# THE MODULAR CLASS OF A LIE ALGEBROID COMORPHISM

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Abstract. We introduce the definition of modular class of a Lie algebroid comorphism and exploit some of its properties.

## 1. Introduction

The modular class of a Poisson manifold $M$ is an element of the first Poisson cohomology group $H^1_\pi(M)$, which measures the obstruction to the existence of a measure in $M$ invariant under all hamiltonian diffeomorphisms ([9, 12]). This notion was extended to Lie algebroids by Evan, Lu and Weinstein [3] who showed that the modular class of the cotangent bundle of a Poisson manifold is twice the modular class of the Poisson structure. Grabowski, Marmo and Michor [6] introduced the modular class of a Lie algebroid morphism and this was more deeply studied by Kosmann-Schwarzbach, Laurent-Gengoux and Weinstein in [7] and [8]. In a recent paper [2], the notion of modular class of a Poisson map was given and some of its properties studied. Even more recently Grabowski [5] generalizes all these definitions introducing the modular class of skew algebroid relations. In this paper we exploit the definition of the modular class of a Lie algebroid comorphism, following the approach in [2].

## 2. The modular class of a Lie algebroid

Let $A \to M$ be a Lie algebroid over $M$, with anchor $\rho : A \to TM$ and Lie bracket $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$. We will denote by $\Omega^k(A) \equiv \Gamma(\wedge^k A^*)$ the $A$-forms and by $\mathfrak{X}^k(A) \equiv \Gamma(\wedge^k A)$ the $A$-multivector fields. Recall that the $A$-differential $\mathrm{d}_A : \Omega^k(A) \to \Omega^{k+1}(A)$ is given by

$$\begin{aligned}\mathrm{d}_A\alpha(X_0, X_1 \dots, X_n) = &\sum_{k=1}^{n} (-1)^i \rho(X_i) \dots \alpha(X_0, \cdots, \hat{X}_i, \dots, X_n)\\ &+ \sum_{0\leq i<j\leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots \hat{X}_j, \dots, X_n)\end{aligned}$$

and turns $\Omega^\bullet(A)$ into a complex whose cohomology is called the Lie algebroid cohomology and will be denoted by $H^\bullet(A)$.

**Example 2.1.** In case $A = TM$, the Lie algebroid cohomology is the De Rham cohomology.

**Example 2.2.** For any Poisson manifold $(M, \pi)$ there is a natural Lie algebroid structure on its cotangent bundle $T^*M$: the anchor is $\rho = \pi^\sharp$ and the Lie bracket on sections of $A = T^*M$, i.e., on one forms, is given by:

$$[\alpha, \beta] = \boldsymbol{\pounds}_{\pi^\sharp\alpha}\beta - \boldsymbol{\pounds}_{\pi^\sharp\beta}\alpha - \mathrm{d}\pi(\alpha, \beta).$$

The Poisson cohomology of $(M, \pi)$ is just the Lie algebroid cohomology of $T^*M$.

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A **morphism** between two Lie algebroids $A \to M$ and $B \to N$ is a vector bundle map $(\Phi, \phi)$

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ \downarrow & & \downarrow \\ M & \xrightarrow[\phi]{} & N \end{array}$$

such that the dual vector bundle map $\Phi^* : (\Omega^\bullet(B), \mathrm{d}_B) \to (\Omega^\bullet(A), \mathrm{d}_A)$ is a chain map.

The most basic example of a Lie algebroid morphism is the tangent map $T\phi$ of a smooth map $\phi : M \to N$.

A **representation of a Lie algebroid** $A$ is a vector bundle $E \to M$ together with a flat $A$-connection $\nabla$ (see, e.g, [4]). The usual operations $\oplus$ and $\otimes$ on vector bundles turn the space of representations $\mathrm{Rep}(A)$ into a semiring. Given a morphism of Lie algebroids $(\Phi, \phi)$, there is a pullback operation on representations $E \mapsto \phi^! E$, which gives a morphism of rings $\phi^! : \mathrm{Rep}(B) \to \mathrm{Rep}(A)$.

