ON DERIVATIONS OF THE TERNARY MALCEV ALGEBRA M_8

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Derivations of the simple ternary Malcev algebra M_8 and its algebra of multiplications are described.

Key Words: Derivation algebra; Filippov algebra; n-Lie algebra; Ternary Malcev algebra.

2000 Mathematics Subject Classification: 17A42; 17B99; 17D99.

1. INTRODUCTION

Ternary Malcev algebras are a particular case of n-ary Malcev algebras, first defined in Pozhidaev (2001), and these naturally arise from the classification of n-ary vector cross product algebras (Brown and Gray, 1967). Indeed, the classification theorem for the latter asserts that, in the case $n = 2$, the only possible algebras are the simple 3-dimensional Lie algebra $sl(2)$ and the simple 7-dimensional Malcev algebra C_7; in the case $n \geq 3$, those are the simple $(n+1)$-dimensional n-Lie algebras (which, in turn, are a natural generalization of Lie algebras to the case of an n-ary multiplication—See Filippov, 1985—, and nowadays called *Filippov algebras*) with vector cross product, being analogs of $sl(2)$, and also some exclusive ternary algebras arising on composition algebras. Explicit formulas for this ternary vector cross product (which is displayed in (4)) and for generalized vector cross products can be found in Zvengrowski (1966), Brown and Gray (1967), and Massey (1983). An interesting overview on the subject (also including an historical approach), can be found in Eckmann (1991), while Ebbinghaus et al. (1969) underlines the relation between vector cross products and composition algebras.

It has been proven Pozhidaev (2001) that these ternary algebras are ternary central simple Malcev algebras, which are not 3-Lie algebras if the characteristic of the ground field is different from 2 and 3 (more generally, the result states that every n-ary vector cross product algebra is an n-ary central simple Malcev algebra).

The class of n-ary Malcev algebras has also the following interesting properties:

1. It is an extension of the class of n-Lie algebras, i.e., every n-Lie algebra is an n-ary Malcev algebra (generalizing the fact that every Lie algebra is a Malcev algebra);
2. Fixing an arbitrary component in the multiplication (i.e., defining a new reduced operation on the vector space A of the n-ary Malcev algebra by the rule $[x_1, \ldots, x_{n-1}]_a = [a, x_1, \ldots, x_{n-1}]$), (*reduced algebra*) we obtain an $(n-1)$-ary Malcev algebra.

At the moment, the only known example of a simple n-ary Malcev algebra which is not an n-Lie algebra is the above mentioned ternary central simple Malcev algebra arising on an 8-dimensional composition algebra.

In this article we continue investigating the properties of this ternary simple Malcev algebra M. We obtain some results on $\mathrm{Der}(M)$ and $\mathrm{Innder}(M)$, that is, its derivation and inner derivation algebras (namely, concluding that these coincide) and its associative and Lie algebras of multiplications. In the case of Malcev algebras we know that the operators of the type $[R_x, R_y] + R_{xy}$ are inner derivations. We prove an analog of this theorem to the case of the ternary Malcev algebra M. Namely, we prove that

$$\mathrm{Der}(M) = \langle [R_{x,y}, R_{x,z}] + R_{x,[y,x,z]} : x, y, z \in M \rangle.$$

The purpose is to use these results in forthcoming investigations, in order to classify the irreducible finite-dimensional representations of this ternary algebra. Further, we describe the algebra of quasi-derivations of M.

Some of the results of this article were previously announced in Pojidaev and Saraiva (2002) and published in Pojidaev and Saraiva (2003) as a preprint. However, most of the proofs in Pojidaev and Saraiva (2003) about derivations, which were based on direct and large computations, have been widely improved. Indeed, here we suggest other proofs based on some symmetries of the canonical basis of a composition algebra.

We start recalling some definitions. Let Φ be an associative, commutative ring with unity. An Ω-*algebra* over Φ is a unital module over Φ, on which we define a system of multilinear algebraic operations $\Omega = \{\omega_i : |\omega_i| = n_i \in N, i \in I\}$, where $|\omega_i|$ denotes the arity of ω_i. Henceforth, an Ω-algebra is sometimes briefly called an algebra.

An *n-Lie algebra* ($n \geq 3$) (*n-ary Filippov algebra*) is an Ω-algebra L with one n-ary operation $[x_1, \ldots, x_n]$ satisfying the identities

$$[x_1, \ldots, x_n] = \mathrm{sgn}(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}], \tag{1}$$

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n], \tag{2}$$

where σ is a permutation in the symmetric group S_n, with sign denoted by $\mathrm{sgn}(\sigma)$. The relation (1) is called the *anticommutativity identity* and (2) is the *generalized Jacobi identity* (or *Filippov identity*).

By an *n-ary Jacobian*, we mean the following function defined on an n-ary algebra:

$$\begin{aligned} &J(x_1, \ldots, x_n; y_2, \ldots, y_n) \\ &\quad = [[x_1, \ldots, x_n], y_2, \ldots, y_n] - \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n]. \end{aligned}$$

Note that, in an n-Lie algebra, $J(x_1, \ldots, x_n; y_2, \ldots, y_n)$ is skew-symmetric with respect to $x_1, \ldots, x_n$ and with respect to $y_2, \ldots, y_n$, but not on all of its arguments. It follows from the definition that A is an n-Lie algebra if and only if

$$J(x_1, \ldots, x_n; y_2, \ldots, y_n) = 0$$

for all $x_1, \ldots, x_n, y_2, \ldots, y_n \in A$.

