VANISHING IDEALS OVER COMPLETE MULTIPARTITE GRAPHS

JORGE NEVES AND MARIA VAZ PINTO

Abstract. We study the vanishing ideal of the parametrized algebraic toric set associated to the complete multipartite graph $\mathcal{G} = \mathcal{K}_{\alpha_1,\dots,\alpha_r}$ over a finite field of order q. We give an explicit family of binomial generators for this lattice ideal, consisting of the generators of the ideal of the torus, (referred to as type I generators), a set of quadratic binomials corresponding to the cycles of length 4 in $\mathcal{G}$ and which generate the *toric algebra of* $\mathcal{G}$ (type II generators) and a set of binomials of degree $q-1$ obtained combinatorially from $\mathcal{G}$ (type III generators). Using this explicit family of generators of the ideal, we show that its Castelnuovo–Mumford regularity is equal to $\max\{\alpha_1(q-2),\dots,\alpha_r(q-2),\lceil (n-1)(q-2)/2\rceil\}$, where $n=\alpha_1+\cdots+\alpha_r$.

1. introduction

The class of vanishing ideals of parameterized algebraic toric sets over a finite field was first studied by Renteria, Simis and Villarreal in [18]. Here we focus on the case when the set is parameterized by the edges of a simple graph. Let K be a finite field of order q and $\mathcal{G}$ a simple graph with n vertices $\{v_1,\dots,v_n\}$ and nonempty edge set. Given a choice of ordering of the edges, given by a bijection $e\colon \{1,\dots,s\}\to E(G)$, and writing $\mathbf{x}^{e(i)} = x_jx_k$ for every $\mathbf{x}=(x_1,\dots,x_n)\in (K^*)^n$ and $e(i)=\{v_j,v_k\}\in E(\mathcal{G})$, we define the associated *algebraic toric set* as the subset of $\mathbb{P}^{s-1}$ given by:

$$X = \left\{(\mathbf{x}^{e_1},\dots,\mathbf{x}^{e_s})\in \mathbb{P}^{s-1} : \mathbf{x}\in (K^*)^n\right\}, \tag{1.1}$$

where we abbreviate the notation $e(i)$ to e_i.

The variety X can also be seen as the subgroup of $\mathbb{T}^{s-1}\subset\mathbb{P}^{s-1}$ given by the image of the group homomorphism $(K^*)^n\to\mathbb{T}^{s-1}$ defined by $\mathbf{x}\mapsto(\mathbf{x}^{e_1},\dots,\mathbf{x}^{e_s})$. The *vanishing ideal of* X, which we denote by $I(X)$, is the ideal generated by all homogeneous forms in $S=K[t_1,\dots,t_s]$ that vanish on X. This ideal is a Cohen–Macaulay, radical, lattice ideal of codimension $s-1$ (*cf.* [18, Theorem 2.1]). One motivation for the study of these ideals lies in the fact that they combine the toric ideal of the edge subring of a graph, $P(\mathcal{G})\subset K[t_1,\dots,t_s]$, with the arithmetic of the finite field. This relation is expressed in the equality:

$$I(X) = \left(\left[P(\mathcal{G}) + (t_2^{q-1}-t_1^{q-1},\dots,t_s^{q-1}-t_1^{q-1})\right] : (t_1\cdots t_s)^\infty\right) \tag{1.2}$$

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which (in particular) holds for any connected *or* bipartite $\mathcal{G}$ (*cf.* [18, Corollary 2.11]). Recall that $P(\mathcal{G})$ is the kernel of the epimorphism $K[t_1,\ldots,t_s] \to K[\mathcal{G}]$ given by $t_i \mapsto y^{e_i}$, where $K[\mathcal{G}]$ is the edge subring of $\mathcal{G}$, *i.e.*, the subring of the polynomial ring $K[y_1,\ldots,y_n]$ given by $K[G] = K[y^{e_i} : i = 1,\ldots,s]$. For a survey on the subject of toric ideals of edge subrings of graphs we refer the reader to [6, Chapter 5].

The properties of $I(X)$ (even in the general case of disconnected graphs) are reflected in $\mathcal{G}$ and *vice versa*. For one, the degree (or multiplicity) of $S/I(X)$ is equal to

$$\begin{cases} \left(\tfrac{1}{2}\right)^{\gamma-1}(q-1)^{n-m+\gamma-1}, \text{ if } \gamma \geq 1 \text{ and } q \text{ is odd,} \\ (q-1)^{n-m+\gamma-1}, \text{ if } \gamma \geq 1 \text{ and } q \text{ is even,} \\ (q-1)^{n-m-1}, \text{ if } \gamma = 0, \end{cases} \tag{1.3}$$

where m is the number of connected components of $\mathcal{G}$, of which exactly γ are non-bipartite (*cf.* [17, Theorem 3.2]). Another invariant of interest is the *index of regularity* of the quotient $S/I(X)$, which, since this quotient is Cohen–Macaulay of dimension 1, coincides with the Castelnuovo–Mumford regularity. To present day knowledge, there is no single general formula expressing the regularity of $S/I(X)$ in terms of the data of $\mathcal{G}$. It is known that when $\mathcal{G} = \mathcal{C}_{2k}$, an even cycle of length $2k$, the regularity of $S/I(X)$ is $(k-1)(q-2)$ (*cf.* [17, Theorem 6.2]). In the case of an odd cycle, X coincides with $\mathbb{T}^{s-1}$ (*cf.* [18, Corollary 3.8]) — another way of seeing this, using (1.2), is that if $\mathcal{G}$ is an odd cycle then $P(\mathcal{G}) = (0)$ (*cf.* [24]); accordingly $I(X) = (t_2^{q-1} - t_1^{q-1}, \ldots, t_s^{q-1} - t_1^{q-1})$ is a complete intersection (see also [8, Theorem 1]). In this case the regularity is $(s-1)(q-2) = (n-1)(q-2)$, where n (odd) is the number of vertices (and edges) of $\mathcal{G}$ (*cf.* [8, Lemma 1]). If $\mathcal{G} = \mathcal{K}_{a,b}$ is a complete bipartite graph, the regularity of $S/I(X)$ is given by

$$\max\{(a-1)(q-2), (b-1)(q-2)\}$$

(*cf.* [7, Corollary 5.4]). Recently, a formula for the regularity of $S/I(X)$ in the case of a complete graph $\mathcal{G} = \mathcal{K}_n$ was given in [9]. In this case, if $n > 3$,

$$\operatorname{reg} S/I(X) = \lceil (n-1)(q-2)/2 \rceil. \tag{1.4}$$

Notice that the case $\mathcal{G} = \mathcal{K}_2$ is trivial and case $\mathcal{G} = \mathcal{K}_3 = \mathcal{C}_3$ was already discussed.

In this work we focus on the case of $\mathcal{G} = \mathcal{K}_{\alpha_1,\ldots,\alpha_r}$, a complete multipartite graph with $\alpha_1 + \cdots + \alpha_r = n$ vertices. One of our main results, Theorem 4.3, states that in this case, if $r \geq 3$ and $n \geq 4$,

$$\operatorname{reg} S/I(X) = \max\{\alpha_1(q-2), \ldots, \alpha_r(q-2), \lceil (n-1)(q-2)/2 \rceil\}. \tag{1.5}$$

This formula generalizes (1.4); it contains the case of the complete graph by setting $\alpha_1 = \cdots = \alpha_r = 1$. However, as far as the proof of Theorem 4.3 is concerned, we restrict to the case when $\mathcal{K}_{\alpha_1,\ldots,\alpha_r}$ is not a complete graph. Moreover, the methods used in this work are distinctly orthogonal to those used in [9]. Our main interest being the lattice ideal $I(X)$, we rely on a precise description of a generating set of binomials to prove the statement on the regularity. In Theorem 3.3, we show that a given set of binomials generates $I(X)$. These binomials are classified into 3 classes: the binomials $t_i^{q-1} - t_j^{q-1}$, for every $i \neq j$, which, by (1.2), belong to $I(X)$ no matter which $\mathcal{G}$ we take, and are referred to as *type I generators*; the binomials $t_it_j - t_kt_l \in P(\mathcal{G})$, for each $e_ie_ke_je_l$ cycle of length 4 contained in $\mathcal{G}$, are referred to as *type II generators*; finally the *type III generators*, obtained from weighted subgraphs of $\mathcal{G}$, are described in full detail in the beginning of Section 3. Theorem 3.3 applies without restrictions on $\alpha_1, \ldots, \alpha_r$. In particular, it yields a generating set for $I(X)$ in the case of a complete bipartite graph, which, despite the result on the regularity of $S/I(X)$ in [7], was missing in the literature.