For an orientable line bundle $L \in \mathrm{Rep}(A)$ the only characteristic class can be obtained as follows: for any nowhere vanishing section $\mu \in \Gamma(L)$,

$$\nabla_X \mu = \langle \alpha_\mu, X \rangle \mu, \quad \forall X \in \mathfrak{X}(A).$$

The 1-form $\alpha_\mu \in \Omega^1(A)$ is $\mathrm{d}_A$-closed and it is called the **characteristic cocycle** of the representation $L$. Its cohomology class is independent of the choice of section $\mu$ and defines the characteristic class of the representation $L$:

$$\mathrm{char}(L) := [\alpha_\mu] \in H^1(A).$$

One checks easily that if $L, L_1, L_2 \in \mathrm{Rep}(A)$, then:

$$\mathrm{char}(L^*) = -\,\mathrm{char}(L), \qquad \mathrm{char}(L_1 \otimes L_2) = \mathrm{char}(L_1) + \mathrm{char}(L_2).$$

Also, if $(\Phi, \phi) : A \to B$ is a morphism of Lie algebroids, and $L \in \mathrm{Rep}(B)$ then:

$$\mathrm{char}(\phi^! L) = \Phi^* \,\mathrm{char}(L),$$

where $\Phi^* : H^\bullet(B) \to H^\bullet(A)$ is the map induced by $\Phi$ at the level of cohomology. If $L$ is not orientable, then one defines its characteristic class to be the one half that of the representation $L \otimes L$, so the formulas above still hold, for non-orientable line bundles.

Every Lie algebroid $A \to M$ has a canonical representation in the line bundle $L_A = \wedge^{\mathrm{top}} A \otimes \wedge^{\mathrm{top}} T^* M$:

$$\nabla_X(\omega \otimes \mu) = \boldsymbol{\mathcal{L}}_X \omega \otimes \mu + \omega \otimes \boldsymbol{\mathcal{L}}_{\rho(X)} \mu.$$

Then we set:

**Definition 2.3.** The **modular cocycle** of a Lie algebroid $A$ relative to a nowhere vanishing section $\omega \otimes \mu \in \Gamma(\wedge^{\mathrm{top}} A \otimes \wedge^{\mathrm{top}} T^* M)$ is the characteristic cocycle $\alpha_{\omega \otimes \mu}$ of the representation $L_A$. The **modular class** of $A$ is the characteristic class:

$$\mathrm{mod}(A) := [\alpha_{\omega \otimes \mu}] \in H^1(A).$$

*Remark* 2.4. Notice that, if $\nu = f\mu$ is another section of $L_A$, for a nonvanishing function $f \in C^\infty(M)$, then

$$\alpha_\nu = \alpha_\mu - \mathrm{d}_A \ln f. \tag{1}$$

**Example 2.5.** The modular class of a tangent bundle is trivial.

**Example 2.6.** Let $(M,\pi)$ be a Poisson manifold. The first Poisson cohomology space $H^1_\pi(M)$, is the space of Poisson vector fields modulo the hamiltonian vector fields.

The Lie derivative of any volume form along hamiltonian vector fields leads to a unique vector field $X_\mu \in \mathfrak{X}(M)$ such that:

$$\boldsymbol{\mathcal{L}}_{X_f}\mu = X_\mu(f)\mu.$$

One calls $X_\mu$ the **modular vector field** of the Poisson manifold $(M,\pi)$ relative to $\mu$. The modular vector field $X_\mu$ is Poisson and, if $\nu = g\mu$ is another volume form, then:

$$X_{g\mu} = X_\mu - \pi^\sharp(\mathrm{d}\ln|g|). \tag{2}$$

This lead to the definition of modular class of a Poisson manifold, which is due to Weinstein [12]:

> The **modular class** of a Poisson manifold $(M,\pi)$ is the Poisson cohomology class
>
> $$\mathrm{mod}(M) := [X_\mu] \in H^1_\pi(M).$$

Note that $\mathrm{mod}(M) = 0$ if and only if we can find a volume form $\mu$ invariant under all hamiltonian flows.Therefore the modular class is the obstruction to the existence of a volume form in $(M,\pi)$ invariant under all hamiltonian flows.