An n-ary *Malcev algebra* ($n \geq 3$) is an Ω-algebra L with one anticommutative n-ary operation $[x_1, \ldots, x_n]$ satisfying the identity

$$\begin{aligned} &\sum_{i=2}^{n} [[z, x_2, \ldots, x_n], x_2, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n] \\ &\quad + \sum_{i=2}^{n} [[z, x_2, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n], x_2, \ldots, x_n] \\ &\quad = [[[z, x_2, \ldots, x_n], x_2, \ldots, x_n], y_2, \ldots, y_n] \\ &\qquad - [[[z, y_2, \ldots, y_n], x_2, \ldots, x_n], x_2, \ldots, x_n]. \end{aligned} \tag{3}$$

In terms of right multiplications, this identity is equivalent to:

$$R_x\left(\sum_{i=2}^{n} R_{x_2, \ldots, x_i R_y, \ldots, x_n}\right) + \left(\sum_{i=2}^{n} R_{x_2, \ldots, x_i R_y, \ldots, x_n}\right) R_x = R_x^2 R_y - R_y R_x^2,$$

where $R_x = R_{x_2, \ldots, x_n}$ and $R_y = R_{y_2, \ldots, y_n}$ are right multiplication operators: $zR_x = [z, x_2, \ldots, x_n]$. Note also that we can rewrite (3) as

$$-J(zR_x, x_2, \ldots, x_n; y_2, \ldots, y_n) = J(z, x_2, \ldots, x_n; y_2, \ldots, y_n) R_x.$$

A version of the ternary case of (3) can be written as follows:

$$\begin{aligned} &[[x, y, z], [y, u, v], z] + [[x, y, z], y, [z, u, v]] + [[x, [y, u, v], z], y, z] \\ &\quad + [[x, y, [z, u, v]], y, z] = [[[x, y, z], y, z], u, v] - [[[x, u, v], y, z], y, z]. \end{aligned}$$

Henceforth, we assume that Φ is a field of characteristic not equal to 2, 3 and denote by A a composition algebra over Φ with an involution $^- : a \mapsto \bar{a}$ and unity 1. The symmetric, bilinear form $\langle x, y \rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$ defined on A is assumed to be nonsingular and we can define the norm of each $a \in A$ by the rule $n(a) = \langle a, a \rangle$. If A is equipped with a ternary multiplication $[\cdot, \cdot, \cdot]$ by the rule

$$[x, y, z] = x\bar{y}z - \langle y, z \rangle x + \langle x, z \rangle y - \langle x, y \rangle z, \tag{4}$$

then A becomes a ternary Malcev algebra (Pozhidaev, 2001) which will be denoted by $M(A)$. If $\dim A = 8$ then $M(A)$ is not a 3-Lie algebra and we denote it by M_8 or simply by M.

Although the properties about composition algebras appear in several places (e.g., Brown and Gray, 1967; Pozhidaev, 2001; Shestakov et al., 1978), we now list some for a complete comprehension of the present article. Being A a composition algebra on the above conditions we have:

$$\begin{gathered} a\bar{a}b = a(\bar{a}b) = n(a)b = b\bar{a}a = b(\bar{a}a); \qquad \bar{a}b\bar{a} = -n(a)\bar{b} + 2\langle a, b\rangle\bar{a}; \\ a\bar{b}c + a\bar{c}b = 2\langle b, c\rangle a; \qquad a(\bar{b}c) + b(\bar{a}c) = 2\langle a, b\rangle c; \\ \langle ab, c\rangle = \langle b, \bar{a}c\rangle = \langle a, c\bar{b}\rangle; \qquad \langle \bar{a}, \bar{b}\rangle = \langle a, b\rangle, \langle \bar{a}, b\rangle = \langle a, \bar{b}\rangle. \end{gathered} \tag{5}$$

Further, if $a, b, c \in A$ are orthonormal, then

$$\bar{a}b\bar{a} = -\bar{b}; \qquad a\bar{b}c = -a\bar{c}b; \qquad a(\bar{b}c) = -b(\bar{a}c). \tag{6}$$

Finally, recall that (4) is anticommutative, since it is a ternary vector cross product (Brown and Gray, 1967).

2. ALGEBRAS OF MULTIPLICATIONS OF *M*

Let A be the above mentioned composition algebra and assume that $1, a, b, c$ are orthonormal vectors in A. Choose the following basis of M:

$$\{e_1 = 1, e_2 = a, e_3 = b, e_4 = ab, e_5 = c, e_6 = ac, e_7 = bc, e_8 = abc\},$$

denoted by $\mathscr{E}$ and called the *canonical basis* of M. For each $i \in \{2, \dots, 8\}$, it is possible to choose j, k, l, m, s, t, all depending on i, such that

$$e_i = e_1 e_i = e_j e_k = e_l e_m = e_s e_t \tag{7}$$

and

$$e_k e_m = e_t. \tag{8}$$

Definition. Any set of equations (7) which satisfies (8) is said to be a *partition of the basis* $\mathscr{E}$. The ordered set of natural numbers $\{i, j, k, l, m, s, t\}$ which corresponds to a given partition (7) of $\mathscr{E}$ is called the *index of partition*. The set of all indexes of the partitions of $\mathscr{E}$ will be denoted by $\mathscr{P}$.

Note that if we have a partition (7), then

$$\begin{gathered} e_i = e_1 e_i = e_s e_t = e_j e_k = e_l e_m, \\ e_i = e_1 e_i = e_l e_m = e_s e_t = e_j e_k \end{gathered} \tag{9}$$

are partitions too. Moreover, we also have the following partitions:

$$\begin{aligned}
e_j &= e_1e_j = e_se_m = e_ke_i = e_te_l,\\
e_k &= e_1e_k = e_ie_j = e_me_t = e_se_l,\\
e_l &= e_1e_l = e_me_i = e_ke_s = e_je_t,\\
e_m &= e_1e_m = e_ie_l = e_te_k = e_je_s,\\
e_s &= e_1e_s = e_le_k = e_te_i = e_me_j,\\
e_t &= e_1e_t = e_ie_s = e_ke_m = e_le_j.
\end{aligned} \tag{10}$$

Let $M = M(A)$ be the simple 8-dimensional ternary Malcev algebra over Φ and $\mathcal{R}$ the vector space spanned by the right multiplications of M. Let $\mathrm{Ass}(\mathcal{R})$ and $\mathrm{Lie}(\mathcal{R})$ denote, respectively, the associative and the Lie algebra generated by $\mathcal{R}$. Let $\mathrm{Der}(M)$ and $\mathrm{Innder}(M)$ be, respectively, the derivation and inner derivation algebras of M. Recall that a derivation is called *inner* if it belongs to the Lie algebra $\mathrm{Lie}(\mathcal{R})$ of transformations. In what follows, $\langle w_v; v \in \Upsilon\rangle$ denotes the linear space over Φ spanned by a family of vectors $\{w_v; v \in \Upsilon\}$.

Proposition 2.1. *With the above notations*:

1. $\mathrm{Ass}(\mathcal{R}) = M_{8,8}(\Phi) = \langle \mathcal{R}^2\rangle$;
2. $\mathrm{Lie}(\mathcal{R}) \cong D_4$ *and* $\mathrm{Lie}(\mathcal{R}) = \mathcal{R}$ *as vector spaces*;
3. $\mathrm{Der}(M) \cong B_3$.

