Efficient Skewness/Semivariance Portfolios[*]

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Abstract

This article proposes a flexible methodology for portfolio selection using a skewness/ semivariance biobjective optimization framework. The solutions of this biobjective optimization problem allow the investor to analyze the efficient tradeoff between skewness and semivariance. This methodology is used empirically on four data sets, collected from the Fama/French data library. The out-of-sample performance of the skewness/semivariance model was assessed by choosing three portfolios belonging to each in-sample Pareto frontier and measuring their performance in terms of skewness per semivariance ratio, Sharpe ratio and Sortino ratio. Both the in-sample and the out-of-sample performance analyses were conducted using three different target returns for the semivariance computations. The results show that the efficient skewness/semivariance portfolios are consistently competitive when compared with several benchmark portfolios.

JEL Classification: C44; C58; C61; C63; C88; G11

Keywords: portfolio selection; semivariance; skewness; multiobjective optimisation; derivative-free optimisation

1 Introduction

As pointed out by Markowitz (1952), an investor that wishes to allocate his wealth to a set of securities must keep in mind not only the maximization of profit, but also the minimization of risk. In his ground-breaking work, Markowitz (1952) proposes a mean-variance optimization model intended to minimize the portfolio risk (measured by its variance) for a given level of expected return, over the set of feasible portfolios. By varying the level of expected return, the model determines the efficient frontier, which is the set of mean-variance efficient portfolios.

The theoretical consistency of the mean-variance model proposed by Markowitz (1952) with the Von Neumann and Morgenstern (1953) axioms of choice is dependent on either the returns following a normal distribution or the investors having a quadratic utility function. However, there is some misunderstanding about these issues. Occasionally it is argued that the mean-variance analysis assumes normal return distributions or quadratic utility functions; however, as recently discussed and clarified by Markowitz (2014) himself, normal return distributions or quadratic utility functions are sufficient but not necessary conditions for the mean-variance

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analysis. A careful choice on a mean-variance efficient frontier can approximately maximize the expected utility for a variety of utility functions. Nevertheless, if the distribution of returns is not normal and if the utility function can take other forms than the quadratic one, another type of analysis may be more suitable.

Several studies, such as Beedles (1979) and Campbell and Hentschel (1992), showed that portfolio returns are not, in general, normally distributed. On the other hand, Arditti (1975), Kraus and Litzenberger (1976) and Harvey and Siddique (2000) found that investors have preferences for skewness, suggesting that usually utility functions are not quadratic. In fact, recent studies suggest that there may be gains from taking into account higher moments in portfolio selection (see, for example,Harvey et al., 2010).

With non-normal return distributions, the use of a downside risk measure is more suitable than the traditional use of the variance (Nawrocki, 1999). Roy (1952) was the first to use a downside risk measure in portfolio selection, in the form of a "safety first" rule, that measures the probability of outcomes falling below a predetermined target return (Sing and Ong, 2000). Markowitz (1959) recognized the importance of the Roy's work, arguing that there are more plausible measures of risk than the variance, and proposed the use of a below-mean semivariance or a target return semivariance. These metrics belong to a more general family of downside risk measures known as lower partial moments (Bawa, 1975; Fishburn, 1977). Quirk and Saposnik (1962) confirmed the theoretical superiority of the semivariance versus the variance, while Ang and Chua (1979) showed the superiority of the target return semivariance relative to the below-mean semivariance. Ang and Chua (1979) also showed that the target return semivariance is the only measure of risk in accordance with the Von Neumann and Morgenstern (1953) axioms of choice. However, the semivariance is seldom used in portfolio selection problems due to the endogeneity of the cosemivariance.

On the basis of assumptions that the semivariance is a more plausible measure of risk than the variance and that investors' preferences and skewness are positively related, we suggest a direct analysis of the efficient tradeoff between skewness and semivariance by means of a biobjective optimization problem. This methodology is flexible, in the sense that the investor is free to choose the target return, required for the semivariance computation. Another strength point is that skewness is interpreted as a third moment tensor and the endogeneity issue of the cosemivariance matrix is addressed explicitly. Owing to the endogeneity of the cosemivariance matrix, we use a derivative-free algorithm (based on direct multisearch) to obtain the solution of the biobjective optimization problem. Direct multisearch is a class of methods used in multiobjective optimization problems that does not use derivatives and does not aggregate or scalarize any of the objective function components. It essentially generalizes all direct-search methods of directional type from single to multiobjective optimization. For a complete description of direct multisearch see the algorithmic framework in Custódio et al. (2011).

The contribution of this article is therefore twofold. First, we suggest a skewness/ semivariance biojective model that allows the investor to directly analyze the efficient tradeoff between skewness and semivariance (regardless of the target return used in the semivariance calculation). Second, through a derivative-free algorithm we overcome the endogeneity problem of the cosemivariance matrix.

The empirical work is conducted on four data sets collected from the Fama/French data library. First, the Pareto frontiers on the skewness/semiariance space are computed using three different target returns, corresponding to the returns of the maximum Sharpe ratio, the minimum variance and the $1/N$ portfolios, respectively. Then, an extensive out-of-sample performance analysis is implemented on three efficient skewness/semivariance portfolios from the in-sample Pareto frontiers: the portfolios with the maximum skewness per semivariance ratio, with the maximum Sharpe ratio and with the maximum Sortino ratio. The out-of-sample performance is measured in terms of skewness per semivariance ratio, Sharpe ratio and Sortino ratio. We conclude that the efficient portfolios exhibit a competitive out-of-sample performance compared with the three benchmark portfolios. One interesting result is that in the four data sets, at least one of the three chosen efficient skewness/semivariance portfolios consistently outperform the $1/N$ portfolio in terms of Sharpe ratio, which is known to be difficult to achieve (DeMiguel et al., 2009).

The rest of the article is organized as follows. In Section 2 we introduce some concepts and notions used thereafter. Section 3 describes the maximum Sharpe ratio portfolio, the minimum variance portfolio and the $1/N$ portfolio. In Section 4 it is shown that the expected utility of a risk averse investor is an increasing function of the skewness and a decreasing function of the semivariance. Section 5 presents the suggested model. Section 6 shows the empirical results and, finally, Section 7 summarizes the main findings and discusses future research.