In fact, the modular class of the Poisson manifold $(M,\pi)$ and the modular class of the Lie algebroid $T^*M$ just differ by a multiplicative factor:

$$\mathrm{mod}(T^*M) = 2\,\mathrm{mod}(M).$$

## 3. The modular class of a Lie algebroid morphism

Let $\Phi : A \to B$ be a morphism of Lie algebroids covering a map $\phi : M \to N$. The induced morphism at the level of cohomology $\Phi^* : H^\bullet(B) \to H^\bullet(A)$, in general, does not map the modular classes to each other. Therefore one sets ([8]):

**Definition 3.1.** The **modular class** of a Lie algebroid morphism $\Phi : A \to B$ is the cohomology class defined by:

$$\mathrm{mod}(\Phi) := \mathrm{mod}(A) - \Phi^*\,\mathrm{mod}(B) \in H^1(A).$$

**Proposition 3.2.** *Let $\Phi : A \to B$ and $\Psi : B \to C$ be Lie algebroid morphisms, then:*

$$\mathrm{mod}(\Psi\circ\Phi) = \mathrm{mod}(\Phi) + \Phi^*\,\mathrm{mod}(\Psi).$$

The basic properties for characteristic classes show that the modular class of a Lie algebroid morphism $(\Phi,\phi) : A \to B$ can be seen as the characteristic class of a representation. Namely, one takes the canonical representations $L_A \in \mathrm{Rep}(A)$ and $L_B \in \mathrm{Rep}(B)$ and forms the representation $L^\phi := L_A \otimes \phi^!(L_B)^*$. Then:

**Proposition 3.3.** *Let $(\Phi,\phi) : A \to B$ be a Lie algebroid morphism. Then:*

$$\mathrm{mod}(\Phi) = \mathrm{char}(L^\phi).$$

## 4. The modular class of a Lie algebroid comorphism

In this section we extend some of the results for Poisson maps in [2] to comorphisms between Lie algebroids. We begin with the definition of a Lie algebroid comorphism. Further details about comorphisms can be seen in [10, 1, 11, 13].

**Definition 4.1.** Let $A \to M$ and $B \to N$ be two Lie algebroids. A **comorphism** between $A$ and $B$ covering $\phi : M \to N$ is a vector bundle map $\Phi : \phi^! B \to A$ from the pullback vector bundle $\phi^! B$ to $A$, such that the following two conditions hold:

$$\left[\bar{\Phi} X, \bar{\Phi} Y\right] = \bar{\Phi}\left[X, Y\right],$$

and

$$\mathrm{d}\phi \circ \rho_A(\bar{\Phi} X) = \rho_B(X),$$

for $X, Y \in \mathfrak{X}(B)$, where $\bar{\Phi} : \mathfrak{X}(B) \to \mathfrak{X}(A)$ is the natural map induced by $\Phi$.

Equivalently, we may say that $(\Phi, \phi)$ is a Lie algebroid comorphism if and only if $\Phi^* : A^* \to B^*$ is a Poisson map for the natural linear Poisson structures on the dual Lie algebroids.

**Proposition 4.2.** *Let $\Phi : \phi^! B \to A$ be a Lie algebroid comorphism. The pullback vector bundle $\phi^! B \to M$ carries a natural Lie algebroid structure characterized by:*

$$\left[X^!, Y^!\right] = \left[X, Y\right]^!$$

*and*

$$\rho(X^!) = \rho_A(\bar{\Phi} X^!),$$

*for $X, Y \in \mathfrak{X}(B)$, $X^! = X \circ \phi \in \Gamma(\phi^! B)$ and $Y^! = Y \circ \phi \in \Gamma(\phi^! B)$.*

*For this structure, the natural maps*

$$\begin{array}{ccc} \phi^! B & \xrightarrow{\Phi} & A \\ & \searrow^{j} & \\ & & B \end{array} \tag{3}$$

*are Lie algebroid morphisms.*

The modular class of a Lie algebroid comorphism is defined as follows:

**Definition 4.3.** Let $\Phi : \phi^! B \to A$ be a Lie algebroid comorphism between the Lie algebroids $A$ and $B$. The **modular class** of $\Phi$ is the cohomology class:

$$\mathrm{mod}(\Phi) := \Phi^* \,\mathrm{mod}(A) - j^* \,\mathrm{mod}(B) \in H^1(\phi^! B).$$

**Example 4.4.** A Poisson map $\phi : M \to N$ defines a comorphism between cotangent bundles: $\Phi : \phi^! T^*N \to T^*M$ such that $\Phi(\alpha^!) = (\mathrm{d}\phi)^* \alpha$, where $\alpha^! = \alpha \circ \phi \in \mathfrak{X}\left(\phi^!(T^*N)\right)$, for all $\alpha \in \Omega^1(N)$. The modular class of the Poisson map $\phi$ was defined in [2] and we see that it is one half the modular class of the comorphism $\Phi$ induced by $\phi$.

