SYMPLECTIC GEOMETRY
OF THE
AOMOTO-FORRESTER SYSTEM

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ABSTRACT. This article is devoted to the study of a general class of Hamiltonian systems which extends the Calogero systems with external quadratic potential associated to any root system. The interest for such a class comes from a previous article of Aomoto and Forrester. We consider first the one-degree of freedom case and compute the Birkhoff series defined near each of its stationary points. In general, the analysis of the system motivates finding some expression for the inverses of a rational map introduced by Aomoto and Forrester. We derive here some diagrammatic expansion series for these inverses.

1. INTRODUCTION

In [13], Gallavotti and Marchioro derived a remarkable integral formula as a consequence of Feynmann-Kac path integral method applied to the quantum Calogero system. They posed the question of finding a purely classical proof. In [9] the question was answered and the formula was derived by showing the existence of an associated symplectic action of the torus. In [2] an integral was computed for a much general setting of an Hamiltonian defined on the complement of an arrangement of hyperplanes. In this article, we call the Aomoto-Forrester Hamiltonian system, the system defined by:

$$(1) \quad H = \frac{1}{2} \sum_{j=1}^{n} \left\{ y_j^2 + \left[x_j - \sum_{h \in A} \frac{\lambda_h u_{h,j}}{\langle u_h, x \rangle + u_{h,0}} \right]^2 \right\}, \quad \omega = \sum_{j=1}^{n} dx_j \wedge dy_j, \lambda_h > 0$$

The index h enumerates an arrangement of *real* hyperplanes defined by the equations:

$$(2) \qquad f_h(x) := \langle u_h, x \rangle + u_{h,0} = u_{h,0} + \sum_{j=1}^{n} u_{h,j} x_j = 0.$$

In the particular case where the arrangement of hyperplanes defines the walls of Weyl chambers, and hence where the complement of these hyperplanes is invariant under the action of a Weyl group, we recover the class of Hamiltonians considered by ([4], [5], [6], [15], [14]) (both in quantum and classical cases). We call this situation, the root system cases. For these root systems, the integral formula was also proved via the existence of an associated symplectic action of the torus in ([7], [8]). In contrast, the Aomoto-Forrester integral formula was proved by a complex

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analytic residue techniques (Griffiths residue). It is remarkable that it covers a much more general situation. For instance, this one-dimensional system:

$$(3) \qquad H = \frac{1}{2}y^2 + \frac{1}{2}\left(x - \sum_{h=1}^{N}\frac{\lambda_h}{x-a_h}\right)^2,$$
$$\omega = dx \wedge dy, \ \lambda_h > 0,$$

provides already a quite meaningful example. It is known that if a (n-degrees of freedom) Hamiltonian System is associated to the symplectic action of an n-torus, it displays critical points of Birkhoff type and the Birkhoff normal form is reduced to linear terms.

After the Aomoto-Forrester article, it was natural to ask the question of the existence of an associated symplectic action of the torus for the Aomoto-Forrester system. This is rather easily disproved here with the one-dimensional system (3) defined above. The system (3) displays $N+1$ critical points of Morse type. We are indeed able to derive a formula for the inverse of the Birkhoff series of each of these points using a Cauchy residue (Lagrange inversion formula). We check on specific examples that in general these series do not reduce to linear terms.

In a second part, we obtain partial results in the general case of the Aomoto-Forrester system. This system is likely to be non-integrable in general. Nevertheless, using some techniques ([1], [3], [16]) introduced around the Jacobian conjecture, we can derive some expansion for the inverses of the Aomoto-Forrester map.

2. The one-dimensional case and the Lagrange inversion formula

In this paragraph, we focus on the one-dimensional Hamiltonian:

$$(4) \qquad H = \frac{1}{2}y^2 + \frac{1}{2}\left(x - \sum_{j=1}^{N}\frac{\lambda_j}{x-a_j}\right)^2.$$

This Hamiltonian is defined on bands $]a_i, a_{i+1}[$. Under the condition $\lambda_j > 0$, this HS displays a unique stationary point $(b_i, 0), i = 0, ..., N$ on each interval $]a_i, a_{i+1}[$ and these stationary points are minima and of Birkhoff type. To each of these points one can associate a Birkhoff normal form. This Birkhoff normal form can be obtained as follows:

In the coordinate system (y, w), the system displays

$$(5) \qquad H = \frac{1}{2}y^2 + \frac{1}{2}w^2, \quad w = x - \sum_{j=1}^{N}\frac{\lambda_j}{x-a_j},$$
$$\omega = dx \wedge dy.$$

$$(6) \qquad w = f(x) = \frac{q(x)}{p(x)}, \quad q(x) = \prod_{j=0}^{N}(x-b_j), \quad p(x) = \prod_{j=1}^{N}(x-a_j).$$

$$(7) \qquad \gamma = \{y = c\sin\theta, w = c\cos\theta, \theta \in [0, 2\pi[\}.$$

Relative cohomology techniques developed around the isochore Morse lemma ([10], [11], [12]) show that the (inverse of) Birkhoff series is essentially given by the following:

$$(8) \qquad \int_\gamma x_k(w)dy,$$

where $x_k(w)$ denotes the unique series obtained by inverting $w = f(x)$ near the point $(b_k, 0)$. This inverse series can be obtained in many different ways.

Here we focus on the Lagrange inversion formula:

Proposition 2.1. *Let*

$$(9) \qquad g_i(z) = \frac{\partial^{i-1}}{\partial z^{i-1}} \left\{ \left[\frac{z - b_k}{f(z)} \right]^i \right\},$$

then

$$(10) \qquad x_k(w) = b_k + \sum_{i=1}^{+\infty} g_i(b_k) \frac{w^i}{i!}.$$

Proof. Let γ_0 be a closed curve around 0 and $\gamma_k = x_k(\gamma_0)$ such that f has no more singularities inside γ_k than b_k. Cauchy integral formula reads,

$$x_k(w) \quad = \quad \frac{1}{2\sqrt{-1}\pi} \int_{\gamma_k} \frac{x_k(z)}{z - w} dz.$$

Making $z = f(x)$ we have

$$\begin{aligned} x_k(w) \quad &= \quad \frac{1}{2\sqrt{-1}\pi} \int_{\gamma_k} \frac{x f'(x)}{f(x) - w} dx \\ &= \quad \frac{1}{2\sqrt{-1}\pi} \int_{\gamma_k} x \frac{f'(x)}{f(x)} \frac{1}{1 - \frac{w}{f(x)}} dx \\ &= \quad \frac{1}{2\sqrt{-1}\pi} \sum_{i \geq 0} \int_{\gamma_k} x \frac{f'(x)}{f(x)^{i+1}} w^i dx \\ &= \quad \frac{1}{2\sqrt{-1}\pi} \int_{\gamma_k} x \frac{f'(x)}{f(x)} dx + \frac{1}{2\sqrt{-1}\pi} \sum_{i \geq 1} w^i \int_{\gamma_k} x \frac{f'(x)}{f(x)^{i+1}} dx. \end{aligned}$$

By residue formula,

$$\begin{aligned} x_k(w) \quad &= \quad b_k + \sum_{i \geq 1} w^i \mathrm{Res}_{x=b_k} (x - b_k) \frac{f'(x)}{f(x)^{i+1}} \\ &= \quad b_k + \sum_{i \geq 1} \frac{w^i}{i} \mathrm{Res}_{x=b_k} \frac{1}{f(x)^i} \\ &= \quad b_k + \sum_{i \geq 1} \frac{w^i}{i} \frac{1}{(i-1)!} \frac{\partial^{i-1}}{\partial x^{i-1}} \left[\frac{(x - b_k)^i}{f(x)^i} \right]_{|_{x=b_k}}, \end{aligned}$$

and Lagrange integral formula follows. $\qquad\square$

Note that the closed formula derived above for the coefficients of this series can be easily implemented. We did it with Mathematica. The final step is to include this formula in the integral:

$$(11) \qquad \int_{\gamma = H^{-1}(c)} x_k(w) dy.$$

$$(12) \qquad \int_{\gamma = H^{-1}(c)} x_k(w) dy = \sum_i \int_0^{2\pi} \frac{1}{i!} g_i(b_k) c^{i+1} (\cos \theta)^{i+1} d\theta$$

$$(13) \qquad = \sum_p \frac{1}{(2p-1)!} g_{2p-1}(b_k) c^{2p} \int_0^{2\pi} \cos^{2p} \theta d\theta = \sum_p 2\pi \frac{2p}{(2^p p!)^2} g_{2p-1}(b_k) c^{2p}.$$