2 Notation

Following Brito and Vicente (2014), suppose that an investor has a certain wealth to invest in a set of N securities. The return of each security i is described by a random variable R_i. On the basis of historical data, a return portfolio matrix, R, can be defined as

$$R = [\ R_1 \mid R_2 \mid \ \dots \ \mid R_N \ ] = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,N} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ r_{T-1,1} & r_{T-1,2} & \cdots & r_{T-1,N} \end{bmatrix},$$

where T corresponds to the total number of price observations. Let w_i, $i = 1, \dots, N$, represent the weights of the total investment allocated to each security. Thus a portfolio can be defined by an $N \times 1$ vector w of weights, which must satisfy the constraint

$$\sum_{i=1}^{N} w_i \ = \ e^\top w \ = \ 1,$$

where e is the $N \times 1$ vector of ones. Lower bounds on the variables, of the form $w_i \geq 0$, $i = 1, \dots, n$, can be also considered if short selling is undesirable. The expected return of the security i is given by $\mu_i = E(R_i)$, for $i = 1, \dots, N$. Let $m = [\ \mu_1 \ \mu_2 \ \dots \ \mu_N \ ]^\top$ be the vector of the expected returns. Thus, the portfolio expected return can be written as

$$\mu(w) \ = \ E(R(w)) \ = \ E\left(\sum_{i=1}^{N} w_i R_i\right) \ = \ \sum_{i=1}^{N} w_i \mu_i \ = \ m^\top w.$$

The portfolio variance, is given by

$$\nu(w) = E\left[(R(w) - \mu(w))^2\right] = w^\top M_2 w,$$

where M_2 is the covariance matrix in which each entry $c_{ij} = \text{COV}(R_i, R_j)$ is computed as

$$c_{ij} = \frac{1}{T-1} \sum_{i,j=1}^{N} \sum_{t=1}^{T-1} (r_{t,i} - \mu_i)(r_{t,j} - \mu_j).$$

M_2 is symmetric and positive semi-definite.

3 Benchmark portfolios

3.1 The maximum Sharpe ratio portfolio

The Markowitz's model (Markowitz, 1952; 1959) is based on the formulation of a mean-variance optimization problem. The solution of that problem is the portfolio of minimum variance for an expected return not below a certain target value r. Therefore, the aim is to minimize the risk for a given level of return. This problem can be defined as:

$$\begin{aligned} \min_{w \in \mathbb{R}^N} \quad & \nu(w) \\ \text{subject to} \quad & \mu(w) \geq r, \\ & e^\top w = 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{1}$$

The classical Markowitz mean-variance model can be reformulated as a biobjective problem which consists of simultaneously minimizing the portfolio risk (variance) and maximizing the portfolio profit (expected return)

$$\begin{aligned} \min_{w \in \mathbb{R}^N} \quad & \nu(w) = w^\top M_2 w \\ \max_{w \in \mathbb{R}^N} \quad & \mu(w) = m^\top w \\ \text{subject to} \quad & e^\top w = 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N. \end{aligned} \tag{2}$$

It is easy to prove that a solution of problem (1) is nondominated, efficient or Pareto optimal for problem (2). Efficient portfolios are thus the ones which have the minimum variance for a given expected return, or, alternatively, those that have the maximum expected return up to a certain variance. The efficient frontier (or Pareto frontier) is typically represented as a 2-dimensional curve in the mean-standard deviation space.

In the efficient frontier the portfolio that maximizes the risk premium per unit of risk deserves a special attention; that is, the portfolio that maximizes the ratio between the reward and the variability of the investment (reward-to-variability ratio). This portfolio is obtained by maximizing the so-called Sharpe ratio

$$\begin{aligned} \max_{w \in \mathbb{R}^N} \quad & \frac{\mu(w) - r_f}{\sqrt{\nu(w)}} \\ \text{subject to} \quad & e^\top w = 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N, \end{aligned} \tag{3}$$

where r_f is the risk-free rate.

3.2 The minimum variance portfolio

The minimum variance portfolio (*mv* portfolio) corresponds to the solution of the following convex QP

$$\begin{array}{rl} \min\limits_{w \in \mathbb{R}^N} & \nu(w) \\ \text{subject to} & e^\top w \;=\; 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N. \end{array} \tag{4}$$

Regarding problem (4), it has been shown in the literature that the estimates of the mean returns are so noisy that it is completely preferable to ignore these estimates and use only the covariance matrix (Jagannathan and Ma, 2003). It is also known that not allowing for short selling on the minimum-variance portfolio has a regularizing effect (Jagannathan and Ma, 2003).

3.3 The $1/N$ portfolio

The $1/N$ rule or naive strategy ($1/N$ portfolio) is the one in which the available investor's wealth is divided equally among the available securities

$$w_i \;=\; \frac{1}{N}, \quad i = 1, \ldots, N. \tag{5}$$

Although a number of theoretical models have been developed in recent years, many investors pursuing diversification revert to the use of the naive strategy to allocate their wealth (Benartzi and Thaler, 2001). DeMiguel et al. (2009) evaluated 14 models across 7 empirical data sets and showed that none is consistently better than the naive strategy. A possible explanation for this evidence lies in the fact that the naive strategy does not involve estimation and promotes "optimal" diversification. The naive strategy is therefore an excellent benchmarking strategy.

4 Investor expected utility

Many authors have argued in favor of using the skewness and the semivariance, based on expected utility reasoning (for example, Kraus and Litzenberger, 1976; Koekebakker and Zakamouline, 2008). When the utility-based framework is used and at the same time moment-based objective functions are considered, it is important to bear in mind some aspects often neglected in the literature. Moment-based objective functions do not allow the study of the dependence of the portfolio weights on the initial wealth. In turn, as observed in Bamberg and Dorfleitner (2013), the utility-based framework allows to investigate the dependence of the portfolio structure on different levels of investment. Thus, in general, moment-based objective functions do not necessarily fit with expected utility, as utility depends on the initial wealth while moments of the return distribution do not (see Bamberg and Dorfleitner, 2013, for further details). Nevertheless, we highlight that a utility-based framework is fully compatible with a framework based on the moments of returns, if the investor considers a CRRA utility (Bamberg and Dorfleitner, 2013).