This problem area has been attracting increasing interest. The field of binomial ideals has been quite explored and its general theory can be found in Eisenbud and Sturmfels article [4]. In our present setting, these binomial ideals have a remarkable application to coding theory. Associating to X an evaluation code, one can relate two of its basic parameters (the length and the dimension) to $I(X)$ by a straightforward application of the Hilbert function (*cf.* [11, 12]); moreover a set of generators of $I(X)$ can make way to computing the Hamming distance of the code (*cf.* [19]). There has been substantial recent research exploring the relation to coding theory (*cf.* [17, 18, 19, 20]) and also focusing on the vanishing ideal of parameterized algebraic toric sets (*cf.* [15, 16]).

Let us describe the structure of this paper. In Section 2 we establish the definitions and notations used in the article. In that section, Lemma 2.3 provides a useful characterization of a binomial in $I(X)$ by a condition on the associated weighted subgraph of $\mathcal{G}$. In Section 3, we describe 3 families of binomials and prove that they form a generating set for $I(X)$ — Theorem 3.3. In Section 4, we prove Theorem 4.3, that states that under the assumption that $r \geq 3$ and $n = \alpha_1 + \cdots + \alpha_r \geq 4$, the regularity of $S/I(X)$ is given by the integer d of formula (1.5). We show this by: i) exhibiting a monomial in $K[t_1, \ldots, t_s]$ of degree d and showing that that monomial does not belong to $I(X)+(t_1)$, where $t_1 \in K[t_1, \ldots, t_s]$ is a variable; ii) and by showing that every monomial in $K[t_1, \ldots, t_s]$ of degree $d+1$ is in $I(X) + (t_1)$.

For any additional information in the theory of monomial ideals and Hilbert functions, we refer to [21, 23], and for graph theory we refer to [1].

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2. Preliminaries

Let K be a finite field of order q. Throughout, $\mathcal{G} = \mathcal{K}_{\alpha_1,\ldots,\alpha_r}$ will denote the complete multipartite graph. More precisely, a graph with vertex set $V_{\mathcal{G}} = \{v_1, \ldots, v_n\}$ endowed with a partition $V_{\mathcal{G}} = P_1 \sqcup \cdots \sqcup P_r$, satisfying $\#P_i = \alpha_i$, for $i = 1, \ldots, r$, and $\alpha_1 + \cdots + \alpha_r = n$, such that $\{v_k, v_l\}$ is an edge of $\mathcal{G}$ if and only if $v_k \in P_i$ and $v_l \in P_j$ with $i \neq j$. An important case to consider is that when all $\alpha_i = 1$, *i.e.*, when $\mathcal{G}$ is the complete graph on r vertices. Fix an ordering of the set of edges of the graph, $E(\mathcal{G})$, given by $e\colon \{1, \ldots, s\} \to E(\mathcal{G})$, where $s = ((\sum_{i=1}^r \alpha_i)^2 - \sum_{i=1}^r \alpha_i^2)/2$. We write e_j for $e(j)$. Let $S = K[t_1, \ldots, t_s]$ be a polynomial ring on $s = \#E(\mathcal{G})$ variables. We fix a bijection between the set of variables of S and $E(\mathcal{G})$, given by the map $t_i \mapsto e_i$. To ease notation, we also use $t_{\{v_k,v_l\}}$ for the variable corresponding to the edge $\{v_k, v_l\}$. Let $X \subset \mathbb{P}^{s-1}$ be the algebraic toric set associated to $\mathcal{G}$, as defined in (1.1), and denote by $I(X) \subset S$ its vanishing ideal.

The identification of a monomial (or a binomial) of S with a weighted subgraph of $\mathcal{G}$ will play an important role in Sections 3 and 4. If $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$ we denote by t^a the monomial of S given by $t_1^{a_1} \cdots t_s^{a_s}$.

Definition 2.1. Given $g = t^a \in S$, where $a \in \mathbb{N}^s$, the *weighted subgraph associated to* t^a, denoted by $\mathcal{H}_g$, is the subgraph of $\mathcal{G}$ with the same vertex set, with the set of edges corresponding to the variables dividing t^a and with weight function given by a, *i.e.*, such that the weight of the edge e_i is a_i for each $i \in \operatorname{supp} a$. The *weighted degree* of a vertex v is the sum of the weights of all edges incident to v in $\mathcal{H}_g$.

Let us denote the weight, a_i, of an edge e_i in $\mathcal{H}_g$ by $\mathrm{wt}_{\mathcal{H}_g}(e_i) = a_i$. We denote the weighted degree of v by $\mathrm{wt}_{\mathcal{H}_g}(v)$. This number is zero if no edge in $\mathcal{H}_g$ is incident to v.

Definition 2.2. Given a binomial, $f = t^a - t^b \in S$, with $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, the *weighted subgraph associated to* f, which we denote by $\mathcal{H}_f$, is defined by $\mathcal{H}_{t^a} \cup \mathcal{H}_{t^b}$.

The edges of $\mathcal{H}_f$ may be colored with two colors using the partition $E(\mathcal{H}_{t^a}) \sqcup E(\mathcal{H}_{t^b})$. We will use a solid line for edges corresponding to variables dividing t^a and a dotted line for the edges corresponding to variables dividing t^b. We refer to the former as black edges and to the latter as dotted edges. Although one can also define the notion of weighted degree of a vertex in this case, we will only use it for monomials. The usual notion of degree of a vertex, *i.e.*, the number of edges incident to it, disregarding weights and coloring of the edges, will be used.

Lemma 2.3. *Let $\mathcal{G}$ be any graph and X its associated algebraic toric set. Let $f = t^a - t^b$ be a binomial (not necessarily homogeneous). Then, f vanishes on X if and only if for all $v \in V_{\mathcal{G}}$, $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) \equiv \mathrm{wt}_{\mathcal{H}_{t^b}}(v) \pmod{q-1}$.*

Proof. Suppose that $f = t^a - t^b$ vanishes on X and let $v = v_{l_0} \in V_{\mathcal{G}}$, for $l_0 \in \{1, \ldots, n\}$. Let $\mathbf{x} = (\ldots, x_l x_{l'}, \ldots) \in X$ be given by $x_{l_0} = u$, where u is a generator of K^*, and $x_l = 1$, for any $l \neq l_0$. Then

$$f(\mathbf{x}) = 0 \implies u^{\mathrm{wt}_{\mathcal{H}_{t^a}}(v_{l_0})} - u^{\mathrm{wt}_{\mathcal{H}_{t^b}}(v_{l_0})} = 0 \iff \mathrm{wt}_{\mathcal{H}_{t^a}}(v_{l_0}) \equiv \mathrm{wt}_{\mathcal{H}_{t^b}}(v_{l_0}) \pmod{q-1}.$$

Conversely, let $\mathbf{x} = (\ldots, x_l x_{l'}, \ldots) \in X$ be any point of X and assume that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) \equiv \mathrm{wt}_{\mathcal{H}_{t^b}}(v) \pmod{q-1}$, for every $v \in V_{\mathcal{G}}$. Then

$$f(\mathbf{x}) = \prod_{l=1}^{n} x_l^{\mathrm{wt}_{\mathcal{H}_{t^a}}(v_l)} - \prod_{l=1}^{n} x_l^{\mathrm{wt}_{\mathcal{H}_{t^b}}(v_l)} = 0. \qquad \square$$

Let $I \subset S = K[t_1, \ldots, t_s]$ be (any) homogeneous ideal and let $H_{S/I}(d) = \dim_K S_d / I_d$, for every $d \geq 0$, be the Hilbert function of the graded ring S/I. There exists a polynomial $h_{S/I}(t)$ in $\mathbb{Z}[t]$ of degree $k-1$, where $k = \dim S/I$, such that $H_{S/I}(d) = h_{S/I}(d)$ for $d \gg 0$. The *degree* or *multiplicity* of S/I is, by definition, the leading coefficient of $h_{S/I}(t)$ multiplied by $(k-1)!$. The *index of regularity* of S/I is the least integer $\ell \geq 0$ such that $H_{S/I}(d) = h_{S/I}(d)$ for $d \geq \ell$. Letting

$b_{ij} = \dim_K \mathrm{Tor}_i(K, S/I)_j$ be the *graded Betti numbers* of S/I, the *Castelnuovo–Mumford regularity* of S/I is, by definition, $\max\{j - i \,|\, b_{ij} \neq 0\}$. If I is Cohen–Macaulay and $\dim S/I = 1$, then the index of regularity of S/I coincides with the Castelnuovo–Mumford regularity of S/I (cf. [23, Corollary 2.5.14 and Proposition 4.2.3]). As this is the context of this work, with $I = I(X)$ (*cf.* [18, Theorem 2.2]), we will denote both by $\mathrm{reg}(S/I(X))$.

Hence, in our case, the Hilbert function of the ring $S/I(X)$ is constant for $d \geq \mathrm{reg}(S/I(X))$. The constant value at which it stabilizes is $|X|$, and since the Hilbert function of $S/I(X)$ is strictly increasing in the range $0 \leq d \leq \mathrm{reg}(S/I(X))$ (cf. [3], [5]), the regularity of $S/I(X)$ is equal to the first d for which it attains the value $|X|$. Note that $|X|$ is equal to $(q-1)^{n-1}$ by the formula (1.3) — recall that $n = \#V_{\mathcal{G}}$.

