb{R}[X]_{\leq 2d}$ are (d_1, d_2)-rsos and converge to f. Then there exist g_i, h_i which are respectively d_1 and d_2-sos and $f_i g_i = h_i$. We may rescale g_i and assume that

$$\frac{1}{m} \sum_{j=1}^{m} g_i(v_j) = 1.$$

The set of d_1-sos polynomials with average 1 on X is compact. Therefore a subsequence of $\{g_i\}$ converges to g, which is also d_1-sos. Then the corresponding subsequence of $f_i g_i$ converges to fg and, since each $f_i g_i$ is d_2-sos, it follows that fg is d_2-sos.

□

We now develop some results about linear functionals on $\mathbb{R}[X]_{\leq 2d}$ that are nonnegative on k-sos polynomials. These results are based on elementary dimension counting, but they will be crucial in the proof of Theorem 1.1 as we will be able to show non-existence of a certain separating linear functional. Let $\ell : \mathbb{R}[X]_{\leq 2d} \to \mathbb{R}$ be a linear functional given as a combination of point evaluations on X:

$$\ell(f) = \sum_{i=1}^{m} \mu_i f(v_i), \quad f \in \mathbb{R}[X]_{\leq 2d}, \, \mu_i \in \mathbb{R}.$$

We assume that the coefficients μ_i are non-zero and let m_+ and m_- be the number of positive and negative μ_i respectively, and let $Q_\ell : \mathbb{R}[X]_{\leq d} \to \mathbb{R}$ be the quadratic form associated to

ℓ given by

$$Q_\ell(f) = \ell(f^2) = \sum_{i=1}^{m} \mu_i f^2(v_i).$$

We can now provide a way of bounding degree of multipliers via a simple sign counting argument, that will prove to be the crucial element in the proof of Theorem 1.1 .

Lemma 2.2. *Let $\ell : \mathbb{R}[X]_{\leq 2d} \to \mathbb{R}$ be given by $\ell(f) = \sum_{i=1}^{m} \mu_i f(v_i)$ with all $\mu_i \neq 0$. Suppose that ℓ is nonnegative on $\Sigma(X)_{\leq 2d}$. Then $m_+ \geq \dim \mathbb{R}[X]_{\leq d}$.*

Proof. Let $\pi_X : \mathbb{R}[X]_{\leq d} \to \mathbb{R}^m$ be the evaluation projection of forms in $\mathbb{R}[X]_{\leq d}$ given by

$$\pi_X(f) = (f(v_1), \dots, f(v_m)), \quad f \in R[X]_{\leq d}.$$

We observe that the map π_X has a trivial kernel and therefore

$$\dim \pi_X(\mathbb{R}[X]_{\leq d}) = \dim \mathbb{R}[X]_{\leq d}.$$

Let $\bar{Q}_\ell$ be the quadratic form on $\mathbb{R}^m$ given by:

$$\sum_{i=1}^{m} \mu_i x_i^2.$$

By its definition, the form Q_ℓ is a composition of π_X and $\bar{Q}_\ell$:

$$Q_\ell = \bar{Q}_\ell \circ \pi_X.$$

The form $\bar{Q}_\ell$ has m_- negative eigenvalues, and thus $\bar{Q}_\ell$ is strictly negative on a subspace of dimension m_-. Recall that the form Q_ℓ is positive semidefinite, which implies that $\bar{Q}_\ell$ is positive semidefinite on the image of π_X. Thus the image of π_X has codimension at least m_- in $\mathbb{R}^m$. Since $m_+ + m_- = m$ the Lemma follows. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose not. By Lemma 2.1, the set of all polynomials in $\mathbb{R}[X]_{\leq 2s}$ that is not $(k+s)$-rsos is open. Thus we can find $p \in \mathbb{R}[X]_{\leq 2s}$ that is strictly positive on X but is not $(k+s)$-rsos. Now consider the pointed, closed convex cones $p\Sigma(X)_{\leq 2k}$ and $\Sigma(X)_{\leq 2k+2s}$ in $\mathbb{R}[X]_{\leq 2k+2s}$. By our assumption

$$p\Sigma(X)_{\leq 2k} \cap \Sigma(X)_{\leq 2k+2s} = \{0\}.$$

Therefore there exists a linear functional $\ell : \mathbb{R}[X]_{\leq 2k+2s} \to \mathbb{R}$ strictly separating the two cones: $\ell(f) > 0$ for all nonzero $f \in \Sigma(X)_{\leq 2k+2s}$ and $\ell(f) < 0$ for all nonzero $f \in p\Sigma(X)_{\leq 2k}$.