4.1 Expected utility based on skewness

Let $u(\cdot)$ be the utility function of a typical investor. Considering the Taylor expansion of $u(R(w))$ around $\mu(w)$, then

$$\begin{aligned} u(R(w)) &= u(\mu(w)) + u'(\mu(w))(R(w) - \mu(w)) \\ &+ \frac{1}{2!}u''(\mu(w))(R(w) - \mu(w))^2 + \frac{1}{3!}u'''(\mu(w))(R(w) - \mu(w))^3 \\ &+ \sum_{j=4}^{\infty} \frac{1}{j!}u^j(\mu(w))(R(w) - \mu(w))^j. \end{aligned}$$

Applying the mathematical expectation operator, E,

$$E\left[u(R(w))\right] = u(\mu(w)) + \frac{1}{2!}u''(\mu(w))\nu(w) + \frac{1}{3!}u'''(\mu(w))\kappa(w) + \sum_{j=4}^{\infty} \frac{1}{j!}u^j(\mu(w))m^j(w),$$

where $\nu(w) = E\left[(R(w) - \mu(w))^2\right]$, $\kappa(w) = E\left[(R(w) - \mu(w))^3\right]$, $m^j(w) = E\left[(R(w) - \mu(w))^j\right]$ are the second, third and j-th (with $j = 4, 5, \ldots$) central moments of $R(w)$, respectively.

Considering the third-order approximation of the expected utility, as in Joro and Na (2006), then

$$E\left[u(R(w))\right] \approx u(\mu(w)) + \frac{1}{2}u''(\mu(w))\nu(w) + \frac{1}{6}u'''(\mu(w))\kappa(w).$$

Since $u'''(\mu(w)) >= 0$ (see Arditti, 1975; Kraus and Litzenberger, 1976), the expected utility, $E\left[u(R(w))\right]$, of a risk averse investor is an increasing function of skewness, $\kappa(w)$, which is consistent with the desirable properties for an investor's utility function ($u'(\cdot) > 0$, $u''(\cdot) < 0$ and $u'''(\cdot) > 0$), as suggested by Arrow (1971).

4.2 Expected utility based on semivariance

The semivariance can be defined as

$$\Sigma_B(w) = E\left\{[\min(R(w) - B, 0)]^2\right\},$$

where B represents a target return and should be independent of the probability distribution being ranked (Ang and Chua, 1979). If, instead of the variance, one considers the semivariance, then the utility function should have a kink at the reference point B. On the basis of Koekebakker and Zakamouline (2008), the utility function has the form

$$u(R(w)) = \begin{cases} u_+(R(w)) & \text{if } R(w) \geq B, \\ u_-(R(w)) & \text{if } R(w) < B, \end{cases}$$

where $u_+(\cdot)$ is the utility function for gains and $u_-(\cdot)$ is the utility function for losses. Considering the second-order Taylor expansion approximation of $u(R(w))$ around B, then

$$u(R(w)) \approx \begin{cases} u_+(B) + u'_+(B)(R(w)-B) + \frac{1}{2}u''_+(B)(R(w)-B)^2 & \text{if } R(w) \geq B, \\ \\ u_-(B) + u'_-(B)(R(w)-B) + \frac{1}{2}u''_-(B)(R(w)-B)^2 & \text{if } R(w) < B. \end{cases}$$

Applying the mathematical expectation operator, E,

$$E[u(R(w))] \approx \begin{cases} u_+(B) + u'_+(B)E[(R(w)-B)] + \frac{1}{2}u''_+(B)E[(R(w)-B)^2] & \text{if } R(w) \geq B, \\ \\ u_-(B) + u'_-(B)E[(R(w)-B)] + \frac{1}{2}u''_-(B)E[(R(w)-B)^2] & \text{if } R(w) < B. \end{cases}$$

If the investor is risk averse in the domain of losses (the utility function for losses is concave, $u''_-(\cdot) < 0$), then the expected utility, $E[u(R(w))]$, is a decreasing function of semivariance, $\Sigma_B(w)$. Again, this is consistent with the desirable properties for an investor's utility function ($u'(\cdot) > 0$, $u''(\cdot) < 0$ and $u'''(\cdot) > 0$).

5 The skewness/semivariance biobjective model

To overcome the limitation of the mean-variance models, some researchers used downside risk measures, which only gauge the negative deviations from a reference return level. One famous downside risk measure was the "safety first" criterion (Roy, 1952). Other downside risk measures were proposed, for example, in Bawa (1975); Fishburn (1977); Harlow and Rao (1989); Nawrocki (1999). See Nawrocki (1999) for a survey on downside risk measures.

Markowitz (1959) favored one of the best-known downside risk measures: the semivariance of returns. The semivariance can be handled by considering an asymmetric cosemivariance matrix (Hogan and Warren, 1974) or considering a symmetric and exogenous cosemivariance matrix (Estrada, 2008). Another way of handling the semivariance is outside the stochastic environment, considering the fuzzy set environment as in Huang (2008).

Following Markowitz (1959), the endogenous cosemivariance matrix is the approach adopted here. Therefore the exact estimation of the semivariance of a portfolio is obtained as

$$\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cs_{ij} = w^\top M_2^-(w) w,$$

where $M_2^-(w)$ is the cosemivariance matrix in which each entry cs_{ij} is given by

$$cs_{ij} = \frac{1}{T-1}\sum_{k\in U}(r_{k,i} - B)(r_{k,j} - B),$$

with

$$U = \{t \mid r_{t,p} < B\} \subseteq \{1, \ldots, T-1\},$$

where $r_{t,p}$ is the portfolio return at time t. In this case the cosemivariance matrix is endogenous in the sense that a change in the portfolio's weights affects the periods when the portfolio underperforms the benchmark, which in turn affects the elements of the cosemivariance matrix (Estrada, 2008). Note that the target return B is a parameter that can be freely chosen by the investor, according to his/her own preferences.

There is an intuitive explanation of why the skewness is important for the investor. Clearly, the investor has a preference for positive skewness in order to have higher probability for extreme profit values and limited loss. Alderfer and Bierman (1970) showed empirically that investors prefer positive skewness, even if this positive skewness is associated with a lower expected return. Arditti (1975); Kraus and Litzenberger (1976); Harvey and Siddique (2000), have shown theoretically that investors should prefer positive skewness. Moreno and Rodríguez (2009) showed that the coskewness is taken into account by funds managers, representing an important factor in the selection of securities. However, skewness is often neglected in the performance evaluation literature, possibly due to computational difficulties (Joro and Na, 2006).