Proof. Let (7) be a partition of $\mathcal{E}$ and

$$\mathcal{S} = \{R_{ij} : i, j = 1, \ldots, 8, i < j\},$$

where $R_{ij} = R_{e_i,e_j}$. We claim that $\mathcal{S}$ is linearly independent, that is,

$$\sum_{i<j} \alpha_{ij} R_{ij} = 0, \tag{11}$$

implies $\alpha_{ij} = 0$ for all $i, j = 1, \ldots, 8$, $i < j$. Consider the following partition of $\mathcal{S}$:

$$\mathcal{S} = \bigcup_{i=2}^{8} \mathcal{S}_i,$$

where for each $i \in \{2, \ldots, 8\}$, $\mathcal{S}_i = \{R_{1i}, R_{jk}, R_{lm}, R_{st}\}$. Fixing $i \in \{2, \ldots, 8\}$ and applying the left part of (11) to e_1, we have:

$$-\alpha_{jk} - \alpha_{lm} - \alpha_{st} = 0. \tag{12}$$

Indeed, it is easy to see that

$$e_1R_{1i} = 0 \qquad \text{and} \qquad e_1R_{jk} = -e_i = e_1R_{lm} = e_1R_{st}.$$

Further, if $i' \neq i$ and we apply the right multiplications of $\mathcal{S}_{i'}$ to e_1, we never obtain an element of $\langle e_i \rangle$ (except zero, of course) as a consequence of the above partition. Analogously, applying the left side of (11) to e_j, we obtain

$$-\alpha_{1i} - \alpha_{lm} - \alpha_{st} = 0, \tag{13}$$

since

$$e_j R_{jk} = 0 \quad \text{ and } \quad e_j R_{1i} = -e_k = e_j R_{lm} = e_j R_{st}$$

and also because no other right multiplication of $\mathcal{S}$ produces a vector of $\langle e_k \rangle$ when applied to e_j. Analogously, proceeding and applying the left side of (11), respectively, to e_l and to e_s we obtain

$$-\alpha_{1i} - \alpha_{jk} - \alpha_{st} = 0 \tag{14}$$

and

$$-\alpha_{1i} - \alpha_{jk} - \alpha_{lm} = 0, \tag{15}$$

respectively. So, from (12)–(15) we conclude that

$$\alpha_{1i} = \alpha_{jk} = \alpha_{lm} = \alpha_{st} = 0$$

and thus $\mathcal{S}_i$ is linearly independent. Since the same reasoning can be applied for all $i \in \{2, \dots, 8\}$, we conclude that $\mathcal{S}$ is linearly independent.

Consider R_{jk} as a linear transformation of the space M with the basis $\mathcal{E}$. It is easy to see that

$$R_{jk} = \Delta_{i1} + \Delta_{ml} + \Delta_{ts}, \tag{16}$$

where $\Delta_{ij} = e_{ij} - e_{ji}$ and e_{ij} are the usual matrix units. Note that $\dim \Delta = 28$, where

$$\Delta = \langle \Delta_{ij} : i, j = 1, \dots, 8 \rangle_\Phi.$$

By (16), $\langle \mathcal{S} \rangle = \mathcal{R}$ is a subspace of Δ, with $\dim \mathcal{R} = 28$. Hence,

$$\mathcal{R} = \Delta. \tag{17}$$

Now, from the development of $\Delta_{ij}\Delta_{kl}$ and after some simple computations, it is possible to obtain each e_{st} as a linear combination of elements $\Delta_{ij}\Delta_{kl}$. This fact and (17) easily lead to what is stated in item 1.

On the other hand, we have

$$[\Delta_{ij}, \Delta_{kl}] = \delta_{jk}\Delta_{il} + \delta_{jl}\Delta_{ki} + \delta_{ik}\Delta_{lj} + \delta_{il}\Delta_{jk}.$$

It is not difficult to obtain each Δ_{st} as linear combination of products like $[\Delta_{ij}, \Delta_{kl}]$. Again recalling (17), we conclude that the vector spaces $\mathrm{Lie}(\mathcal{R})$ and $\mathcal{R}$ coincide.

Further, it is known (see Humphreys, 1972) that the simple Lie algebra $so(8, \Phi)$—which is spanned by $\{\Delta_{ij} : i, j = 1, \dots, 8, i < j\}$—is a realization of D_4. Thus, the Lie algebras Lie($\mathcal{R}$) and D_4 are isomorphic.

To prove that $\mathrm{Der}(M) \cong B_3$ let D be a derivation of M. Then D is a linear mapping of M such that

$$[x, y, z]D = [xD, y, z] + [x, yD, z] + [x, y, zD], \tag{18}$$

for all $x, y, z \in M$. Let $D = \left[a_{ij}\right]_{\mathscr{C}}$. Fix $i \in \{1, \dots, 8\}$ and consider the partition (7). Taking $x = e_i$, $y = e_j$ and $z = e_k$ in (18), we obtain

$$e_1 D = \sum_{p=1}^{8} (a_{ip}[e_p, e_j, e_k] + a_{jp}[e_i, e_p, e_k] + a_{kp}[e_i, e_j, e_p]),$$

which, due to (7) and (8), is equivalent to

$$\begin{aligned}\sum_{p=1}^{8} a_{1p} e_p = {} & a_{ii}e_1 - a_{il}e_m + a_{im}e_l - a_{is}e_t + a_{it}e_s - a_{i1}e_i \\ & + a_{jj}e_1 - a_{js}e_m - a_{jt}e_l + a_{jl}e_t + a_{jm}e_s - a_{j1}e_j \\ & + a_{kk}e_1 + a_{kt}e_m - a_{ks}e_l - a_{km}e_t + a_{kl}e_s - a_{k1}e_k\end{aligned}$$

(for example, $[e_i, e_l, e_k] = (e_i\bar{e}_l)e_k = (e_l e_i)e_k = -e_m e_k = e_k e_m = e_t$). Thus, we have

$$\begin{aligned} a_{11} &= a_{ii} + a_{jj} + a_{kk}, \qquad a_{1p} = -a_{p1}, \quad p \in \{i, j, k\} \\ a_{1l} &= a_{im} - a_{jt} - a_{ks}, && (19) \\ a_{1m} &= -a_{il} - a_{is} + a_{kt}, && (20) \\ a_{1s} &= a_{it} + a_{jm} + a_{kl}, && (21) \\ a_{1t} &= -a_{is} + a_{jl} - a_{km}. && (22)\end{aligned}$$

