

AN EXTENSION OF MARKOV'S THEOREM

A. BRANQUINHO, U. FIDALGO PRIETO AND A. FOULQUIÉ MORENO

ABSTRACT: We give a general sufficient condition for the uniform convergence of sequences of type II Hermite-Padé approximants associated with Nikishin systems of functions.

KEYWORDS: Hermite-Padé approximation, Nikishin systems.

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1. Introduction

Let $\Delta \subset \mathbb{R}$ be a compact interval and $\mathcal{M}(\Delta)$ the set of finite Borel measures with constant sign whose support $S(\mu)$ is a subset of Δ such that Δ is the smallest interval which contains $S(\mu)$; we write $\text{Co}(S(\mu)) = \Delta$. Given $\mu \in \mathcal{M}(\Delta)$, the associated Markov function is defined by

$$\widehat{\mu}(z) = \int \frac{d\mu(x)}{z-x} \in \mathcal{H}(\overline{\mathbb{C}} \setminus S(\mu))$$

which is holomorphic in $\overline{\mathbb{C}} \setminus S(\mu)$.

Fix a measure $\sigma \in \mathcal{M}(\Delta)$ and a system of m weights $\mathbf{r} = (\rho_1, \dots, \rho_m)$ with respect to σ ; that is, each $\rho_k \in L_1(\sigma)$ and has constant sign. Consider the system of measures $\mathbf{s} = (s_1, \dots, s_m)$, where $ds_j = \rho_j d\sigma$, and the corresponding system of Markov functions $\widehat{\mathbf{s}} = (\widehat{s}_1, \dots, \widehat{s}_m)$. Take a multi-index $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}_+^m$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. There exist polynomials $Q_{\mathbf{n}}$ and $P_{\mathbf{n},j}$, $j = 1, \dots, m$, such that

$$\begin{aligned} i) \quad & \deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_1 + \dots + n_m, \quad Q_{\mathbf{n}} \not\equiv 0, \\ ii) \quad & (Q_{\mathbf{n}} \widehat{s}_j - P_{\mathbf{n},j})(z) = \mathcal{O}(1/z^{n_j+1}), \quad z \rightarrow \infty, \quad j = 1, \dots, m. \end{aligned} \quad (1)$$

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In the sequel we assume that $Q_{\mathbf{n}}$ is monic.

For each $j = 1, \dots, m$, $Q_{\mathbf{n}}$ annihilates the terms corresponding to the powers between -1 and $-n_j$ of the Laurent expansion of $Q_{\mathbf{n}}\widehat{s}_j$ whereas $P_{\mathbf{n},j}$ represents the polynomial part of $Q_{\mathbf{n}}\widehat{s}_j$. Hence, $Q_{\mathbf{n}}$ determines univocally $P_{\mathbf{n},j}$ and, consequently, the rational fraction $P_{\mathbf{n},j}/Q_{\mathbf{n}}$.

The vector rational fractions $\mathbf{R}_{\mathbf{n}} = (P_{\mathbf{n},1}/Q_{\mathbf{n}}, \dots, P_{\mathbf{n},m}/Q_{\mathbf{n}})$ is called type II Hermite-Padé approximant corresponding to the system $\widehat{\mathbf{s}}$ and the multi-index \mathbf{n} .

When $m = 1$, $\mathbf{R}_{\mathbf{n}} = P_{\mathbf{n},1}/Q_{\mathbf{n}} = P_n/Q_n$, $\mathbf{n} = n$, is the n th diagonal Padé approximant of $\widehat{s}_1 = \widehat{s}$. It is well known (for example, see [15, Chapter II]), that in this case Q_n is the n th monic orthogonal polynomial with respect to the measure s . Usually, monic orthogonal polynomials are defined for positive measures, however, the definition is trivially extended to measures with constant sign. Q_n has n simple zeros in the interior of $\text{Co}(S(s))$ (see [16, Lemma 1.1.3]).

In [13], A.A. Markov proved that given an arbitrary measure $s \in \mathcal{M}(\Delta)$ the sequence $\{P_n/Q_n\}_{n \in \mathbb{Z}_+}$ converges uniformly to \widehat{s} on every compact subset contained in the domain $\overline{\mathbb{C}} \setminus \Delta$. We write

$$\frac{P_n}{Q_n} \underset{n \rightarrow \infty}{\rightrightarrows} \widehat{s}, \quad \text{on} \quad \overline{\mathbb{C}} \setminus \Delta.$$

In the present paper, we extend Markov's Theorem to the context of type II Hermite-Padé approximation.

The first drawback in extending Markov's Theorem to the context of Hermite-Padé approximation is that in the vector case, in general, $Q_{\mathbf{n}}$ is not uniquely determined by (1). However, in [10] it is shown that uniqueness takes place for the so called Nikishin systems of measures which we introduce below. In this case, $Q_{\mathbf{n}}$ also has $|\mathbf{n}|$ simple zeros in the interior of Δ .

Nikishin systems of measures were introduced by E.M. Nikishin in his famous article [14]. Take two compact intervals Δ_{α} and Δ_{β} of the real line such that $\Delta_{\alpha} \cap \Delta_{\beta} = \emptyset$ and two measures $\sigma_{\alpha} \in \mathcal{M}(\Delta_{\alpha})$ and $\sigma_{\beta} \in \mathcal{M}(\Delta_{\beta})$. We define a third measure $\langle \sigma_{\alpha}, \sigma_{\beta} \rangle$ whose differential expression is

$$d\langle \sigma_{\alpha}, \sigma_{\beta} \rangle(x) = \int \frac{d\sigma_{\beta}(t)}{x-t} d\sigma_{\alpha}(x) = \widehat{\sigma}_{\beta}(x) d\sigma_{\alpha}(x).$$

Observe that $\langle \sigma_{\alpha}, \sigma_{\beta} \rangle \in \mathcal{M}(\Delta_{\alpha})$.