Notice that the map $j^* : \Omega^k(B) \to \Omega^k(\phi^! B)$ is simply defined by

$$j^*(\alpha) = \alpha \circ \phi, \quad \alpha \in \Omega^k(B).$$

Taking this into account we can give an explicit description of a representative of the modular class of a comorphism $\Phi$:

**Proposition 4.5.** *Let $\Phi : \phi^! B \to A$ be a Lie algebroid comorphism over $\phi : M \to N$ and fix non-vanishing sections $\mu \in \Gamma(L_A)$, $\nu \in \Gamma(L_B)$. The modular class $\mathrm{mod}(\Phi)$ is represented by:*

$$\alpha_{\mu,\nu} = \Phi^*(\alpha_\mu) - \alpha_\nu \circ \phi,$$

*where $\alpha_\mu$ and $\alpha_\nu$ are the modular cocycle of $A$ and $B$ relative to $\mu$ and $\nu$, respectively.*

We will refer to $\alpha_{\mu,\nu}$ as the **modular cocycle** of $\Phi$ relative to $\mu$ and $\nu$.

**Corollary 4.6.** *The class* $\mathrm{mod}(\Phi)$ *is the obstruction to the existence of modular cocycles* $\alpha \in \Omega^1(A)$ *and* $\beta \in \Omega^1(B)$*, such that*

$$\Phi^*\alpha = \beta \circ \phi.$$

*Proof.* The Lie algebroid morphism $\Phi$ has trivial modular class if its modular cocycles are exact in the Lie algebroid cohomology of $\phi^! B$, i.e., if for each $\mu \in \Gamma(L_A)$ and $\nu \in \Gamma(L_B)$,

$$\alpha_{\mu,\nu} = \mathrm{d}_{\phi^! B} f = \Phi^*(\mathrm{d}_A f), \quad \text{ for some } f \in C^\infty(M)$$

By definition $\alpha_{\mu,\nu} = \Phi^*(\alpha_\mu) - \alpha_\nu \circ \phi$, hence we have $\Phi^*(\alpha_\mu + \mathrm{d}_A f) = \alpha_\nu \circ \phi$, and taking into account equation (1), we conclude that $\alpha_\mu + \mathrm{d}_A f = \alpha_{e^{-f}\mu}$ and $\Phi^* \alpha_{e^{-f}\mu} = X_\nu$. □

**Corollary 4.7.** *Let* $\Phi : \phi^! B \to A$ *be a comorphism between Lie algebroids. If there exists a Lie algebroid morphism* $\widehat{\Phi} : A \to B$ *making the diagram commutative*

$$\begin{array}{ccc} \phi^! B & \xrightarrow{\Phi} & A \\ & \searrow^{j} & \downarrow{\scriptstyle \widehat{\Phi}} \\ & & B \end{array}$$

*then*

$$\mathrm{mod}\,\Phi = \Phi^* \,\mathrm{mod}\,\widehat{\Phi}.$$

*Proof.* Since $j = \widehat{\Phi} \circ \Phi$ we have $j^* = \Phi^* \circ \widehat{\Phi}^*$ and

$$\Phi^* \,\mathrm{mod}\,\widehat{\Phi} = \Phi^*(\mathrm{mod}\,A - \widehat{\Phi}^* B) = \Phi^* \,\mathrm{mod}\,A - j^* \,\mathrm{mod}\,B = \mathrm{mod}\,\Phi.$$

□

**Proposition 4.8.** *Let* $\Phi : A \to B$ *be a comorphism between Lie algebroids. There is a natural representation of* $\phi^! B$ *on the line bundle* $L^\phi := L_A \otimes \phi^! L_B^*$*, and we have:*

$$\mathrm{mod}(\Phi) = \mathrm{char}(L^\phi).$$

*Proof.* We define a representation of $\phi^! B$ on the line bundle $L_A$ by setting:

$$\nabla_{X^!}(\mu \otimes \nu) := [X^!, \mu]_A \otimes \nu + \mu \otimes \boldsymbol{\pounds}_{\rho_A \bar{\Phi} X} \nu$$

and another representation on $\phi^! L_B$ by setting:

$$\nabla_{X^!}(\mu^! \otimes \nu^!) := [\alpha, \mu]_B^! \otimes \nu^! + \mu^! \otimes \left(\boldsymbol{\pounds}_{\rho_B(X)} \nu\right)^!,$$