Since the index i can be arbitrarily chosen in $\{2, \dots, 8\}$, by (9) we obtain

$$a_{pp} = a_{qq} \qquad \text{for all } p, q \in \{2, \dots, 8\}.$$

Taking $x = e_i$, $y = e_j$, and $z = e_l$ in (18) and proceeding analogously, we obtain

$$e_s D = \sum_{p=1}^{8} (a_{ip}[e_p, e_j, e_l] + a_{jp}[e_l, e_i, e_p] + a_{lp}[e_i, e_j, e_p]),$$

which, after the development of each term, is equivalent to

$$\begin{aligned}\sum_{p=1}^{8} a_{sp} e_p = {} & -a_{it}e_1 + a_{ik}e_m - a_{im}e_k + a_{i1}e_t + a_{ii}e_s - a_{is}e_i \\ & - a_{jm}e_1 + a_{j1}e_m + a_{jt}e_k - a_{jk}e_t + a_{jj}e_s - a_{js}e_j \\ & + a_{lk}e_1 + a_{lt}e_m - a_{l1}e_k - a_{lm}e_t + a_{ll}e_s - a_{ls}e_l.\end{aligned}$$

It follows from here that $a_{ss} = a_{ii} + a_{jj} + a_{ll}$ and $a_{sp} = -a_{ps}$, where $p \in \{i, j, l\}$. Again the possibility of choosing arbitrarily the index i, together with the relations (9) and (10), allows us to conclude that

$$a_{pq} = -a_{qp}, \qquad a_{pp} = 0 \tag{23}$$

for all $p, q \in \{1, \ldots, 8\}$. Note that, as in the previous case, we obtain four other relations, but these easily follow from (19)–(22) and from the arbitrary choice of the partition of the basis. Moreover, (20)–(22) can be deduced from (19), (23), and again by choosing arbitrarily the partition of the basis.

Next, considering the cases $x = e_i, y = e_k, z = e_l$ and $x = e_1, y = e_i, z = e_j$ in (18), we see that we have not any new relation. It is easy to observe that applying different partitions of the basis to the cases considered above, we obtain all possible cases.

Thus, relations (19) and (23) exhaust all possible relations imposed to the elements of the matrix of D (note that the indexes in (19) are a consequence of the arbitrary choice of the index of the partition). Therefore,

$$\mathrm{Der}(M) = \langle \Delta_{1l} - \Delta_{mi}, \Delta_{1l} - \Delta_{jt}, \Delta_{1l} - \Delta_{ks}; i, j, k, l, m, s, t \in \mathscr{P} \rangle. \tag{24}$$

It is now easy to exhibit a basis of $\mathrm{Der}(M)$, e.g.:

$$\begin{aligned} \mathscr{B} = \big\{ &\Delta_{23} - \Delta_{14}, \Delta_{24} + \Delta_{13}, \Delta_{25} - \Delta_{16}, \Delta_{26} + \Delta_{15}, \Delta_{27} + \Delta_{18}, \Delta_{28} - \Delta_{17}, \\ &\Delta_{34} - \Delta_{12}, \Delta_{35} - \Delta_{17}, \Delta_{36} - \Delta_{18}, \Delta_{37} + \Delta_{15}, \Delta_{38} + \Delta_{16}, \Delta_{45} - \Delta_{18}, \\ &\Delta_{46} + \Delta_{17}, \Delta_{47} - \Delta_{16}, \Delta_{48} + \Delta_{15}, \Delta_{56} - \Delta_{12}, \Delta_{57} - \Delta_{13}, \Delta_{58} - \Delta_{14}, \\ &\Delta_{67} + \Delta_{14}, \Delta_{68} - \Delta_{13}, \Delta_{78} + \Delta_{12} \big\}. \end{aligned} \tag{25}$$

Further, is is possible to show that the 21-dimensional Lie algebra $\mathrm{Der}(M)$ is simple. Taking into account this fact together with the dimension of $\mathrm{Der}(M)$, we may conclude that $\mathrm{Der}(M) \cong B_3$. □

From the proof of Proposition 2.1. we may explicitly describe all derivations of M in the basis $\mathscr{E}$.

Proposition 2.2. *The derivation algebra* $\mathrm{Der}(M)$ *is spanned over* Φ *by* $\mathscr{B}$, *as described in* (25).

Theorem 2.3. *All derivations of M are inner.*

Proof. By Proposition 2.1, $\mathscr{R} = \Delta = \mathrm{Lie}(\mathscr{R})$. By Proposition 2.2, we have that every $D \in \mathrm{Der}(M)$ belongs to $\mathrm{Lie}(\mathscr{R})$ and thus D is an inner derivation of M. □

The following lemma states some relations in the Lie algebra $\mathrm{Lie}(\mathscr{R})$.

Lemma 2.4. *Let $\mathscr{E}$ be the canonical basis of M. Then,*

$$[[R_{x,y}, R_{x,z}], R_{y,z}] = 0$$

for any $x, y, z \in \mathscr{E}$.

Proof. In order to prove that the lemma is true, it is enough to show that

$$t[[R_{x,y}, R_{x,z}], R_{y,z}] = 0$$

for all $x, y, z, t \in \mathscr{E}$. This identity is equivalent to

$$\begin{aligned}&[[[t,x,y],x,z],y,z] - [[[t,x,z],x,y],y,z]\\ &\quad -[[[t,y,z],x,y],x,z] + [[[t,y,z],x,z],x,y] = 0.\end{aligned} \tag{26}$$

Denoting the left-hand side of (26) by $f(t, x, y, z)$, it is clear that f is symmetric on y, z. If two of the arguments are equal, then (26) is satisfied. Indeed, the cases $x = y$, $x = z$ and $y = z$ are trivial. Suppose now that $t = x$. By using some of the properties listed in (5) and (6), if t, y, z are distinct in $\mathscr{E}$, then:

$$\begin{aligned}f(t,t,y,z) &= -[[[t,y,z],t,y],t,z] + [[[t,y,z],t,z],t,y]\\ &= [z,t,z] + [y,t,y] = 0.\end{aligned}$$

If $t = y$, then $f(t, x, t, z) = -[[[t, x, z], x, t], t, z]$. Since t, x, z are distinct in $\mathscr{E}$, we have $[[t, x, z], x, t] = z$, and thus $f(t, x, t, z) = -[z, t, z] = 0$. By the symmetry of f on y, z, the case $t = z$ follows from the previous case.

