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Monotone insertion and monotone extension of frame homomorphisms[☆]

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Abstract

The purpose of this paper is to introduce monotonization in the setting of pointfree topology. More specifically, monotonically normal locales are characterized in terms of monotone insertion and monotone extensions theorems.

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1. Introduction

The purpose of this paper is to introduce monotonization into pointfree topology. We first recall that pointfree topology deals with complete lattices in which finite meets distribute over arbitrary joins. These lattices are called frames or locales. A map between two frames is a frame homomorphism if it preserves arbitrary joins and finite meets. The resulting category is denoted by **Frm**. One source of frames is given by the lattice $\mathcal{O}X$ of all open subsets of a topological space X. The assignment $X \mapsto \mathcal{O}X$ gives rise to a contravariant functor $\mathcal{O} : \mathbf{Top} \to \mathbf{Frm}$ which makes a continuous map $f : X \to Y$ into the frame homomorphism $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ determined by $\mathcal{O}f(U) = f^{-1}(U)$ for all $U \in \mathcal{O}Y$.

What is then meant by a monotonization? Suppose we have a concept consisting of sets P, Q and a specific map $\Delta : P \to Q$. Suppose further that we can enrich the concept by claiming that both P and Q carry partial orderings $\leq_P$ and $\leq_Q$ and then require the map $\Delta : (P, \leq_P) \to (Q, \leq_Q)$ to be monotone, i.e., order-preserving. In this way we have arrived at a new concept which is just the monotonization of the former concept. Usually, monotonization

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yields a specialization of the original concept. It should be remarked that a particular concept may have different monotonizations (cf. [7]).

To illustrate the monotonization procedure, let X be topological space with topology $\mathcal{O}X$ (and $\mathcal{C}X$ being the family of all closed sets of X), let $P = \{(K, U) \in \mathcal{C}X \times \mathcal{O}X : K \subseteq U\}$ and $Q = \mathcal{O}X$. Then X is normal if and only if there exists a map $\Delta : P \to Q$ with $K \subseteq \Delta(K, U) \subseteq \overline{\Delta(K, U)} \subseteq U$ for all $(K, U) \in P$. Such a map is called a normality operator. Now observe that both P and Q carry natural orderings. Namely, P is ordered by componentwise inclusion $\leq_P$, i.e., $(K_1, U_1) \leq_P (K_2, U_2)$ if and only if $K_1 \subseteq K_2$ and $U_1 \subseteq U_2$, while $\leq_Q$ is the usual inclusion. One may ask what happens if one requires $\Delta : (P, \leq_P) \to (Q, \leq_Q)$ to be monotone. A space X for which there exists a monotone normality operator is called monotonically normal. We refer to [9] for a survey of those spaces.

It is of interest to know whether a particular result does have – after the monotonization procedure – its monotone variant. Indeed, a monotone version of Urysohn's lemma is given in Borges [5]. It is good to recall that Urysohn's lemma is actually an insertion-type theorem, for it can be stated as follows:

> *A space X is normal if and only if for each closed subset K, contained in any open subset U, there is a continuous function $h : X \to \mathbb{R}$ such that $\chi_K \leq h \leq \chi_U$.*

By replacing the characteristic functions by arbitrary upper and lower semicontinuous real-valued functions one arrives at the classical Katětov–Tong insertion theorem ([14,29]; see also [15] and [17]) which we recall here for the reader's convenience:

> *A space X is normal if and only if for each upper semicontinuous function $f : X \to \mathbb{R}$ and lower semicontinuous function $g : X \to \mathbb{R}$ such that $f \leq g$, there is a continuous function $h : X \to \mathbb{R}$ such that $f \leq h \leq g$.*

The monotonization of the insertion property in the Katětov–Tong theorem was first investigated in [16] where it has been proved to characterize monotonically normal spaces. Later on, it turned out that monotonizing the insertion properties of the two other classical insertion theorems (viz., Michael insertion theorem [21] and Dowker insertion theorem [6]) does characterize stratifiable spaces, i.e. monotonically perfectly normal spaces (see Good and Stares [8] and Lane, Nyikos and Pan [18]). Stares, in [28], characterized monotone normality by a counterpart of the Tietze–Urysohn extension theorem (see Theorem 6.3).

The recognition that the notions of upper and lower semicontinuous real functions in frames provide a pointfree axiomatization of semicontinuity in spaces [11] naturally raised the question of pointfree insertion theorems and the corresponding monotone versions. The first of these was the pointfree version of Katětov–Tong theorem [19,22]. The purpose of this paper is to introduce monotonization in the pointfree context and, in particular, to present the monotone pointfree version of Katětov–Tong insertion theorem and the pointfree counterpart of Stares' extension theorem.