Now, take m compact intervals $\Delta_1, \dots, \Delta_m$ with the property that for each $j = 1, \dots, m-1$, $\Delta_j \cap \Delta_{j+1} = \emptyset$. Let $(\sigma_1, \dots, \sigma_m)$ be a system of measures

such that $\sigma_j \in \mathcal{M}(\Delta_j)$, $j = 1, \dots, m$. The system of measures (s_1, \dots, s_m) given by

$$s_1 = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \quad s_3 = \langle \sigma_1, \langle \sigma_2, \sigma_3 \rangle \rangle = \langle \sigma_1, \sigma_2, \sigma_3 \rangle, \\ \dots, s_m = \langle \sigma_1, \dots, \sigma_m \rangle,$$

is the so called Nikishin system of measures generated by $(\sigma_1, \dots, \sigma_m)$. For short, we write $(s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ whereas $\widehat{\mathbf{s}} = (\widehat{s}_1, \dots, \widehat{s}_m) = \widehat{\mathcal{N}}(\sigma_1, \dots, \sigma_m)$ is the corresponding Nikishin system of functions. Nikishin systems have received a great deal of attention in the recent past and have found numerous applications, see for example [1, 2, 3, 4, 6, 7, 8, 11, 12, 17].

In order to state our main result we need to review some concepts. Given two disjoint compact sets K_1 and K_2 of \mathbb{R} , $\text{dist}(K_1, K_2)$ denotes the distance between K_1 and K_2 i.e. $\text{dist}(K_1, K_2) = \min\{|x_1 - x_2| : (x_1, x_2) \in K_1 \times K_2\}$ whereas $\text{diam}(K_1) = \max\{|x_1 - x_2| : x_1, x_2 \in K_1\}$ denotes the diameter of K_1 .

The main result of this paper is the following theorem.

Theorem 1. *Let $\{\mathbf{R}_n = (P_{n,1}/Q_n, \dots, P_{n,m}/Q_n)\}_{n \in \Lambda}$ be the sequence of type II Hermite-Padé approximants corresponding to a sequence of distinct multi-indices $\Lambda \subset \mathbb{Z}_+^m$ and a system $(\widehat{s}_1, \dots, \widehat{s}_m) = \widehat{\mathcal{N}}(\sigma_1, \dots, \sigma_m)$. Assume $\text{diam}(\Delta_k) < \text{dist}(\Delta_1, \Delta_2)$. Then, for each compact set $K \subset \overline{\mathbb{C}} \setminus \Delta_1$*

$$\limsup_{n \in \Lambda} \left\| \widehat{s}_j - \frac{P_{n,j}}{Q_n} \right\|_K^{1/(|\mathbf{n}|+n_j)} \leq \|\phi_\infty\|_K < 1, \quad j = 1, \dots, m,$$

where $\|\cdot\|_K$ denotes the sup-norm on K and ϕ_∞ denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta_1$ onto the open unit disk such that $\phi_\infty(\infty) = 0$ and $\phi'_\infty(\infty) > 0$.

Notice that the sequence of multi-indices may be completely arbitrary. In Markov's Theorem, there is no assumption on the measure. This is also true in our case whenever $\text{diam}(\Delta_k) < \text{dist}(\Delta_1, \Delta_2)$, $k = 1, 2$. We have imposed no restrictions on the measures $\sigma_1, \dots, \sigma_m$ at all. Another extension of Markov's Theorem was given in [10, Corollary 1.1] without any assumption on the measures, but the indices are required to satisfy $n_j \geq |\mathbf{n}|/m - c|\mathbf{n}|^\kappa$, $j = 1, \dots, m$, for $c > 0$ and $\kappa < 1$. We believe that a complete analogue of Markov's Theorem should hold.

The following result extends [9, Corollary 2] to a larger class of multi-indices.

Theorem 2. *Let $\Lambda \subset \mathbb{Z}_+^m$ be a sequence of multi-indices such that either there exists $k \in \{2, \dots, m\}$ such that for every $\mathbf{n} = (n_1, \dots, n_m) \in \Lambda$, $n_k = \max\{n_1 + 1, n_2, \dots, n_m\}$, or $n_1 = \max\{n_1, n_2 - 1, \dots, n_m - 1\}$ (in which case we take $k = 1$). Then, for each compact set $K \subset \overline{\mathbb{C}} \setminus \Delta_1$,*

$$\limsup_{\mathbf{n} \in \Lambda} \left\| \widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}} \right\|_K^{1/2|\mathbf{n}|} \leq \kappa(K) < 1, \quad (2)$$

where

$$\kappa(K) = \sup\{\|\phi_t\|_K : t \in \Delta_2 \cup \{\infty\}\}$$

and ϕ_t denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta_1$ onto the open unit disk such that $\phi_t(t) = 0$ and $\phi_t'(t) > 0$.

In the first three sections we give some preliminary results which are necessary for the proof of the Theorems above. Section 2 includes some properties of multiple orthogonal polynomials corresponding to Nikishin systems of measures. In Section 3 we study properties of Fourier series of functions expanded in terms of orthogonal polynomials with respect to varying measures. Theorem 2 is proved in Section 4 as a first step to the proof of Theorem 1 which is completed in Section 5.

2. Multiple orthogonality in Nikishin systems

Let $\mathbf{s} = (s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ and $\mathbf{n} = (n_1, \dots, n_m)$ be given. It is well known and easy to verify that the conditions (1) imply

$$0 = \int x^\nu Q_{\mathbf{n}}(x) ds_j(x), \quad \nu = 0, \dots, n_j - 1, \quad j = 1, \dots, m. \quad (3)$$

For each $j = 1, \dots, m$, let h be an arbitrary polynomial such that $\deg h \leq n_j$. Then

$$0 = \int \frac{h(z) - h(x)}{z - x} Q_{\mathbf{n}}(x) ds_j(x)$$

hence

$$\int \frac{Q_{\mathbf{n}}(x)}{z - x} ds_j(x) = \frac{1}{h(z)} \int \frac{h(x) Q_{\mathbf{n}}(x)}{z - x} ds_j(x) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \quad \text{as } z \rightarrow \infty.$$

Define

$$P(z) = \int \frac{Q_{\mathbf{n}}(z) - Q_{\mathbf{n}}(x)}{z - x} ds_j(x).$$

Thus

$$(Q_{\mathbf{n}}\widehat{s}_j - P)(z) = \int \frac{Q_{\mathbf{n}}(x)}{z-x} ds_j(x) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \quad \text{as } z \rightarrow \infty.$$