Let $X' \subseteq X$ be a subset of X such that point evaluations on X' form a basis of the dual space of linear functionals $\mathbb{R}[X]^*_{\leq 2k+2s}$. We note that

$$|X'| = \dim \mathbb{R}[X]_{\leq 2k+2s} \quad \text{and} \quad \dim \mathbb{R}[X']_{\leq d} = \mathbb{R}[X]_{\leq d} \quad \text{for all} \quad d \leq 2k+2s.$$

Therefore the separating functional ℓ can be written as

$$\ell = \sum_{v_i \in X'} \mu_i \ell_{v_i}, \quad \mu_i \in \mathbb{R},$$

where ℓ_{v_i} are point evaluation functionals on points of X'. Let p' be the image of p under the canonical projection from $\mathbb{R}[X]$ to $\mathbb{R}[X'] = \mathbb{R}[X]/I(X')$. It follows that ℓ also strictly separates $p'\Sigma_{\leq 2k}(X')$ from $\Sigma_{\leq 2k+2s}(X')$ and p' is strictly positive on X'. Since ℓ strictly separates the two cones we may assume without loss of generality that all coefficients μ_i are non-zero. Let m_+ and m_- be the number of positive and negative μ_i respectively. Then by Lemma 2.2 we know that $m_+ \geq \dim \mathbb{R}[X']_{\leq k+s} = \dim \mathbb{R}[X]_{\leq k+s}$.

Now define $\ell' : \mathbb{R}[X']_{\leq 2k} \to \mathbb{R}$ by

$$\ell' = \sum_{v_i \in X'} \mu_i p'(v_i) \ell_{v_i}.$$

The functional ℓ' is nonnegative on $\Sigma_{\leq 2k}(X')$, therefore, by applying Lemma 2.2, we see that $m_- \geq \dim \mathbb{R}[X']_{\leq k} = \dim \mathbb{R}[X]_{\leq k}$, since $p'(v_i) > 0$ for all $v_i \in X'$. Combining, we see that

$$H_X(2k+2s) = |X'| = m_+ + m_- \geq H_X(k+s) + H_X(k),$$

which is a contradiction. ◻

Corollary 2.3. *Let $p \in \mathbb{R}[C]_{\leq 2}$ be a quadratic polynomial nonnegative on C and let $k = \lfloor \frac{n}{2} \rfloor$. Then p is $(k+1)$-rsos.*

Proof. This follows immediately from Theorem 1.1 since $H_C(t) = \sum_{i=0}^{t} \binom{n}{i}$. ◻

2.1. **Strictly Positive Multipliers.** We observe that having a k-rsos representation of a polynomial $p \in \mathbb{R}[X]$ is not in general a certificate of nonnegativity of p. This is due to the fact that X is a reducible variety and the multiplier h may vanish on some points of X. On these points nonnegativity of p is not certified.

Therefore we are interested in showing existence of strictly positive sum of squares multipliers. More specifically we will be interested in multipliers h of the form

$$h = 1 + \sum q_i^2, \quad q_i \in \mathbb{R}[X]_{\leq k}.$$

We note that, up to multiplication by a positive constant, such sums of squares correspond precisely to the interior points of the cone $\Sigma(X)_{\leq 2k}$. We will concentrate on the case of a quadratic polynomial nonnegative on a subset X of the hypercube C. We first show that the bound of $d = \lfloor \frac{n}{2} \rfloor$ suffices also for any strictly positive quadric $p \in \mathbb{R}[X]_{\leq 2}$.

In the proof we will make use of *Cayley-Bacharach duality*. From the point of view of algebraic geometry the hypercube C is a 0-dimensional transverse complete intersection of n quadratic polynomials. Intuitively we expect that polynomials in $\mathbb{R}[C]_{\leq t}$ vanishing on X form a linear subspace L_X of $\mathbb{R}[C]_{\leq t}$ of codimension equal to the size $|X|$ of X. In the language of algebraic geometry we say that in this case points of X impose independent conditions on polynomials of degree at most t. However, this does not necessarily occur, and *the failure of X to impose independent conditions* on polynomials of degree at most t is defined as the difference $|X| - \operatorname{codim} L_X$. There is a general duality which equates the dimension of the vector space of polynomials of degree at most t vanishing on X with the failure of the complementary subset $\bar{X}$ to impose independent conditions on polynomials of at most complementary degree. We refer the reader to [2] for more details.