Skewness can be computed as a third moment tensor and can be visualized as a $N \times N \times N$ cube in the three-dimensional space. It is possible to transform the skewness tensor into a $N \times N^2$ matrix by slicing each $(N \times N)$ layer and pasting them, in the same order, sideways (Athayde and Flôres, 2004). Following this idea the skewness of a portfolio can be computed as

$$\kappa(w) = E\left[(R(w) - \mu(w))^3\right] = w^\top M_3(w \otimes w),$$

where $\otimes$ denotes the Kronecker product and M_3 is the coskewness matrix. The coskewness matrix of dimension $N \times N^2$ can be represented by N matrixes A_{ijk}, each with dimension $N \times N$, such that

$$M_3 = [\ A_{1jk}\ A_{2jk}\ \ldots\ A_{Njk}\ ],$$

where $j, k = 1, \ldots, N$. The individual elements of the coskewness matrix can by obtained as

$$a_{ijk} = \frac{1}{T-1} \sum_{i,j,k=1}^{N} \sum_{t=1}^{T-1} (r_{t,i} - \mu_i)(r_{t,j} - \mu_j)(r_{t,k} - \mu_k).$$

We propose the simultaneous consideration of the two investor's objectives

- maximizing the skewness $\kappa(w) = w^\top M_3(w \otimes w)$,
- minimizing the semivariance $\Sigma_B(w) = w^\top M_2^-(w)w$,

over the set of feasible portfolios. The skewness/semivariance biobjective optimization model can be written as

$$\begin{array}{rl} \max\limits_{w \in \mathbb{R}^N} & \kappa(w) = w^\top M_3(w \otimes w) \\ \min\limits_{w \in \mathbb{R}^N} & \Sigma_B(w) = w^\top M_2^-(w)w \\ \text{subject to} & e^\top w = 1, \\ & w_i \geq 0, \quad i = 1, \ldots, N. \end{array} \tag{6}$$

By solving (6), we identify a skewness/semivariance Pareto frontier. A portfolio in this frontier is such that there exists no other feasible one which simultaneously presents a higher skewness and a lower semivariance. Given such an efficient frontier and a semivariance level, an investor may directly find the answers to the questions of what is the optimal (higher) skewness level that can be chosen and what are the portfolios leading to such a skewness level. The biojective problem (6) has two objective functions, a linear constraint and N inequality constraints. The first objective, $\kappa(w) = w^\top M_3(w \otimes w)$, is nonlinear but smooth. However, the second objective, $\Sigma_B(w) = w^\top M_2^-(w)w$, is nonlinear and nonsmooth due to the endogeneity problem on the cosemivariance matrix $M_2^-(w)$. We have thus decided to solve the problem through a derivative-free solver, based on direct multisearch (see Custódio et al., 2011). This derivative-free solver was previously and for the first time used in the portfolio selection framework for solving a cardinality constrained problem (Brito and Vicente, 2014).

The skewness/semivariance biobjective model can be extensively explored by the investor. It is clear that the gains of this approach in relation to the mean-variance one is, in great extent, a function of the degree of asymmetry in the distribution of the portfolio returns. Given this asymmetric property, a careful choice on a skewness/semivariance efficient frontier can approximately maximize the expected utility for a variety of utility functions, as we demonstrate in Section 4. Moreover, the proposed model is quite flexible since it makes use of a general definition of the semivariance, by not restricting the target return (in the related literature, the target return is often set to the mean of the distribution). In fact, giving its specific preferences, the investor can choose this parameter freely.

6 Empirical analysis

The empirical analysis is conducted on four data sets collected from the Fama/French data library, which is publicly available at: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. The data sets are constructed according to different criteria, and each one is composed by portfolios, rebalanced annually at the end of June.

The SBM6 data set corresponds to six portfolios based on size and book to market ratio. These portfolios are the intersections of 2 portfolios based on size (market equity, ME) and 3 portfolios based on the ratio of book equity to market equity (BE/ME).

The FF10 data set corresponds to 10 industry portfolios.

The SOP25 data set corresponds to 25 portfolios based on size and operating profitability. These portfolios, are the intersections of 5 portfolios based on size (market equity, ME) and 5 portfolios based on operating profitability (OP).

The BMOP25 data set corresponds to 25 portfolios formed on book-to-market and operating profitability. These portfolios, are the intersections of 5 portfolios formed on the ratio of book equity to market equity (BE/ ME) and 5 portfolios formed on profitability (OP).

The overall sample for all four data sets is formed by monthly data from 07/1964 to 06/ 2014 (600 months). Table 1 reports some descriptive statistics for each data set. The monthly continuous returns showed, on average, negative skewness and a kurtosis well above that of normal distribution. The application of the Jarque-Bera test for normality to all the portfolios of each data set (SBM6, FF10, SOP25 and BMOP25) showed that the null hypothesis of normality was rejected with p-values lower than 1 per cent.

Table 1: Descriptive statistics

Data set	Mean	Variance	Skewness	Kurtosis
SBM6	0.0105	0.0035	-0.5401	6.3148
FF10	0.0103	0.0043	-0.4678	6.5228
SOP25	0.0098	0.0034	-0.6280	6.3244
BMOP25	0.0114	0.0039	-0.5324	6.6087

Note: This table reports some descriptive statistics for the four data sets collected from the Fama/French data library. These statistics are averaged cross-sectionally.

6.1 In-sample analysis

We applied direct multisearch (see Custódio et al., 2011) to determine the Pareto frontier of the skewness/semivariance biobjective optimization Problem (6) for three different values of B: B_{ms}, the return of the maximum Sharpe ratio portfolio[1] (solution of Problem (3)); B_{mv}, the return of the minimum variance portfolio (solution of Problem (4)); and $B_{1/N}$, the return of the $1/N$ portfolio (given by Equation (5)).

Let $P_{t,i}$ be the price at time t of security i. The discrete return rate, at time t of security i, is given by

$$R_{t,i} = \frac{P_{t,i} - P_{t-1,i}}{P_{t-1,i}} = \frac{P_{t,i}}{P_{t-1,i}} - 1.$$

While the corresponding continuous return rate is given by

$$r_{t,i} = \ln\left(\frac{P_{t,i}}{P_{t-1,i}}\right)$$

Thus, the equivalence between these rates imply is given by

$$r_{t,i} = \ln(1 + R_{t,i}) \tag{7}$$

The monthly returns available in the Fama/French data library are discrete returns. Since discrete returns allow for an accurate cross-sectional aggregation while continuous returns[2] allow for an accurate time aggregation, the initial returns are converted into continuous returns using Equation (7) in order to obtain better parameter estimates.