The paper is organized as follows. Section 2 sets up the basic terminology of pointfree topology. In Section 3 we characterize normal locales in terms of certain operators. We introduce hereditarily normal locales as those locales in which every sublocale is normal and prove that this is equivalent to the requirement that each open sublocale be normal. Section 4 deals with monotonically normal locales. These are characterized in several ways in terms of monotone normality-type operators. We show that every metrizable locale is monotonically normal. Section 5 provides the monotone localic Katětov–Tong insertion theorem. On the one hand it is a monotonization of the localic insertion theorem of [22], while on the other it is a pointfree variant of the monotone insertion theorem of [16]. In Section 6, the monotone localic insertion theorem is used to characterize monotonically normal locales in terms of monotone extenders. When applied to topological spaces it gives the result proved directly in [28, Theorem 2.3]. We point out that, in contrast to [28], our argument is free of the T_1-axiom.

2. Background in locales

Here we gather some basic frame-theoretic terminology that we shall need in what follows. Some other specific concepts will be defined when actually needed. Our main references for frames and locales are [13] and [24].

A *frame* L is a complete lattice satisfying the frame distributive law

$$a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$$

for all $a \in L$ and $B \subseteq L$. A *frame homomorphism* is a map $f : L \to M$ which preserves finite meets (including the top 1) and arbitrary joins (including the bottom 0). The bounds of L may occasionally be denoted by 1_L and 0_L. The resulting category is denoted **Frm**. The set of all frame homomorphisms from L to M is denoted by $\mathbf{Frm}(L, M)$. The category of locales is the dual category $\mathbf{Loc} = \mathbf{Frm}^{op}$ of **Frm**.

Due to the frame distributive law, all the maps $a \wedge (\cdot) : L \to L$ preserve arbitrary joins and, thus, have right (Galois) adjoints $a \to (\cdot) : L \to L$, which means that $a \wedge c \le b$ iff $c \le a \to b$. Thus, in a frame L we have

$$a \to b = \bigvee \{c \in L : a \wedge c \le b\},$$

and for $a, b \in L$ and $A \subseteq L$ the following hold:

(H1) $a \le b \to a$,
(H2) $a \wedge (a \to b) = a \wedge b$,
(H3) $a \to b = 1$ iff $a \le b$,
(H4) $\left(\bigvee A\right) \to b = \bigwedge_{a \in A} (a \to b)$.

The *pseudocomplement* of $a \in L$ is $a^* = a \to 0$. Then $a \wedge a^* = 0$, $a \le a^{**}$ and $\left(\bigvee A\right)^* = \bigwedge_{a \in A} a^*$. In particular, $a \le b$ implies $b^* \le a^*$.

The subobjects in **Loc** (equivalently, the quotients in **Frm**) have been described in several equivalent ways in the literature. The definition of a sublocale that we adopt here is taken from [13, Exercise II.2.3]. It follows the lines of [23].

A subset $S \subseteq L$ is a *sublocale* of L if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,
(S2) For every $a \in L$ and $s \in S$, $a \to s \in S$.

Partially ordered by inclusion, the set of all sublocales of L is a complete lattice (more precisely, a co-frame, i.e. a dual of a frame) in which $\{1\}$ is the bottom and L is the top. The sets

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a = \{b \in L : a \le b\}$$

are sublocales of L for all $a \in L$. They will be referred to as the *open* and *closed* sublocales of L, respectively.

Each sublocale $S \subseteq L$ is a frame with the same meets as in L and with the same Heyting operation $\to$, since the relevant properties of the latter merely depend on the meet operation. However, the joins in S may differ from those of L. One has, $1_S = 1$, but in general $0_S \ne 0$. In particular, $0_{\mathfrak{o}(a)} = a^*$ and $0_{\mathfrak{c}(a)} = a$.

A sublocale $S \subseteq L$ determines the surjection $c_S \in \mathbf{Frm}(L, S)$ given by

$$c_S(a) = \bigwedge (S \cap \mathfrak{c}(a))$$

for all $a \in L$. In particular, $c_S(s) = s$ for all $s \in S$. This correspondence yields a canonical representation of the sublocales of L by the *quotients* of L (in the category of frames), that is, the codomains M of frame surjections $m : L \to M$ (m is referred to as the *quotient map*): The quotients of L are naturally preordered by $M \le N$ if the quotient map $n : L \to N$ factors through the quotient map $m : L \to M$. The equivalence relation induced by this preorder identifies two quotients M and N if each is larger than the other, and this happens precisely when there is a frame isomorphism $h : M \to N$ such that $h \circ m = n$. The collection of all equivalence classes of quotients of a given frame L forms a frame, dually isomorphic to the co-frame of all sublocales of L.