for $X \in \mathfrak{X}(B)$ and $\mu \otimes \nu \in \Gamma(L_A)$. The tensor product of the first representation with the dual of the second representation defines a representation of $\phi^! B$ on the line bundle

$$L^\phi := L_A \otimes \phi^! L_B^*.$$

□

Let us consider two Lie algebroids morphisms $\Phi : \phi^! B \to A$ and $\Psi : \psi^! C \to B$ over $\phi : M \to N$ and $\psi : N \to P$, respectively. The restriction $\widetilde{\Psi} = \Psi_{|(\psi \circ \phi)^! C}$ maps $(\psi \circ \phi)^! C$ to $\phi^! B$ and defines a map at the cohomology level:

$$\widetilde{\Psi}^* : H^\bullet(\phi^! B) \to H^\bullet((\psi \circ \phi)^! C).$$

The function $\Psi \circ \Phi : (\psi \circ \phi)^! C \to A$ defined by:

$$\Psi \circ \Phi(X_{\psi \circ \phi(m)}) = \Phi\left(\widetilde{\Psi}\left(X_{\psi \circ \phi(m)}\right)\right), \quad (\forall m \in M),$$

is a Lie algebroid comorphism.

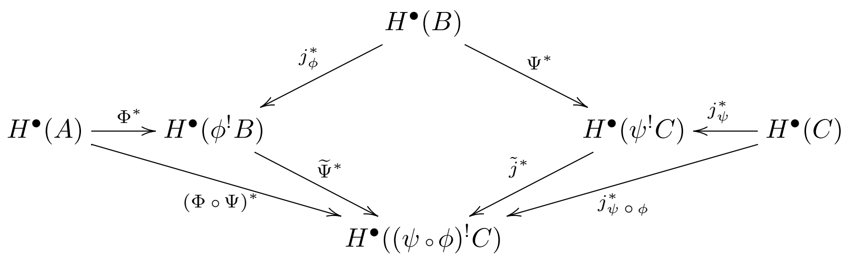
We also have the natural Lie algebroid morphism $\tilde{j} : (\psi \circ \phi)^! C = \phi^! \psi^! C \to \psi^! C$ that defines a map at the cohomology level

$$\tilde{j}^* : H^\bullet(\psi^! C) \to H^\bullet((\psi \circ \phi)^! C), \; \alpha \mapsto \alpha \circ \phi.$$

**Proposition 4.9.** *Let $\Phi : \phi^! B \to A$ and $\Psi : \psi^! C \to B$ be Lie algebroid comorphisms. Then:*

$$\operatorname{mod}(\Psi \circ \Phi) = \widetilde{\Psi}^* \operatorname{mod}(\Phi) + \tilde{j}^* \operatorname{mod}(\Psi).$$

*Proof.* The following diagram commutes:




Hence, we find:

$$\begin{aligned}
\operatorname{mod}(\Phi \circ \Psi) &= (\Phi \circ \Psi)^* \operatorname{mod}(A) - j_{\psi\circ\phi}^* \operatorname{mod}(C) \\
&= \widetilde{\Psi}^* \circ \Phi^* \operatorname{mod}(A) - \tilde{j}^* \circ j_\psi^* \operatorname{mod}(C) \\
&= \widetilde{\Psi}^* \circ \Phi^* \operatorname{mod}(A) - \widetilde{\Psi}^* \circ j_\phi^* \operatorname{mod}(B) + \widetilde{\Psi}^* \circ j_\phi^* \operatorname{mod}(B) - \tilde{j}^* \circ j_\psi^* \operatorname{mod}(C) \\
&= \widetilde{\Psi}^* (\Phi^* \operatorname{mod}(A) - j_\phi^* \operatorname{mod}(B)) + \tilde{j}^* (\Psi^* \operatorname{mod}(B) - j_\psi^* \operatorname{mod}(C)) \\
&= \widetilde{\Psi}^* \operatorname{mod}(\Phi) + \tilde{j}^* \operatorname{mod}(\Psi).
\end{aligned}$$

□

## 5. Generalization to Dirac structures

The modular class of a Lie algebroid morphism and the modular class of a Lie algebroid comorphism fit together into the notion of modular class of a skew algebroid relation, given by Grabowski in [5]. As a particular case we have the modular class of a Dirac map but very few was said about this particular case. The study of these structures will be exposed in a future work.

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