From (1) we see that

$$P(z) - P_{\mathbf{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \in \mathcal{H}(\overline{\mathbb{C}}) \quad z \rightarrow \infty.$$

Consequently,

$$P_{\mathbf{n},j}(z) \equiv \int \frac{Q_{\mathbf{n}}(z) - Q_{\mathbf{n}}(x)}{z-x} ds_j(x), \quad (Q_{\mathbf{n}}\widehat{s}_j - P_{\mathbf{n},j})(z) = \int \frac{Q_{\mathbf{n}}(x)}{z-x} ds_j(x). \quad (4)$$

From [10] we know that the conditions (3) imply that $Q_{\mathbf{n}}$ has $|\mathbf{n}|$ simple zeros which lie in the interior of Δ_1 . Let $x_{\mathbf{n},1} < \dots < x_{\mathbf{n},|\mathbf{n}|}$ be the zeros of $Q_{\mathbf{n}}$. Decomposing into simple fractions, we get

$$\frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} = \sum_{i=1}^{|\mathbf{n}|} \frac{\lambda_{i,j,\mathbf{n}}}{z-x_{\mathbf{n},i}}, \quad j = 1, \dots, m. \quad (5)$$

The coefficients $\lambda_{i,j,\mathbf{n}}$, $i = 1, \dots, |\mathbf{n}|$ and $j = 1, \dots, m$, were called Nikishin-Christoffel coefficients in [9, Definition 2]. Taking into account the equality in (4), we have that

$$\lambda_{i,j,\mathbf{n}} = \lim_{z \rightarrow x_{\mathbf{n},i}} (z - x_{\mathbf{n},i}) \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} = \int \frac{Q_{\mathbf{n}}(x) ds_j(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})}. \quad (6)$$

For each $j = 1, \dots, m$,

$$\begin{aligned} \left| \sum_{i=1}^{|\mathbf{n}|} \lambda_{i,j,\mathbf{n}} \right| &= \left| \sum_{i=1}^{|\mathbf{n}|} \int \frac{Q_{\mathbf{n}}(x) ds_j(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})} \right| \\ &= \left| \int \sum_{i=1}^{|\mathbf{n}|} \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})} ds_j(x) \right| = \left| \int ds_j(x) \right| = \|s_j\| < +\infty, \end{aligned} \quad (7)$$

where $\|s\|$ represents the total variation of the measure s . In this chain of equalities we have used that $\mathcal{P}(x) = \sum_{i=1}^{|\mathbf{n}|} Q_{\mathbf{n}}(x) / (Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x_{\mathbf{n},i} - x))$ is the polynomial of degree $\leq |\mathbf{n}| - 1$ which interpolates the constant function 1 at the zeros of $Q_{\mathbf{n}}$. Thus $\mathcal{P} \equiv 1$.

From [10, Lemma 3.2] one can state the following result. We wish to point out that the measure denoted here with τ are products of those in [10].

Lemma 1. *Let $(\widehat{s}_{2,2}, \dots, \widehat{s}_{2,m}) = \widehat{\mathcal{N}}(\sigma_2, \dots, \sigma_m)$, there is a system of $m - 1$ measures $(\tau_{2,1}^k, \dots, \tau_{2,k-1}^k, \tau_{2,k+1}^k, \dots, \tau_{2,m}^k)$ where $\text{Co}(S(\tau_{2,j}^k)) \subset \Delta_2$, $j = 1, \dots, k - 1, k + 1, \dots, m$, such that*

$$\frac{1}{\widehat{s}_{2,k}(z)} = \ell_{2,k}(z) + \widehat{\tau}_{2,1}^k(z), \quad (8)$$

where $\ell_{2,k}$ denotes a polynomial with degree one, and

$$\frac{\widehat{s}_{2,j}(z)}{\widehat{s}_{2,k}(z)} - \frac{|s_{2,j}|}{|s_{2,k}|} = \widehat{\tau}_{2,j}^k(z), \quad j = 2, \dots, k - 1, k + 1, \dots, m. \quad (9)$$

In [10, Theorem 1.4] refers to so called mixed type multiple orthogonal polynomials of two Nikishin systems. When reduced to type II multiple orthogonal polynomials of a Nikishin system it may be restated in the following form.

Lemma 2. *Let $(s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ and $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}_+^m$ be given. Set $k = 1$ if $n_1 + 1 = M = \max\{n_1 + 1, n_2, \dots, n_m\}$, otherwise k is equal to the subscript of the first component of \mathbf{n} such that $M = n_k$. Then, there exists a permutation λ of $\{1, \dots, m\}$ which reorders the components of \mathbf{n} such that $n_{\lambda(1)} + \delta_{\lambda(1),1} \geq n_{\lambda(2)} \geq \dots \geq n_{\lambda(m)}$ with $n_k = n_{\lambda(1)}$ and $\delta_{\lambda(1),1}$ denoting the known Kronecker delta function, and an associated Nikishin system $\widetilde{\mathbf{s}} = (r_1, \dots, r_m) = \mathcal{N}(\rho_1, \dots, \rho_m)$, where $s_k = r_1 = \rho_1$ and $\text{Co}(S(\rho_j)) \subset \Delta_j$, $j = 1, \dots, m$, such that if $\widetilde{\mathbf{n}} = (n_{\lambda(1)}, \dots, n_{\lambda(m)})$, the pairs (\mathbf{s}, \mathbf{n}) and $(\widetilde{\mathbf{s}}, \widetilde{\mathbf{n}})$ have the same type II multiple orthogonal polynomial. That is, $Q_{\mathbf{n}}$ satisfies (3) and*

$$0 = \int x^\nu Q_{\mathbf{n}}(x) \widehat{r}_{2,j}(x) ds_k(x), \quad \nu = 0, \dots, n_{\lambda(j)} - 1, \quad j = 1, \dots, m,$$

where $r_{2,j} = \langle \rho_2, \dots, \rho_j \rangle$, $j = 2, \dots, m$, and $\widehat{r}_{2,1} \equiv 1$.