Figures 1-3 contain the plots of the Pareto frontiers, computed using the overall sample period, for the SBM6 data set. Figures 1-3 correspond to the cases in which the target returns are B_{ms}, B_{mv} and $B_{1/N}$, respectively. We obtained all the Pareto frontiers for the remaining data sets in a similar way, but we do not report all the plots for reasons of space. For each

[1] We considered as a risk-free asset the 90-day Treasury-Bills US. Such data is public and made available by the Federal Reserve, at the site `www.federalreserve.gov`.

[2] We have used the common approach of estimating the model parameters that depend partly or totally on time-series aggregation (such as the coskewness and cosemivariance) through continuous returns (see, for example, Alexander, 2008, p. 97). The proper procedure consists by using the continuous returns (invariants) for the estimation and then mapping from continuous returns to discrete returns (see Meucci, 2001, for a detailed description). Nevertheless, for a short horizon (as in our case) the differences that arise from mispecifying the returns are negligible (Meucci, 2001, p. 2).

data set the efficient skewness/ semivariance frontier differs according to the choice of the target return B, emphasizing the importance of this parameter in the proposed approach.

As it is clear from the visualization of the plots, direct multisearch was able to determine the Pareto frontiers for the biobjective skewness/semivariance optimization problems (solutions of Problem 6) for all the instances considered, and that these frontiers are smooth and "well-behaved". Thus, this methodology offers a direct way for analyzing the efficient tradeoff between skewness and semivariance, regardless of the choice of the target return. We point out that because of the convergence properties of direct multisearch, the investor can determine the Pareto frontier for the biojective skewness/semivariance optimization problem for any other data set that he or she may choose.

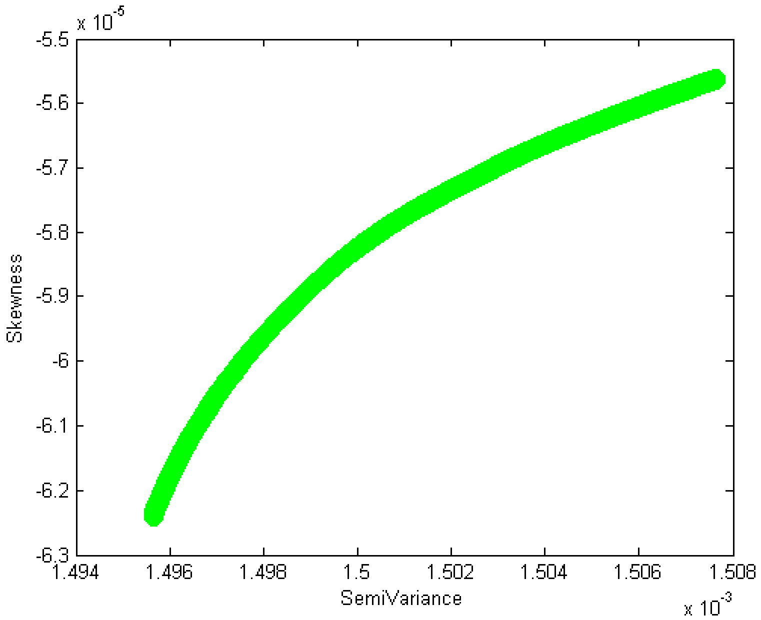

Figure 1: SBM6 skewness/semivariance efficient frontier, with B_{ms} as the target return for semivariance computation.

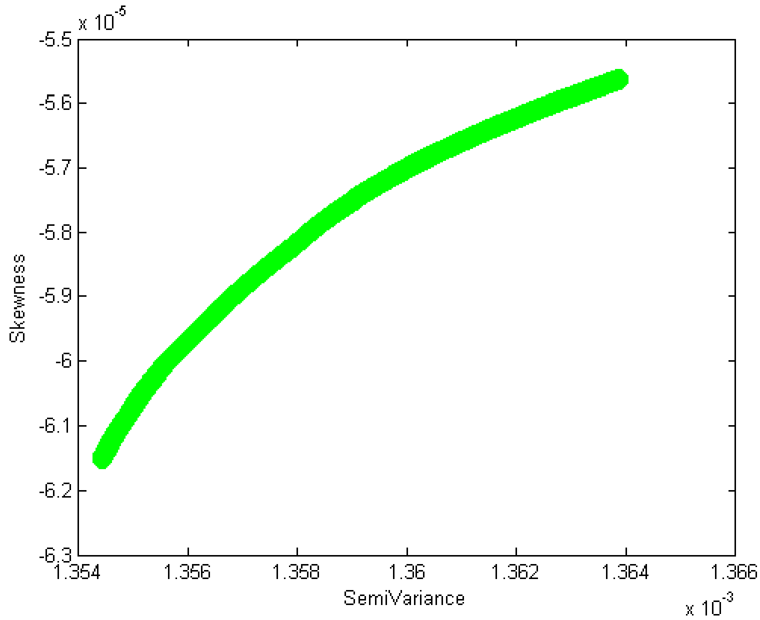

Figure 2: SBM6 skewness/semivariance efficient frontier, with B_{mv} as the target return for semivariance computation.

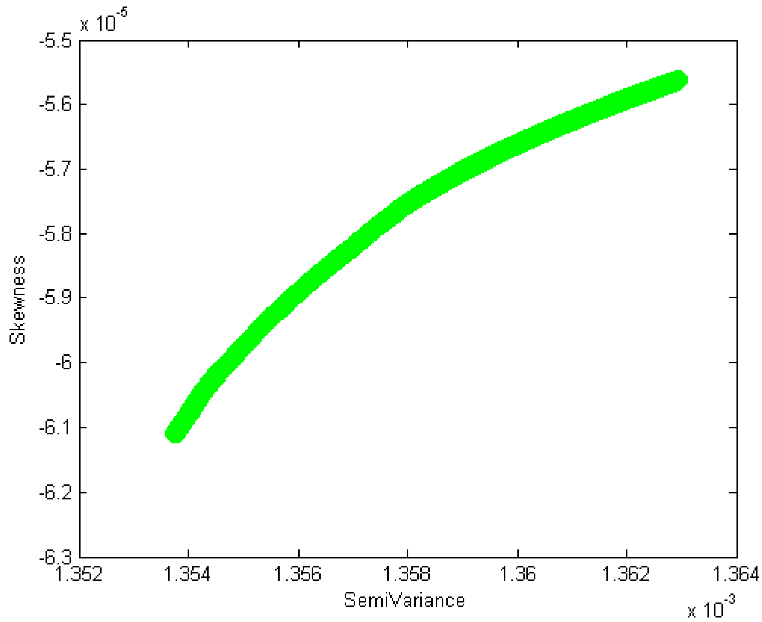

Figure 3: SBM6 skewness/semivariance efficient frontier, with $B_{1/N}$ as the target return for semivariance computation.

