

Optimal design problems with fractional diffusions

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Abstract

In this article we study optimization problems ruled by fractional diffusion operators with volume constraints. By means of penalization techniques we prove existence of solutions. We also show that every solution is locally of class $C^{0,\alpha}$ (optimal regularity), and that the free boundary is a $C^{1,\gamma}$ surface, up to a \mathcal{H}^{n-1} -negligible set.

1 Introduction

The goal of this paper is to establish existence and geometric properties of nonlocal optimal design problems. This class of problems arises in the study of best insulation devices. Motivations also come from semi-conductor theory, plasma physics, flame propagations etc. When taking into account eventual turbulence or long-range integrations, the model becomes more accurate when ruled by nonlocal operators, such as $(-\Delta)^\alpha$. In particular, the model becomes sensible to interior changes in the temperature; local versions of the problem can only feel changes on the boundary of the body to be insulated.

Let us recall that the fractional Laplacian $(-\Delta)^\alpha$ is given by

$$(-\Delta)^\alpha u(x) = C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} dy,$$

where PV is the Cauchy principal value and $C_{n,\alpha}$ is a normalization constant. The free boundary optimization problem we study here takes the following formulation: given a smooth domain $D \subset \mathbb{R}^n$, a nonnegative function $\varphi: D \rightarrow \mathbb{R}$, and a positive number $\omega > 0$, minimize the α -energy

$$J_\alpha(u) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dx dy \quad (1.1)$$

among competing functions u within the functional set

$$K(\alpha, \omega, \varphi) := \{u \in H^\alpha(\mathbb{R}^n) \mid u = \varphi \text{ in } D \text{ and } \mathcal{L}^n(\{u > 0\} \cap D^c) = \omega\}, \quad (1.2)$$

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where \mathcal{L}^n is the n -dimensional Lebesgue measure, and $H^\alpha(\mathbb{R}^n)$ is the α -fractional Sobolev space (see, for example, [8]), i.e. the set of functions u for which

$$\|u\|_{H^\alpha} = \sqrt{\int_{\mathbb{R}^n} (1 + |\xi|^{2\alpha}) |\hat{u}(\xi)|^\alpha d\xi} < \infty.$$

By usual methods in the calculus of variations it is hard to perform volume preserving perturbations as to derive energy estimates. In turn, proving existence of a minimizer u as well as regularity properties of u and its free boundary $\partial\{u > 0\} \cap D$ is in general difficult tasks from the mathematical view point.

Local versions of the problem have been well studied in the literature, see [1, 2, 5, 15, 19, 20] among others. A celebrate approach for tackling problems involving volume constrains is based on penalization techniques. The idea is to introduce an artificial parameter in the energy functional which charges for configurations that exceed the volume budget. For fixed values of the penalization parameter, the penalized functional can then be analyzed by free boundary variational methods. Nonlocal free boundary variational tools were first introduced in [6]. Thus, the starting point of this current paper is to obtain regularity results for minimizers of nonlocal free boundary problems with fixed penalized parameter. This requires adjustments of existing free boundary methods from [6]; some extra work is needed though. Of course all analytic and geometric estimates obtained depend upon the penalized parameter, and they blow-up as the penalization term goes to infinity.

The auxiliary penalization problem we consider here takes the following set-up. Fix an $\varepsilon > 0$, we define the ε -energy functional

$$J_\varepsilon(v) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} dx dy + f_\varepsilon(\mathcal{L}^n(\{u > 0\} \cap D^c)), \quad (1.3)$$

where

$$f_\varepsilon(s) := \begin{cases} \frac{1}{\varepsilon}(s - \omega) & \text{for } s \geq \omega, \\ \varepsilon(s - \omega) & \text{for } s \leq \omega. \end{cases}$$

Define the functional set

$$K := \{u \in H^\alpha(\mathbb{R}^n) \mid u = \varphi \text{ in } D\}, \quad (1.4)$$

and then, the penalized problem becomes

$$\text{find } u \in K \text{ such that } J_\varepsilon(u) = \inf_{v \in K} J_\varepsilon(v). \quad (1.5)$$

As previously mentioned, the minimization problem (1.5) is similar to the one treated in [6]. However, *there's no such thing as a free lunch* – the key and in general hard issue, though, is to prove that the aimed volume is attained for small (but still positive) values of the penalization parameter. This is only possible by means of a refined control on the rate between volume decreasing versus energy increasing, of competing shapes. The appropriate tool for such a control

is the so-called Hadamard's variational formula, whose local (smooth) version is known for over one-hundred years, [13]. Hence, one of the main difficulties we handle in this work is to derive a measure-theoretic, α -fractional Hadamard's formula for domain variations. This is accomplished in section 4.

Another difficulty in dealing with nonlocal optimal design problems is the lack of *local* information on the unknowns. To bypass this inconvenience, we will make use of the extension property discovered in [7]. We consider the upper half-space

$$\mathbb{R}_+^{n+1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}_+\},$$

and set $\beta = 1 - 2\alpha$. For $u \in C^2(\mathbb{R}^n)$ we solve the Dirichlet problem

$$\begin{aligned} -\operatorname{div}(y^\beta \nabla v) &= 0 \text{ in } \mathbb{R}_+^{n+1}, \\ v(x, 0) &= u(x). \end{aligned} \tag{1.6}$$

A solution to such a problem can be obtained by convolution with the Poisson kernel $P_{n,\alpha}(x, y)$ of the operator $\operatorname{div}(y^\beta \nabla)$ in \mathbb{R}_+^{n+1} , see [7]. We have

$$P_{n,\alpha}(x, y) = q_{n,\alpha} \frac{y^{2\alpha}}{(x^2 + y^2)^{(n+2\alpha)/2}},$$

where $q_{n,\alpha}$ is such that $\int P_{n,\alpha}(x, 1) dx = 1$. From [7] we also know that

Theorem 1.1. *We have $(-\Delta)^\alpha u(x) = -\lim_{y \rightarrow 0} y^\beta v_y(x, y)$.*

Because of the divergence form of the elliptic operator in (1.6), a Dirichlet integral is available. We also observe that if u solves (1.6) and $\partial_y u = 0$ on the hyperplane $\{y = 0\}$, one can extend it evenly across $\{y = 0\}$, and the new equation satisfied by u is

$$\begin{aligned} -\operatorname{div}(|y|^\beta \nabla v) &= 0 \text{ in } \mathbb{R}^{n+1}, \\ v(x, 0) &= u(x). \end{aligned}$$