Theorem 2.4. *Let $d = \lfloor \frac{n}{2} \rfloor$ and let X be a subset of C. If $p \in \mathbb{R}[X]_{\leq 2}$ is a quadratic polynomial that is strictly positive on X then there exists h in the interior of $\Sigma(X)_{\leq 2d}$ such that $p \cdot h$ lies in the interior of $\Sigma(X)_{\leq 2d+2}$.*

Proof. Suppose not. Then the pointed convex cones $p\Sigma(X)_{\leq 2d}$ and $\Sigma(X)_{\leq 2d+2}$ can be weakly separated. Therefore there exists a linear functional $\ell \in \mathbb{R}[X]^*_{\leq 2d+2}$ such that $\ell(s) \geq 0$ for all

$s \in \Sigma(X)_{\leq 2d+2}$ and $\ell(s) \leq 0$ for all $s \in p\Sigma(X)_{\leq 2d}$. We can write

$$\ell = \sum_{v_i \in X} \mu_i \ell_{v_i}, \quad \mu_i \in \mathbb{R}.$$

Let X' be the subset of X corresponding to non-zero coefficients μ_i. Let p' be the image of p under the canonical projection from $\mathbb{R}[X]$ to $\mathbb{R}[X'] = \mathbb{R}[X]/I(X')$. It follows that ℓ also separates $p'\Sigma(X')_{\leq 2d}$ from $\Sigma(X')_{\leq 2d+2}$ and p' is strictly positive on X'.

Let m_+ and m_- be the number of positive and negative μ_i respectively. Using Lemma 2.2 we see that $m_+ \geq \dim \mathbb{R}[X']_{\leq d+1}$. On the other hand we may define $\ell' : \mathbb{R}[X']_{\leq 2d} \to \mathbb{R}$ by

$$\ell'(q) = \ell(p'q), \qquad \ell' = \sum_{v_i \in X'} \mu_i p'(v_i) \ell_{v_i}.$$

Since p' is strictly positive on X' and ℓ' is nonpositive on squares we can apply Lemma 2.2 to see that $m_- \geq \dim \mathbb{R}[X']_{\leq d}$.

We now claim that

$$\dim \mathbb{R}[X']_{\leq d} + \dim \mathbb{R}[X']_{\leq d+1} > |X'|. \tag{1}$$

Let $\bar{X}'$ denote the complement of X' in C. Using Cayley-Bacharach duality [2, Theorem CB6], we see that $|X'| - \dim \mathbb{R}[X']_{\leq d} = \dim \mathbb{R}[C]_{\leq n-d-1} - \dim \mathbb{R}[\bar{X}']_{\leq n-d-1}$. We observe that $d+1 > n-d-1$ and we must have $\dim \mathbb{R}[X']_{\leq d+1} > \dim \mathbb{R}[X']_{\leq n-d-1}$, otherwise $\dim \mathbb{R}[X']_{\leq d+1} = \dim \mathbb{R}[X']_{\leq d} = |X'|$ and (1) is proved. Thus we have

$$\dim \mathbb{R}[X']_{\leq d+1} + \dim \mathbb{R}[X']_{\leq d} - |X'| = \dim \mathbb{R}[X']_{\leq d+1} + \dim \mathbb{R}[\bar{X}']_{\leq n-d-1} - \dim \mathbb{R}[C]_{\leq n-d-1} >$$
$$\dim \mathbb{R}[X']_{\leq n-d-1} + \dim \mathbb{R}[\bar{X}']_{\leq n-d-1} - \dim \mathbb{R}[C]_{\leq n-d-1} \geq 0.$$

This finishes the proof of the claim, and now we observe that since $m_+ + m_- = |X'|$ we have reached a contradiction. □

We now show that if $p \in \mathbb{R}[X]_{\leq 2}$, is nonnegative on $X \subseteq C$ then there are interior sum of squares multipliers of degree at most $\lfloor \frac{n}{2} \rfloor + 1$, i.e. we may need to increase the degree by 1 in order to certify nonnegativity of a quadric. It is not clear to us whether this is truly necessary, or perhaps there exist interior sum of squares multipliers of degree at most $\lfloor \frac{n}{2} \rfloor$.