3. A generating set for $I(X)$

In this section we describe a generating set for the vanishing ideal $I(X)$ of the algebraic toric set associated to a complete multipartite graph $\mathcal{G} = K_{\alpha_1,\ldots,\alpha_r}$. We will distinguish 3 types of generators. The *type I generators* are of the form $t_i^{q-1} - t_j^{q-1}$, for $1 \leq i, j \leq s$. The *type II generators* are in 1-to-1 correspondence with the cycles of length 4, $\{e_i, e_j, e_k, e_l\}$, in $\mathcal{G}$ (*cf.* see Figure 1). Each cycle yields the generator: $t_i t_k - t_j t_l$.

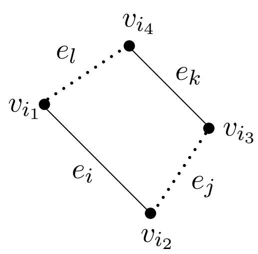

FIGURE 1. A 4-cycle yielding type *II* generator.

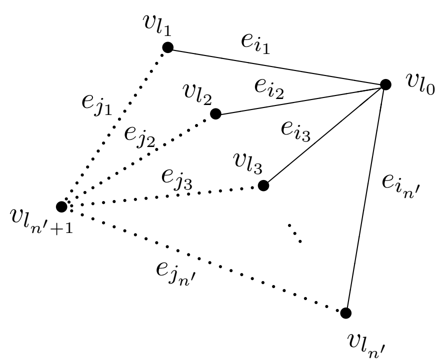

FIGURE 2. A subgraph yielding a type *III* generator.

A *type III generator* is specified by the choice of $n' \geq 2$ distinct vertices, $v_{l_1}, \ldots, v_{l_{n'}} \in V_{\mathcal{G}}$, plus 2 additional vertices v_{l_0} and $v_{l_{n'+1}}$ such that the edges $\{v_{l_0}, v_{l_k}\}$ and $\{v_{l_{n'+1}}, v_{l_k}\}$ exist, for all $1 \leq k \leq n'$ and, furthermore, by the choice of positive integers $1 \leq d_k \leq q-2$ such that $d_1 + \cdots + d_{n'} = q-1$. We associate to this data the binomial $\prod_{k=1}^{n'} t_{i_k}^{d_k} - \prod_{k=1}^{n'} t_{j_k}^{d_k}$, where, for each $k \in \{1, \ldots, n'\}$, i_k and j_k are such that $e_{i_k} = \{v_{l_0}, v_{l_k}\}$ and $e_{j_k} = \{v_{l_{n'+1}}, v_{l_k}\}$. The associated weighted subgraph of $\mathcal{G}$ is depicted in Figure 2. By a straightforward application of Lemma 2.3, it is clear that the homogeneous binomials of the 3 types listed earlier belong to $I(X)$.

We will need the following lemma.

Lemma 3.1. *Let $v_{l_0}, v_{l_1}, \ldots, v_{l_{n'}}, v_{l_{n'+1}} \in V_{\mathcal{G}}$, be such that $n' \geq 2$ and the edges $\{v_{l_0}, v_{l_k}\}$ and $\{v_{l_{n'+1}}, v_{l_k}\}$ exist in $\mathcal{G}$, for all $1 \leq k \leq n'$. Let $1 \leq d_k \leq q-2$ be such that $d_1 + \cdots + d_{n'} = \alpha(q-1)$, where α is a positive integer. Consider $f = \prod_{k=1}^{n'} t_{i_k}^{d_k} - \prod_{k=1}^{n'} t_{j_k}^{d_k}$, where, for each $k \in \{1, \ldots, n'\}$, i_k and j_k are such that $e_{i_k} = \{v_{l_0}, v_{l_k}\}$ and $e_{j_k} = \{v_{l_{n'+1}}, v_{l_k}\}$. Then f belongs to the ideal generated by the type III binomials.*

Proof. We proceed by induction on α. If $\alpha = 1$, f is exactly a type III generator. Assume $\alpha \geq 2$. Choose $m \in \mathbb{N}$ such that $2 \leq m \leq n'$ and, for each $k = 1, \ldots, m$, choose $d'_k \in \mathbb{N}$ such that $1 \leq d'_k \leq d_k \leq q-2$ and $d'_1 + \cdots + d'_m = q-1$. Then $\prod_{k=1}^{m} t_{i_k}^{d'_k} - \prod_{k=1}^{m} t_{j_k}^{d'_k}$ is a type III generator. Write $f = t^a \prod_{k=1}^{m} t_{i_k}^{d'_k} - t^b \prod_{k=1}^{m} t_{j_k}^{d'_k}$, for appropriate $a, b \in \mathbb{N}^s$. Then:

$$f = \prod_{k=1}^{n'} t_{i_k}^{d_k} - \prod_{k=1}^{n'} t_{j_k}^{d_k} = \Big(\prod_{k=1}^{m} t_{i_k}^{d'_k} - \prod_{k=1}^{m} t_{j_k}^{d'_k}\Big)\, t^a + (t^a - t^b) \prod_{k=1}^{m} t_{j_k}^{d'_k}.$$

By induction, $t^a - t^b$ is in the ideal generated by the type III binomials, hence so is f. □

Lemma 3.2. *Let $f = t^a - t^b$ be a (homogeneous) binomial in $I(X)$. If $\mathcal{H}_f$ contains one of the subgraphs depicted in Figure 3, with $v_1 \neq v_2$, and either there is an edge in $\mathcal{G}$ through v_1 and v_2 or one of v_1, v_2 is dichromatic, then there exists $j \in \{1, \ldots, s\}$ and a (homogeneous) binomial $g \in K[t_1, \ldots, t_s]$ such that $f - t_j g$ belongs to the ideal of $K[t_1, \ldots, t_s]$ generated by the binomials of type I,II and III.*

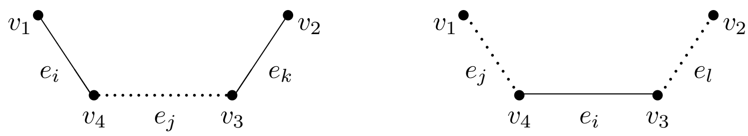

FIGURE 3. Two special dichromatic edge arrangements

Proof. Let J be the ideal of $K[t_1, \ldots, t_s]$ generated by the binomials of type I, II and III.

Case 1. Suppose, without loss of generality, that it is the case of the graph on the left of Figure 3 and that there exists an edge in $\mathcal{G}$ through v_1 and v_2. Denote this edge by e_l. Then $t_i t_k - t_j t_l$ is a type II generator. Let $f = t^a - t^b = t_i t_k t^{a'} - t_j t^{b'}$, for appropriate $a', b' \in \mathbb{N}^s$ and consider $g = t_l t^{a'} - t^{b'}$. Then,

$$f - t_j g = t_i t_k t^{a'} - t_j t^{b'} - t_j(t_l t^{a'} - t^{b'}) = (t_i t_k - t_j t_l) t^{a'} \in J.$$

Case 2. Assume now that v_1 and v_2 have no edge in $\mathcal{G}$ between them, (in other words, that they belong to the same part of the partition of $V_{\mathcal{G}}$) and that v_1 is dichromatic. Let e_l be a dotted edge incident to this vertex. Let v_5 be its other endpoint. If there exists an edge in $\mathcal{G}$ between v_3 and v_5, we reduce to Case 1 (graph on the right of Figure 3). Assume v_3 and v_5 belong to the same part of the partition of $V_{\mathcal{G}}$. Observe that v_1 and v_3 must be in distinct parts of the partition of $V_{\mathcal{G}}$ (otherwise v_2 and v_3 would be in that same part, but we are assuming that there is an edge e_k through v_2 and v_3). Therefore there is an edge in $\mathcal{G}$, denote it by e_λ, through v_1 and v_3. By the same type of reasoning, we deduce that there is an edge in $\mathcal{G}$, denote it by e_μ, through v_2 and v_4.