For any open subset Ω of \mathbb{R}^{n+1} , we introduce the weighted Hilbert space

$$H^1(\beta, \Omega) := \{u \in L^2(\Omega_+); |y|^{\beta/2} \nabla u \in L^2(\Omega)\},$$

where $\Omega_+ = \Omega \cap \mathbb{R}_+^{n+1}$. We set

$$I_\varepsilon(u, \Omega) := \int_\Omega |y|^\beta |\nabla u|^2 dx dy + f_\varepsilon(\mathcal{L}^n(\{u > 0\} \cap \mathbb{R}^n \cap \Omega)), \quad u \in H^1(\beta, \Omega). \tag{1.7}$$

The study of penalized problem (1.5) is now replaced by the study of local minimizers of I_ε , i.e. functions u that are in $H^1(\beta, B_1)$ and satisfy

$$\forall B \subset B_1, \forall v \in H^1(\beta, B) \text{ with } v = u \text{ on } \partial B, \quad I_\varepsilon(u, B) \leq I_\varepsilon(v, B). \tag{1.8}$$

The paper is organized as follows: in section 2 we list few analytic and geometric properties we will need for the study of the (fixed) penalized functional

(1.7). Still in section 2 we show that a minimum of I_ε is α -Hölder continuous, which corresponds to the optimal regularity for the free boundary problem. We also show that it is non-degenerate. The constants though do depend on the penalized parameter ε and they blow-up as $\varepsilon \rightarrow 0$. In section 3 we obtain measure estimates on the free boundary. In section 4 we derive a fractional Hadamard's variational formula and in section 5 we prove that for $\varepsilon > 0$ small enough – but still positive – the volume constraint is verified. This finally provides the existence of a minimum for the fractional optimization problem with volume constraint.

2 Preliminaries

We start off by listing few properties we will need along the article. Initially, let us set, for the sake of notations, that a minimizer of (1.8) will be denoted by u instead of u_ε . Here and afterwards by $B_r(x, y)$ we denote the ball in \mathbb{R}^{n+1} centered at (x, y) and of radius r . When $x = 0$, we simply write B_r . We will also write $B_r^n(x)$ for the ball in \mathbb{R}^n centered at x with radius r . We now prove existence of minimizers for the ε -penalized problem.

Lemma 2.1. *Given a smooth boundary datum, $\varphi(x, y)$ defined on $\partial B_1 \cap \mathbb{R}_+^{n+1}$, problem (1.8) has an absolute minimum u .*

Proof. Take u_0 with $\mathcal{L}^n(\{u_0 > 0\}) \leq \omega$, then $I_\varepsilon(u_0) \leq C$ (uniformly in ε), also $I_\varepsilon \geq -\omega$. Therefore, there exists a minimizing sequence $\{u_k\}_{k \in \mathbb{N}}$. The sequence is bounded in $H^\alpha(B_1)$ and, because H^α is continuously embedded into $L^{2n/(n-2\alpha)}$, the sequence $\{u_k\}_{k \in \mathbb{N}}$ converges (up to a subsequence) to a function u strongly in $L^{2n/(n-2\alpha)-\epsilon}$, for any $\epsilon > 0$, and almost everywhere in \mathbb{R}^n . Thus,

$$\mathcal{L}^n(\{u > 0\}) \leq \liminf_{k \rightarrow \infty} \mathcal{L}^n(\{u_k > 0\})$$

and

$$\int_{B_1} |y|^\beta |\nabla u|^2 \leq \liminf_{k \rightarrow \infty} \int_{B_1} |y|^\beta |\nabla u_k|^2.$$

Hence, $u \in H^1(\beta, B_1)$ and since f_ε is a continuous and increasing function, one has

$$I_\varepsilon(u) \leq \liminf_{k \rightarrow \infty} I_\varepsilon(u_k) = \inf_{v \in H^1(\beta, B_1)} I_\varepsilon(v).$$

Therefore u is an absolute minimizer of I_ε in $H^1(\beta, B_1)$. \square

For the behavior of a minimizer in its positivity set, we state the following proposition and refer the reader for its proof to Proposition 3.1 of [6].

Proposition 2.1. *Let u be a local minimizer in (1.8), and $x_0 \in \mathbb{R}^n$ be such that $u(x_0, 0) > 0$. Then*

$$\lim_{y \rightarrow 0} |y|^\beta \nabla u(x_0, y) = 0.$$

Moreover, if u is defined in \mathbb{R}^{n+1} , is positive outside the hyperplane $\{y = 0\}$ and satisfies $\operatorname{div}(|y|^\beta \nabla u) = 0$ in its positivity set, together with the estimate $u(x, y) = O(|(x, y)|^\alpha)$, then $(-\Delta)^\alpha u(\cdot, 0) = 0$ in $\mathbb{R}^n \cap \{u > 0\}$.

We now turn our attention to optimal Hölder estimates for minimizers. In the proof of the following theorem we will use the characterization of Hölder functions, [14]: given $\alpha \in (0, 1)$, if B is a ball in \mathbb{R}^{n+1} , and if there are $C > 0$ and $p \in (1, n + 1)$ such that

$$\forall x \in B, \forall r < \operatorname{dist}(x, \partial B), \int_{B_r(x)} |\nabla u|^p \leq Cr^{n+1-p+p\alpha}, \quad (2.1)$$

then $u \in C^{0,\alpha}(B)$.