Theorem 2.5. *Let $d = \lfloor \frac{n}{2} \rfloor$ and let X be a subset of C. If $p \in \mathbb{R}[X]_{\leq 2}$ is a non-zero quadratic function nonnegative on X, then there exists h in the interior of $\Sigma(X)_{\leq 2d+2}$ such that $p \cdot h \in \Sigma(X)_{\leq 2d+4}$.*

Proof. It is equivalent to show that any linear functional in $\mathbb{R}[X]^*_{\leq 2d+4}$ which separates $p\Sigma(X)_{\leq 2d+2}$ and $\Sigma(X)_{\leq 2d+4}$ is identically zero on $p\Sigma(X)_{\leq 2d+2}$. Let ℓ be such a functional. We can write

$$\ell = \sum_{v_i \in X} \mu_i \ell_{v_i}, \quad \mu_i \in \mathbb{R}.$$

Let $V \subsetneq X$ be the variety of p in X and let $X' = X \setminus V$. Let p' be the image of p under the canonical projection from $\mathbb{R}[X]$ to $\mathbb{R}[X'] = \mathbb{R}[X]/I(X')$. Let $\ell' \in \mathbb{R}[X']^*_{\leq 2d+2}$ be the linear functional given by

$$\ell' = \sum_{v_i \in X} \mu_i p(v_i) \ell_{v_i} = \sum_{v_i \in X'} \mu_i p'(v_i) \ell_{v_i}.$$

We claim that ℓ' separates $p'\Sigma(X')_{\leq 2d}$ from $\Sigma(X')_{\leq 2d+2}$. Indeed for any $q \in \Sigma(X)_{\leq 2d}$ we have

$$\ell'(p'q) = \ell(p^2 q) \geq 0,$$

while for any $q \in \Sigma(X')_{\leq 2d+2}$ we have

$$\ell'(q) = \ell(pq) \leq 0.$$

By Theorem 2.4 it follows that ℓ' must be identically zero, which implies that ℓ is defined only in terms of evaluations on points of V. Thus ℓ vanishes identically on $p\Sigma(X)_{\leq 2d+2}$. $\square$

3. Lower Bound on Multipliers

In this section we prove the lower bound on the degree of rational function representations for polynomials on the hypercube. We deal with S_n-invariant polynomials which vanish on a *level* $T = \{x \in C : \sum x_i = t\}$ of the hypercube $C = \{0,1\}^n$. Such functions come up naturally in combinatorial optimization, where we are counting objects subject to some symmetric restrictions; see Section 4.1. We will show that such functions do not have rational sums of squares representations with multipliers of low degree.

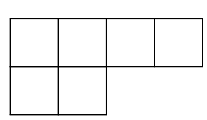
It will simplify the notation to use subsets of $[n]$ as exponents: $x^{\{1,4\}} = x_1 x_4$. The vector space $\mathbb{R}[C]$ of functions on the hypercube has a basis $\{x^m : m \subseteq [n]\}$ of squarefree monomials. Thus we can write any function $f \in \mathbb{R}[C]$ as $f = \sum_{m \subset [n]} c_m x^m$, and we have $\deg(f) = \max\{|m| : c_m \neq 0\}$. We define $\mathbb{R}[C]_d$ to be the collection of homogeneous degree-d functions, and $\mathbb{R}[C]_{\leq d} = \oplus_{i=0}^{d} \mathbb{R}[C]_i$ the collection of functions of degree at most d.

We also need to discuss the notion of divisibility in a coordinate ring. For instance, we may have $f, g, h \in \mathbb{R}[X]$ with $f = gh$ but $\deg(f) < \deg(g) + \deg(h)$; in the case of the hypercube, $x \cdot x = x$. To fix this, for $f, g \in \mathbb{R}[X]$, we say that *g properly divides f* if there exists $h \in \mathbb{R}[X]$ such that $f = gh$ and $\deg(f) = \deg(g) + \deg(h)$. We will also say that *g properly divides f to order m* if g^m properly divides f, but g^{m+1} does not.

We note that the symmetric group S_n acts on $\mathbb{R}[C]$ by permuting the variables directly: $(123)x_1 = x_2$. To start, we decompose $\mathbb{R}[C]$ into *irreducible S_n-modules.* We introduce the necessary background in representation theory of the symmetric group below. For further information see the introduction by Sagan [17], whose notation we adopt here.