The open sublocale $\mathfrak{o}(a)$ viewed as a quotient of L (with quotient map $b \mapsto a \to b$) is isomorphic to the quotient ${\downarrow} a$ (with quotient map $b \mapsto a \wedge b$), where the former is regarded as a frame in the order inherited from L. The isomorphism $h : \mathfrak{o}(a) \to {\downarrow} a$ is given by $h(a \to b) = a \wedge (a \to b) = a \wedge b$.

Let us also recall that $A \subseteq L$ *generates* L if each element of L is a join of a family of meets of finite subsets of A.

Following [1] and [2], the *locale of reals* is the locale $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q) where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \le r < q \le s$,
(R3) $(p, q) = \bigvee \{(r, s) : p < r < s < q\}$,
(R4) $1 = \bigvee_{p, q \in \mathbb{Q}} (p, q)$.

By (R3), $(p, q) = \bigvee \varnothing = 0$ if $p \geq q$. We write:

$$(p, -) = \bigvee_{q \in \mathbb{Q}} (p, q) \quad \text{and} \quad (-, q) = \bigvee_{p \in \mathbb{Q}} (p, q).$$

Then $(p, -) \wedge (-, q) = (p, q)$.

An obvious equivalent representation of the locale of reals is the following [19]: $\mathfrak{L}(\mathbb{R})$ is the locale generated by the elements $(p, -)$ and $(-, q)$ where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1$'$) $(p, -) \vee (-, q) = 1$ whenever $p < q$,
(R2$'$) $(p, -) \wedge (-, q) = (p, q)$ whenever $q \leq p$,
(R3$'$) $(p, -) = \bigvee_{r > p} (r, -)$,
(R4$'$) $(-, q) = \bigvee_{r < q} (-, r)$,
(R5$'$) $1 = \bigvee_{p \in \mathbb{Q}} (p, -) = \bigvee_{q \in \mathbb{Q}} (-, q)$.

Following [11] (cf. also [19]), we denote by $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ the sublocales of $\mathfrak{L}(\mathbb{R})$ generated by all the elements $(p, -)$ and $(-, q)(p, q \in \mathbb{Q})$, respectively. As in [11], we let

$$\mathrm{USC}(L) = \left\{ f \in \mathbf{Frm}(\mathfrak{L}_l(\mathbb{R}), L) : \bigcap_{q \in \mathbb{Q}} \mathfrak{o}(f(-, q)) = \{1\} \right\}$$

and

$$\mathrm{LSC}(L) = \left\{ g \in \mathbf{Frm}(\mathfrak{L}_u(\mathbb{R}), L) : \bigcap_{p \in \mathbb{Q}} \mathfrak{o}(g(p, -)) = \{1\} \right\}.$$

Members of USC(L) [resp., LSC(L)] are called *upper* [resp., *lower*] *semicontinuous real functions* on L.

Remark 2.1. The two extra algebraic conditions defining USC(L) and LSC(L) are just translations in terms of sublocales of the original conditions of [11] stated in terms of congruences. We recall that the reason for these extra conditions is that when $L = \mathcal{O}X$, there is a one-to-one correspondence between USC$(\mathcal{O}X)$ and the set USC$(X, \mathbb{R})$ of all upper semicontinuous real functions on X as well as between LSC$(\mathcal{O}X)$ and the set LSC$(X, \mathbb{R})$ of all lower semicontinuous real functions on X (see [11, Corollary 4.3]). With these conditions our pointfree insertion and extension theorems become true generalizations of their topological counterparts (see also [22] for a related discussion).

Partial orders: (1) USC(L) is partially ordered by:

$$f_1 \leq f_2 \Leftrightarrow f_2(-, q) \leq f_1(-, q) \quad \text{for all } q \in \mathbb{Q}.$$

(2) LSC(L) is partially ordered by:

$$g_1 \leq g_2 \Leftrightarrow g_1(p, -) \leq g_2(p, -) \quad \text{for all } p \in \mathbb{Q}.$$

(3) $\mathrm{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ is partially ordered by:

$$h_1 \leq h_2 \Leftrightarrow h_{1|\mathfrak{L}_u(\mathbb{R})} \leq h_{2|\mathfrak{L}_u(\mathbb{R})} \Leftrightarrow h_{2|\mathfrak{L}_l(\mathbb{R})} \leq h_{1|\mathfrak{L}_l(\mathbb{R})}.$$

Members of C(L) are called *continuous real functions* [1] on L.