If $v_3 = v_5$, then $e_\lambda = e_l$, which is a dotted edge in $\mathcal{H}_f$. Since v_2 and v_4 are in different parts of the partition of $V_\mathcal{G}$, we are reduced (with e_i, e_l and e_k) to Case 1 (graph on the left of Figure 3). So we may assume $v_3 \neq v_5$. Write $f = t_i t_k t^{a'} - t^b$ and consider $f' = t_\lambda t_\mu t^{a'} - t^b$. Notice that e_λ is a black edge of the subgraph $\mathcal{H}_{f'}$, and since t_l and t_j divide t^b, e_l and e_j are dotted edges of the subgraph $\mathcal{H}_{f'}$. Observe also that v_4 and v_5 are not in the same part of the partition of $V_\mathcal{G}$ (otherwise v_3 and v_4 would be in that same part, but we are assuming that there is an edge e_j through v_3 and v_4). Then there must be an edge in $\mathcal{G}$ through v_4 and v_5. Therefore, $\mathcal{H}_{f'}$ contains a subgraph with edges e_l, e_λ and e_j, satisfying the assumption of Case 1 (graph on the right of Figure 3). Thus, there exists $g \in K[t_1, \ldots, t_s]$ such that $f' - t_\lambda g$ is a multiple of a type II generator. To complete the proof it suffices to observe that $f = f' + (t_i t_k - t_\lambda t_\mu) t^{a'}$ and that $t_i t_k - t_\lambda t_\mu$ is a type II generator. $\square$

Theorem 3.3. *The binomials of type I, II and III generate $I(X)$.*

Proof. Denote by J the ideal of $K[t_1, \ldots, t_s]$ generated by the binomials of type I, II and III. Clearly $J \subseteq I(X)$. By [17, Theorem 4.5] there exists a set of generators of $I(X)$ consisting of the generators of type I, plus a finite set of homogeneous binomials $t^a - t^b$ with $\operatorname{supp}(a) \cap \operatorname{supp}(b) = \emptyset$ and such that the degree of $t^a - t^b$ in each variable is $\leq q-2$. Hence it will suffice to show that any homogeneous binomial $f = t^a - t^b \in I(X)$ satisfying the latter conditions belongs to the ideal generated by the elements in the 3 classes described, or, equivalently, is congruent to 0 modulo J. Let $f = t^a - t^b$ be such a binomial and, as defined above, let $\mathcal{H}_f$ be the induced subgraph of $\mathcal{G}$. We will argue by induction on $\# V^+_{\mathcal{H}_f} + \deg(f)$, where $V^+_{\mathcal{H}_f}$ denotes the subset of $V_{\mathcal{H}_f} = V_\mathcal{G}$ of vertices with positive degree.

By Lemma 2.3, no vertex of $\mathcal{H}_f$ has (standard) degree equal to 1. Hence $\# V^+_{\mathcal{H}_f} \geq 3$ and if $\# V^+_{\mathcal{H}_f} = 3$ then $\mathcal{H}_f$ reduces to a triangle. This situation is impossible, since f is homogeneous and the condition given in the statement of Lemma 2.3 must be satisfied. The base case is thus $\# V^+_{\mathcal{H}_f} = 4$ and $\deg(f) = 2$. $\mathcal{H}_f$ must reduce to a square, and using Lemma 2.3 we deduce that f is a type II generator if $q > 3$ and a type II or a type III generator if $q = 3$.

Suppose that $\# V^+_{\mathcal{H}_f} + \deg(f) \geq 7$. If all vertices of $\mathcal{H}_f$ are dichromatic then we can find a subgraph of $\mathcal{H}_f$ satisfying the assumptions of Lemma 3.2. Then there exists $j \in \{1, \ldots, s\}$ such that $f - t_j g \in J$, for some homogeneous binomial $g \in K[t_1, \ldots, t_s]$. Since $J \subset I(X)$, $f \in I(X)$ and t_j does not vanish on any point of X, we deduce that $g \in I(X)$. It is clear that $\deg(g) < \deg(f)$. We may assume that $g = t^{a'} - t^{b'}$ with $\operatorname{supp}(a') \cap \operatorname{supp}(b') = \emptyset$ (dividing through by an appropriate monomial, if necessary). We may also assume that $0 \leq a'_i, b'_i \leq q-2:$ suppose that $t_l^{a'_l}$ divides $t^{a'}$ and $a'_l \geq q-1$; let t_k divide $t^{b'}$; then $g = t_l^{q-1} t^{a''} - t_k t^{b''} + t_k^{q-1} t^{a''} - t_k^{q-1} t^{a''} = (t_l^{q-1} - t_k^{q-1}) t^{a''} + t_k h$, where $h = t_l^{a'_l - (q-1)} t^c - t^d$, for appropriate $a'', b'', c, d \in \mathbb{N}^s$; since $t_l^{q-1} - t_k^{q-1}$ is a type I generator, $g \in I(X)$ if and only if $h \in I(X)$. Repeating the argument, we may indeed assume that $0 \leq a'_i, b'_i \leq q-2$. Therefore, by the induction assumption, $g \equiv 0 \mod J$, which implies that $f \equiv 0 \mod J$.

Let now v_{l_0} be a monochromatic vertex of $\mathcal{H}_f$. Assume, without loss of generality, that all of its incident edges are black. Denote by $v_{l_1},\ldots,v_{l_{n'}}$ $(n'\geq 2)$ their endpoints and let $e_{i_k}=\{v_{l_0},v_{l_k}\}$, for $k=1,\ldots,n'$.

Special condition: Suppose there exists $v_{l_{n'+1}}\in V^+_{\mathcal{H}_f}$ with $v_{l_{n'+1}}\neq v_{l_0}$ and such that, for all $k=1,\ldots,n'$, $\{v_{l_{n'+1}},v_{l_k}\}\in E_{\mathcal{G}}$. Denote these edges by e_{j_k}. By Lemma 2.3, $a_{i_1}+\cdots+a_{i_{n'}}\equiv 0 \pmod{q-1}$ and therefore, by Lemma 3.1, $\prod_{k=1}^{n'} t_{i_k}^{a_{i_k}}-\prod_{k=1}^{n'} t_{j_k}^{a_{i_k}}\in J$. Writing $t^a=t^{a'}\prod_{k=1}^{n'} t_{i_k}^{a_{i_k}}$, for suitable $a'\in\mathbb{N}^s$,

$$f=t^{a'}\textstyle\prod_{k=1}^{n'} t_{i_k}^{a_{i_k}}-t^b\equiv t^{a'}\prod_{k=1}^{n'} t_{j_k}^{a_{i_k}}-t^b \mod J.$$

Write $g=f-t^{a'}(\prod_{k=1}^{n'} t_{i_k}^{a_{i_k}}-\prod_{k=1}^{n'} t_{j_k}^{a_{i_k}})=t^{a'}\prod_{k=1}^{n'} t_{j_k}^{a_{i_k}}-t^b=t^c-t^b$, for appropriate $c\in\mathbb{N}^s$. Then $g\in I(X)$, the graph induced by g has one vertex of positive weighted degree fewer than the graph $\mathcal{H}_f$, and g has degree equal to $\deg f$. As above, by using a type I generator and dividing through by an appropriate monomial (in which case the resulting monomial would have degree strictly less that $\deg f$), we may also assume that no variable in g occurs to a higher power than $q-2$ and that $\operatorname{supp}(c)\cap\operatorname{supp}(b)=\emptyset$. By induction, we deduce that $f\equiv g\equiv 0 \mod J$.

Denote now by $P_{\mu_0},P_{\mu_1},\ldots,P_{\mu_m}$ the parts of the partition of $V_{\mathcal{G}}$ that have nonempty intersection with $V^+_{\mathcal{H}_f}$, with $v_{l_0}\in P_{\mu_0}$. If $\#(P_{\mu_0}\cap V^+_{\mathcal{H}_f})\geq 2$, we are in the special condition case, and therefore $f\equiv 0 \mod J$. We may assume that $P_{\mu_0}\cap V^+_{\mathcal{H}_f}=\{v_{l_0}\}$. We may also assume that $P_{\mu_k}\cap\{v_{l_1},\ldots,v_{l_{n'}}\}\neq\emptyset$, for all $k=1,\ldots,m$, for otherwise, if for some $k\in\{1,\ldots,m\}$, this intersection is empty, choosing $v_{l_{n'+1}}\in P_{\mu_k}\cap V_{\mathcal{H}^+_f}$, we are again in the special condition case. Another observation is the following: if there exists $k\in\{1,\ldots,m\}$ such that $\#(P_{\mu_k}\cap V_{\mathcal{H}^+_f})\geq 2$ and one of the vertices of $P_{\mu_k}\cap V_{\mathcal{H}^+_f}$ is monochromatic, then, using the same argument we used for $P_{\mu_0}\cap V_{\mathcal{H}^+_f}$, we conclude that $f\equiv 0 \mod J$. We may therefore assume that for all $k\in\{1,\ldots,m\}$ such that $\#(P_{\mu_k}\cap V_{\mathcal{H}^+_f})\geq 2$, the vertices of $P_{\mu_k}\cap V_{\mathcal{H}^+_f}$ are all dichromatic.