Type II multiple orthogonal polynomials of Nikishin systems with respect to decreasing multi-indices satisfy other orthogonality relations. In particular, from [11, Propositions 2 and 3] (see also [2, relations (5)-(7)]), we have

Lemma 3. *Let $\mathbf{s} = (s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ and $\mathbf{n} = (n_1, \dots, n_m)$ be given. Let $k \in \{1, \dots, m\}$ be as in Lemma 2. Then, there exist two monic polynomial $Q_{\mathbf{n},2}$, $\deg Q_{\mathbf{n},2} = |\mathbf{n}| - n_k$, and $Q_{\mathbf{n},3} = |\mathbf{n}| - n_k - n_{\lambda(2)}$, whose zeros are simple and lie in the interior of Δ_2 and Δ_3 , respectively, such that:*

$$\left(\frac{Q_{\mathbf{n}} \widehat{s}_k - P_{\mathbf{n},k}}{Q_{\mathbf{n},2}} \right) (z) = \mathcal{O} \left(\frac{1}{z^{|\mathbf{n}|+1}} \right) \in \mathcal{H}(\overline{\mathbb{C}} \setminus S(\sigma_1)),$$

$$0 = \int x^\nu Q_{\mathbf{n}}(x) \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)}, \quad \nu = 0, \dots, |\mathbf{n}| - 1 \quad (10)$$

and

$$0 = \int t^\nu Q_{\mathbf{n},2}(t) \int \frac{Q_{\mathbf{n}}^2(x) ds_k(x)}{t-x} \frac{d\rho_2(t)}{Q_{\mathbf{n}}(t)Q_{\mathbf{n},3}(t)}, \quad \nu = 0, \dots, |\mathbf{n}| - n_k - 1. \quad (11)$$

(Here, ρ_2 is the measure coming from Lemma 2.)

Formulas (10) and (11) state that $Q_{\mathbf{n}}$ and $Q_{\mathbf{n},2}$ are, respectively, the $|\mathbf{n}|$ th and $(|\mathbf{n}| - n_k)$ th monic orthogonal polynomials with respect to the varying measures

$$\frac{ds_k}{Q_{\mathbf{n},2}} \quad \text{and} \quad \int \frac{Q_{\mathbf{n}}^2(x) ds_k(x)}{t-x} \frac{d\rho_2(t)}{Q_{\mathbf{n}}(t)Q_{\mathbf{n},3}(t)}. \quad (12)$$

There are other full orthogonality relations with respect to varying measures satisfied deeper in the system, but we will not need them.

3. Varying measures and associated Fourier series

Let $\text{sign} : \mathbb{R} \setminus \{0\} \rightarrow \{-1, 1\}$ denote the sign function. Analogously, $\text{sign}(\mu)$ will denote the sign of a given measure $\mu \in \mathcal{M}(\Delta)$. Notice that $\text{sign}(\mu) \cdot \mu$ is a positive measure. Given a measurable function $f : \Delta \rightarrow \mathbb{R}$,

$$\|f\|_{2,\mu} = \sqrt{\text{sign}(\mu) \int f^2(x) d\mu(x)},$$

denotes the L_2 norm with respect to μ . If $\|f\|_{2,\mu} < +\infty$ we write $f \in L_2(\mu)$.

Let $\{q_{\mu,n}\}_{n \in \mathbb{Z}_+}$ be the family of monic orthogonal polynomials with respect to μ . For each $n \in \mathbb{Z}_+$ let $p_{\mu,n}(z) \equiv q_{\mu,n} / \|q_{\mu,n}\|_{2,\mu}$ denote the n th orthonormal polynomial with respect to the measure μ . That is

$$\int p_{\mu,n}(x) p_{\mu,k}(x) d\mu(x) = \delta_{n,k} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}, \quad (n, k) \in \mathbb{Z}_+^2.$$

Fix $n \in \mathbb{Z}_+$, for each polynomial h of degree $\leq n$ we have the identity

$$0 = \int \frac{h(z) - h(x)}{z-x} p_{\mu,n}(x) d\mu(x),$$

thus

$$\int \frac{p_{\mu,n}(x) d\mu(x)}{z-x} = \frac{1}{p_{\mu,n}(z)} \int \frac{p_{\mu,n}^2(x) d\mu(x)}{z-x}. \quad (13)$$

From (10) we see that $q_{\mu,|\mathbf{n}|} \equiv Q_{\mathbf{n}}$ when $d\mu = ds_k/Q_{\mathbf{n},2}$, and $p_{\mu,|\mathbf{n}|} \equiv Q_{\mathbf{n}}/\|Q_{\mathbf{n}}\|_{2,\mu}$.

Lemma 4. *Let $\{d\mu_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{M}(\Delta)$ be given. Then for each $t \in \mathbb{C} \setminus \Delta$ we have that*

$$\left| \frac{q_{\mu_n,n}(x)}{q_{\mu_n,n}(t)} \right|^{1/n} \leq \frac{\text{diam}(\Delta)}{\text{dist}(t, \Delta)}, \quad n \in \mathbb{Z}_+ \quad (14)$$

uniformly in $\{x \in \Delta\}$.

Proof: Fix $n \in \mathbb{Z}_+$. Since $q_{\mu_n,n}$ has its n zeros in the interior of Δ then

$$\left| \frac{q_{\mu_n,n}(x)}{q_{\mu_n,n}(z)} \right| \leq \left(\frac{\text{diam}(\Delta)}{\text{dist}(K, \Delta)} \right)^n$$

This proves immediately (14). ■

Fix two integers $n, \nu \in \mathbb{Z}_+$ and a function $f \in L_2(\mu_\nu)$. The sum

$$S_{f,n,\mu_\nu}(z) = \sum_{i=0}^n \gamma_{i,\nu} p_{\mu_\nu,i}(z), \quad (15)$$

where

$$\gamma_{i,\nu} = \text{sign}(\mu_\nu) \int f(x) p_{\mu_\nu,i}(x) d\mu_\nu(x), \quad i = 0, \dots, n,$$

defines the n th partial sum of the Fourier series corresponding to f in terms of the orthonormal system $\{p_{\mu_n,i}\}_{i \in \mathbb{Z}_+}$.