3. Normal and hereditarily normal locales

Recall that a locale L is called *normal* if, given $a, b \in L$ with $a \vee b = 1$, there exist $u, v \in L$ such that $a \vee u = 1 = b \vee v = 1$ and $u \wedge v = 0$. Clearly, one can select $v = u^*$. Thus, L is normal if and only if, whenever $a \vee b = 1$, there exists a $u \in L$ satisfying $a \vee u = 1 = b \vee u^*$. For a future monotonization (in Section 4), it will be convenient to restate the definition of normality in the following terms. Let

$$\mathcal{D}_L = \{(a, b) \in L \times L : a \vee b = 1\}.$$

A locale L is normal if and only if there exists a function $\Delta : \mathcal{D}_L \to L$ such that

$$a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$$

for all $(a, b) \in \mathcal{D}_L$. The function Δ is called a *normality operator*.

Notation and terminology. For an arbitrary function $\Delta : \mathcal{D}_L \to L$ we let $\Delta^{op}(a, b) = \Delta(b, a)$. The function Δ will be called *self-disjoint* whenever the pointwise meet $\Delta \wedge \Delta^{op}$ is equal to 0.

Remarks 3.1. (1) If $\Delta : \mathcal{D}_L \to L$ is a normality operator, then so is $\Delta^{\circledast}$ defined by

$$\Delta^{\circledast}(a, b) = \Delta(b, a)^*.$$

(2) If Δ_1 and Δ_2 are normality operators, then so is $\Delta_1 \wedge \Delta_2$ (the pointwise meet).

(3) In particular, each normal locale L admits a self-disjoint normality operator Θ. Indeed, if Δ is a normality operator, then $\Theta = \Delta \wedge \Delta^{\circledast}$ has the required property.

Proposition 3.2. *Let L be a locale. The following are equivalent:*

(1) *L is normal.*
(2) *There exists a self-disjoint $\Theta : \mathcal{D}_L \to L$ such that $a \vee \Theta(a, b) = 1$ for all $(a, b) \in \mathcal{D}_L$.*

Proof. (1) implies (2) by Remark 3.1(3). If (2) holds true, then $b \vee \Theta(a, b)^* \geq b \vee \Theta(b, a) = 1$, hence Θ is a normality operator. $\square$

A locale will be called *hereditarily normal* if every its sublocale is normal.

Proposition 3.3. *A locale L is hereditarily normal if and only if every open sublocale of L is normal.*

Proof. The "only if" part is obvious. To prove the "if" part, let L be a locale whose open sublocales are normal. Let $S \subseteq L$ be an arbitrary sublocale of L, given by the surjective homomorphism $c_S : L \to S$. In order to prove that S is normal, let $a, b \in S$ be such that $a \vee_S b = 1$ and consider $\overline{a}, \overline{b} \in L$ such that $c_S(\overline{a}) = a$ and $c_S(\overline{b}) = b$. By hypothesis, $\downarrow(\overline{a} \vee \overline{b})$ is normal thus there exist $c, d \in L$ such that $\overline{a} \vee c = \overline{a} \vee \overline{b} = \overline{b} \vee d$ and $c \wedge d = 0$. Now $a \vee_S c_S(c) = 1 = b \vee_S c_S(d)$ and $c_S(c) \wedge c_S(d) = 0$, showing S is normal. $\square$

4. Monotonically normal locales

Conventions 4.1. (1) For $(P, \leq)$ a partially ordered set, any subset $\mathcal{P} \subseteq P \times P$ will always be assumed to have the componentwise order inherited from $P^{op} \times P$, i.e.,

$$(a, b) \leq_{\mathcal{P}} (c, d) \Leftrightarrow c \leq_P a \text{ and } b \leq_P d.$$

In what follows, P will either be L, $L^{\mathbb{N}}$ or $L^{\mathbb{Q}}$, where the latter two sets are ordered componentwise.

(2) Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partially ordered sets. A map $\phi : (P, \leq_P) \to (Q, \leq_Q)$ is called *monotone* [resp, *antitone*] iff: $x \leq_P y \Rightarrow \phi(x) \leq_Q \phi(y)$ [resp., $\phi(y) \leq_Q \phi(x)$] for all $x, y \in P$.