Finally, assume that all arguments of f are distinct in $\mathscr{E}$. Again applying (5) and (6), each summand of $f(t, x, y, z)$ is easily computable, and we obtain:

$$\begin{aligned}&[[[t,x,y],x,z],y,z] = t + \langle t\bar{y}, x\bar{z}\rangle[x,y,z] = [[[t,y,z],x,y],x,z];\\ &[[[t,x,z],x,y],y,z] = -t - \langle t\bar{y}, x\bar{z}\rangle[x,y,z] = [[[t,y,z],x,z],x,y].\end{aligned}$$

Replacing in (26), we have $f(t, x, y, z) = 0$, which concludes the proof. □

3. INNER DERIVATIONS OF *M*

Let A be a ternary algebra, $a \in A$ and $D \in \mathrm{Der}(A)$ such that $D(a) = 0$. It is easy to see that D is a derivation of the reduced algebra A_a. It is also easy to observe that, even in the case when $A = M$, there exists $D \in \mathrm{Der}(M)$ such that $D(a) \neq 0$ for all $a \in M$, $a \neq 0$. Take for instance $D = (\Delta_{14} - \Delta_{23}) + (\Delta_{56} + \Delta_{78})$.

Let $A = M$ and $a \in A$ such that the quotient algebra $A_a/\langle a\rangle$ is a simple Malcev algebra (we know Saraiva, 2003, that this happens when a is an element of the canonical basis, for example). Let $D \in \mathrm{Der}(M)$ ([1]) and $D(a) \neq 0$. Since D is a derivation of the reduced simple Malcev algebra M_a, and every derivation of such algebra is inner (i.e., it belongs to $\langle [R_x, R_y] + R_{xy}\rangle$), we have

$$D = \sum_i ([R_{a,x_1^i}, R_{a,x_2^i}] + R_{a,[x_1^i,a,x_2^i]}).$$

[1]Despite being $\mathrm{Der}(M) = \mathrm{Innder}(M)$ (which justifies the title of this section), we will rather use the notation $\mathrm{Der}(M)$ instead of $\mathrm{Innder}(M)$.

We know that in the general case a derivation of the type

$$[R_x, R_y] + R_{x \circ R_y},$$

where $x = (x_2, \ldots, x_n) \in L^{\times(n-1)}$, $y = (y_2, \ldots, y_n) \in L^{\times(n-1)}$ and

$$x \circ R_y = \sum_{i=2}^{n} (x_2, \ldots, x_i R_y, \ldots, x_n) \in L^{\times(n-1)},$$

is not a derivation of an n-ary Malcev algebra L. Therefore, a natural question arises: Are the operators $[R_{z,x}, R_{z,y}] + R_{z,[x,z,y]}$ derivations of the ternary Malcev algebra M? The answer is given by the following result.

Theorem 3.1. *Let $M(A)$ be a ternary Malcev algebra. For any $x, y, z \in A$*

$$[R_{z,x}, R_{z,y}] + R_{z,[x,z,y]} \in \mathrm{Der}(M(A)).$$

Proof. Linearizing the operator $[R_{z,x}, R_{z,y}] + R_{z,[x,z,y]}$ by z, we obtain

$$[R_{u,x}, R_{v,y}] + [R_{v,x}, R_{u,y}] + R_{u,[x,v,y]} + R_{v,[x,u,y]}.$$

In order to prove the theorem, it is enough to show that the last operator is a derivation for any $x, y, u, v \in \mathscr{E}$. To do this we need some auxiliary results.

Lemma 3.2. *For any distinct $x, y, u, v \in \mathscr{E} \backslash \{e_1\}$,*

$$[R_{x,y}, R_{u,v}] \in \mathrm{Der}(M).$$

Proof. Fix $i \in \{2, \ldots, 8\}$ and consider the partition (7) of the basis. Note that it is enough to prove the assertion in the case $x = e_i$, $y = e_j$, since the other cases will follow from this by using different partitions of $\mathscr{E}$. Recalling that each R_{e_u, e_v} can be denoted by R_{uv}, it is possible to deduce that

$$R = R_{ij} = \Delta_{k1} + \Delta_{tm} + \Delta_{ls}$$

by a suitable adjustment of indexes in (16). Doing the same for the other needed right multiplications, simple computations allow us to obtain the following expressions:

$$\begin{aligned}
[R, R_{kl}] &= -[R, \Delta_{s1} + \Delta_{it} + \Delta_{jm}] = (\Delta_{im} + \Delta_{1l}) + (\Delta_{ks} + \Delta_{tj}),\\
[R, R_{ks}] &= -[R, \Delta_{l1} + \Delta_{im} + \Delta_{tj}] = (\Delta_{ti} - \Delta_{1s}) + (\Delta_{mj} - \Delta_{lk}),\\
[R, R_{km}] &= [R, \Delta_{t1} + \Delta_{si} + \Delta_{jl}] = (\Delta_{1m} - \Delta_{il}) + (\Delta_{tk} - \Delta_{js}),\\
[R, R_{kt}] &= [R, \Delta_{m1} + \Delta_{sj} + \Delta_{li}] = (\Delta_{t1} + \Delta_{lj}) + (\Delta_{mk} + \Delta_{is}),\\
[R, R_{lm}] &= [R, \Delta_{i1} + \Delta_{ts} + \Delta_{kj}] = (\Delta_{j1} + \Delta_{sm}) + (\Delta_{ik} + \Delta_{tl}),\\
[R, R_{ls}] &= [R, \Delta_{k1} + \Delta_{ji} + \Delta_{tm}] = 0,
\end{aligned}$$

$$[R, R_{lt}] = [R, \Delta_{j1} + \Delta_{ms} + \Delta_{ik}] = (\Delta_{jk} + \Delta_{ts}) + (\Delta_{1i} + \Delta_{ml}),$$
$$[R, R_{ms}] = [R, \Delta_{j1} + \Delta_{lt} + \Delta_{ik}] = (\Delta_{jk} + \Delta_{ts}) + (\Delta_{1i} + \Delta_{ml}),$$
$$[R, R_{mt}] = [R, \Delta_{k1} + \Delta_{ji} + \Delta_{ls}] = 0,$$
$$[R, R_{st}] = [R, \Delta_{i1} + \Delta_{kj} + \Delta_{ml}] = (\Delta_{j1} + \Delta_{sm}) + (\Delta_{ik} + \Delta_{tl}).$$