6.2 Out-of-sample analysis

The validation of a new methodology for portfolio selection must be based on out-of-sample performance analysis. This subsection deals with an extensive out-of-sample analysis of the efficient skewness/semivariance portfolios, constructed according to the proposed model in Section 5, and compared with each of the benchmark portfolios presented in Section 3.

The out-of-sample analysis relies on a rolling-sample approach. We considered an estimation window of 120 months, with the initial estimation period going from 07/1964 to 06/1974. The evaluation period comprised 480 months, from 07/1974 to 06/2014. For each estimation window, the benchmark portfolios (the ms portfolio, the mv portfolio and the $1/N$ portfolio) were computed first. Then, the Pareto frontiers of the skewness/semivariance biobjective optimization Problem 6 were determined, considering B_{ms}, B_{mv} and $B_{1/N}$ as target returns for the semivariance computation. Finally, three efficient skewness/ semivariance portfolios were selected in each of the in-sample Pareto frontiers. The first one, w_{SSR}, was the portfolio that maximizes the skewness per semivariance ratio (SSR):

$$SSR = \frac{\kappa(w_{SSR})}{\Sigma_B(w_{SSR})}.$$

The second one, w_{SR}, was the portfolio that maximizes the Sharpe ratio (SR)

$$SR = \frac{\mu(w_{SR}) - r_f}{\sqrt{\nu(w_{SR})}}.$$

The third one, w_{SOR}, was the portfolio that maximizes the Sortino ratio (SOR)

$$SOR = \frac{\mu(w_{SOR}) - B}{\sqrt{\Sigma_B(w_{SOR})}}.$$

The numerator of these ratios may be negative. Thus, in order to have correct rankings, the denominators are modified as proposed by Israelsen (2005). Finally, each portfolio was held fixed and its returns (discrete returns) were observed over the next month. The estimation window was then moved forward 1 month, and the returns were calculated for the next month of the evaluation period. The process was thus repeated until the end of the evaluation period was reached. Notice that, since continuous returns allow for an accurate time aggregation of returns, the monthly out-of-sample discrete returns are converted into continuous returns (according to Equation (7)).

6.2.1 Performance measured by a skewness per semivariance ratio

We computed an out-of-sample skewness per semivariance ratio, defined as the sample skewness, $\hat{\kappa}_B$, divided by the sample semivariance, $\hat{\Sigma}_B$:

$$\widehat{\text{SSR}} = \frac{\widehat{\kappa}_B}{\widehat{\Sigma}_B}.$$

Then, we computed the bootstrap p-values of the difference between the skewness per semivariance ratio of each efficient skewness/semivariance portfolio and those of the benchmarks. Since none of the differences were statistically significant (all the bootstrap p-values were higher than 5 per cent), we decided not to report these results here. Because the computed skewness per semivariance ratios are negative (we are in the presence of negative skewness), in order to achieve a correct ranking of the portfolios, the ratios were refined according to Israelsen (2005). Thus we computed the refined skewness per semivariance ratio as:

$$\widehat{\mathrm{SSR}}_{\mathrm{ref}} = \frac{\widehat{\kappa}_B}{\widehat{\Sigma}_B^{\widehat{\kappa}_B/\,\mathrm{abs}(\widehat{\kappa}_B)}},$$

where abs(·) is the absolute value function.

Table 2 reports the refined skewness per semivariance ratios, when we choose as a target return the $1/N$ portfolio return ($B_{1/N}$). We can see that the efficient skewness/ semivariance portfolios w_{SR} and w_{SOR} have a higher refined skewness per semivariance ratio than two of the three benchmarks portfolios (the ms and the $1/N$), for all the data sets. For the data sets (SOP25 and BMOP25) all the efficient skewness/ semivariance portfolios (w_{SSR}, w_{SR} and w_{SOR}) have a higher refined skewness per semivariance ratio than two (the ms and the $1/N$ portfolios) of the three benchmarks portfolios.

We also computed the refined skewness per semivariance ratios for the cases in which we choose the maximum Sharpe ratio portfolio return (B_{ms}) and the minimum variance portfolio return (B_{mv}) as the target return. The results are similar and therefore are not reported here. All these results suggests the robustness of the efficiency provided by the skewness/semivariance model.

Table 2: Portfolio refined skewness per semivariance ratios for the target return $B_{1/N}$

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	-1.6162E-07	4	-6.5469E-08	4	-1.8097E-07	5	-1.8555E-07	4
mv portfolio	-1.1985E-07	1	-9.2942E-09	1	-7.1022E-08	1	-1.2056E-07	1
$1/N$ portfolio	-2.2200E-07	6	-2.7991E-07	6	-2.6171E-07	6	-2.7021E-07	6
Efficient skewness/semivariance portfolios								
w_{SSR}	-1.9538E-07	5	-9.6091E-08	5	-1.4698E-07	4	-2.0206E-07	5
w_{SR}	-1.3590E-07	2	-2.9696E-08	2	-1.0937E-07	2	-1.5525E-07	2
w_{SOR}	-1.5376E-07	3	-6.0198E-08	3	-1.2314E-07	3	-1.7617E-07	3

Note: This table reports, for each of the data sets, the monthly benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The target return used in the computation of the semivariance is the $1/N$ portfolio return ($B_{1/N}$). This table also reports the monthly refined skewness per semivariance ratios for the efficient skewness/semivariance portfolios referred in Section 6: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). The correct rank of each portfolio according to the refined skewness per semivariance ratios is also reported.