3.1. Representation theory of S_n.

A *partition* of a positive integer n is an ordered tuple $(\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \ldots \geq \lambda_k$, and $\lambda_1 + \ldots + \lambda_k = n$. Corresponding to each partition is its diagram, where we draw k rows of boxes, with λ_i boxes in the ith row. For example, the partition $(4,2)$ of $n = 6$ has the following diagram:



A *tableau* of shape λ is an assignment of numbers $\{1, \ldots, n\}$ to the boxes in the diagram of λ. A *standard* tableau has strictly increasing rows and columns. Here is an example of a tableau and a standard tableau of shape $(4,2)$:

A *tabloid* is an equivalence class of tableaux, where we identify two tableaux if the fillings of their rows are the same as subsets of $\{1, \ldots, n\}$.

1	6	3	2
4	5		

1	3	4	5
2	6		

FIGURE 1. A tableau and a standard tableau of shape $(4,2)$.

For a tableau T and an element $\sigma \in S_n$, let σ act on T by permuting the numbers in T. Then the action of S_n can be extended to tabloids and formal linear combination of tabloids. Formal linear combinations of tabloids of shape λ form the *permutation module* M^λ.

Let C_T be the *column group* of T; that is, the subgroup of S_n fixing the columns of T. Now we can define the *polytabloid* $e_T = \sum_{\sigma \in C_T} \operatorname{sign}(\sigma) \cdot [\sigma(T)]$, where $[\sigma(T)]$ is the tabloid equivalence class of $\sigma(T)$. Now, define the *Specht module* S^λ:

$$S^\lambda := \operatorname{span}(\{e_T : T \text{ is a standard tableau of shape } \lambda\}),$$

which is a submodule of M^λ. Irreducible representations (irreducible S_n-modules) of S_n are precisely given by the Specht modules S^λ, where λ is a partition of n.

3.2. **Functions on the hypercube C and S_n-representations.** Recall that S_n acts on $\mathbb{R}[C]$ by permuting the variables. In the following we treat $(n,0)$ as an alias for the partition (n) to simplify our notation. We now define an isomorphism between tabloids and monomials. For $k \leq n/2$ let $M^{(n-k,k)}$ and $S^{(n-k,k)}$ denote the permutation and the Specht modules respectively, corresponding to the partition $(n-k,k)$.

Define $\phi_k : M^{(n-k,k)} \to \mathbb{R}[C]$ by $\phi_k([m^c, m]) = x^m$, and extend ϕ_k linearly. For example, $\phi_3([12345, 678]) = x_6 x_7 x_8$. The image of ϕ_k is the subspace $\mathbb{R}[C]_k$ of homogeneous functions of degree k. We also have $\mathbb{R}[C]_k \cong \mathbb{R}[C]_{n-k}$ as S_n-modules, since we can take complements in the exponent: if $n = 6$, then $x_1 x_2 \in \mathbb{R}[C]_2 \leftrightarrow x_3 x_4 x_5 x_6 \in \mathbb{R}[C]_4$.

Proposition 3.1. *The S_n-module $\mathbb{R}[C]$ decomposes into $n+1-2k$ copies of $S^{(n-k,k)}$, for $0 \leq k \leq \frac{n}{2}$.*

Proof. By Young's rule (Theorem 2.11.2 in [17]), $M^{(n-k,k)}$ splits into direct sum of $S^{(n-i,i)}$ for $0 \leq i \leq k$, each coming with multiplicity 1. By the above, if $k \leq n/2$, $\mathbb{R}[C]_{n-k} \cong \mathbb{R}[C]_k \cong M^{(n-k,k)}$. If n is odd, then

$$\begin{aligned}
\mathbb{R}[C] &= \bigoplus_{0 \leq k < n/2} (\mathbb{R}[C]_k \oplus \mathbb{R}[C]_{n-k}) \\
&\cong 2 \bigoplus_{0 \leq k < n/2} M^{(n-k,k)} \\
&\cong 2 \bigoplus_{0 \leq k < n/2} \left(\bigoplus_{i=0}^{k} S^{(n-i,i)} \right) \\
&\cong 2 \bigoplus_{i=0}^{\lfloor n/2 \rfloor} \left(\frac{n-1}{2} - i + 1 \right) S^{(n-i,i)},
\end{aligned}$$

which gives the result. For even n just add the single copy of $\mathbb{R}[C]_{n/2} \cong M^{(n/2,n/2)}$. □