By monotonizing the concept of a normal locale one arrives at the concept of a monotonically normal locale. Specifically, a locale L is called *monotonically normal* if there exists a monotone function $\Delta : \mathcal{D}_L \to L$ such that

$$a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$$

for all $(a, b) \in \mathcal{D}_L$. We call Δ a *monotone normality operator*.

We start with a monotone variant of Remarks 3.1.

Remarks 4.2. (1) If $\Delta : \mathcal{D}_L \to L$ is a monotone normality operator, then so is $\Delta^{\circledast} : \mathcal{D}_L \to L$ defined by $\Delta^{\circledast}(a, b) = \Delta(b, a)^*$. Indeed, $\Delta^{\circledast}$ is a normality operator (Remark 3.1(1)) and is monotone, because $(a, b) \leq (c, d)$ in $\mathcal{D}_L$ if and only if $(d, c) \leq (b, a)$ in $\mathcal{D}_L$ and $(\cdot)^*$ is antitone.

(2) If Δ_1 and Δ_2 are monotone normality operators, then so is the pointwise meet $\Delta_1 \wedge \Delta_2$.

(3) Each monotonically normal locale L admits a self-disjoint monotone normality operator Θ (cf. [12, Lemma 2.2]). Indeed, if Δ is a monotone normality operator, then with $\Theta = \Delta \wedge \Delta^{\circledast}$ one has $\Theta(a,b) \wedge \Theta^{op}(a,b) \leq \Delta(a,b) \wedge \Delta(a,b)^* = 0$. Notice that if Θ is self-disjoint, then $\Theta \leq \Theta^{\circledast}$.

(4) A self-disjoint and monotone function $\Delta : \mathcal{D}_L \to L$ such that $a \vee \Delta(a,b) = 1$ for all $(a,b) \in \mathcal{D}_L$ is clearly a monotone normality operator (cf. Proposition 3.2).

Before moving to a canonical example of a monotonically normal locale, viz. metrizable locales, we give a number of obvious examples.

Examples 4.3. (1) A topological space X is monotonically normal if and only if $\mathcal{O}X$ is monotonically normal (note that here, as in [16] and [10], we do not assume the T_1-axiom to be part of the definition of monotone normality, contrary to what is usual in the literature – see [12], for example, which was the first paper on the topic – where T_1 is always taken as part of the monotone normality). When T_1-separation is part of the definition of monotone normality, then any monotonically normal space is hereditarily monotonically normal. It has already been pointed out in [10] that this need not be the case without T_1 (see Example 4.8).

(2) If 1 is coprime (i.e., $a \vee b = 1$ implies $a = 1$ or $b = 1$), then $\mathcal{D}_L = (\{1\} \times L) \cup (L \times \{1\})$ and L is monotonically normal. In fact, $\Delta : \mathcal{D}_L \to L$ defined by

$$\Delta(a,b) = \begin{cases} 0 & \text{if } a = 1 \text{ and } b \neq 1, \\ 1 & \text{if } b = 1 \end{cases}$$

is a monotone normality operator (see also [22, Example 4.2]).

We shall now show that metrizable locales [25] (further developed, among others, in [26,27] and [4]) are monotonically normal. Before doing this, some preparatory material is needed which is taken from the just cited papers. Given $A \subseteq L$ and $b \in L$, we put

$$A(b) = \bigvee \{a \in A : a \wedge b \neq 0\}.$$

Clearly, $A(\cdot) : L \to L$ preserves arbitrary joins and – as such – admits a right (Galois) adjoint $\alpha_A : L \to L$ given by

$$\alpha_A(a) = \bigvee \{b \in L : A(b) \leq a\}.$$

Consequently, $A(b) \leq a$ iff $b \leq \alpha_A(a)$. In particular, $A(\alpha_A(a)) \leq a$.

A *metric diameter* on a locale L is a map $d : L \to [0, \infty]$ satisfying the following conditions:

(D1) $d(0) = 0$,
(D2) d is monotone,
(D3) $d(a \vee b) \leq d(a) + d(b)$ whenever $a \wedge b \neq 0$,
(D4) for each $\varepsilon > 0$ the set $B_\varepsilon = \{a \in L : d(a) < \varepsilon\}$ is a cover of L (i.e. $\bigvee B_\varepsilon = 1$),
(D5) $a = \bigvee \{b \in L : B_\varepsilon(b) \leq a \text{ for some } \varepsilon > 0\}$ for all $a \in L$,
(D6) $d(a) = \bigvee \{d(b \vee c) : b, c \leq a \text{ and } d(b) \vee d(c) < \varepsilon\}$ for all $a \in L$ and $\varepsilon > 0$.