Applying Lemma 2.3 for each vertex in $V^+_{\mathcal{H}_f}\setminus\{v_{l_0}\}$ and summing all the congruences obtained,

$$\textstyle\sum_{k=1}^{n'} a_{i_k}+2\sum_{i^*}a_{i^*}=2\sum_{i^*}b_{i^*}+\delta(q-1), \tag{3.1}$$

where i^* varies on the subset of $\{1,\ldots,s\}$ corresponding to edges of $\mathcal{H}_f$ except for $e_{i_1},\ldots,e_{i_{n'}}$, and $\delta\in\mathbb{Z}$ is given by $\delta=\sum_{v\in V^+_{\mathcal{H}_f}\setminus\{v_{l_0}\}}\delta_v$, where $\delta_v\in\mathbb{Z}$ yields the congruence in of Lemma 2.3 for the vertex v. Since f is homogeneous,

$$\textstyle\sum_{k=1}^{n'} a_{i_k}+\sum_{i^*}a_{i^*}=\sum_{i^*}b_{i^*}. \tag{3.2}$$

Together, (3.1) and (3.2) imply that $\sum_{k=1}^{n'} a_{i_k}+\delta(q-1)=0$. We deduce that $\delta<0$. This means that for some vertex $v_{\lambda_0}\in V^+_{\mathcal{H}_f}\cap\big(P_{\mu_1}\cup\cdots\cup P_{\mu_m}\big)$, $\delta_{\lambda_0}<0$, and, in particular, the sum of the weights of dotted edges incident to v_{λ_0} is $\geq q-1$. Denote by $e_{\nu_1},\ldots,e_{\nu_{\tilde n}}$ the dotted edges incident to v_{λ_0} and by $v_{\lambda_1},\ldots,v_{\lambda_{\tilde n}}$ their endpoints, so that $e_{\nu_k}=\{v_{\lambda_0},v_{\lambda_k}\}$ and $\sum_{k=1}^{\tilde n} b_{\nu_k}\geq q-1$. Notice that, necessarily $\tilde n\geq 2$. We claim that v_{λ_0} must be dichromatic. This is clear if v_{λ_0} coincides with one of $v_{l_1},\ldots,v_{l_{n'}}$. Otherwise, since v_{λ_0} belongs to some P_{μ_k} and $P_{\mu_k}\cap\{v_{l_1},\ldots,v_{l_{n'}}\}\neq\emptyset$, we get $\#(P_{\mu_k}\cap V^+_{\mathcal{H}_f})\geq 2$ and hence v_{λ_0} is not monochromatic. The same argument can be used to show that $v_{\lambda_1},\dots,v_{\lambda_{\tilde n}}$ are all dichromatic. For each of the vertices $v_{\lambda_0},v_{\lambda_1},\dots,v_{\lambda_{\tilde n}}$ choose a black edge incident to it, denote it by e_{γ_i}, for $i=0,\dots,\tilde n$, and let v_{β_i} be its endpoint.

Suppose there exists $r\in\{1,\dots,\tilde n\}$ such that $v_{\beta_0}\neq v_{\beta_r}$. Either there is an edge in $\mathcal{G}$ through v_{β_0} and v_{β_r}, or v_{β_0} and v_{β_r} are in the same part of the partition of $V_{\mathcal{G}}$. Then there exists $k\in\{1,\dots,m\}$ such that $v_{\beta_0},v_{\beta_r}\in P_{\mu_k}\cap V^+_{\mathcal{H}_f}$, and since $\#(P_{\mu_k}\cap V^+_{\mathcal{H}_f})\geq 2$, v_{β_0} and v_{β_r} are dichromatic. In any case, we may use Lemma 3.2 and argue as previously to show that $f\equiv 0 \mod J$. Suppose then that for all $r\in\{0,\dots,\tilde n\}$, the endpoints v_{β_r} all coincide. As in the proof of Lemma 3.1, choose $m'\in\mathbb{N}$ such that $2\leq m'\leq\tilde n$ and, for each $k=1,\dots,m'$, choose $b'_{\nu_k}\in\mathbb{N}$ such that $1\leq b'_{\nu_k}\leq b_{\nu_k}\leq q-2$ and $b'_{\nu_1}+\cdots+b'_{\nu_{m'}}=q-1$. Then

$$\textstyle\prod_{k=1}^{m'} t_{\gamma_k}^{b'_{\nu_k}}-\prod_{k=1}^{m'} t_{\nu_k}^{b'_{\nu_k}}$$

is a type III generator, and therefore belongs to J. Writing

$$f=t^a-t^b=t^{a'}\textstyle\prod_{k=1}^{m'} t_{\gamma_k}-t^{b'}\prod_{k=1}^{m'} t_{\nu_k}^{b'_{\nu_k}},$$

for appropriate $a',b'\in\mathbb{N}^s$, we deduce that

$$f\equiv t^{a'}\textstyle\prod_{k=1}^{m'} t_{\gamma_k}-t^{b'}\prod_{k=1}^{m'} t_{\gamma_k}^{b'_{\nu_k}}=\big(t^{a'}-t^{b'}\prod_{k=1}^{m'} t_{\gamma_k}^{b'_{\nu_k}-1}\big)\prod_{k=1}^{m'} t_{\gamma_k}\mod J.$$

The homogeneous binomial $g=t^{a'}-t^{b'}\prod_{k=1}^{m'} t_{\gamma_k}^{b'_{\nu_k}-1}=t^{a'}-t^{b''}$, which by the above belongs to $I(X)$, is such that $\deg(g)=\deg(f)-m'<\deg(f)$. We may assume $\operatorname{supp}(a')\cap\operatorname{supp}(b'')=\emptyset$. By induction, $g\equiv 0\mod J$, which implies that $f\equiv 0\mod J$ and completes the proof. □

4. Regularity of $S/I(X)$

Let $\mathcal{G}=\mathcal{K}_{\alpha_1,\dots,\alpha_r}$ be a complete multipartite graph with $r\geq 3$. We assume $\mathcal{G}$ does not coincide with $\mathcal{K}_{1,1,1}$, the complete graph on 3 vertices. In that case, the associated toric set X coincides with the ambient torus, $\mathbb{T}^2\subset\mathbb{P}^2$ and by *cf.* [8, Lemma 1], the regularity of $S/I(X)$ is $2(q-2)$. In this section we will show that in all other cases,

$$\operatorname{reg} S/I(X)=\max\{\alpha_1(q-2),\alpha_2(q-2),\dots,\alpha_r(q-2),\lceil(n-1)(q-2)/2\rceil\},$$

where, $n=\#V_{\mathcal{G}}=\alpha_1+\cdots+\alpha_r\geq 4$. If, without loss in generality, we assume that $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_r$, then this formula takes on the simpler form:

$$\operatorname{reg} S/I(X)=\max\{\alpha_1(q-2),\lceil(n-1)(q-2)/2\rceil\}.$$

The case of the complete graph, *i.e.*, when $\alpha_1=\cdots=\alpha_r=1$, is treated in [9].

In the proof of this result we will need to show that $\operatorname{reg} S/I(X)\geq\lceil(n-1)(q-2)/2\rceil$. This inequality was shown to hold for any graph for which $\#X=(q-1)^{n-1}$ in [10]. Indeed [10, Corollary 3.13] implies that if X is the algebraic toric set associated to a k-uniform clutter on n vertices then

$$\operatorname{reg} S/I(X)\geq\left\lceil\frac{(\#X)(n-1)(q-2)}{k(q-1)^{n-1}}\right\rceil.$$

In the proof of Theorem 4.3 we argue on an Artinian reduction of $S/I(X)$. More precisely, if $f \in S$ is a regular element on $S/I(X)$ of degree 1, the short exact sequence

$$0 \to S/I(X)[-1] \xrightarrow{f} S/I(X) \to S/(I(X)+(f)) \to 0$$

yields $\operatorname{reg} S/I(X) = \operatorname{reg} S/(I(X)+(f)) - 1$. Accordingly, showing that $\operatorname{reg} S/I(X) = d$ amounts to proving that every monomial of degree $d+1$ belongs to $I(X)+(f)$ and that there exists a monomial of degree d that does not. For a detailed explanation of this fact, see [17, Theorem 6.2]. We begin with two lemmas.