Theorem 2.1 (Optimal regularity). *If u is a local minimizer of I_ε posed in B_1 , then $u \in C_{\text{loc}}^{0,\alpha}(B_1)$.*

Proof. For every $r \in (0, 1)$ and $(x_0, y_0) \in B_1$, let us consider the harmonic replacement of u in $B_r(x_0, y_0)$ (we have chosen $r < 1 - |x_0|$), i.e. the solution of

$$-\operatorname{div}(|y|^\beta \nabla h_r^{x_0, y_0}) = 0 \text{ in } B_r(x_0, y_0), \quad h_r^{x_0, y_0} = u \text{ on } \partial B_r(x_0, y_0). \quad (2.2)$$

From the translation invariance in x , we may assume $x_0 = 0$. We will use the notation h_r for the solution of (2.2). Note that u is an admissible Dirichlet datum (see Theorems 2.2 and 2.3 of [6]). Note also, that for all $r > 0$ one has $I_\varepsilon(u, B_r) \leq I_\varepsilon(h_r, B_r)$. The latter implies

$$\int_{B_r} |y|^\beta |\nabla u|^2 \leq \int_{B_r} |y|^\beta |\nabla h_r|^2 + Cr^n. \quad (2.3)$$

Note that although the constant C depends on ε , but this does not bother us, because ε will always be fixed (maybe very small but fixed).

The rest of the proof is the same as of Theorem 1.1 in [6]. We bring it here for reader's convenience.

Due to the identity $\int_{B_r} |y|^\beta \nabla h_r \cdot \nabla(u - h_r) = 0$, we get from (2.3)

$$\int_{B_r} |y|^\beta |\nabla(u - h_r)|^2 \leq Cr^n.$$

Therefore, if $r < \rho < 1$,

$$\begin{aligned}
\int_{B_r} |y|^\beta |\nabla u|^2 &= \int_{B_r} |y|^\beta |\nabla(u - h_\rho + h_\rho)|^2 \\
&\leq 2 \left(\int_{B_\rho} |y|^\beta |\nabla(u - h_\rho)|^2 + \int_{B_r} |y|^\beta |\nabla h_\rho|^2 \right) \\
&\leq C\rho^n + 2 \int_{B_r} |y|^\beta |\nabla h_\rho|^2 \\
&\leq C\rho^n + C \left(\frac{r}{\rho} \right)^{n+1+\beta} \int_{B_\rho} |y|^\beta |\nabla h_\rho|^2 \quad \text{by Theorem 2.6 of [6]} \\
&\leq C\rho^n + C \left(\frac{r}{\rho} \right)^{n+1+\beta} \int_{B_\rho} |y|^\beta |\nabla u|^2. \tag{2.4}
\end{aligned}$$

Let now $\delta < 1/2$. If $\rho = \delta^k$, $r = \delta^{k+1}$, $\mu = \delta^n$, then (2.4) gives

$$\int_{B_{\delta^{k+1}}} |y|^\beta |\nabla u|^2 \leq C\mu^k + C\mu\delta^{2(1-\alpha)} \int_{B_{\delta^k}} |y|^\beta |\nabla u|^2,$$

which in turn, by choosing δ such that $q = C\delta^{2(1-\alpha)} < 1$ and using induction, gives

$$\int_{B_{\delta^k}} |y|^\beta |\nabla u|^2 \leq \frac{C^2}{1-q} \mu^{k-1}.$$

Therefore, for all $r < 1/2$

$$\int_{B_r} |y|^\beta |\nabla u|^2 \leq Cr^n. \tag{2.5}$$

Now, if $\alpha \leq 1/2$, then $\beta \geq 0$, and

$$\int_{B_r} |\nabla u| \leq \left(\int_{B_r} |y|^{-\beta} \right)^{1/2} \left(\int_{B_r} |y|^\beta |\nabla u|^2 \right)^{1/2} \leq Cr^{n+\alpha},$$

which is (2.1) with $p = 1$, and so $u \in C^{0,\alpha}(B_{1/2})$.

In case of $\alpha > 1/2$, we get from (2.5)

$$\int_{B_r} |\nabla u|^2 \leq r^{-\beta} \int_{B_r} |y|^\beta |\nabla u|^2 \leq Cr^{n-\beta} = Cr^{n-1+2\beta},$$

which is (2.1) with $p = 2$. □

Next we prove the non-degeneracy of a minimizer.

Theorem 2.2 (Non-degeneracy). *If u is a local minimizer of (1.8), then there exists a constant $C_0 > 0$ such that for all $x \in B_{1/2}^n(0) \cap \{u > 0\}$,*

$$u(x, 0) \geq C_0 \text{dist}(x, \partial\{u > 0\})^\alpha.$$

Proof. Assume that the conclusion of the theorem is not true. It means that for every $C_0 > 0$ there is a point $z \in B_{1/2}^n(0) \cap \{u > 0\}$ such that $u(z, 0) < C_0 \text{dist}(z, \partial\{u > 0\})^\alpha$. We denote by d the distance of the point $(z, 0)$ from the free boundary. We also define $\delta := u(z, 0)$. The contradictory assumption implies that δ can be made as small as we wish.

From the Harnack inequality, [11], there exists $c > 0$ such that $u \leq c\delta$ in $B_d(z, 0)$. Now if γ is a smooth nonnegative function such that

$$\gamma(x, y) = 0 \text{ in } B_{d/2}(z, 0), \quad \gamma(x, y) = 2c \text{ in } B_{7d/8}(z, 0) \setminus B_{3d/4}(z, 0),$$

we define

$$v(x, y) := \min(u(x, y), \delta\gamma(x, y)).$$

Note that $v \in H^1(\beta, B_d(z, 0))$, and $v = u$ at the boundary of the ball, so it is an admissible test function for (1.8). Therefore

$$I_\varepsilon(u, B_d(z, 0)) \leq I_\varepsilon(v, B_d(z, 0)). \quad (2.6)$$

On the other hand, from the definition of v one has

$$\int_{B_d(z, 0)} |y|^\beta |\nabla v|^2 \leq \int_{B_d(z, 0)} |y|^\beta |\nabla u|^2 + O(\delta),$$

and since $v = 0$ in $B_{d/2}(z, 0)$, then

$$\mathcal{L}^n(\{v > 0\}) \leq \mathcal{L}^n(\{u > 0\}) - \mathcal{L}^n(B_{d/2}(z, 0) \cap \mathbb{R}^n),$$

which implies that

$$f_\varepsilon(\mathcal{L}^n(\{v > 0\})) < f_\varepsilon(\mathcal{L}^n(\{u > 0\})).$$

Therefore,

$$I_\varepsilon(u, B_d(z, 0)) > I_\varepsilon(v, B_d(z, 0)),$$

which contradicts (2.6). We remark that the constant C_0 may depend on ε . \square

Now, arguing as in [6, Proposition 3.3], it is possible to show that u is in fact strongly non degenerate.