A *metric locale* is a pair (L, d) where d is a metric diameter on L. A locale that admits a metric diameter is called *metrizable*. We note that a topological space X is metrizable if and only if $\mathcal{O}X$ is metrizable as a locale. The passage from a metric ρ to a metric diameter d is provided by the usual diameter $d(U) = \sup\{\rho(x,y) : x, y \in U\}$ for all $U \in \mathcal{O}X$.

In what follows, we write α_ε instead of α_{B_ε}. The following facts are standard (cf. [26, Lemma 1.10]):

Lemma 4.4. *For each $a \in L$ the following hold:*

(1) $a \vee \alpha_\varepsilon(a)^* = 1$ *for all* $\varepsilon > 0$.
(2) $a = \bigvee_{\varepsilon > 0} \alpha_\varepsilon(a)$.

Proposition 4.5. *Each metrizable locale is monotonically normal.*

Proof. Let L be metrizable. Define $\Delta : \mathcal{D}_L \to L$ by

$$\Delta(a, b) = \bigvee_{\varepsilon>0} (\alpha_\varepsilon(a)^* \wedge \alpha_\varepsilon(b))$$

for all $(a, b) \in \mathcal{D}_L$. Since α_ε is monotone and $(\cdot)^*$ is antitone, Δ is easily seen to be monotone. By Lemma 4.4, we have

$$\begin{aligned} a \vee \Delta(a, b) &= \bigvee_{\varepsilon>0} ((a \vee \alpha_\varepsilon(a)^*) \wedge (a \vee \alpha_\varepsilon(b))) \\ &= \bigvee_{\varepsilon>0} (a \vee \alpha_\varepsilon(b)) \\ &= a \vee \bigvee_{\varepsilon>0} \alpha_\varepsilon(b) \\ &= a \vee b = 1. \end{aligned}$$

Also,

$$\begin{aligned} (\Delta \wedge \Delta^{op})(a, b) &= \bigvee_{\varepsilon_1,\varepsilon_2>0} (\alpha_{\varepsilon_1}(a)^* \wedge \alpha_{\varepsilon_2}(a) \wedge \alpha_{\varepsilon_2}(b)^* \wedge \alpha_{\varepsilon_1}(b)) \\ &\le \bigvee_{\varepsilon_1\ge\varepsilon_2} (\alpha_{\varepsilon_1}(a)^* \wedge \alpha_{\varepsilon_2}(a)) \vee \bigvee_{\varepsilon_1<\varepsilon_2} (\alpha_{\varepsilon_2}(b)^* \wedge \alpha_{\varepsilon_1}(b)) = 0. \end{aligned}$$

Hence, by Remark 4.2(4), Δ is a monotone normality operator. □

Remarks 4.6. (a) We note that the proof of Proposition 4.5 merely uses (D4), (D5) and the fact that $\{B_\varepsilon : \varepsilon > 0\}$ is a chain under inclusion (the latter property is used in proving that Δ is self-disjoint). According to [25] and [26], a system $\mathcal{C}$ of covers of a locale L is called *admissible* if it satisfies condition (D5), i.e.,

$$a = \bigvee\{b \in L : \exists_{C\in\mathcal{C}}\, C(b) \le a\} = \bigvee_{C\in\mathcal{C}} \alpha_C(x),$$

i.e., we have (2) of Lemma 4.4. Also, we have $a \vee \alpha_C(a)^* = 1$ for all $a \in L$. Consequently: *Each locale that has an admissible chain of covers is monotonically normal.*

Note that any nearness [3] with a countable basis is of this kind. It is also worth mentioning that since this general condition is preserved by taking homomorphic images it is automatic that the locales in question are hereditarily monotonically normal.

(b) Both $\mathfrak{L}(\mathbb{R})$ and $\mathfrak{L}[p, q] = \uparrow ((-, p) \vee (q, -))$ (where $p < q$), being examples of metrizable locales, are monotonically normal.

We now have the following equivalent formulation of monotone normality (cf. [20, Proposition 3], and a part of Theorem 2.4 in [5]; see also [10, Proposition 3.1]).