3. The general case and the inverses of the Aomoto-Forrester map

Let A be a finite arrangement of hyperplanes in the n-dimensional complex affine space $\mathbb{C}^n$. Let $N(A)$ be the union of hyperplanes of A in $\mathbb{C}^n$ and $M(A)$ be its complement. It is further assumed that A is real, meaning that the defining function of every hyperplane $f_h(z) = u_{h,0} + \sum_{i=1}^n u_{h,i} z_i$ has real coefficients. We define the Aomoto-Forrester system on $T^*(M(A) \cap \mathbb{R}^n)$ by the Hamiltonian function:

$$(14) \qquad H = \frac{1}{2}\sum_{i=1}^n y_i^2 + \frac{1}{2}\sum_{i=1}^n \left(x_i - \sum_{h \in A} \frac{\lambda_h u_{h,i}}{f_h(x)} \right)^2.$$

The isochore geometry of this Hamiltonian system is, in particular, devoted to the study of the canonical partition function

$$(15) \qquad \int \exp(-\beta H)\Omega, \quad \Omega = d^n x \wedge d^n y.$$

In the purpose of computing this function, Aomoto and Forrester introduced the mapping:

$$(16) \qquad F : z \mapsto w, \quad w_i = z_i - \sum_{h \in A} \frac{\lambda_h u_{h,i}}{f_h(z)}.$$

A connected component of $M(A) \cap \mathbb{R}^n$ is called a chamber. Aomoto and Forrester proved that the mapping F displays in restriction to any chamber Δ a unique analytic inverse. Note that this map displays a special form which seems to have not been exploited before. It writes:

$$(17) \qquad F : z \mapsto w, \quad w_i = z_i - \frac{\partial P}{\partial z_i},$$

with:

$$(18) \qquad P = \log\left(\prod_{h \in A} f_h(z)^{\lambda_h} \right).$$

It is possible to show, for instance by using the Abyankhar formula for the inverse of a map ([1], [3]) that the inverse of a map of this type is also a map of this type:

$$(19) \qquad G : w \mapsto z, z_i = w_i + \frac{\partial Q}{\partial w_i}.$$

Inspired on the recent contributions to the Jacobian conjecture ([16]), we propose to characterize these inverses as follows. We consider for that purpose the mapping

$$(20) \qquad F_t : z \mapsto F_t(z) = z - t\nabla P.$$

For t small, this mapping is invertible. We give an expansion for this inverse which exists for t small. Aomoto-Forrester showed that the map F_t is invertible till $t = 1$ when it is restricted to a chamber Δ. These local inverses are local restrictions of extensions of the inverse we described below till $t = 1$.

Theorem 3.1. *Let t be a small parameter, consider the deformation $F_t(z) = z - t\nabla P$. The inverse map of $z \mapsto F_t(z)$ can be written:*

$$(21) \qquad G_t(z) = z + t\nabla Q_t(z),$$

where $Q_t(z)$ is the unique solution of the Cauchy problem for the Hamilton-Jacobi equation:

$$(22) \qquad \begin{aligned} \frac{\partial Q_t(z)}{\partial t} &= \frac{1}{2}\langle \nabla Q_t, \nabla Q_t \rangle, \\ Q_{t=0}(z) &= P(z) \end{aligned}$$

We include a proof to be self-contained although it follows the lines of ([16]). Note that with $U_t = \nabla Q_t$, the equation can be alternatively written:

$$(23) \qquad \frac{\partial U_t(z)}{\partial t} = J(U_t(z))U_t(z),$$

where J is the Jacobian matrix, which is the inviscid n-dimensional Burgers equation (cf. [16]).

Proof. More generally, the formal inverse of $F_t(z) = z - tM(z)$ is the formal series $G_t(z) = z + tN_t(z)$ if and only if:

$$(24) \qquad \begin{aligned} N_t(F_t(z)) &= M(z), \\ M(G_t(z)) &= N_t(z). \end{aligned}$$