Lemma 2.2. *If u is a local minimizer of (1.8) in B_1 , and $(0, 0)$ is a free boundary point, then there is $C > 0$ such that for $r \in (0, 1/2)$,*

$$\sup_{B_r^n} u \geq Cr^\alpha.$$

As a consequence of the Hölder regularity result and the non-degeneracy, we get the positive density result below.

Theorem 2.3 (Positive density). *Let $(0, 0)$ be a free boundary point. If u is a local minimizer of (1.8) in B_1 , then there is a constant $C_1 > 0$ such that for every $r > 0$*

$$\mathcal{L}^n(\{u = 0\} \cap B_r^n) \geq C_1 r^n, \quad \mathcal{L}^n(\{u > 0\} \cap B_r^n) \geq C_1 r^n. \quad (2.7)$$

Proof. In fact, from the non-degeneracy we know that there is $y \in B_r^n$ such that $u(y) \geq Cr^\alpha > 0$. By Hölder continuity, $u > 0$ in $B_{\delta r}^n(y)$ for a small $\delta > 0$, which gives us the second estimate of (2.7).

To prove the first estimate of (2.7), it is enough to consider the case $r = 1$. Assume the contrary. There is a sequence of minimizers u_k defined in B_1 , such that $0 \in \partial\{u_k > 0\}$ and

$$\lim_{k \rightarrow +\infty} \mathcal{L}^n(\{u_k = 0\}) = 0. \quad (2.8)$$

Recall that u_k is uniformly Hölder continuous. We may assume that the sequence converges to u_0 uniformly. Moreover, one has

$$\int_{B_1} |y|^\beta |\nabla u_0|^2 dx dy \leq \liminf_{k \rightarrow +\infty} \int_{B_1} |y|^\beta |\nabla u_k|^2 dx dy.$$

For every v agreeing with u_k on ∂B_1 one has $I_\varepsilon(u_k, B_1) \leq I_\varepsilon(v, B_1)$. Together with (2.8) this implies for every $v \in H^1(\beta, B_1)$ which agrees with u_0 on ∂B_1 ,

$$\begin{aligned} f_\varepsilon(\mathcal{L}^n(B_1^n)) + \int_{B_1} |y|^\beta |\nabla u_0|^2 dx dy &\leq I_\varepsilon(v, B_1) \\ &\leq f_\varepsilon(\mathcal{L}^n(B_1^n)) + \int_{B_1} |y|^\beta |\nabla v|^2 dx dy. \end{aligned}$$

Therefore, u_0 minimizes the Dirichlet integral over the unit ball of \mathbb{R}^n , and as such, satisfies $\operatorname{div}(|y|^\beta \nabla u_0) = 0$ in B_1 . Since $u_0(0) = 0$ and $u_0 \geq 0$, then the strong maximum principle, see [4], implies that $u_0 \equiv 0$, which is a contradiction on the non-degeneracy property. Once again we remark, that the constant C_1 may depend on ε . \square

3 Further properties of solutions

The ultimate goal of this section is to prove that the free boundaries of local minimizers of (1.8) have local finite parameter. The results in this section are the analogue of the ones from [3].

Proposition 3.1. *For a local minimizer u in Ω , $\mu(u) := -(-\Delta)^\alpha u$ is a non-negative Radon measure with support in $\Omega \cap \partial\{u > 0\}$.*

Proof. Once again we recall the extension result from [7] (see also Theorem 1.1 above). The proof is now the same as the one of Remark 4.2 of [3]. \square

In the spirit of [3] a representation theorem can be proven. It plays an important role in the study of the free boundary.

Theorem 3.1 (Representation theorem). *If u is a local minimizer in Ω , then*

1. $\mathcal{H}^{n-1}(\mathcal{K} \cap \partial\{u > 0\} \cap \mathbb{R}^n) < \infty$, for every compact set $\mathcal{K} \subset \Omega$.

2. There exists a Borel function q_ε such that

$$\mu(u) = q_\varepsilon \mathcal{H}^{n-1} \llbracket \partial\{u > 0\} \rrbracket,$$

that is for any $\psi \in C_0^\infty(\Omega)$ there holds

$$-\int_{\Omega} |y|^\beta \nabla u \cdot \nabla \psi = \int_{\{u > 0\}} \psi q_\varepsilon d\mathcal{H}^{n-1}.$$

3. For any compact set $\mathcal{K} \subset \Omega$, there exist constants $0 < c < C < \infty$ depending on n, Ω and ε such that for $B_r(x) \subset \Omega$ and $x \in \partial\{u > 0\}$ one has $c \leq q_\varepsilon(x) \leq C$ and

$$cr^{n-1} \leq \mathcal{H}^{n-1}(B_r(x) \cap \partial\{u > 0\} \cap \mathbb{R}^n) \leq Cr^{n-1}.$$

4. For \mathcal{H}^{n-1} almost everywhere in $\partial\{u > 0\}$ an outward normal $\nu = \nu(x_0)$ is defined and furthermore

$$u(x_0 + x) = q_\varepsilon(x_0)(x \cdot \nu(x_0))_+^\alpha + o(|x|), \quad \text{as } x \rightarrow 0.$$

5. $\mathcal{H}^{n-1}((\partial\{u > 0\} \cap \mathbb{R}^n) \setminus (\partial^*\{u > 0\} \cap \mathbb{R}^n)) = 0$, where ∂^* is the reduced boundary.

6. The reduced free boundary $\partial^*\{u > 0\} \cap \mathbb{R}^n$ is locally a $C^{1,\gamma}$ surface.

Proof. The first three assertions of the theorem follow as those of Theorem 4.5 of [3]. For the first assertion we also refer the reader to Theorem 1.1 of [17]. Note that $\Omega \cap \{u > 0\} \cap \mathbb{R}^n$ has finite perimeter, thus, the reduced boundary is defined as well as the measure theoretic normal $\nu(x)$, for $x \in \partial^*\{u > 0\}$ (see, for example [10]). The 5th assertion of the theorem is a consequence of the 3rd one and properties of minimizers proved above (see [10]). $C^{1,\gamma}$ regularity of the reduced free boundary follows by [18], see also [16].