Proposition 4.7. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *There exists a self-disjoint $\Sigma : \mathcal{D}_L \to L$ such that $\Sigma(a, b) \le b$, $a \vee \Sigma(a, b) = 1$, and $\Sigma(a, \cdot)$ is monotone on $\{b \in L : (a, b) \in \mathcal{D}_L\}$.*

Proof. (1) ⇒ (2): By Remark 4.2(3), let Θ be a self-disjoint monotone normality operator of L. Further, for each $(a, b) \in \mathcal{D}_L$, let

$$\Sigma(a, b) = b \wedge \bigvee_{(a,c)\le(a,b)} \Theta(a, c).$$

Clearly, $\Sigma(a, b) \le b \wedge \Theta(a, b)$, so that $\Sigma(a, b) \le b$ and $\Sigma \le \Theta$. Consequently $\Sigma \wedge \Sigma^{op} \le \Theta \wedge \Theta^{op} = 0$. Also,

$$a \vee \Sigma(a, b) = (a \vee b) \wedge \left(\bigvee_{(a,c)\le(a,b)} (a \vee \Theta(a, c)) \right) = 1.$$

Finally, if $(a, b) \leq (a, d)$ in $\mathcal{D}_L$, then

$$\Sigma(a, b) = b \wedge \bigvee_{(a,c)\leq(a,b)} \Theta(a, c) \leq d \wedge \bigvee_{(a,c)\leq(a,d)} \Theta(a, c) = \Sigma(a, d).$$

(2) $\Rightarrow$ (1): Let Σ be the operator of (2). Define $\Delta : \mathcal{D}_L \to L$ by

$$\Delta(a, b) = \bigvee_{(c,b)\leq(a,b)} \Sigma(c, b)$$

for all $(a, b) \in \mathcal{D}_L$. Clearly, $a \vee \Delta(a, b) \geq a \vee \Sigma(a, b) = 1$. Since $\Sigma \wedge \Sigma^{op} = 0$, we have $\Sigma \leq \Sigma^{\circledast}$. Thus

$$\Delta(a, b)^* = \bigwedge_{(c,b)\leq(a,b)} \Sigma(c, b)^* = \bigwedge_{(b,a)\leq(b,c)} \Sigma^{\circledast}(b, c) \geq \bigwedge_{(b,a)\leq(b,c)} \Sigma(b, c) \geq \Sigma(b, a).$$

Therefore

$$b \vee \Delta(a, b)^* \geq b \vee \Sigma(b, a) = 1.$$

Finally, if $(a, b) \leq (a_1, b_1)$, then $\Delta(a, b) \leq \bigvee_{(c,b)\leq(a_1,b_1)} \Sigma(c, b) = \Delta(a_1, b_1)$. $\square$

It is easy to show that any closed sublocale of a (monotonically) normal locale is (monotonically) normal. However, in contrast to the topological situation (with the T_1-separation axiom), the localic monotone normality fails to be a hereditary property. The following example shows that a monotonically normal locale may have an open sublocale which fails to be normal.

Example 4.8. Let L be a non-normal locale. Add a new element $\infty \notin L$ to L and consider $M = L \cup \{\infty\}$ with its natural ordering, i.e., $a \leq \infty$ for all $a \in L$. Then ∞ is coprime in M, $\mathcal{D}_M = (\{\infty\} \times M) \cup (M \times \{\infty\})$ and M becomes a monotonically normal locale (see Example 4.3(2)). Finally, the open sublocale $\mathfrak{o}(1) \cong {\downarrow}1 = L$ fails to be normal.

An obvious question posed by this and Proposition 3.3 is whether a locale is hereditarily monotonically normal whenever all its open sublocales are monotonically normal. We have not yet been able to prove or disprove that statement.

It is well-known that a topological space is normal if and only if every two separated F_σ-sets have disjoint open neighborhoods. A monotone variant of that statement is in [16, Proposition 3.1], while its localic variant is in [22, Lemma 3.2]. The following provides its monotone localic counterpart.

The statement involves the set

$$\mathcal{U}_L = \left\{(a, b) \in L^{\mathbb{N}} \times L^{\mathbb{N}} : \left(a(n), \bigwedge b(\mathbb{N})\right), \left(b(n), \bigwedge a(\mathbb{N})\right) \in \mathcal{D}_L\right\}$$

partially ordered according to Conventions 4.1. For a sequence $a \in L^{\mathbb{N}}$, we define $a^{\downarrow}(n) = \bigwedge_{i\leq n} a(n)$ (clearly, $a^{\downarrow}(\cdot)$ is antitone).