Proposition 3.1 gives the decomposition of $\mathbb{R}[C]$ into irreducible submodules. To analyze a specific function $f \in \mathbb{R}[C]$, we now give an explicit decomposition of $\mathbb{R}[C]$. We choose a slightly idiosyncratic description which will be useful for our purposes. Fix $t \in \mathbb{R}$ and let $\ell = t - \sum x_i$. Recalling that $S^{(n-k,k)} \subset M^{(n-k,k)}$, define $H_{k0} = \phi(S^{(n-k,k)}) \subseteq \mathbb{R}[C]_k$. Since ϕ is an S_n-module isomorphism, we have $H_{k0} \cong S^{(n-k,k)}$. Then for $i = 1, \ldots, n-2k$, define $H_{ki} = (t - \sum_j x_j)^i \cdot H_{k0}$. Note that no element of H_{k0} is properly divisible by ℓ.

Theorem 3.2. *$\mathbb{R}[C]$ has the following decomposition into irreducibles:*

$$\mathbb{R}[C] = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \left(\bigoplus_{i=0}^{n+1-2k} H_{ki} \right).$$

This decomposition respects degree: for any d,

$$\mathbb{R}[C]_{\leq d} = \bigoplus_{k+i \leq d} H_{ki}.$$

Proof. By Proposition 3.1, the above decomposition contains the correct number of each irreducible S_n-module. Therefore, it remains to show that the summands are linearly independent.

By Corollary 2.11 in [19], the map $U : \mathbb{R}[C]_k \to \mathbb{R}[C]_{n-k}$ given by $U(f) = (\sum x_j)^{n-2k} f$ is a bijection. Therefore, the map $U' : \mathbb{R}[C]_k \to \mathbb{R}[C]_{\leq k+i}$ given by $f \mapsto (t - \sum x_j)^i f$ is injective for $i \leq n - 2k$, by consideration of the top degree terms of $U'(f)$. Since $H_{ki} = U'(H_{k0})$, we have that $\deg(f) = k + i$ for each nonzero $f \in H_{ki}$; in particular, $H_{ki} \neq 0$. Since S_n acts trivially on $(t - \sum_j x_j)^i$, we have $H_{ki} \cong H_{k0}$. By irreducibility, we know that vectors in H_{ki} and $H_{k'i'}$ are linearly independent if $k \neq k'$. It remains to consider H_{ki} for varying i; but since each nonzero $f_i \in H_{ki}$ has degree exactly $k + i$, these are linearly independent as well.

The expression for $\mathbb{R}[C]_{\leq d}$ now follows from the linear independence of the modules H_{ki}. $\square$

We now show that proper divisibility holds for functions of low degree vanishing on a level T, i.e. on the subset of the hypercube where the sum of coordinates is equal to a fixed number t.

Lemma 3.3. *Let $T = \{x \in C : \sum_i x_i = t\}$, for fixed $t \in \{0, \ldots, n\}$. Suppose $f \in \mathbb{R}[C]_{\leq d}$, and f vanishes on T. If $d \leq t \leq n - d$, then f is properly divisible by $\ell = t - \sum x_i$.*

Proof. Let V be the S_n-submodule of $\mathbb{R}[C]_{\leq d}$ consisting of polynomials that are properly divisible by ℓ and let

$$W = H_{00} \oplus \ldots \oplus H_{d0} \cong S^{(n)} \oplus \cdots \oplus S^{(n-d,d)}.$$

By Theorem 3.2 we have $\mathbb{R}[C]_{\leq d} = V \oplus W$. Let $U \subset W$ be the S_n-submodule of polynomials vanishing on T. Since W contains exactly one copy of each irreducible submodule of $\mathbb{R}[C]_{\leq d}$ it suffices to show that $U = 0$. Since the H_{i0} are nonisomorphic irreducible S_n-modules, it follows that

$$U = \bigoplus_{i \in I} H_{i0},$$

where I is a subset of $\{0, \ldots, d\}$. Now we claim that polynomials in H_{i0} do not identically vanish on T for all $0 \leq i \leq d$. Since H_{i0} is an irreducible S_n-module it suffices to exhibit a single polynomial $p \in H_{i0}$ not vanishing on T.