The proof of 4 is similar to the corresponding one in [9] (Theorem 5.5), but since in our case we are dealing with the fractional Laplacian, some modifications need to be done (similar to [6]).

For a minimizer u we will denote by $\Omega_-(u) \subset \mathbb{R}^n$ the set where it is 0, and by $\Omega_+(u) \subset \mathbb{R}^n$ its positivity set. We will also use $\Gamma(u) \subset \mathbb{R}^n$ to denote the free boundary of u , and $\Gamma^*(u)$ - the reduced free boundary.

The reduced free boundary is the set of points x_0 at which the following holds (see [12]): given the half ball $(B_r^n)^+(x_0) := \{(x - x_0) \cdot \nu \geq 0\} \cap B_r^n(x_0)$, one has

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n((B_r^n)^+(x_0) \Delta \Omega_+(u))}{\mathcal{L}^n(B_r^n(x_0))} = 0.$$

Note that from the uniform density of Ω^\pm one has, as $r \rightarrow 0$ at the free boundary point x_0

$$B_r^n(x_0) \cap \Gamma^*(u) \subset \{|(x - x_0) \cdot \nu(x_0)| \leq o(r)\}. \quad (3.1)$$

Indeed, if $u(x) = 0$ for $(x - x_0) \cdot \nu(x_0) \geq \delta r$, there is $\gamma > 0$ such that $\mathcal{L}^n(B_{\delta r}^n(x) \cap \{u = 0\}) \geq \gamma \delta r^n$, which implies

$$\liminf_{r \rightarrow 0} \frac{\mathcal{L}^n((B_r^n)^+(x_0) \Delta \Omega_+(u))}{\mathcal{L}^n(B_r^n(x_0))} \geq \gamma \delta;$$

a contradiction.

The same argument is valid, if $x \in \Omega_-(u)$ is such that $(x - x_0) \cdot \nu \leq -\delta r$.

In order to prove 4, we remark, that in fact it follows from the fact that blow-up limits at regular points are one-dimensional. To prove it, without loss of generality we assume that $\nu(x_0) = e_n$. Let

$$u_r(x, y) := \frac{1}{r^\alpha} u(x_0 + rx, ry)$$

be a blow-up of u , and let u_0 be a blow-up limit. We need to prove that $u_0(x, 0) = q(x_n)_+^\alpha$, where q is a constant. There exists a coordinate system (x', x_n) centered at 0 such that

- $\Omega_+(u_0) = \mathbb{R}_+^n$ (this is because of (3.1)),
- $(-\Delta)^\alpha u_0 = 0$ in $\Omega_+(u)$.

Define $v(x) = (x_n)_+^\alpha$. By optimal regularity and non-degeneracy, there are positive constants C_1, C_2 such that $0 < C_1 v \leq u_0 \leq C_2 v$. By applying the oscillation lemma (see [11] or Theorem 2.5 of [6]), one has that there exists $\lambda \in (0, 1)$ such that for all small enough $r > 0$,

$$\text{osc}_{B_r} \frac{u_0}{v} \leq \lambda \text{osc}_{B_{2r}} \frac{u_0}{v}.$$

Since Harnack constants are invariant under the blow-up scaling, the oscillation lemma holds at every scale, the solutions being global. Thus, one may apply it all the way down from a ball of radius $2^N r$ (N being arbitrary large) to a ball of radius r . Therefore, $\frac{u_0}{v}$ is a constant. Of course, that constant depends on ε and also on the blow-up point. Although for our further work this fact is not very important, but by following the argument of [3], which proves 2nd assertion of the theorem, one can see that the constant is actually q_ε appearing in 2. \square

Next, we see that points, at which the free boundary has a tangent ball, are regular points.

Definition 3.1. *We say that a free boundary point $x_0 \in \Gamma(u)$ has a tangent ball from outside, if there is a ball $B \subset \Omega_-(u)$ such that $x_0 \in B \cap \Gamma(u)$. A point $x_0 \in \Gamma(u)$ is said to be regular, if $\Gamma(u)$ has a tangent hyperplane at x_0 .*

The following result is from [6]. The proof can also be concluded from the 4th assertion of the Theorem 3.1 together with Theorem 4.1 below.

Theorem 3.2. *(The free boundary condition). Let $x_0 \in \Gamma(u)$ be a regular point. There exists a constant $\lambda_\varepsilon(\alpha)$ such that*

$$\lim_{x \rightarrow x_0} \frac{u(x)}{\left((x - x_0) \cdot \nu(x_0)\right)_+^\alpha} = \lambda_\varepsilon(\alpha).$$

4 A fractional Hadamard formula

In this section we prove a fractional Hadamard's variational formula, which will provide a rate control upon volume decreasing versus energy increasing, for competing shapes.

Let u be a local minimizer. We recall the notation $\Gamma^*(u)$ for its reduced free boundary. For given two points $x_1, x_2 \in \Gamma^*(u)$, the idea is to make an inward perturbation around x_1 , and outward perturbation around x_2 in such a way, that we do not change very much the original volume, and then compare the optimal configuration to the perturbed one in terms of the functional I_ε . We proceed as follows.

Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative function from $C_0^\infty[0, 1]$ with $\int \rho = 1$. For any $r \in (0, \frac{\text{dist}(x_1, x_2)}{100})$ and $\gamma > 0$, we define

$$P_r(x, y) := \begin{cases} (x, y) + \gamma r \rho\left(\frac{|(x-x_1, y)|}{r}\right) \nu(x_1, 0), & \text{when } (x, y) \in B_r(x_1, 0) \\ (x, y) - \gamma r \rho\left(\frac{|(x-x_2, y)|}{r}\right) \nu(x_2, 0), & \text{when } (x, y) \in B_r(x_2, 0) \\ (x, y) & \text{elsewhere.} \end{cases}$$

If v is any vector in \mathbb{R}^{n+1} , direct computations show that in $B_r(x_i, 0)$

$$DP_r(x, y) \cdot v = v + (-1)^{i+1} \left\{ \gamma \rho' \left(\frac{|(x-x_i, y)|}{r} \right) \frac{\langle (x-x_i, y), v \rangle}{|(x-x_i, y)|} \right\} \nu(x_i, 0), \quad (4.1)$$

where $\langle \cdot, \cdot \rangle$ the the inner product in \mathbb{R}^{n+1} and i takes the values 1 and 2. Note, that if γ is small enough, then P_r is a diffeomorphism that maps $B_r(x_i, 0)$ onto itself. Indeed, if $\gamma \sup_{t \in [0, 1]} \rho'(t) < 1$, then P_r is a local injective diffeomorphism.