Using the description (24), (7), and (10), it is easy to see that all of these commutators belong to $\mathrm{Der}(M)$. □

Lemma 3.3. *For any distinct $x, y, u, v \in \mathscr{E}$,*

$$[R_{x,y}, R_{u,v}] \in \mathrm{Der}(M).$$

Proof. Taking into account Lemma 3.2 and since the partition of the basis is arbitrary, it is enough to prove that

$$\langle [R_{1i}, R_{jk}], [R_{1i}, R_{jl}], [R_{1i}, R_{jm}], [R_{1i}, R_{kl}], [R_{1i}, R_{km}] \rangle \subseteq \mathrm{Der}(M).$$

We have

$$R_{1i} = \Delta_{kj} + \Delta_{ml} + \Delta_{ts}, \qquad R_{jk} = \Delta_{i1} + \Delta_{ml} + \Delta_{ts},$$
$$R_{jl} = -(\Delta_{t1} + \Delta_{si} + \Delta_{mk}), \qquad R_{jm} = -(\Delta_{s1} + \Delta_{kl} + \Delta_{it}),$$
$$R_{kl} = -(\Delta_{s1} + \Delta_{it} + \Delta_{jm}), \qquad R_{km} = \Delta_{t1} + \Delta_{si} + \Delta_{jl}.$$

From here and (24) the lemma follows. □

Remark. The condition that the elements are distinct is essential. For example, we have $[R_{1i}, R_{1j}] = \Delta_{ji} + 2\Delta_{sl} + 2\Delta_{mt} \notin \mathrm{Der}(M)$ (in the opposite case $\Delta_{sl} \in \mathrm{Der}\, M$, which is impossible).

Lemma 3.4. *For any distinct $x, y, z \in \mathscr{E}$,*

$$[R_{x,y}, R_{x,z}] + R_{x,[y,x,z]} \in \mathrm{Der}(M).$$

Proof. Due to the arbitrariness of the partition of $\mathscr{E}$, the following six cases exhaust all possibilities.

1. $x = e_1, y = e_i, z = e_j$. In this case

$$[R_{1i}, R_{1j}] = [\Delta_{kj} + \Delta_{ml} + \Delta_{ts}, \Delta_{ms} + \Delta_{ik} + \Delta_{lt}] = \Delta_{ji} + 2\Delta_{sl} + 2\Delta_{mt},$$
$$R_{1,[i,1,j]} = R_{1k} = \Delta_{ji} - \Delta_{sl} - \Delta_{mt}$$

(if there is no ambiguity we denote e_i by i). Thus, $[R_{1i}, R_{1j}] + R_{1,[i,1,j]} = 2\Delta_{ji} + \Delta_{sl} + \Delta_{mt}$, which, by (24), is a derivation of M.

2. $x = e_i, y = e_j, z = e_k$. We have

$$[R_{ij}, R_{ik}] = -[\Delta_{k1} + \Delta_{tm} + \Delta_{ls}, \Delta_{j1} + \Delta_{ms} + \Delta_{lt}] = \Delta_{kj} - 2\Delta_{ts} - 2\Delta_{ml},$$
$$R_{i,[j,i,k]} = -R_{i1} = R_{1i} = \Delta_{kj} + \Delta_{ml} + \Delta_{ts}.$$

Thus, $[R_{ij}, R_{ik}] + R_{i,[j,i,k]} = -2\Delta_{jk} + \Delta_{st} + \Delta_{lm}$, which is a derivation of M by (24).

3. $x = e_i, y = e_j, z = e_l$. We have

$$[R_{ij}, R_{il}] = -2\Delta_{km} - 2\Delta_{1t} + \Delta_{lj}, R_{i,[j,i,l]} = -R_{is} = \Delta_{1t} + \Delta_{km} + \Delta_{lj}.$$

Summarizing, we see that the lemma is true in this case.

4. $x = e_i, y = e_j, z = e_m$. We have

$$[R_{ij}, R_{im}] = -[\Delta_{k1} + \Delta_{tm} + \Delta_{ls}, \Delta_{l1} + \Delta_{sk} + \Delta_{tj}] = -2\Delta_{lk} - 2\Delta_{1s} + \Delta_{mj},$$
$$R_{i,[j,i,m]} = R_{it} = \Delta_{1s} + \Delta_{lk} + \Delta_{mj}.$$

Summarizing, we obtain $-\Delta_{1s} - \Delta_{lk} + 2\Delta_{mj}$, which is a derivation of M by (24).

5. $x = e_i, y = e_k, z = e_l$. We have

$$[R_{ik}, R_{il}] = -[\Delta_{j1} + \Delta_{ms} + \Delta_{lt}, \Delta_{m1} + \Delta_{kt} + \Delta_{sj}] = -2\Delta_{mj} - 2\Delta_{1s} + \Delta_{lk},$$
$$R_{i,[k,i,l]} = R_{it} = \Delta_{1s} + \Delta_{lk} + \Delta_{mj}.$$

Summarizing, we obtain $-\Delta_{1s} - \Delta_{mj} + 2\Delta_{lk}$, which is a derivation of M by (24).