This yields the equations:

$$(25) \qquad 0 = \frac{\partial}{\partial t}[N_t(F_t(z))],$$

$$(26) \qquad 0 = \frac{\partial N_t}{\partial t}(F_t(z)) + J(N_t)(F_t(z)))\frac{\partial F_t}{\partial t},$$

$$(27) \qquad 0 = \frac{\partial N_t}{\partial t}(F_t(z)) - J(N_t)(F_t(z)))M.$$

After composing with $G_t(z)$ from the right, this displays:

$$(28) \qquad \frac{\partial N_t}{\partial t} = J(N_t)M(G_t) = J(N_t)N_t.$$

The theorem is proved in particular in the gradient case where, $M = \nabla P$, $N_t = \nabla Q_t$, $\nabla Q_t(F_t) = \nabla P$ and $\nabla P(G_t) = \nabla Q_t$ as a consequence of:

$$(29) \qquad \frac{\partial}{\partial z_i}\left(\frac{\partial Q_t}{\partial t}\right) = \sum_j \frac{\partial^2 Q_t}{\partial z_i \partial z_j}\frac{\partial Q_t}{\partial z_j} = \frac{\partial}{\partial z_i}\frac{1}{2}\langle \nabla Q_t, \nabla Q_t\rangle.$$

$\square$

This yields an expansion series for the inverse map obtained by solving the Hamilton-Jacobi equation (3.8) as follows. Write:

$$(30) \qquad \begin{aligned} Q_t(z) &= \sum_{m=0}^{+\infty} t^m Q_m(z), \\ Q_0(z) &= P(z), \end{aligned}$$

and identify terms of equal powers in t displays the recurrence relation:

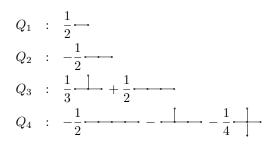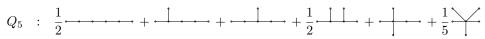$$(31) \qquad mQ_m(z) = \frac{1}{2}\sum_{i+j=m-1}\langle \nabla Q_i, \nabla Q_j\rangle.$$

Introduce at this point the notation u_h for the n-dimensional vector $u_{h,i}, i = 1, ..., n$ where the index h enumerates the hyperplanes of the arrangement A. We find successively:

$$(32) \qquad Q_1(z) = \frac{1}{2}\sum_{(h,k)\in A\times A}\frac{\lambda_h \lambda_k \langle u_h, u_k\rangle}{f_h(z)f_k(z)},$$

$$(33) \qquad Q_2(z) = -\frac{1}{2} \sum_{(h,k,l) \in A \times A \times A} \frac{\lambda_h \lambda_k \lambda_l \langle u_h, u_k \rangle \langle u_h, u_l \rangle}{f_k(z) f_h^2(z) f_k(z)}.$$

Each function Q_m, $m \geq 1$, may be represented by a graph. Each connected component of the graph is a non-labelled tree and represents a different summand of Q_m. All the trees have $m+1$ vertices connected by m edges. At a vertex h_i a factor λ_{h_i} is attached and an edge connecting vertices h_i and h_j gives a factor $\dfrac{\langle u_{h_i}, u_{h_j} \rangle}{f_{h_i}(z) f_{h_j}(z)}$ to the corresponding summand. The whole expression is summed over $\sum_{(h_1,\ldots,h_{m+1}) \in A^{m+1}}$. arrangement A, trees are non-labeled. With this representation we see that:

$Q_1 \quad : \quad \dfrac{1}{2} \, \rule[2pt]{24pt}{0.4pt}$

$Q_2 \quad : \quad -\dfrac{1}{2} \, \rule[2pt]{40pt}{0.4pt}$

$Q_3 \quad : \quad \dfrac{1}{3} \, \rule[2pt]{40pt}{0.4pt} + \dfrac{1}{2} \, \rule[2pt]{56pt}{0.4pt}$

$Q_4 \quad : \quad -\dfrac{1}{2} \, \rule[2pt]{72pt}{0.4pt} - \rule[2pt]{56pt}{0.4pt} - \dfrac{1}{4} \, \rule[2pt]{40pt}{0.4pt}$

$Q_5 \quad : \quad \dfrac{1}{2} \, \rule[2pt]{88pt}{0.4pt} + \rule[2pt]{72pt}{0.4pt} + \rule[2pt]{72pt}{0.4pt} + \dfrac{1}{2} \, \rule[2pt]{56pt}{0.4pt} + \rule[2pt]{56pt}{0.4pt} + \dfrac{1}{5} \, \rule[2pt]{40pt}{0.4pt}$

Further analysis of this diagrammatic expansion will be developed in relation with finding integrable cases in another publication.

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