Now, if $\gamma \rho(t) \leq 1 - t$, for $t \in [0, 1]$,

$$|P_r(x, y) - (x_i, 0)| \leq |(x - x_i, y)| + \gamma r \rho\left(\frac{|(x - x_i, y)|}{r}\right) \leq r,$$

for any $(x, y) \in B_r(x_i, 0)$. Finally, note that $P_r = Id$ on $\partial B_r(x_i, 0)$, therefore P_r has to be onto.

For each $r > 0$ small enough, we consider the r -perturbed configuration by

$$v_r(P_r(x, y)) = u(x, y).$$

The idea is to compare our optimal configuration $\{u > 0\}$ to its perturbation $\{v_r > 0\}$ in terms of the penalized problem (1.8). For any $r > 0$ small enough

and $i = 1, 2$, we consider the blow-up sequence $u_r^i : B_1(0) \rightarrow \mathbb{R}$ given by

$$u_r^i(x, y) := \frac{1}{r^\alpha} u(x_i + rx, ry).$$

From the blow-up analysis (see [9]) we know that the set $B_1 \cap \{u_r^i > 0\} \cap \mathbb{R}^n$ approaches to $\{(x, 0) \in B_1, \langle (x, 0) \cdot \nu(x_i, 0) \rangle < 0\}$, as $r \rightarrow 0$. In order to compute the change on the volume of the perturbation, we use the Change of Variables Theorem to obtain

$$\begin{aligned} & \frac{\mathcal{L}^n(B_r(x_i, 0) \cap \{v_r > 0\} \cap \mathbb{R}^n)}{r^n} = \frac{1}{r^n} \int_{B_r(x_i, 0) \cap \{v_r > 0\} \cap \mathbb{R}^n} dz \\ &= \int_{B_1 \cap \{v_r(x_i + rx, ry) > 0\} \cap \mathbb{R}^n} dx dy \\ &= \int_{B_1 \cap \{u_r^i(x_i + rx, ry) > 0\} \cap \mathbb{R}^n} \det(DP_r(x_i + rx, ry)) dx dy \quad (4.2) \\ &\rightarrow \int_{B_1 \cap \{\langle (x, 0), \nu(x_i, 0) \rangle < 0\} \cap \mathbb{R}^n} 1 + (-1)^{i+1} \gamma \rho'(|(x, y)|) \left\langle \frac{(x, y)}{|(x, y)|}, \nu(x_i, 0) \right\rangle dx dy, \end{aligned}$$

as $r \rightarrow 0$. Note that there is a constant $C(\rho)$ such that for any unit vector $\nu \in \mathbb{S}^n$

$$C(\rho) = \int_{B_1 \cap \{\langle (x, 0), \nu \rangle < 0\} \cap \mathbb{R}^n} \rho'(|(x, y)|) \left\langle \frac{(x, y)}{|(x, y)|}, \nu \right\rangle dx dy. \quad (4.3)$$

A similar computation shows that

$$\frac{\mathcal{L}^n(B_r(x_i, 0) \cap \{u > 0\} \cap \mathbb{R}^n)}{r^n} \rightarrow \int_{B_1 \cap \{\langle (x, 0), \nu(x_i, 0) \rangle < 0\} \cap \mathbb{R}^n} dx, \quad (4.4)$$

as $r \rightarrow 0$. From (4.2), (4.3) and (4.4) we get

$$\frac{1}{r^n} \left[\mathcal{L}^n(B_r(x_i, 0) \cap \{v_r > 0\} \cap \mathbb{R}^n) - \mathcal{L}^n(B_r(x_i, 0) \cap \{u > 0\} \cap \mathbb{R}^n) \right] \rightarrow 0,$$

as $r \rightarrow 0$. From the Lipschitz continuity of the penalization f_ε , we obtain

$$\begin{aligned} f_\varepsilon(\mathcal{L}^n(B_r(x_i, 0) \cap \{v_r > 0\} \cap \mathbb{R}^n)) &- f_\varepsilon(\mathcal{L}^n(B_r(x_i, 0) \cap \{u > 0\} \cap \mathbb{R}^n)) \\ &\leq \frac{1}{\varepsilon} o(r^n). \end{aligned} \quad (4.5)$$

Now we should check what happens with the integral part of the functional. Initially we observe that

$$\begin{aligned} & \frac{1}{r^n} \int_{B_r(x_i, 0)} |y|^\beta |\nabla u(x, y)|^2 dx dy = \int_{B_1} |y|^\beta |\nabla u_r^i(x, y)|^2 dx dy \\ &= \int_{B_1 \cap \{u_r^i > 0\}} |y|^\beta |\nabla u_r^i(x, y)|^2 dx dy, \end{aligned} \quad (4.6)$$

once $\Gamma^*(u_r^i)$ is smooth. Next we apply twice the Change of Variables Theorem and take into account that P_r maps $B_r(x_i, 0)$ diffeomorphically onto itself to write

$$\begin{aligned}
& \frac{1}{r^n} \int_{B_r(x_i, 0)} |y|^\beta |\nabla v_r(x, y)|^2 dx dy \\
&= \frac{1}{r^n} \int_{B_r(x_i, 0)} |y|^\beta |DP_r(P_r^{-1}(x, y))^{-1} \cdot \nabla u(P_r^{-1}(x, y))|^2 dx dy \\
&= \frac{1}{r^n} \int_{B_r(x_i, 0)} |y|^\beta |DP_r(z, y)^{-1} \cdot \nabla u(z, y)|^2 |\det(DP_r(z, y))| dz dy \quad (4.7) \\
&= \int_{B_1 \cap \{u_r^i > 0\}} |y|^\beta |DP_r(x_i + rh, ry)^{-1} \cdot \nabla u_r^i(h, y)|^2 |\det(DP_r(x_i + rh, ry))| dh dy.
\end{aligned}$$