Lemma 4.1. *Let $t^a \in S$ be a monomial and let $\mathcal{H}_{t^a}$ be the associated weighted subgraph of $\mathcal{G}$. Let $v_0 \in V_{\mathcal{G}}$ and $\{e_{i_k} = \{v_0, v_{i_k}\} : k = 1, \ldots, n'\}$, $n' \geq 2$, be a subset of the edges of $\mathcal{H}_{t^a}$ incident to v_0, such that $\sum_{k=1}^{n'} \mathrm{wt}_{\mathcal{H}_{t^a}}(e_{i_k}) \geq \alpha(q-1)$, for some positive integer α. Let $w_0 \in V_{\mathcal{G}} \setminus \{v_0\}$ be such that $\{w_0, v_{i_k}\} \in E(\mathcal{G})$, for all $k = 1, \ldots, n'$. Then there exists a monomial $t^b \in S$ such that $t^a - t^b \in I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_0) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_0) - \alpha(q-1)$, $\mathrm{wt}_{\mathcal{H}_{t^b}}(w_0) \geq \sum_{k=1}^{n'} \mathrm{wt}_{\mathcal{H}_{t^b}} \{w_0, v_{i_k}\} \geq \alpha(q-1)$ and, for all $v \in V_{\mathcal{G}} \setminus \{v_0, w_0\}$, $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v)$.*

Proof. We may write $t^a = \big(\prod_{k=1}^{n'} t_{i_k}^{d_k}\big) t^{a'}$, where $\sum_{k=1}^{n'} d_k = \alpha(q-1)$, for some $a' \in \mathbb{N}^s$. Denote $\{w_0, v_{i_k}\}$ by $e_{j_1}, \ldots, e_{j_{n'}}$. Consider $t^b = \big(\prod_{k=1}^{n'} t_{j_k}^{d_k}\big) t^{a'}$. Then, by Lemma 3.1,

$$t^a - t^b = \big(\textstyle\prod_{k=1}^{n'} t_{i_k}^{d_k} - \prod_{k=1}^{n'} t_{j_k}^{d_k}\big) t^{a'} \in I(X).$$

Additionally, $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_0) = \mathrm{wt}_{\mathcal{H}_{t^{a'}}}(v_0) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_0) - \alpha(q-1)$ and

$$\mathrm{wt}_{\mathcal{H}_{t^b}}(w_0) \geq \textstyle\sum_{k=1}^{n'} \mathrm{wt}_{\mathcal{H}_{t^b}} \{w_0, v_{i_k}\} \geq \sum_{k=1}^{n'} d_k = \alpha(q-1) \geq q-1.$$

Finally, if v is not an endpoint of $e_{i_1}, \ldots, e_{i_{n'}}, e_{j_1}, \ldots, e_{j_{n'}}$ then $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) = \mathrm{wt}_{\mathcal{H}_{t^{a'}}}(v) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v)$, and otherwise, $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_{i_k}) = d_k + \mathrm{wt}_{\mathcal{H}_{t^{a'}}}(v_{i_k}) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v_{i_k})$, for all $k = 1, \ldots, n'$. $\square$

Lemma 4.2. *Let $t^a \in S$ be a monomial and let $\mathcal{H}_{t^a}$ be the associated weighted subgraph of $\mathcal{G}$. Given $i, j \in \{1, \ldots, r\}$ such that $i \neq j$, let Δ_j^a be the total weight of edges between the vertices of $V_{\mathcal{G}} \setminus P_j$, let $v_i \in P_i$ be a vertex, and let $\delta_{ij}^a \leq \Delta_j^a$ be the total weight of edges between v_i and the vertices of $V_{\mathcal{G}} \setminus P_j$. Suppose that there exists an edge $\{w_1, w_2\} \in E(\mathcal{H}_{t^a})$ with $w_1, w_2 \notin P_j$ and $w_1, w_2 \neq v_i$. Then, there exists a monomial $t^b \in S$ such that $t^a - t^b \in I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v)$, for all $v \in V_{\mathcal{G}}$, and such that the total weight of edges between v_i and the vertices of $V_{\mathcal{G}} \setminus P_j$, δ_{ij}^b, is equal to* $\min\{\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i), \Delta_j^a\}$.

Proof. Notice that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) \geq \delta_{ij}^a$. We argue by induction on $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) - \delta_{ij}^a$. If $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) = \delta_{ij}^a$, we choose $t^b = t^a$ and there is nothing to prove. Suppose that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) > \delta_{ij}^a$. Then there exists an edge $\{v_i, w_3\} \in E(\mathcal{H}_{t^a})$ with $w_3 \in P_j$. Since $\{w_1, w_2\}$ is an edge, one of its vertices does not belong to P_i. Assume that $w_1 \notin P_i$. Then $t_{\{v_i,w_3\}} t_{\{w_1,w_2\}} - t_{\{v_i,w_1\}} t_{\{w_2,w_3\}}$ is a generator of $I(X)$ of type II. Writing $t^a = t_{\{v_i,w_3\}} t_{\{w_1,w_2\}} t^{a'}$, for suitable $a' \in \mathbb{N}^s$, and $t^c = t_{\{v_i,w_1\}} t_{\{w_2,w_3\}} t^{a'}$, it is clear that $t^a - t^c \in I(X)$; moreover the numbers Δ_j^c and $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_i)$ are the same as they were for $\mathcal{H}_{t^a}$, however $\delta_{ij}^c = \delta_{ij}^a + 1$. By induction, there exists a monomial $t^b \in S$ such that $t^c - t^b \in I(X)$, hence $t^a - t^b \in I(X)$, and such that $\delta_{ij}^b = \min\{\mathrm{wt}_{\mathcal{H}_{t^c}}(v_i), \Delta_j^c\} = \min\{\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i), \Delta_j^a\}$. Also by induction,

$\mathrm{wt}_{\mathcal{H}_{t^c}}(v) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v)$, for all $v \in V_{\mathcal{G}}$. On the level of the graph, the induction step merely takes two edges and swaps a pair of their endpoints. This does not change the weighted degree; therefore $\mathrm{wt}_{\mathcal{H}_{t^c}}(v) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v)$, for all $v \in V_{\mathcal{G}}$, and hence, $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v)$, for all $v \in V_{\mathcal{G}}$. □

Theorem 4.3. *Let X be the algebraic toric set associated to an r-partite complete graph $\mathcal{G} = \mathcal{K}_{\alpha_1,\dots,\alpha_r}$, with $r \geq 3$ and $n = \alpha_1 + \cdots + \alpha_r \geq 4$. Then*

$$\operatorname{reg} S/I(X) = \max\left\{\alpha_1(q-2), \alpha_2(q-2), \dots, \alpha_r(q-2), \lceil (n-1)(q-2)/2 \rceil\right\}.$$

Proof. Fix vertices $v_i \in P_i \subset V_{\mathcal{G}}$. Without loss in generality, let t_1 be the variable $t_{\{v_1,v_2\}}$. We assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$. If $\alpha_1 = 1$ then $\mathcal{G}$ is the complete graph of n vertices and $\operatorname{reg} S/I(X) = \lceil (n-1)(q-2)/2 \rceil$, as was proved in [9]. From now on we assume that $\alpha_1 \geq 2$. By [10, Corollary 3.13], $\operatorname{reg} S/I(X) \geq \lceil (n-1)(q-2)/2 \rceil$. Let us show that $\operatorname{reg} S/I(X) \geq \alpha_1(q-2)$. Consider the monomial of degree $\alpha_1(q-2)$ given by

$$t^a = t_{\{v_2,v_3\}}^{q-2} \prod_{w \in P_1 \setminus \{v_1\}} t_{\{w,v_2\}}^{q-2},$$

and consider $\mathcal{H}_{t^a}$, the weighted subgraph of $\mathcal{G}$ induced by t^a, which is depicted in Figure 4.

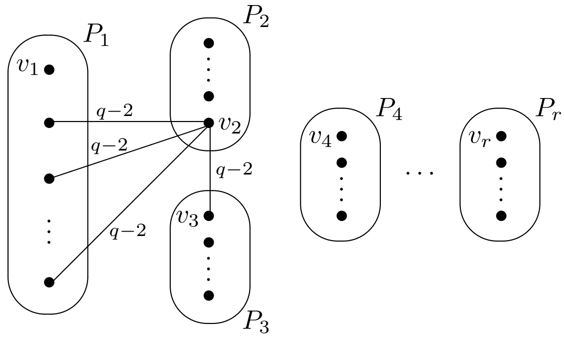

FIGURE 4. The weighted subgraph associated to t^a.

Let us show that $t^a \notin I(X) + (t_1)$. Let $\{f_1, \dots, f_m\}$ be a Gröbner basis of $I(X)$. Since, by Theorem 3.3, $I(X)$ is generated by binomials, we may assume that f_i is a binomial, for every $i = 1, \dots, m$ (*cf.* [4, Proposition 1.1 (a)], a straighforward consequence of Buchberger's algoritm). Since a variable t_i never vanishes on a point of X, we may also assume that each $f_i = t^a - t^b$ with $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$. Choose a monomial order in which t_1 is the least variable. Then, since $f_1, \dots, f_m, t_1$ generate $I(X) + (t_1)$, and t_1 does not divide the leading term of any $f_1, \dots, f_m$, we conclude, by Buchberger's algorithm (*cf.* [23, Theorem 2.4.15] and [2]), that $\{t_1, f_1, \dots, f_m\}$ is a Gröbner basis of $I(X) + (t_1)$.