6.2.2 Performance measured by the Sharpe ratio

Given the time series of monthly out-of-sample returns for each portfolio, we computed the out-of-sample Sharpe ratio, defined as the sample mean of excess returns (over the risk-free asset), $\hat{m}$, divided by its sample standard deviation, $\hat{\sigma}$:

$$\widehat{\text{SR}} = \frac{\widehat{m}}{\widehat{\sigma}}.$$

The results are presented in Table 3. This table also reports the bootstrap p-values (Ledoit and Wolf, 2008) for the statistical significance of the difference between the Sharpe ratios of the benchmarks and the efficient skewness/semivariance portfolios.

For the SBM6 data set, the efficient skewness/semivariance portfolio w_{SOR} has a higher Sharpe ratio than all the three benchmark portfolios (the ms portfolio, the mv portfolio and the $1/N$ portfolio), independently of the target return used in the computation of the semivariance (B_{ms}, B_{mv} or $B_{1/N}$). For all the target returns (B_{ms}, B_{mv} and $B_{1/N}$), the difference between the Sharpe ratios of the efficient skewness/semivariance portfolio w_{SOR} and the $1/N$ portfolio is statistically significant.

In the case of the FF10 data set, for all the target returns (B_{ms}, B_{mv} and $B_{1/N}$), the efficient skewness/ semivariance portfolios w_{SR} and w_{SOR} have a higher Sharpe ratio than two of the three benchmark portfolios (the ms portfolio and the $1/N$ portfolio). The difference between the Sharpe ratio of the efficient skewness/semivariance portfolio w_{SR} and the benchmark $1/N$ portfolio is always statistically significant.

We do not observe statistically significant differences between the Sharpe ratios of the efficient skewness/semivariance portfolios and the benchmark portfolios for the SOP25 data set. However, we can see that in the case in which we use a target return B_{ms} to compute the semivariance, the efficient skewness/semivariance portfolio w_{SOR} has a higher Sharpe ratio than two of the three benchmark portfolios (the ms portfolio and the $1/N$ portfolio).

Table 3: Out-of-sample Sharpe ratios

Benchmark	Strategy	SBM6	FF10	SOP25	BMOP25
	ms portfolio	0.2572	0.2736	0.2113	0.2993
	mv portfolio	0.2480	0.3188	0.2130	0.2935
	$1/N$ portfolio	0.2235	0.2115	0.1976	0.2357
	Efficient skewness/semivariance portfolios				
B_{ms}	w_{SSR}	0.2541	0.2689	0.2041	0.2624
		$(0.89)^1(0.74)^2(0.15)^3$	$(0.92)^1(0.11)^2(0.16)^3$	$(0.75)^1(0.63)^2(0.74)^3$	$(0.16)^1(0.25)^2(0.26)^3$
	w_{SR}	0.2472	0.3078	0.2071	0.3151
		$(0.52)^1(0.95)^2(0.19)^3$	$(0.35)^1(0.59)^2(0.02)^3$	$(0.84)^1(0.70)^2(0.61)^3$	$(0.23)^1(0.14)^2(0.00)^3$
	w_{SOR}	0.2640	0.2829	0.2125	0.3019
		$(0.73)^1(0.38)^2(0.03)^3$	$(0.81)^1(0.21)^2(0.07)^3$	$(0.95)^1(0.98)^2(0.44)^3$	$(0.89)^1(0.69)^2(0.00)^3$
B_{mv}	w_{SSR}	0.2539	0.2676	0.2018	0.2596
		$(0.88)^1(0.74)^2(0.14)^3$	$(0.89)^1(0.11)^2(0.17)^3$	$(0.68)^1(0.54)^2(0.84)^2$	$(0.13)^1(0.20)^2(0.31)^3$
	w_{SR}	0.2485	0.3057	0.2077	0.3109
		$(0.58)^1(0.96)^2(0.17)^3$	$(0.38)^1(0.51)^2(0.02)^3$	$(0.86)^1(0.74)^2(0.57)^3$	$(0.37)^1(0.22)^2(0.00)^3$
	w_{SOR}	0.2629	0.2842	0.2097	0.3051
		$(0.77)^1(0.38)^2(0.03)^3$	$(0.80)^1(0.22)^2(0.06)^3$	$(0.94)^1(0.86)^2(0.51)^3$	$(0.71)^1(0.55)^2(0.00)^3$
$B_{1/N}$	w_{SSR}	0.2540	0.2658	0.2004	0.2566
		$(0.89)^1(0.75)^2(0.15)^3$	$(0.87)^1(0.11)^2(0.19)^3$	$(0.63)^1(0.51)^2(0.89)^3$	$(0.10)^1(0.17)^2(0.37)^3$
	w_{SR}	0.2488	0.3050	0.2057	0.3127
		$(0.61)^1(0.94)^2(0.17)^3$	$(0.39)^1(0.48)^2(0.02)^3$	$(0.78)^1(0.66)^2(0.65)^3$	$(0.28)^1(0.19)^2(0.00)^3$
	w_{SOR}	0.2634	0.2806	0.2099	0.3073
		$(0.76)^1(0.39)^2(0.03)^3$	$(0.87)^1(0.19)^2(0.09)^3$	$(0.95)^1(0.86)^2(0.51)^3$	$(0.61)^1(0.50)^2(0.00)^3$

Note: This table reports, for each of the data sets, the monthly Sharpe ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. This table also reports, for three different scenarios (three different values for the target return of the semivariance), the monthly Sharpe ratios for the efficient skewness/semivariance portfolios referred in Section 6: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). The computation of the semivariance it carried out using three different values for the target return: the maximum Sharpe ratio portfolio return (B_{ms}), the minimum variance portfolio return (B_{mv}) and the $1/N$ portfolio return ($B_{1/N}$). In parenthesis are the bootstrap $p-values$ of the difference between the Sharpe ratio of each efficient skewness/semivariance portfolio from those of the benchmarks: from the ms portfolio, from the mv portfolio and from the $1/N$ portfolio; respectively in the first, second and third parenthesis. These p-values are computed according the Ledoit and Wolf (2008) methodology.

Finally, for the BMOP25 data set, the efficient skewness/semivariance portfolios w_{SR} and w_{SOR} have a higher Sharpe ratio than all the benchmark portfolios, independently of the target return used in the computation of the semivariance (B_{ms}, B_{mv} or $B_{1/N}$). The differences between the Sharpe ratios of these two efficient skewness/semivariance portfolios (w_{SR} and w_{SOR}) and the benchmark $1/N$ portfolio are always statistically significant.