From (4.1), using the fact that for any matrix A with $|A| < 1$,

$$(Id + A)^{-1} = Id + \sum_{i=1}^{\infty} (-1)^i A^i,$$

we have

$$\begin{aligned}
& DP_r(x_i + rh, ry)^{-1} \cdot \nabla u_r^i(h, y) \quad (4.8) \\
&= \nabla u_r^i(h, y) - (-1)^{i+1} \gamma \frac{\rho'(|(h, y)|)}{|(h, y)|} \langle (h, y), \nabla u_r^i(h, y) \rangle \nu(x_i, 0) + o(\gamma).
\end{aligned}$$

On the other hand,

$$|\det(DP_r(x_i + rh, ry))| = 1 + (-1)^{i+1} \gamma \frac{\rho'(|(h, y)|)}{|(h, y)|} \langle (h, y), \nu(x_i, 0) \rangle. \quad (4.9)$$

Combining (4.6), (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}
& \frac{1}{r^n} \int_{B_r(x_i, 0)} |y|^\beta [|\nabla v_r(x, y)|^2 - |\nabla u(x, y)|^2] dx dy \\
&= (-1)^{i+1} \gamma \int_{B_1 \cap \{u_r^i > 0\}} |y|^\beta |\nabla u_r^i(h, y)|^2 \frac{\rho'(|(h, y)|)}{|(h, y)|} \langle (h, y), \nu(x_i, 0) \rangle dh dy \quad (4.10) \\
&+ (-1)^i 2\gamma \int_{B_1 \cap \{u_r^i > 0\}} |y|^\beta \frac{\rho'(|(h, y)|)}{|(h, y)|} \langle (h, y), \nabla u_r^i(h, y) \rangle \langle \nabla u_r^i(h, y), \nu(x_i, 0) \rangle dh dy \\
&+ o(\gamma).
\end{aligned}$$

Now we recall that in the proof of 4 of Theorem 3.1, we verified that $u_r^i(h, y) = q_\varepsilon(x_i)(h_n)_+^\alpha + o(r)$, as $r \rightarrow 0$, i.e. blow-up limits are one dimensional. Therefore, from the blow-up analysis, we have, as $r \rightarrow 0$

$$\int_{B_1 \cap \{u_r^i > 0\}} |y|^\beta |\nabla u_r^i|^2 \rightarrow \alpha^2 q_\varepsilon^2(x_i) \int_{B_1 \cap \{z, \nu_i < 0\}} |y|^\beta |h_n|^{2(\alpha-1)} dh dy.$$

Hence, letting $r \rightarrow 0$ in (4.10) leads to

$$\frac{1}{r^n} \int_{B_r(x_i, 0)} |y|^\beta [|\nabla v_r|^2 - |\nabla u|^2] \rightarrow (-1)^{i+1} \gamma \alpha^2 q_\varepsilon^2(x_i) c(\rho) + o(\gamma), \quad (4.11)$$

where

$$c(\rho) := \lim_{r \rightarrow 0} \int_{B_1 \cap \{u_r^i > 0\}} \frac{|y|^\beta}{|h_n|^{1+\beta}} \frac{\rho'(|(h, y)|)}{|(h, y)|} \langle (h, y), \nu(x_i, 0) \rangle dh dy$$

is a positive constant. To check that indeed, $0 < c(\rho) < \infty$ is a positive constant, we argue as follows: since

$$\operatorname{div}(\rho(|z|)) = \frac{\rho'(|z|)}{|z|} \langle z, \nu(x_i, 0) \rangle,$$

so the divergence theorem together with the blow-up analysis provides

$$\int_{B_1 \cap \{u_r^i > 0\}} \frac{\rho'(|z|)}{|z|} \langle z, \nu(x_i, 0) \rangle dz \rightarrow \int_{B_1 \cap \{z, \nu(x_i, 0) = 0\}} \rho(|z|) d\mathcal{H}^{n-1}(z) = \text{const.}$$

Recalling that the function ρ is compactly supported in $[0, 1]$, we conclude, that $0 < c(\rho) < \infty$. Returning to (4.11), we can write

$$\int_{\Omega} |y|^\beta [|\nabla v_r|^2 - |\nabla u|^2] dx dy = r^n \gamma \alpha^2 c(\rho) (q_\varepsilon^2(x_1) - q_\varepsilon^2(x_2)) + r^n o(\gamma). \quad (4.12)$$

Combining (4.12) with the minimality property of u , we get

$$\begin{aligned} 0 \leq I_\varepsilon(v_r) - I_\varepsilon(u) &\leq r^n \gamma \alpha^2 c(\rho) (q_\varepsilon^2(x_1) - q_\varepsilon^2(x_2)) \\ &\quad + r^n o(\gamma) + \frac{1}{\varepsilon} o(r^n), \end{aligned} \quad (4.13)$$

which gives after dividing by r^n and letting $r \rightarrow 0$

$$0 \leq \gamma \alpha^2 c(\rho) (q_\varepsilon^2(x_1) - q_\varepsilon^2(x_2)) + o(\gamma).$$

Now dividing by γ and letting $\gamma \rightarrow 0$, and afterwards reversing the places of x_1 and x_2 , we obtain

$$q_\varepsilon(x_1) = q_\varepsilon(x_2).$$

Since x_1 and x_2 were arbitrary in $\Gamma^*(u)$, we actually proved

Theorem 4.1. *On the reduced free boundary, we have $q_\varepsilon \equiv \lambda_\varepsilon(\alpha)$.*

Note also that (4.10) provides the Hadamard's variational formula:

$$\int_{\Omega} |y|^\beta [|\nabla v_r|^2 - |\nabla u|^2] = \lambda_\varepsilon^2(\alpha) V + o(V), \quad (4.14)$$

where V is the volume change.

5 Recovering the original problem

Here we shall relate a solution to the penalized problem to a (possible) solution of our original problem. The idea is that the function f_ε will charge a lot for those configurations that have a volume bigger than ω . We hope that if the charge is too big optimal configuration of problem (1.8) will prefer to have volume ω than pay for the penalization.

Proposition 5.1. *There exist $C, c > 0$ constants such that*

$$0 < c \leq \mathcal{L}^n(\{u_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega) \leq \omega + C\varepsilon,$$

where u_ε is a solution to (1.8).