Suppose that $t^a \in I(X) + (t_1)$. Then the division algorithm of t^a by $t_1, f_1, \dots, f_m$ produces zero remainder. Since t_1 does not divide t^a, there exists i such that $\mathrm{Lt}(f_i)$ divides t^a. Write $\mathrm{Lt}(f_i) = t^b$ and, without loss in generality, $f_i = t^b - t^c$, for some $b, c \in \mathbb{N}^s$. Then there exists $a' \in \mathbb{N}^s$ such that $t^a = t^b t^{a'}$. Let

$$t^{a_1} = t^a - t^{a'} f_i = t^{a'} t^c$$

be the first partial reduction of t^a modulo $t_1, f_1, \ldots, f_m$. Denote by $t^a = t^{a_0}, t^{a_1}, \ldots, t^{a_p}$ the full sequence of partial reductions of t^a modulo $t_1, f_1, \ldots, f_m$. Since division of a monomial by one of f_i always yields a nonzero monomial, we conclude that there must exist $p > 0$ such that t^{a_p} is divisible by t_1 (and, accordingly, $t^{a_{p+1}} = 0$). Write $t^{a_p} = t_1 t^w$, for some $w \in \mathbb{N}^s$. From the fact that $t^{a_{i+1}} - t^{a_i} \in I(X)$, for all $i = 0, \ldots, p-1$, we deduce that

$$t^a - t_1 t^w \in I(X).$$

Consider $\mathcal{H}_{t^a}$ and $\mathcal{H}_{t_1 t^w}$ the weighted subgraphs of $\mathcal{G}$ induced by t^a and $t_1 t^w$, respectively. By the definition of t^a (see also Figure 4), $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_1) = 0$ and, if $v \in P_1 \setminus \{v_1\}$, $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) = q-2$. Accordingly, using Lemma 2.3, we deduce that $\mathrm{wt}_{\mathcal{H}_{t_1 t^w}}(v_1) \geq q-1$ and, if $v \in P_1 \setminus \{v_1\}$, that $\mathrm{wt}_{\mathcal{H}_{t_1 t^w}}(v) \geq q-2$. We get:

$$\alpha_1(q-2) = \deg(t^a) = \deg(t_1 t^w) \geq \textstyle\sum_{v \in P_1} \mathrm{wt}_{\mathcal{H}_{t_1 t^w}}(v) \geq \alpha_1(q-2) + 1,$$

which is a contradiction. We just showed that $t^a \notin I(X) + (t_1)$, hence $\operatorname{reg} S/I(X) \geq \alpha_1(q-2)$.

Let $d = \max\{\alpha_1(q-2), \lceil (n-1)(q-2)/2 \rceil\}$. Let us now show that any monomial $t^a \in S$ of degree $d+1$ belongs to $I(X) + (t_1)$, or, equivalently, that there exists $t^b \in (t_1)$ such that $t^a \equiv t^b \mod I(X)$. As earlier, we will argue on $\mathcal{H}_{t^a}$, the weighted subgraph of $\mathcal{G}$ induced by t^a.

Let $w_1, w_2 \in V_{\mathcal{G}}$ such that $\{w_1, w_2\} \in E(\mathcal{H}_{t^a})$. Write $t_{i_0} = t_{\{w_1, w_2\}}$ and $t^a = t_{i_0}^b t^{a'}$, where $b \geq 1$ and $a' \in \mathbb{N}^s$. Suppose $b \geq q-1$. If $t_{i_0} = t_1$, then $t^a \in (t_1) + I(X)$. If $t_{i_0} \neq t_1$, then $g = t_{i_0}^{q-1} - t_1^{q-1}$ is a type I generator of $I(X)$. Writing $t^a = t_{i_0}^{q-1} t^{a''}$, for some $a'' \in \mathbb{N}^s$, we have $t^a = t_1^{q-1} t^{a''} + g t^{a''} \in (t_1) + I(X)$. Therefore, we may assume that if $t_{i_0}^b$ divides t^a, then $b \leq q-2$. This implies we may also assume that if $v \in V_{\mathcal{G}}$ is such that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v) \geq q-1$, then there exist at least two edges in $\mathcal{H}_{t^a}$ that have v as an endpoint.

We start by deriving a basic inequality. Suppose that $v_0 \in P_i \setminus \{v_1, \ldots, v_r\}$ has $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_0) \geq q-1$. Then using Lemma 4.1 with $w_0 = v_i$ and α a positive integer such that $(\alpha+1)(q-1) > \mathrm{wt}_{\mathcal{H}_{t^a}}(v_0) \geq \alpha(q-1)$, we see that there exists $t^b \in S$ such that $t^a \equiv t^b \mod I(X)$ and $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_0) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_0) - \alpha(q-1) \leq q-2$. Accordingly, we may assume that

$$\text{for all } v \notin \{v_1, \ldots, v_r\},\ \mathrm{wt}_{\mathcal{H}_{t^a}}(v) \leq q-2. \tag{4.1}$$

Hence, $\sum_{v \in V_{\mathcal{G}}} \mathrm{wt}_{\mathcal{H}_{t^a}}(v) \leq \sum_{j=1}^r \mathrm{wt}_{\mathcal{H}_{t^a}}(v_j) + (n-r)(q-2)$.

Since $\sum_{v \in V_{\mathcal{G}}} \mathrm{wt}_{\mathcal{H}_{t^a}}(v) = 2(d+1) \geq 2\left\lceil \frac{n-1}{2}(q-2) \right\rceil + 2 \geq (n-1)(q-2) + 2$, we get

$$\textstyle\sum_{j=1}^r \mathrm{wt}_{\mathcal{H}_{t^a}}(v_j) \geq (r-1)(q-2) + 2. \tag{4.2}$$

Case 1. Suppose that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_1), \mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) > 0$.

Subcase 1.1. Suppose v_1 and v_2 have edges in $\mathcal{H}_{t^a}$ connecting them to 2 vertices in distinct P_i, say $\{v_1, w_1\}, \{v_2, w_2\}$, with $w_1 \in P_i$ and $w_2 \in P_j$, and $i \neq j$. If $w_1 = v_2$ or $w_2 = v_1$, then t_1 divides t^a and we are done. If $w_1 \neq v_2$ and $w_2 \neq v_1$, then since $\{w_1, w_2\}$ belongs to $E(\mathcal{G})$, we have that $t_{\{v_1, w_1\}} t_{\{v_2, w_2\}} - t_1 t_{\{w_1, w_2\}}$ is a type II generator of $I(X)$. As, by assumption, $t_{\{v_1, w_1\}} t_{\{v_2, w_2\}}$ divides

t^a, we may write $t^a = t_{\{v_1,w_1\}}t_{\{v_2,w_2\}}t^{a'}$, for a suitable $a' \in \mathbb{N}^s$. Then, $t^a \equiv t_1 t_{\{w_1,w_2\}}t^{a'} \mod I(X)$, as required.

Subcase 1.2. Suppose there exists $i \in \{1,\dots,r\}$ such that all edges in $\mathcal{H}_{t^a}$ from v_1 and v_2 have endpoints in P_i. Clearly $i \in \{3,\dots,r\}$. Assume, additionally, that there exists and edge $\{w_1,w_2\} \in E(\mathcal{H}_{t^a})$, such that $w_1, w_2 \notin P_i$. Then, necessarily $w_1, w_2 \neq v_1$ and $w_1, w_2 \neq v_2$ and at least one of $\{v_2,w_1\}$ or $\{v_2,w_2\}$ belongs to $E(\mathcal{G})$. Assume this is the case with $\{v_2,w_1\}$. Let $z \in P_i$ be an endpoint of an edge in $\mathcal{H}_{t^a}$ incident to v_2. Then $t_{\{v_2,z\}}t_{\{w_1,w_2\}} - t_{\{v_2,w_1\}}t_{\{z,w_2\}}$ is a type II generator of $I(X)$ and writing $t^a = t_{\{v_2,z\}}t_{\{w_1,w_2\}}t^{a'}$, for a suitable $a' \in \mathbb{N}^s$, we get $t^a \equiv t_{\{v_2,w_1\}}t_{\{z,w_2\}}t^{a'}$ mod $I(X)$. Denote $t^e = t_{\{v_2,w_1\}}t_{\{z,w_2\}}t^{a'}$. Then t^e satisfies the assumptions of Subcase 1.1 as $w_1 \notin P_i$, $\{v_2,w_1\} \in E(\mathcal{H}_{t^e})$, and all the edges in $E(\mathcal{H}_{t^e})$ starting at v_1 have the other endpoint in P_i.