Substituting in (15) the well known Christoffel-Darboux identity (in [5, Theorem 4.5, page 23]) we obtain

$$S_{f,n,\mu_\nu}(z) = a_{\mu_\nu,n+1} \int \frac{p_{\mu_\nu,n+1}(z)p_{\mu_\nu,n}(x) - p_{\mu_\nu,n+1}(x)p_{\mu_\nu,n}(z)}{z-x} f(x) d\mu_\nu(x), \quad (16)$$

where

$$a_{\mu_\nu,n+1} = \int x p_{\mu_\nu,n+1}(x) p_{\mu_\nu,n}(x) d\mu_\nu(x).$$

Notice that $\text{sign}(a_{\mu_n,n+1}) = \text{sign}(\mu_n)$. For an arbitrary polynomial \mathcal{P} of degree less or equal to n , we have $S_{\mathcal{P},n,\mu_n} \equiv \mathcal{P}$.

Proposition 1. *Let $\{\mu_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{M}(\Delta)$ be given. Fix $t \in \mathbb{C} \setminus \Delta$ such that $\text{dist}(t, \Delta) > \text{diam}(\Delta)$. Then*

$$S_{1/(z-t),n,\mu_n} \rightrightarrows \frac{1}{z-t}, \quad \text{for } z \in \Delta. \quad (17)$$

Proof: Fix $N \in \mathbb{Z}_+$. We start by proving

$$S_{1/(z-t),n,\mu_N} \rightrightarrows \frac{1}{z-t}, \quad \text{for } z \in \Delta. \quad (18)$$

For two nonnegative integers $n > n'$ we analyze the difference

$$\varepsilon_{N,n,n'} = \left| S_{1/(z-t),n',\mu_N} - S_{1/(z-t),n,\mu_N} \right| = \left| \sum_{i=n'+1}^n \gamma_{i,N} p_{\mu_N,i}(z) \right|$$

where $\gamma_{i,N} = \int p_{\mu_N,i}(x)/(x-t)d\mu_N(x)$. So

$$\varepsilon_{N,n,n'} = \left| \sum_{i=n'+1}^n p_{\mu_N,i}(z) \int \frac{p_{\mu_N,i}(x)}{\rho_N(x)} \frac{d\mu_N(x)}{x-t} \right|.$$

Taking into account the identity given in (13) we have that

$$\begin{aligned} \varepsilon_{N,n,n'} &= \left| \sum_{i=n'+1}^n \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \int \frac{p_{\mu_N,i}^2(x)d\mu_N(x)}{x-t} \right| \\ &\leq \sum_{i=n'+1}^n \left| \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \right| \left| \int \frac{p_{\mu_N,i}^2(x)d\mu_N(x)}{x-t} \right| \leq \sum_{i=n'+1}^n \left| \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \right| \frac{\left| \int p_{\mu_N,i}^2(x)d\mu_N(x) \right|}{\text{dist}(t, \Delta)}. \end{aligned}$$

Hence we obtain that

$$\varepsilon_{N,n,n'} \leq \frac{1}{\text{dist}(t, \Delta)} \sum_{i=n'+1}^n \left| \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \right|.$$

Lemma 4 implies that there exists a nonnegative integer N' such that for every pair (n, n') , with $n \geq n' \geq N'$

$$\varepsilon_{N,n,n'} \leq \varepsilon_{N,n,n'} \leq \frac{1}{\text{dist}(t, \Delta)} \sum_{i=n'+1}^n M^i \rightarrow 0 \quad \text{as } n, n' \rightarrow \infty,$$

where $M = \text{diam}(\Delta)/\text{dist}(\Delta, t) < 1$. This proves (18).

So, for each $n \in \mathbb{Z}_+$ fixed we can write

$$\frac{1}{z-x} = \sum_{i=0}^{\infty} p_{\mu_n,i}(z) \int \frac{p_{\mu_n,i}(x)}{x-t} d\mu_n(x) = \sum_{i=0}^{\infty} \frac{p_{\mu_n,i}(z)}{p_{\mu_n,i}(t)} \int \frac{p_{\mu_n,i}^2(x)d\mu_n(x)}{x-t}.$$

Then

$$\varepsilon_{n,n,\infty} = \left| S_{1/(z-t),n,\mu_n} - \frac{1}{z-t} \right| = \left| \sum_{i=n+1}^{\infty} \frac{p_{\mu_n,i}(z)}{p_{\mu_n,i}(t)} \int \frac{p_{\mu_n,i}^2(x)d\mu_n(x)}{x-t} \right|$$

Taking again into account Lemma 4 we see that there exists a nonnegative integer N' such that for all $n \geq N'$

$$\varepsilon_{n,n,\infty} \leq \frac{1}{\text{dist}(t, \Delta)} \sum_{i=n}^{\infty} M^i \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This proves (17) and completes the proof of Proposition 1. \blacksquare

Recall the definition of Nikishin-Christoffel coefficients introduced in Section 2.

Proposition 2. *Let $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}_+^m$ and $(s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ be given. Set $k = 1$ if $n_1 + 1 = M = \max\{n_1 + 1, n_2, \dots, n_m\}$, otherwise k is equal to the subscript of the first component of \mathbf{n} such that $M = n_k$. For each $n \in \mathbb{Z}_+$, denote $d\mu_{\mathbf{n}} = ds_k/Q_{\mathbf{n},2}$. Then, for each $j = 1, \dots, m$, the Nikishin-Christoffel coefficients can be written as follows*

$$\lambda_{i,j,\mathbf{n}} = \frac{\|Q_{\mathbf{n}}\|_{2,\mu_{\mathbf{n}}} S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n}}}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}, \quad i = 1, \dots, |\mathbf{n}|. \quad (19)$$

When $j = k$, the Nikishin-Christoffel coefficients acquire the following form

$$\lambda_{i,k,\mathbf{n}} = \frac{\|Q_{\mathbf{n}}\|_{2,\mu_{\mathbf{n}}}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}, \quad i = 1, \dots, |\mathbf{n}|. \quad (20)$$

Thus

$$\text{sign}(\lambda_{i,k,\mathbf{n}}) = \text{sign}(s_k), \quad i = 1, \dots, |\mathbf{n}|. \quad (21)$$