These results show that at least one of the three efficient skewness/semivariance portfolios consistently outperform the $1/N$ benchmark portfolio (something that the literature has shown to be hard to achieve). In fact, this analysis suggests that, in order to achieve a higher Sharpe ratio, one should choose carefully the target return for the computation of the semivariance.

6.2.3 Performance measured by the Sortino ratio

We computed the out-of-sample Sortino ratio, defined as the sample mean of out-of-sample excess returns (over the target return B), $\hat{m}_B$, divided by their sample standard semideviation, $\hat{\sigma}_B$:

$$\widehat{\text{SOR}} = \frac{\widehat{m}_B}{\widehat{\sigma}_B}.$$

Then, we computed the bootstrap p-values of the difference between the Sortino ratio of each efficient skewness/semivariance portfolio and those of the benchmarks. Since none of the differences were statistically significant (all the bootstrap p-values were higher than 5 per cent), we decided not to report these results here. The out-of-sample period showed negative excess returns, thus, in order to achieve a correct rank we modified the denominator according to Israelsen (2005). The refined Sortino ratio was computed as:

$$\widehat{\text{SOR}}_{\text{ref}} = \frac{\widehat{m}_B}{\widehat{\sigma}_B^{\widehat{m}_B/\,\text{abs}(\widehat{m}_B)}},$$

where abs$(\cdot)$ is the absolute value function.

Table 4 presents the refined Sortino ratios for the case in which the target return is the $1/N$ portfolio return ($B_{1/N}$).

Table 4: Portfolio refined Sortino ratios for the benchmark return $B_{1/N}$

Strategy	SBM6	#Rank	FF10	#Rank	SOP25	#Rank	BMOP25	#Rank
ms portfolio	4.6247E-04	2	-3.6611E-05	1	-6.3687E-05	2	0.0297	3
mv portfolio	-5.5112E-05	5	-7.0521E-05	3	-9.1409E-05	6	-3.6877E-05	5
$1/N$ portfolio	-6.0225E-05	6	-7.1192E-05	4	-6.3626E-05	1	-6.7692E-05	6
Efficient skewness/semivariance portfolios								
w_{SSR}	1.6298E-04	3	-7.3915E-05	5	-8.2534E-05	4	0.0062	4
w_{SR}	-4.0554E-05	4	-5.8940E-05	2	-8.5081E-05	5	0.0335	2
w_{SOR}	0.0052	1	-7.5954E-05	6	-7.3910E-05	3	0.0339	1

Note: This table reports, for each of the data sets, the monthly refined Sortino ratios for the benchmark portfolios referred in Section 3: the maximum Sharpe ratio portfolio (ms portfolio), the minimum variance portfolio (mv portfolio) and the $1/N$ portfolio. The target return used in the computation of the semivariance is the $1/N$ portfolio return ($B_{1/N}$). This table also reports the monthly refined Sortinos ratios for the efficient skewness/ semivariance portfolios referred in Section 6: the maximum skewness per semivariance ratio portfolio (w_{SSR}), the maximum Sharpe ratio portfolio (w_{SR}) and the maximum Sortino ratio portfolio (w_{SOR}). The correct rank of each portfolio according to the refined Sortino ratios is also reported.

The efficient skewness/ semivariance portfolio w_{SOR} has the highest refined Sortino ratio, among all the portfolios, in two cases (for the SBM6 and BMOP25 data sets). For the FF10 data set, the efficient skewness/semivariance portfolio w_{SR} has a higher refined Sortino ratio than two of the three benchmark portfolios (the mv and the $1/N$ portfolios).

In the case of the SOP25 data set, the efficient skewness/semivariance portfolios have a higher refined Sortino ratio than one of the benchmark portfolios (the mv portfolio).

We also computed the refined Sortino ratios for the cases in which we choose the maximum Sharpe ratio portfolio return (B_{ms}) and the minimum variance portfolio return (B_{mv}) as the target return. The results are similar and for reasons of space they are not reported here.

Once again, these results suggests the robustness of the efficiency provided by the skewness/semivariance model.

7 Conclusions and future work

In this article we propose a direct analysis of the efficient tradeoff between skewness and semivariance through a skewness/ semivariance biobjective optimization problem. We computed skewness as a third moment tensor and overcame the endogeneity problem of the cosemivariance matrix using a derivative-free algorithm. To the best of our knowledge, this is the first time such an algorithm is used in this context. The solver chosen for solving the skewness/semivariance biobjective optimization problem is based on direct multisearch. Direct-search methods based on polling are known to be extremely robust due their directional properties. We have observed the robustness of direct multisearch in four empirical data sets collected from the Fama/French data library, since direct multisearch was capable of determining in-sample the Pareto frontier for the biobjective skewness/ semivariance problem, using three different target returns for the computation of the semivariance.

In addition, we performed an extensive out-of-sample analysis. The results showed that the efficient skewness/semivariance portfolios are consistently competitive when compared with the benchmark portfolios, in terms of out-of-sample Sharpe ratio. A surprising fact was that at least one of the three chosen efficient skewness/semivariance portfolios, consistently outperforms the $1/N$ portfolio in terms of out-of-sample Sharpe ratio. The efficient skewness/semivariance portfolios, also exhibited a consistently good performance in terms of skewness per semivariance ratio and Sortino ratio, which suggests the robustness of the efficiency provided by the skewness/semivariance model.

In the empirical analyses performed here we have just used three different target returns for the computation of the semivariance. Other choices could have been made (for example, the risk-free return or an index return). It is clear that the choice of this parameter, depending on the nature of the data, has a profound impact in the results. Within the skewness/semivariance model, the investor has the freedom to choose this parameter according to any criteria that may suit him/her. Besides, we only evaluate the performance of three efficient skewness/semivariance portfolios (chosen according to three different criteria). Other efficient portfolios could have been chosen according to the investor's preferences.

The only constraint that we considered in the proposed model was the absence of short-selling. However, other constraints may be able to improve portfolio performance. One may introduce, for example, turnover constraints in order to control the transaction costs. Another promising possibility is to introduce constraints to explore information about the cross-sectional characteristics of securities, as in Brandt et al. (2009). The proposed skewness/semivariance model can readily incorporate such constraints.

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