6. $x = e_i, y = e_k, z = e_m$. We have

$$[R_{ik}, R_{im}] = -[\Delta_{j1} + \Delta_{ms} + \Delta_{lt}, \Delta_{l1} + \Delta_{sk} + \Delta_{tj}] = 2\Delta_{lj} + 2\Delta_{1t} - \Delta_{km},$$
$$R_{i,[k,i,m]} = R_{is} = -\Delta_{1t} - \Delta_{km} - \Delta_{lj}.$$

Summarizing, we obtain $\Delta_{1t} + \Delta_{lj} - 2\Delta_{km}$, which is a derivation of M by (24). □

Lemma 3.5. *For any distinct $x, y, u, v \in \mathscr{E}$,*

$$f(u, x, v, y) = R_{u,[x,v,y]} + R_{v,[x,u,y]} \in \mathrm{Der}(M).$$

Proof. Fix $u \in \mathscr{E}$. Note that if $f(u, x, v, y) \in \mathrm{Der}(M)$, for any distinct $x, v, y \in \mathscr{E}\backslash\{u\}$, then

$$\{f(v, x, u, y), f(x, u, v, y), f(x, v, y, u)\} \subseteq \mathrm{Der}(M)$$

for any distinct $x, v, y \in \mathscr{E}\backslash\{u\}$. By the symmetry on u and v and the skewsymmetry on x and y, it is enough to prove this for $f(x, u, v, y)$. We have

$$f(x, u, v, y) = R_{x,[u,v,y]} + R_{v,[u,x,y]}$$

$$= -(R_{x,[v,u,y]} + R_{u,[v,x,y]}) - (R_{u,[x,v,y]} + R_{v,[x,u,y]})$$
$$= -f(x, v, u, y) - f(u, x, v, y) \in \mathrm{Der}(M).$$

Fix $i \in \{2, \dots, 8\}$. First, consider the case when $u = e_1$. Using (10), the arbitrariness of partition of the basis and the skewsymmetry on x and y, we may suppose that $x = e_i$ and $y = e_j$. Denote $f(p, w, z, h)$ by f_{pwzh}. Thus, we have the following five cases:

1. $f_{1ilj} = -R_{1s} - R_{kl} = -\Delta_{kl} - \Delta_{it} - \Delta_{jm} + \Delta_{s1} + \Delta_{it} + \Delta_{im} \in \mathrm{Der}(M)$.

In what follows we will use the obtained inclusion $R_{1s} + R_{kl} \in \mathrm{Der}(M)$ (see case 1) and the arbitrariness of the partition of the basis without mentioning it.

2. $f_{1ikj} = -R_{11} + R_{kk} = 0$;
3. $f_{1imj} = R_{1t} + R_{mk}$;
4. $f_{1isj} = R_{1l} + R_{sk}$;
5. $f_{1itj} = -R_{1m} + R_{tk}$;

Suppose now that $u = e_i$. Using the arbitrariness of the partition of the basis, we may assume that $v = e_j$. We have:

$$\begin{array}{lll}
f_{ikjl} = R_{im} + R_{jt}; & f_{ikjm} = -R_{il} + R_{js}; & f_{ikjs} = R_{it} - R_{jm}; \\
f_{ikjt} = -R_{is} - R_{jl}; & f_{iljk} = -R_{im} - R_{jt}; & f_{iljm} = R_{ik} - R_{j1}; \\
f_{iljs} = R_{ii} - R_{jj}; & f_{iljt} = R_{i1} + R_{jk}; & f_{imjk} = R_{il} - R_{js}; \\
f_{imjl} = -R_{ik} + R_{j1}; & f_{imjs} = R_{i1} + R_{jk}; & f_{imjt} = -R_{ii} + R_{jj}; \\
f_{isjk} = -R_{it} + R_{jm}; & f_{isjl} = -R_{ii} + R_{jj}; & f_{isjm} = -R_{i1} - R_{jk}; \\
f_{isjt} = R_{ik} - R_{j1}; & f_{itjk} = R_{is} + R_{jl}; & f_{itjl} = -R_{i1} - R_{jk}; \\
f_{itjm} = R_{ii} - R_{jj}; & f_{itjs} = -R_{ik} + R_{j1}. &
\end{array}$$

Using (24), it is easy to see that all these operators belong to $\mathrm{Der}(M)$. □

Finally, the conclusion of Theorem 3.1 follows from Lemmas 3.2–3.5.
The theorem is proven. □

Let $\mathscr{E}_d^4$ be the set of all distinct 4-tuples $(x, y, u, v) \in \mathscr{E}^{\times 4}$.

Corollary 1. $\mathrm{Der}(M) = \langle [R_{x,y}, R_{u,v}] : (x, y, u, v) \in \mathscr{E}_d^4 \rangle$.

Proof. Let

$$D_1 = \Delta_{1i} - \Delta_{jk}, D_2 = \Delta_{1i} - \Delta_{lm}, D_3 = \Delta_{1i} - \Delta_{st}.$$

From the proof of Lemma 3.2 we have

$$[R_{ij}, R_{lt}] = -D_1 + D_2 + D_3 \in \langle [\mathscr{R}, \mathscr{R}] \rangle,$$

where

$$\langle[\mathscr{R}, \mathscr{R}]\rangle = \langle[R_{x,y}, R_{u,v}] : (x, y, u, v) \in \mathscr{E}_d^4\rangle.$$

Using (9), we also obtain

$$-D_3 + D_1 + D_2 \in \langle[\mathscr{R}, \mathscr{R}]\rangle \qquad \text{and} \qquad - D_2 + D_3 + D_1 \in \langle[\mathscr{R}, \mathscr{R}]\rangle,$$

from where the conclusion follows. □

Corollary 2. $\mathrm{Der}(M) = \langle[R_{x,y}, R_{x,z}] + R_{x,[y,x,z]} : x, y, z \in M\rangle$.

Proof. It follows from the proof of Lemma 3.4. □

Corollary 3. *For any distinct $x, y, u, v \in \mathscr{E}$,*

$$R_{u,[v,x,y]} + R_{v,[u,x,y]} \in \mathrm{Der}(M),$$
$$R_{u,[x,y,v]} + R_{v,[x,y,u]} \in \mathrm{Der}(M).$$

Proof. It follows from the proof of Lemma 3.5. □

4. QUASI-DERIVATIONS OF *M*

In this section we describe the algebra of quasi-derivations of M. According to Block (1969), a linear operator $D : A \longrightarrow A$ is called a *quasi-derivation* of a ring A, if it satisfies

$$[D, T] \in T(A), \qquad \text{for all } T \in T(A),$$

where $T(A)$ stands for the Lie ring generated by the right and left multiplications by elements of A.

In the case of n-ary algebras we have the following definition.