In particular

$$\sum_{i=1}^{|\mathbf{n}|} |\lambda_{i,k,\mathbf{n}}| = \|s_k\| < +\infty. \quad (22)$$

Proof: Let us rewrite (6) for each $j = 1, \dots, m$ and each $i = 1, \dots, |\mathbf{n}|$ as

$$\begin{aligned} \lambda_{i,j,\mathbf{n}} &= \int \frac{Q_{\mathbf{n}}(x) ds_j(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})} = \int \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})} \frac{\widehat{s}_{2,j}(x)}{\widehat{s}_{2,k}(x)} Q_{\mathbf{n},2}(x) \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)} \\ &= \frac{\|Q_{\mathbf{n}}\|_{2,\mu_{\mathbf{n}}}}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})} \\ &\quad \times a_{\mu_{\mathbf{n}},|\mathbf{n}|} \int \frac{p_{\mu_{\mathbf{n}},|\mathbf{n}|}(x) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i}) \widehat{s}_{2,j}(x)}{x - x_{\mathbf{n},i} \widehat{s}_{2,k}(x)} Q_{\mathbf{n},2}(x) \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)}. \end{aligned}$$

Using the formula given in (16) it follows that

$$\lambda_{i,j,\mathbf{n}} = \frac{\|Q_{\mathbf{n}}\|_{2,\mu_{\mathbf{n}}} S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n}}}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|-1} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}.$$

When $j = k$, since $\widehat{s}_{2,j}/\widehat{s}_{2,k} \equiv 1$ and $\deg Q_{\mathbf{n},2} = |\mathbf{n}| - n_k$

$$\lambda_{i,k,\mathbf{n}} = \frac{\|Q_{\mathbf{n}}\|_{2,\mu_{\mathbf{n}}} Q_{\mathbf{n},2}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}.$$

So (19) and (20) have been proved. It is well known (see [5, Theorem 5.3]) that the zeros two two consecutive elements of a family of orthogonal polynomials interlace, then $Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})$ must be positive. Hence for each $i = 1, \dots, |\mathbf{n}|$ the equalities (20) imply

$$\begin{aligned} \text{sign}(\lambda_{i,k,\mathbf{n}}) &= \text{sign}(a_{\mu_{\mathbf{n}},|\mathbf{n}|}) \text{sign}(Q_{\mathbf{n},2}) \\ &= \text{sign}(s_k) \text{sign}(Q_{\mathbf{n},2}) \text{sign}(Q_{\mathbf{n},2}) = \text{sign}(s_k). \end{aligned}$$

Combining (7) and (21) we obtain (22). ■

4. Proof of Theorem 2

We proceed as in the proof of (34) in [9, Corollary 2]. Fix $\mathbf{n} \in \Lambda$. Taking into account (22), from (5) we have that for each compact set $K \subset \overline{\mathbb{C}} \setminus \Delta_1$

$$\left\| \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}} \right\|_K \leq \frac{\|s_k\|}{\text{dist}(K, \Delta_1)}.$$

Therefore, the family of functions $\{\widehat{s}_k - P_{\mathbf{n},k}/Q_{\mathbf{n}}\}_{\mathbf{n} \in \Lambda}$, is uniformly bounded on each compact $K \subset \overline{\mathbb{C}} \setminus \Delta_1$ by $2\|s_k\|/\text{dist}(K, \Delta_1)$.

Let $t_{\mathbf{n},1} < \dots < t_{\mathbf{n},|\mathbf{n}|-n_k}$ denote the zeros of $Q_{\mathbf{n},2}$. From Lemma 3 we know that $\{t_{\mathbf{n},1}, \dots, t_{\mathbf{n},|\mathbf{n}|-n_k}\} \subset \Delta_2$ and the zeros of $Q_{\mathbf{n}}$ lie in Δ_1 , and

$$\left(\frac{\widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}}}{Q_{\mathbf{n},2}} \right) (z) = \mathcal{O} \left(\frac{1}{z^{2|\mathbf{n}|+1}} \right), \quad z \rightarrow \infty.$$

So

$$\frac{\widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}}}{\phi_{\infty}^{|\mathbf{n}|+n_k+1} \prod_{i=1}^{|\mathbf{n}|-n_k} \phi_{t_{\mathbf{n},i}}} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta_1).$$

Take $\rho \in (0, 1)$ such that $\gamma_\rho = \{z : |\phi_\infty(z)| = \rho\}$ satisfies that $\Delta_2 \subset \text{Ext}(\gamma_\rho)$, where $\text{Ext}(\gamma_\rho)$ denotes the unbounded connected component of the complement of γ_ρ . We have then

$$\left\| \frac{\widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}}}{\phi_\infty^{|\mathbf{n}|+n_k+1} \prod_{i=1}^{|\mathbf{n}|-n_k} \phi_{t_{\mathbf{n},i}}} \right\|_{\gamma_\rho} \leq \frac{2|s_k|}{\text{dist}(\gamma_\rho, \Delta_1) \psi^{2|\mathbf{n}|+1}(\gamma_\rho)},$$

where

$$\psi(\gamma_\rho) = \inf\{|\phi_t(z)| : z \in \gamma_\rho, t \in \Delta_2 \cup \{\infty\}\}.$$

Considered as a function of the two variables z and t , $\phi_t(z)$ is a continuous function in $\overline{\mathbb{C}^2}$. Since $\gamma_\rho \cap \Delta_2 = \emptyset$ then $\psi(\gamma_\rho) > 0$. Fix a compact $K \subset \overline{\mathbb{C}} \setminus \Delta_1$ and take ρ sufficient by close to 1 so that $K \subset \text{Ext}(\gamma_\rho)$. Since the function under the norm sign is analytic in $\overline{\mathbb{C}} \setminus \Delta_1$, from the maximum principle it follows that the same bound holds for all $z \in K$. Consequently,

$$\left\| \widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}} \right\|_K \leq \frac{2|s_k| \phi_\infty^{|\mathbf{n}|+n_k+1} \prod_{i=1}^{|\mathbf{n}|-n_k} \phi_{t_{\mathbf{n},i}}}{\text{dist}(\gamma_\rho, \Delta_1) \psi^{2|\mathbf{n}|+1}(\gamma_\rho)} \leq \frac{2|s_k|}{\text{dist}(\gamma_\rho, \Delta_1)} \left(\frac{\kappa(K)}{\psi(\gamma_\rho)} \right)^{2|\mathbf{n}|+1},$$

taking $\kappa(K)$ as in the statement of the theorem. Therefore,

$$\limsup_{|\mathbf{n}| \rightarrow \infty} \left\| \widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}} \right\|_K^{1/2|\mathbf{n}|} \leq \frac{\kappa(K)}{\psi(\gamma_\rho)}.$$