Subcase 1.3. Suppose there exists $i \in \{3,\dots,r\}$ such that all edges in $\mathcal{H}_{t^a}$ from v_1 and v_2 have endpoints in P_i. Suppose in addition that all other edges in $\mathcal{H}_{t^a}$ are also incident to vertices of P_i. Using the assumption on $\deg(t^a)$ and (4.1), we get

$$\left\{ \begin{array}{l} \sum_{v\in P_i} \mathrm{wt}_{\mathcal{H}_{t^a}}(v) = d+1 \geq \alpha_1(q-2)+1 \geq 2(q-2)+1 = (q-2)+(q-1) \\ \mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) = d+1-\sum_{v\in P_i\setminus\{v_i\}} \mathrm{wt}_{\mathcal{H}_{t^a}}(v) \geq \alpha_i(q-2)+1-(\alpha_i-1)(q-2) = q-1. \end{array} \right. \tag{4.3}$$

The first inequality implies that either, Δ_1^a, the total weight of the edges in $\mathcal{H}_{t^a}$ between the vertices of $V_{\mathcal{G}} \setminus P_1$ is $\geq q-1$, or Δ_2^a, the total weight of edges in $\mathcal{H}_{t^a}$ between the vertices of $V_{\mathcal{G}} \setminus P_2$ is $\geq q-1$. Assume, without loss of generality that the latter is true.

Let $\{w_1,w_2\} \in E(\mathcal{H}_{t^a})$ be an edge with $w_1, w_2 \notin P_2$ and assume $w_1, w_2 \neq v_i$. Then, by Lemma 4.2 (with $j=2$), there exists $t^b \in S$ such that $t^a \equiv t^b \mod I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_1) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_1) > 0$, and such that the total weight of the edges of $\mathcal{H}_{t^b}$ between v_i and $V_{\mathcal{G}} \setminus P_2$, δ_{i2}^b, is equal to $\min\{\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i), \Delta_2^a\}$. From the second inequality of (4.3) and $\Delta_2^a \geq q-1$, we get $\delta_{i2}^b \geq q-1$. Observe also that the use of Lemma 4.2 guarantees that all edges in $\mathcal{H}_{t^b}$ are still incident to vertices of P_i.

As done in a previous argument, if $t^b = t_k^l t^{b'}$, where $l \geq q-1$ and $b' \in \mathbb{N}^s$, then $t^b \in (t_1)+I(X)$, and so, we may assume that if t_k^l divides t^b, then $l \leq q-2$. Since $\delta_{i2}^b \geq q-1$, there exist at least two edges in $\mathcal{H}_{t^b}$ that have v_i as an endpoint and such that the other endpoints are in $V_{\mathcal{G}} \setminus (P_2 \cup P_i)$.

We now use Lemma 4.1 with v_i and v_2 instead of v_0 and w_0. Then there exists a monomial $t^c \in S$ such that $t^b \equiv t^c \mod I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_1) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v_1) > 0$, and $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_2) \geq q-1$.

Choose $z \in V_{\mathcal{G}} \setminus (P_2 \cup P_i)$ as one of the endpoints mentioned above ($t_{\{v_i,z\}}$ dividing t^b), and such that $t_{\{v_i,z\}}$ is used in the "transfer of weight" from v_i to v_2 of Lemma 4.1. As a consequence, $t_{\{z,v_2\}}$ divides t^c.

If $z = v_1$, then $t^c \in (t_1)$, and we are done.

Consider the case when $z \neq v_1$. Since $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_1) > 0$, there exists $u \in V_{\mathcal{G}}$ such that $t_{\{v_1,u\}}$ divides t^c. If $u = v_2$, we are again done. If $u \neq v_2$, and since all edges in $\mathcal{H}_{t^b}$ are incident to vertices of P_i, we must have $u \in P_i$.

Now, $\mathcal{H}_{t^c}$ satisfies the assumptions of Subcase 1.1.

We still need to consider the case when all edges $\{w_1, w_2\} \in E(\mathcal{H}_{t^a})$ with $w_1, w_2 \notin P_2$ are such that $w_1 = v_i$ or $w_2 = v_i$. In this situation, $\delta^a_{i2} = \Delta^a_2 \geq q-1$, and we repeat the above argument, using Lemma 4.1 for t^a instead of t^b.

Case 2. Suppose that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_1)\,\mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) = 0$.

Assume, without loss of generality, that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_1) = 0$ (if $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) = 0$, we argue in the same way, exchanging v_1 and v_2). Then from (4.2) we get

$$\textstyle\sum_{i=2}^r \mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) \geq (r-1)(q-2) + 2\ . \tag{4.4}$$

Denote by Δ^a_1 the total weight of the edges in $\mathcal{H}_{t^a}$ between the vertices of $V_\mathcal{G} \setminus P_1$. By (4.1),

$$\alpha_1(q-2) + 1 \leq d+1 = \textstyle\sum_{v\in P_1} \mathrm{wt}_{\mathcal{H}_{t^a}}(v) + \Delta^a_1 \leq (\alpha_1 - 1)(q-2) + \Delta^a_1\ ,$$

and thus we deduce that $\Delta^a_1 \geq q-1$.

Subcase 2.1. Assume $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) \geq (q-1)+1$.

Let $\{w_1, w_2\} \in E(\mathcal{H}_{t^a})$ be an edge with $w_1, w_2 \notin P_1$ and assume $w_1, w_2 \neq v_2$. Then, Lemma 4.2 gives a monomial t^b such that $t^a - t^b \in I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_1) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_1) = 0$, $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_2) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) \geq (q-1)+1$, and such that the total weight of the edges from v_2 to the vertices of $V_\mathcal{G} \setminus P_1$, $\delta^b_{21} = \min\{\mathrm{wt}_{\mathcal{H}_{t^a}}(v_2), \Delta^a_1\}$, is $\geq q-1$. By Lemma 4.1 (with $\alpha = 1$, and v_2 and v_1 instead of v_0 and w_0), we get a monomial t^c such that $t^b - t^c \in I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_2) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v_2) - (q-1) \geq 1$ and $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_1) \geq (q-1)$. Therefore, $\mathcal{H}_{t^c}$ satisfies the assumptions of Case 1.

If all edges $\{w_1, w_2\} \in E(\mathcal{H}_{t^a})$ with $w_1, w_2 \notin P_1$ are such that $w_1 = v_2$ or $w_2 = v_2$, then $\delta^a_{21} = \Delta^a_1 \geq q-1$, and repeating the argument (using Lemma 4.1 for t^a instead of t^b), we fall again in Case 1.

Subcase 2.2. Suppose that $1 \leq \mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) \leq q-1$. Then (4.4) implies that there exists $i \in \{3, \ldots, r\}$ such that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) \geq q-1$. We argue as in the previous subcase.

Let $\{w_1, w_2\} \in E(\mathcal{H}_{t^a})$ be an edge with $w_1, w_2 \notin P_1$ and assume $w_1, w_2 \neq v_i$. Then, Lemma 4.2 gives a monomial t^b such that $t^a - t^b \in I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_1) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_1) = 0$, $\mathrm{wt}_{\mathcal{H}_{t^b}}(v_2) = \mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) \geq 1$, and such that $\delta^b_{i1} = \min\{\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i), \Delta^a_1\} \geq q-1$. By Lemma 4.1 (with $\alpha = 1$, and v_i and v_1 instead of v_0 and w_0), we get a monomial t^c such that $t^b - t^c \in I(X)$, $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_i) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v_i) - (q-1)$, $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_1) \geq (q-1)$, and $\mathrm{wt}_{\mathcal{H}_{t^c}}(v_2) = \mathrm{wt}_{\mathcal{H}_{t^b}}(v_2) \geq 1$. Once again $\mathcal{H}_{t^c}$ satisfies the assumptions of Case 1.

If all edges $\{w_1, w_2\} \in E(\mathcal{H}_{t^a})$ with $w_1, w_2 \notin P_1$ are such that $w_1 = v_i$ or $w_2 = v_i$, then $\delta^a_{i1} = \Delta^a_1 \geq q-1$, and repeating the argument (using Lemma 4.1 for t^a instead of t^b), we fall again in Case 1.

Subcase 2.3. Suppose that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) = 0$. Then, as in Subcase 2.2, there exists $i \in \{3, \ldots, r\}$ such that $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_i) \geq q-1$. Since $\mathrm{wt}_{\mathcal{H}_{t^a}}(v_2) = 0$, we can repeat for Δ^a_2 what we did for Δ^a_1:

$$\alpha_2(q-2) + 1 \leq d+1 = \textstyle\sum_{v\in P_2} \mathrm{wt}_{\mathcal{H}_{t^a}}(v) + \Delta^a_2 \leq (\alpha_2 - 1)(q-2) + \Delta^a_2\ ,$$

and conclude that $\Delta^a_2 \geq q-1$. Using Lemmas 4.2 and 4.1, this time moving the weight of v_i towards v_2, we can reduce to either Subcase 2.1 or Subcase 2.2.

This completes the case analysis. We conclude that every monomial t^a of degree $d+1$, where $d = \max\{\alpha_1(q-2), \ldots, \alpha_r(q-2), \lceil (n-2)(q-2)/2 \rceil\}$, belongs to $I(X)+(t_1)$; in other words, that $\operatorname{reg} S/I(X) \leq \max\{\alpha_1(q-2), \ldots, \alpha_r(q-2), \lceil (n-2)(q-2)/2 \rceil\}$; which completes the proof of the theorem. $\square$

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