Definition. Let A be an n-ary anticommutative algebra with multiplication $[\cdot, \dots, \cdot]$. Let $\mathscr{R}$ be the vector space spanned by the right multiplications

$$R_a = R_{a_2,\dots,a_n}, \qquad a_2, \dots, a_n \in A,$$

and let $\mathrm{Ass}(\mathscr{R})$ and $\mathrm{Lie}(\mathscr{R})$ be, respectively, the associative and the Lie algebra generated by $\mathscr{R}$. Every operator $D : A \longrightarrow A$ such that

$$[D, R_a] \in \mathrm{Lie}(\mathscr{R}), \qquad \text{for all } R_a \in \mathrm{Lie}(\mathscr{R}), \tag{27}$$

is said to be a *quasi-derivation* of A. The set of all quasi-derivations of the algebra A is denoted by $Q\,\mathrm{Der}(A)$.

Consider the simple 8-dimensional ternary Malcev algebra M over Φ and let $\mathscr{R}$, $\mathrm{Ass}(\mathscr{R})$, and $L = \mathrm{Lie}(\mathscr{R})$ stand with the above described meaning (with M

instead of A). Under these assumptions, the following result describes the quasi-derivations of M.

Theorem 4.1. $Q\,\mathrm{Der}(M) = \langle Id\rangle_{\Phi} \oplus L$.

Proof. Let $D = [d_{kl}]_{\mathscr{E}} \in M_{8\times 8}(\Phi)$ be a linear transformation considered in the basis $\mathscr{E}$ of Proposition 2.1. By (27) and Proposition 2.1, $D \in Q\,\mathrm{Der}(M)$ if and only if $[D, \Delta_{ij}] \in \langle \Delta_{ij}, i, j = 1, \ldots, 8, i < j\rangle_{\Phi}$.

We have

$$\left[\sum_{k,l=1}^{8} d_{kl}e_{kl}, e_{ij} - e_{ji}\right] = \sum_{k=1}^{8} d_{ki}e_{kj} - \sum_{l=1}^{8} d_{jl}e_{il} - \sum_{k=1}^{8} d_{kj}e_{ki} + \sum_{l=1}^{8} d_{il}e_{jl},$$

which may be equivalently written in the matrix form as

$$[D, \Delta_{ij}] = \begin{array}{cc} \begin{array}{ccccccc} & & \downarrow i & & \downarrow j & & \end{array} & \\ \left(\begin{array}{ccccccc} 0 & \cdots & -d_{1j} & 0 & d_{1i} & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ -d_{j1} & \cdots & -d_{ij} - d_{ji} & \cdots & d_{ii} - d_{jj} & \cdots & -d_{j8} \\ 0 & & \vdots & \ddots & \vdots & & 0 \\ d_{i1} & \cdots & -d_{jj} + d_{ii} & \cdots & d_{ji} + d_{ij} & \cdots & d_{i8} \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & -d_{8j} & 0 & d_{8i} & \cdots & 0 \end{array}\right) & \begin{array}{c} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array} \end{array}$$

Therefore,

$$[D, \Delta_{ij}] \in \mathrm{Lie}(\mathscr{R}) \Leftrightarrow \begin{cases} d_{kj} = -d_{jk}, \ k = 1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, 8, \\ d_{ki} = -d_{ik}, \ k = 1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, 8, \\ d_{ji} + d_{ij} = -d_{ij} - d_{ji}, \\ d_{ii} - d_{jj} = -(d_{ii} - d_{jj}). \end{cases}$$

Since $char\,\Phi \neq 2$, the last two identities imply $d_{ii} = d_{jj}$ and $d_{ij} = -d_{ji}$. Finally, recalling that $i, j = 1, \ldots, 8$, $i < j$, the theorem is proven. $\square$

ACKNOWLEDGMENTS

The authors are very grateful to the referee by his suggestions, towards a clarification of several aspects of the paper. The authors would also like to express gratitude to Prof. F. Leite for her attention to this work, careful reading of the paper and useful remarks which allowed improving the exposition of the article. The first author was supported by Russian Science Support Foundation, by the Russian Foundation for Basic Research (Grant 05-01-00230), and by a SB RAS grants for Young Scientists (Presidium decree \No 29 of 26/01/2006) and Complex Integration Projects 2006.1.9, and the second author was supported by Institute for Systems and Robotics (Coimbra, Portugal).

REFERENCES

Block, R. E. (1969). Determination of the differentiable simple rings with a minimal ideal. *Ann. Math.* 90(3):433–459.

Brown, R. B., Gray, A. (1967). Vector cross products. *Comment. Math. Helv.* 42:222–236.

Ebbinghaus, H.-D., Hermes, H. et al. (1969). *Numbers*. Graduate Texts in Mathematics, 123. Springer-Verlag.

Eckmann, B. (1991). Continuous solutions of linear equations—An old problem, its history and its solution. *Exposition. Math.* 9:351–365.

Filippov, V. T. (1985). n-Lie algebras. *Sib. Math. J.* 26(6), 879–891. Translation from *Sib. Mat. Zh.* 26(6):126–140 (Russian).

Humphreys, J. E. (1972). *Introduction to Lie Algebras and Representation Theory*. New York: Springer-Verlag.

Massey, W. S. (1983). Cross products of vectors in higher dimensional Euclidean spaces. *Am. Math. Monthly* 90:697–701.

Pojidaev, A. P., Saraiva, P. (2002). *On Derivations of Simple* 8*-dimensional Ternary Malcev Algebra*. Abstracts of International Conference on Lie and Jordan algebras, their Representations and Applications. Guaruja, Sao Paulo, Brazil, May 13–18, 44–45.

Pojidaev, A. P., Saraiva, P. (2003). *On Some Algebras Related with the Simple* 8*-dimensional Ternary Malcev Algebra* M_8. Centro de Matematica da U. Coimbra, Dep. de Mat. U. Coimbra, Preprint 03–19, 1–34.

Pozhidaev, A. P. (2001). n-ary Mal'tsev algebras. *Algebra and Logic* 40(3):170–182. Translation from *Algebra i Logika* 40(3):309–329 (Russian).

Saraiva, P. (2003). On some generalizations of Malcev algebras. *Int. J. of Math., Game Th. and Algebra* 13(2):89–108.

Shestakov, I. P., Shirshov, A. I., Slin'ko, A. M., Zhevlakov, K. A. (1978). *Rings that are Nearly Associative*. M. Nauka. English Translation. NY: Acad. Press.

Zvengrowski, P. (1966). A 3-fold vector cross product in $\mathcal{R}^8$. *Comm. Math. Helv.* 40:149–152.