So, the continuity of $|\phi_t(z)|$ in $\overline{\mathbb{C}^2}$ and the fact that $\lim_{\rho \rightarrow 1} \psi(\gamma_\rho) = 1$ prove (2). That $\kappa(K) < 1$ is also a consequence of the continuity of $|\phi_t(z)|$ in $\overline{\mathbb{C}^2}$. ■

5. Proof of Theorem 1

We will use the following auxiliary result.

Proposition 3. *Let $(s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m$ be given. Assume that $\text{diam}(\Delta_k) < \text{dist}(\Delta_1, \Delta_2)$, $k = 1, 2$. Then there exists $N \geq 0$ such that for each $\mathbf{n} \in \Lambda$, where $|\mathbf{n}| \geq N$, every coefficient $\lambda_{i,j,\mathbf{n}}$, $i = 1, \dots, |\mathbf{n}|$, $j = 1, \dots, m$ has the same sign as its corresponding measure s_j .*

Proof: Fix an arbitrary permutation λ of $\{1, \dots, m\}$. Define Λ_λ as the set of all $\mathbf{n} \in \Lambda$ such that there exists $\tilde{\mathbf{s}} = (r_1, \dots, r_m) = \mathcal{N}(\rho_1, \dots, \rho_m)$ for which $Q_{\mathbf{n}}$ is orthogonal with respect to (\mathbf{s}, \mathbf{n}) and $(\tilde{\mathbf{s}}, \tilde{\mathbf{n}})$ (recall that $\tilde{\mathbf{n}} = (n_{\lambda(1)}, \dots, n_{\lambda(m)})$) in such a way that $n_{\lambda(1)} + \delta_{\lambda(1),1} \geq n_{\lambda(2)} \geq \dots \geq n_{\lambda(m)}$. According to Lemma 2 we have that $\cup_\lambda \Lambda_\lambda = \Lambda$. Some of the sets Λ_λ may be empty or have a finite number of elements. Since the group of permutations

of $\{1, \dots, m\}$ is finite it is sufficient to prove that the result holds true for all λ such that Λ_λ has an infinite number of multi-indices. In the sequel we restrict our attention to such λ 's and fix one of them.

Fix $\mathbf{n} \in \Lambda_\lambda$. Let us denote the measures introduced in (12) as

$$d\mu_{\mathbf{n},1} = \frac{d\rho_1}{Q_{\mathbf{n},2}} = \frac{ds_k}{Q_{\mathbf{n},2}} \quad \text{and} \quad d\mu_{\mathbf{n},2}(t) = \int \frac{Q_{\mathbf{n}}^2(x)}{t-x} \frac{d\rho_1(x)}{Q_{\mathbf{n},2}(x)} \frac{d\rho_2(t)}{Q_{\mathbf{n}}(t)Q_{\mathbf{n},3}(t)}.$$

We call $k = \lambda(1)$. From identities (19) in Proposition 2 it is sufficient to show that for each $j = 1, \dots, k-1, k+1, \dots, m$ the sequence of functions $\{S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},1}}\}_{\mathbf{n} \in \Lambda_\lambda}$ converges uniformly to $\widehat{s}_{2,j}/\widehat{s}_{2,k}$ on Δ_1 because this function has constant and constant sign and no zero on Δ_1 .

Denote

$$\mathcal{K}(z, x, |\mathbf{n}| - 1) = \frac{p_{\mu_{\mathbf{n},1},|\mathbf{n}|}(z)p_{\mu_{\mathbf{n},1},|\mathbf{n}|-1}(x) - p_{\mu_{\mathbf{n},1},|\mathbf{n}|}(x)p_{\mu_{\mathbf{n},1},|\mathbf{n}|-1}(z)}{z - x}.$$

Let us start by analyzing the case when $j = 1$. Taking into account the formula (16) and using the identity (8) in Lemma 1 we have that

$$\begin{aligned} & \left| \frac{S_{Q_{\mathbf{n},2}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},1}}(z)}{Q_{\mathbf{n},2}(z)} - \frac{1}{\widehat{s}_{2,k}(z)} \right| \\ &= \left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{Q_{\mathbf{n},2}(x)}{\widehat{s}_{2,k}(x)} - \frac{Q_{\mathbf{n},2}(z)}{\widehat{s}_{2,k}(z)} \right) d\mu_{\mathbf{n},1}(x) \right| \\ &= \left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) (Q_{\mathbf{n},2}(x)\ell_{2,k}(x) - Q_{\mathbf{n},2}(z)\ell_{2,k}(z)) d\mu_{\mathbf{n},1}(x) \right. \\ & \quad \left. + \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) (Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,k}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,k}(z)) d\mu_{\mathbf{n},1}(x) \right|. \end{aligned}$$

Since $\deg Q_{\mathbf{n},2}\ell_{2,k} \leq |\mathbf{n}| - n_k + 1 < |\mathbf{n}| - 1$ ($n_k = \max\{n_1, \dots, n_m\}$), then

$$\begin{aligned} & \left| \frac{S_{Q_{\mathbf{n},2}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},1}}(z)}{Q_{\mathbf{n},2}(z)} - \frac{1}{\widehat{s}_{2,k}(z)} \right| = |\ell_{2,k}(z) - \widehat{\tau}_{2,k}(z)| \\ & \quad + \left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) (Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,k}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,k}(z)) d\mu_{\mathbf{n},1}(x) \right| \\ &= \left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) (Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,k}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,k}(z)) d\mu_{\mathbf{n},1}(x) \right|. \end{aligned}$$