Proof. Let Ω_* be any smooth domain such that its complement contains Ω^c with $\mathcal{L}^n((\Omega_*^c \cap \mathbb{R}^n) \setminus (\Omega^c \cap \mathbb{R}^n)) = \omega$. From the minimality of u_ε we have

$$I_\varepsilon(u_\varepsilon, \Omega) = \int_\Omega |y|^\beta |\nabla u_\varepsilon|^2 + f_\varepsilon(\mathcal{L}^n(\{u_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega)) \leq I_\varepsilon(u_*, \Omega_*) = C, \quad (5.1)$$

where u_* is the α -harmonic function in $\Omega_*^c \setminus \Omega^c$ taking "boundary data" equal to φ in Ω^c and 0 on $\partial\Omega_*^c$. Thus

$$\frac{1}{\varepsilon}(\mathcal{L}^n(\{u_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega) - \omega) \leq f_\varepsilon(\mathcal{L}^n(\{u_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega)) \leq C.$$

This proves the estimate from above. In order to prove the estimate from below, we first note that since the weight $|y|^\beta$ is in the second Muckenhoupt class A_2 for $\beta \in (-1, 1)$, we have a Poincaré inequality (see, for example, [6]), which together with (5.1) provides

$$\int_\Omega |y|^\beta [|\nabla u_\varepsilon|^2 + |u_\varepsilon|^2] \leq C, \quad (5.2)$$

for some C independent of ε . Recalling the fact that u_ε takes values φ outside of the domain, where it is α -harmonic, recalling also the mean value inequality and (5.2), we obtain (by integrating along layers with $E := \Omega \cap B_\delta(\partial\Omega)$)

$$\left(\int_{\partial\Omega} \varphi \right)^2 \leq C(\delta) \mathcal{L}^n(\{u_\varepsilon > 0\} \cap \mathbb{R}^n \cap E) \int_E |y|^\beta [|\nabla u_\varepsilon|^2 + |u_\varepsilon|^2]$$

the last integral being bounded uniformly in ε . Hence, the estimate from below is proved. \square

Lemma 5.1. *There exists $C > 0$ depending on the domain and φ , but independent of ε , such that $\lambda_\varepsilon(\alpha) \leq C$.*

The proof of this lemma is a consequence of Proposition 5.1, isoperimetric inequality and 2nd assertion of the Theorem 3.1 (see the proof of the corresponding result in [1], [5], [9], or [20]).

Lemma 5.2. *There exists $c > 0$ depending on the domain and φ but independent of ε such that $c \leq \lambda_\varepsilon(\alpha)$.*

Proof. As in the above mentioned references, the proof is based on the Hopf's Lemma, which is true also for the fractional Laplacian case (see [4] or Proposition 2.7 in [6]).

In fact, let $z_1 \in \Omega$ be such that $u_\varepsilon(z_1) > 0$ for all $\varepsilon > 0$. Let $\delta := \text{dist}(z_1, \partial\Omega)$. We then consider the smooth family of domains $\Gamma_t := B_{\frac{\delta}{2}+t}(z_1) \cap \Omega$. Let t_ε be the first t such that Γ_t touches $\partial\{u_\varepsilon > 0\}$. Let x_0 be that touching point. Define ψ_ε to be α -harmonic in $\Gamma_{t_\varepsilon} \setminus \Gamma_0$, with the following "boundary data":

$$\psi_\varepsilon|_{\partial\Gamma_0} = \min \varphi \quad \text{and} \quad \psi_\varepsilon|_{\Gamma_{t_\varepsilon}^c} = 0.$$

By the maximum principle we have $u_\varepsilon \geq \psi_\varepsilon$ in $\Gamma_{t_\varepsilon} \setminus \Gamma_0$. From the (generalized) Hopf's Lemma we also know that there exists a constant $c > 0$ depending only on the domain and φ , but independent of ε , such that

$$\psi_\varepsilon(x) \geq c((x - x_0) \cdot \nu(x_0))^\alpha. \quad (5.3)$$

On the other hand we have the following asymptotic development around x_0

$$\psi_\varepsilon(x) \leq u_\varepsilon(x) = \lambda_\varepsilon(\alpha)((x - x_0) \cdot \nu(x_0))^\alpha + o(|x - x_0|). \quad (5.4)$$

Letting $x \rightarrow x_0$ in (5.4) and taking into account (5.3), we obtain

$$c \leq \lambda_\varepsilon(\alpha)$$

as desired. □

Now we are ready to prove the main theorem of this section.

Theorem 5.1. *If ε is small enough, then any solution of (1.8) is a solution of (1.1).*

Proof. Basically, we just need to show that $S := \mathcal{L}^n(\{u_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega) = \omega$, for ε small enough.

Suppose $S > \omega$. In the spirit of the previous section, we consider an inward perturbation of the positivity set of u_ε with the volume change V , such that for the new function \tilde{u}_ε we still have $\mathcal{L}^n(\{\tilde{u}_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega) > \omega$. Thus

$$f_\varepsilon(\mathcal{L}^n(\{\tilde{u}_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega)) - f_\varepsilon(\mathcal{L}^n(\{u_\varepsilon > 0\} \cap \mathbb{R}^n \cap \Omega)) = -\frac{1}{\varepsilon}V. \quad (5.5)$$

From Theorem 4.1 and Lemma 5.1 we have

$$\begin{aligned} \int_{\Omega} |y|^\beta [|\nabla \tilde{u}_\varepsilon|^2 - |\nabla u_\varepsilon|^2] &= \lambda_\varepsilon^2(\alpha)V + o(V) \\ &\leq C^2V + o(V). \end{aligned} \quad (5.6)$$

Using the fact that $I_\varepsilon(u_\varepsilon, \Omega) \leq I_\varepsilon(\tilde{u}_\varepsilon, \Omega)$, from (5.5) and (5.6) we get

$$0 \leq C^2V + o(V) - \frac{1}{\varepsilon}V,$$

therefore (by dividing on V and letting $V \rightarrow 0$) it provides us with

$$\varepsilon \geq \frac{1}{C^2},$$

which is a contradiction, when ε is small enough. Hence $S \leq \omega$, for small ε . If $S < \omega$, arguing the same way and using Lemma 5.2, we obtain another lower bound for ε . Thus, when ε is small enough, $S = \omega$. \square

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