To see this, let q be the standard tableau of shape $(n-i, i)$ where the first row contains $\{1, \ldots, n-i\}$ and the second row contains $\{n-i+1, \ldots, n\}$. Let $\hat{x} \in C$ be given by

$$\hat{x} = e_{n-t+1} + \cdots + e_n,$$

where e_j denotes the j-th standard basis vector. Since $i \leq t \leq n-i$, the support of $\hat{x}$ contains the second row of q and does not contain any of the first i entries of the first row of q. Consider $p = \phi(e_q)$, $p \in H_{i0}$, where e_q is the polytabloid corresponding to q. It follows that $p(\hat{x}) = 1$, since only the monomial $\phi(q)$ is nonzero on $\hat{x}$ in $\phi(e_q)$ and $\phi(q)(\hat{x}) = 1$. See Figure 2 for an example. $\square$

$$q = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 8 & 9 \\ \cline{1-2} \end{array}$$

$$\hat{x} = (0, 0, 0, 0, 0, 0, 1, 1, 1)$$

$$p = \phi(e_q) = x_8x_9 - x_1x_9 - x_8x_2 + x_1x_2$$

FIGURE 2. A standard tableau q with sorted rows, and the associated vector $\hat{x}$. Here $n = 9, i = 2, t = 3$. We have $p(\hat{x}) = 1$.

Now we can prove our main result on lower bounds for the degree of denominators in $\mathbb{R}[C]$.

Theorem 3.4. *Suppose $f \in \mathbb{R}[C]_{\leq t}$ with $t \leq n/2$ is an S_n-invariant polynomial and f is properly divisible by $\ell = t - (x_1 + \cdots + x_n)$ to odd order. Then f is not (d_1, d_2)-rsos for $d_1 \leq \min\left\{\frac{n-\deg f}{2}, t\right\}$, $d_2 \leq t$.*

Proof. Suppose that $f \sum g_i^2 = \sum h_j^2$ with $g_i \in \mathbb{R}[C]_{\leq d_1}$, $g_i \neq 0$ and $h_j \in \mathbb{R}[C]_{\leq d_2}$. Let $g = \sum g_i^2$ and $h = \sum h_j^2$. Without loss of generality we may assume that g and h are S_n-invariant polynomials, otherwise we may replace them by their S_n-symmetrizations.

Since $d_2 \leq t$ by Lemma 3.3 we can write $h_j = \ell^{a_j} q_j$ with $\deg q_j = \deg h_j - a_j$ and q_j not vanishing on all of T. Therefore, after symmetrizing $h = \sum \ell^{2a_j} q_j^2$ we see that $h = \ell^{2a} q$ where $a = \min a_j$ and q is an S_n-invariant polynomial, $\deg q = \deg h - 2a$, and q is strictly positive on T.

Similarly, since $d_1 \leq t$ we argue that $g = \ell^{2b} r$, where r is an S_n-invariant polynomial strictly positive on T, and $\deg r = \deg g - 2b$. Finally, $f = \ell^c p$ where c is odd and p is an S_n-invariant polynomial not identically zero on T with $\deg p = \deg f - c$. Combining, we see that

$$\ell^{2b+c} pr - \ell^{2a} q = 0.$$

Let $\alpha = \min\{2a, 2b+c\}$. By factoring out ℓ^α in the equation above we obtain

$$\ell^\alpha s = 0,$$

for an S_n-invariant polynomial $s \in \mathbb{R}[C]$ of degree strictly less than n since $d_1 \leq \min\left\{\frac{n-t}{2}, t\right\}$ and $d_2 \leq t$. Since q and r are strictly positive on T and p is not identically zero on T, it follows that s does not vanish on T. Thus s is a non-zero symmetric polynomial in $\mathbb{R}[C]$ vanishing on $C \setminus T$. Therefore $s = \beta \chi_T$ for some constant $\beta \neq 0$, where $\chi_T \in \mathbb{R}[C]$ is the polynomial vanishing on $C \setminus T$ and equal to 1 on T. However, it is not hard to check that $\deg \chi_T = n$ for any level T and therefore we arrive at a contradiction.

$\square$

Corollary 3.5. *Fix $t \leq n/2$ and let $f \in \mathbb{R}[C]_{\leq t}$ be an S_n-invariant polynomial. Suppose that f is properly divisible by $\ell = t - (x_1 + \cdots + x_n)$ to odd order. Then f is not d-sos for $d \leq t$.*

Proof. Apply Theorem 3.4 with $d_1 = 0$. □

Theorem 1.2 also follows immediately:

Proof of Theorem 1.2. Apply Theorem 3.4. □

4. Applications

We give two applications of our results. Section 4.1 deals with the MAXCUT problem on K_n, and is an application to combinatorial optimization. Section 4.2 deals with lower degree bounds in Hilbert's 17th problem.