Proceeding analogously as above, for $j = 2, \dots, k-1, k+1, \dots, m$, and taking into account (16) and (9), we obtain

$$\begin{aligned} & \left| \frac{S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},1}}(z)}{Q_{\mathbf{n},2}(z)} - \frac{\widehat{s}_{2,j}(z)}{\widehat{s}_{2,k}(z)} \right| \\ &= \left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{Q_{\mathbf{n},2}(x)\widehat{s}_{2,j}(x)}{\widehat{s}_{2,k}(x)} - \frac{Q_{\mathbf{n},2}(z)\widehat{s}_{2,j}(z)}{\widehat{s}_{2,k}(z)} \right) d\mu_{\mathbf{n},1}(x) \right| \\ &= \left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) (Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,j}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,j}(z)) d\mu_{\mathbf{n},1}(x) \right|. \end{aligned}$$

Summarizing, for each $j = 1, \dots, k-1, k+1, \dots, m$, we need to analyze the expression:

$$\left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) (Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,j}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,j}(z)) d\mu_{\mathbf{n},1}(x) \right|.$$

Using Fubini's Theorem we obtain the following chain of equalities

$$\begin{aligned} & \left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) (Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,j}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,j}(z)) d\mu_{\mathbf{n},1}(x) \right| \\ &= \left| \int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{Q_{\mathbf{n},2}(x)}{x-t} - \frac{Q_{\mathbf{n},2}(z)}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau_{2,j}^k(t) \right| \\ &= \left| \int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{Q_{\mathbf{n},2}(x) - Q_{\mathbf{n},2}(t)}{x-t} \right) d\mu_{\mathbf{n},1}(x) d\tau_{2,j}^k(t) \right. \\ &\quad \left. - \int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{Q_{\mathbf{n},2}(z) - Q_{\mathbf{n},2}(t)}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau_{2,j}^k(t) \right. \\ &\quad \left. + \int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{Q_{\mathbf{n},2}(t)}{x-t} \right) d\mu_{\mathbf{n},1}(x) d\tau_{2,j}^k(t) \right. \\ &\quad \left. - \int \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{Q_{\mathbf{n},2}(t)}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau_{2,j}^k(t) \right| \\ &= \left| \int \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} a_{\mu_{\mathbf{n},1},|\mathbf{n}|} \int \mathcal{K}(z, x, |\mathbf{n}| - 1) \left(\frac{1}{x-t} - \frac{1}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau_{2,j}^k(t) \right| \\ &= \left| \int \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \left(S_{1/(z-t),|\mathbf{n}|-1,\mu_{\mathbf{n},1}} - \frac{1}{z-t} \right) d\tau_{2,j}^k(t) \right| \\ &\leq \left\| \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \right\|_{S(\sigma_2)} \left\| S_{1/(z-t),|\mathbf{n}|-1,\mu_{\mathbf{n},1}} - \frac{1}{z-t} \right\|_{\Delta_1} \|\tau_{2,j}^k\|. \end{aligned}$$

Combining the requirement $\text{diam}(\Delta_k) < \text{dist}(\Delta_1, \Delta_2)$, $k = 1, 2$, Lemma 4 and Proposition 1 we obtain that

$$\left\| \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \right\|_{S(\sigma_2)} \rightarrow 0 \quad \text{and} \quad \left\| S_{1/(z-t), |\mathbf{n}|-1, \mu_{\mathbf{n},1}} - \frac{1}{z-t} \right\|_{\Delta_1} \rightarrow 0.$$

So this completes the proof. \blacksquare

Now we are ready to prove Theorem 1. As in Section 4, we take $\rho \in (0, 1)$ and $\gamma_\rho = \{z : |\phi_\infty(z)| = \rho\}$. For each $j = 1, \dots, k-1, k+1, \dots, m$ we have that

$$\|\widehat{s}_j\|_{\gamma_\rho} = \frac{|s_j|}{\text{dist}(\gamma_\rho, \Delta_1)} \quad \text{and} \quad \left\| \frac{P_j}{Q_{\mathbf{n}}} \right\|_{\gamma_\rho} = \left\| \sum_{i=1}^{|\mathbf{n}|} \frac{\lambda_{i,j,\mathbf{n}}}{z - x_{\mathbf{n},i}} \right\|_{\gamma_\rho} \leq \frac{|s_j|}{\text{dist}(\gamma_\rho, \Delta_1)}$$

The second inequality can be deduced easily from Proposition 3. Combining the above inequalities we have that

$$\left\| \frac{\widehat{s}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}}}{\phi_\infty^{|\mathbf{n}|+n_j+1}} \right\|_{\gamma_\rho} \leq \frac{2|s_j|}{\text{dist}(\gamma_\rho, \Delta_1)\rho^{|\mathbf{n}|+n_j+1}}.$$

Let us fix a compact $K \subset \bar{\mathbb{C}} \setminus \Delta_1$ and take ρ sufficient close to 1. From the maximum principle it follows that the same bound holds for all $z \in K$. Consequently,

$$\left\| \widehat{s}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} \right\|_K \leq \frac{2|s_j| \|\phi_\infty\|_K^{|\mathbf{n}|+n_j+1}}{\text{dist}(\gamma_\rho, \Delta_1)\rho^{|\mathbf{n}|+n_j+1}}$$

Therefore,

$$\limsup_{|\mathbf{n}| \rightarrow \infty} \left\| \widehat{s}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} \right\|_K^{1/(|\mathbf{n}|+n_j)} \leq \frac{\|\phi_\infty\|}{\rho},$$

and the result readily follows making $\rho \rightarrow 1$.

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A. BRANQUINHO

CMUC, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COIMBRA, LARGO D. DINIS, 3001-454 COIMBRA, PORTUGAL.

E-mail address: ajplb@mat.uc.pt

U. FIDALGO PRIETO

DEPARTAMENTO DE MATEMÁTICAS, ESCUELA POLITÉCNICA SUPERIOR, UNIVERSIDAD CARLOS III, AVENIDA DE LA UNIVERSIDAD 30, 28911 LEGANÉS-MADRID, SPAIN.

E-mail address: ulisesfidalgoprieto@yahoo.es

A. FOULQUIÉ MORENO

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE AVEIRO, CAMPUS DE SANTIAGO 3810, AVEIRO, PORTUGAL.

E-mail address: foulquie@ua.pt