4.1. **The maxcut problem.**

A *cut* in a graph arises from a partition of the vertices into two sets S_1, S_2, the cut being the collection of all edges from S_1 to S_2. Note that switching S_1 and S_2 gives the same cut. We write $C = [S_1, S_2] = [S_2, S_1]$, and let $|S| =$ the number of edges from S_1 to S_2. A *maximal cut* is a cut maximizing $|S|$.

In the complete graph K_n, the maximal cuts come from any partition of $[n]$ into two sets of $n/2$ vertices when n is even, or $(n \pm 1)/2$ when n is odd. We note that a point $v \in C$ naturally defines a cut $S^v = [S_1, S_2]$ via $S_1 = \{i \mid v_i = 0\}$ and $S_2 = \{i \mid v_i = 1\}$.

Let n be odd, Let $k = \lfloor \frac{n}{2} \rfloor$ and let $q \in \mathbb{R}[C]$ be given by

$$q = (x_1 + \cdots + x_n - k)(x_1 + \cdots + x_n - k - 1).$$

We note that for all $v \in C$ we have $q(v) = |S^v|$.

Note that the q defined above is the same polynomial as in Theorem 1.2. This allows us to reprove and strengthen a result of Laurent. In [10], Theorem 4, it is shown that the Lasserre rank of the cut polytope of K_n, for n odd, is at least $\frac{n+1}{2}$. This implies that there exists a quadratic polynomial $q \in \mathbb{R}[C]_{\leq 2}$ such that q is not $\frac{n-1}{2}$-sos. In fact the proof by Laurent established this for the same q as above. However from Theorem 1.2 we know that in fact q is not $\frac{n-1}{2}$-rsos. Further, it was conjectured in [10] that the Lasserre rank is precisely $\frac{n+1}{2}$ in this case. This is equivalent to saying that any nonnegative quadratic $q \in \mathbb{R}[C]_{\leq 2}$ that can be written as $q(x) = q_0 + \sum_{i \neq j} q_{ij} x_i x_j$ is $\frac{n+1}{2}$-sos.While we are not able to show this conjecture, it follows from Corollary 2.3 that any quadratic $q \in \mathbb{R}[C]_{\leq 2}$ is $\frac{n+1}{2}$-rsos, and from Theorem 2.5 that even if we demand positive multipliers, $\frac{n+3}{2}$-rsos is enough.

4.2. **Globally nonnegative function with large multipliers.**

We finish with an application to Hilbert's 17th problem.

Theorem 4.1. *Let $k = \lfloor \frac{n}{2} \rfloor$. There exists a polynomial p of degree 4 nonnegative on $\mathbb{R}^n$ which is not k-rsos in $\mathbb{R}[x_1, \ldots, x_n]$.*

Proof. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be given by

$$f = (x_1 + \cdots + x_n - k)(x_1 + \cdots + x_n - k - 1).$$

By Corollary 1.2 we know that f is not k-rsos in $\mathbb{R}[C]$. Using Lemma 2.1 with $X = C$ it follows that $f + \epsilon$ is not k-rsos in $\mathbb{R}[C]$ for all sufficiently small $\epsilon > 0$. Let $f' = f + \epsilon$ for a fixed such ϵ.

Let $r = \sum_{i=1}^{n}(x_i^2 - x_i)^2$. For sufficiently large $\lambda > 0$ the polynomial $p = f' + \lambda r$ is strictly positive on $\mathbb{R}^n$. Suppose that p is k-rsos in $\mathbb{R}[x_1, \ldots, x_n]$: we have $ph = g$ with $(k-2)$-sos non-zero polynomial h, and k-sos polynomial g.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ let C_α be the hypercube given by equations $(x_i - \alpha_i)(x_i - \alpha_i - 1) = 0$. By Lemma 2.1 it follows that p is not $(k-1,k)$-sos in $\mathbb{R}[C_\alpha]$ for all α sufficiently close to 0, since by linear change of variables it suffices to consider a small perturbation of p in $\mathbb{R}[C]$. However, there exist α arbitrarily close to 0 such that $h \not\equiv 0$ in $\mathbb{R}[C_\alpha]$. This is a contradiction since it follows that p is k-rsos in $\mathbb{R}[C_\alpha]$ for such α. □

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