Proposition 4.9. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *There exists a monotone function $\Upsilon : \mathcal{U}_L \to L$ such that $a(n) \vee \Upsilon(a, b) = 1$ and $b(n) \vee \Upsilon(a, b)^* = 1$ for all $(a, b) \in \mathcal{U}_L$ and $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2): Let $\Delta : \mathcal{D}_L \to L$ be a monotone normality operator. Recall that so is $\Delta^{\circledast}$. Define $\Upsilon : \mathcal{U}_L \to L$ by

$$\Upsilon(a, b) = \bigvee_{n\in\mathbb{N}} \left(\Delta\left(a^{\downarrow}(n), \bigwedge b(\mathbb{N})\right) \wedge \Delta^{\circledast}\left(\bigwedge a(\mathbb{N}), b^{\downarrow}(n)\right)\right).$$

It is easy to see that Υ is monotone. Let $(a, b) \in \mathcal{U}_L$. Then with $u_n = \Delta(a^{\downarrow}(n), \bigwedge b(\mathbb{N}))$ and $v_n = \Delta^{\circledast}(\bigwedge a(\mathbb{N}), b^{\downarrow}(n))$ one has

$$a^{\downarrow}(n) \vee u_n = \bigwedge b(\mathbb{N}) \vee u_n^* = 1 = \bigwedge a(\mathbb{N}) \vee v_n = b^{\downarrow}(n) \vee v_n^*$$

for all n. Thus

$$a(n) \vee \Upsilon(a, b) \geq a^{\downarrow}(n) \vee (u_n \wedge v_n) \geq (a^{\downarrow}(n) \vee u_n) \wedge \left(\bigwedge a(\mathbb{N}) \vee v_n\right) = 1.$$

Since both (u_n) and (v_n^*) are monotone, it follows that for all n and m one has

$$u_n \wedge v_n \wedge u_m^* \wedge v_m^* \leq \begin{cases} u_n \wedge u_n^* & \text{if } n \leq m \\ v_n \wedge v_n^* & \text{if } n > m \end{cases} = 0,$$

i.e., $u_m^* \wedge v_m^* \leq \bigwedge_n (u_n \wedge v_n)^* = \Upsilon(a, b)^*$ for all m. So, we get

$$b(n) \vee \Upsilon(a, b)^* \geq b^{\downarrow}(n) \vee (u_n^* \wedge v_n^*) \geq \left(\bigwedge b(\mathbb{N}) \vee u_n^*\right) \wedge (b^{\downarrow}(n) \vee v_n^*) = 1.$$

(2) $\Rightarrow$ (1): This is obvious. In fact, if $(a, b) \in \mathcal{D}_L$, then for $a(n) = a$ and $b(n) = b$ one has $(a, b) \in \mathcal{U}_L$ and $\Theta(a, b) = \Upsilon(a, b)$ defines a monotone normality operator for L. $\square$

We still need a more specific normality-type operator. For this purpose, for each $\alpha \in L^{\mathbb{Q}}$ and $r \in \mathbb{Q}$, define $\alpha_r = \alpha(r)$ and let $\mathcal{S}_L$ denote the collection of all ordered pairs $(\alpha, \beta) \in L^{\mathbb{Q}} \times L^{\mathbb{Q}}$ where α is monotone, β is antitone and

$$\alpha_s \vee \beta_r = 1 \quad \text{whenever } r < s.$$

The set $\mathcal{S}_L$ is partially ordered according to Conventions 4.1.

Proposition 4.10. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *There exists a monotone function $\Gamma : \mathcal{S}_L \to L^{\mathbb{Q}}$ such that for all $(\alpha, \beta) \in \mathcal{S}_L$ and $r < s$ the following holds:*

$$\Gamma(\alpha, \beta)_r \vee \alpha_s = \Gamma(\alpha, \beta)_r \vee \Gamma(\alpha, \beta)_s^* = \beta_r \vee \Gamma(\alpha, \beta)_s^* = 1.$$

Proof. (1) $\Rightarrow$ (2): Let $\{r_n : n \in \mathbb{N}\}$ be an indexation of $\mathbb{Q}$. For each $(\alpha, \beta) \in \mathcal{S}_L$ we will inductively define a family $\{\gamma_{r_i} = \Gamma(\alpha, \beta)_{r_i} : i \in \mathbb{N}\}$ such that

$$\begin{cases} \gamma_{r_j} \vee \alpha_s = \gamma_{r_i} \vee \gamma_{r_j}^* = \beta_r \vee \gamma_{r_i}^* = 1 & \text{if } r < r_i < r_j < s \ \ (i, j < n), \\ \Gamma(\alpha, \beta) \leq \Gamma(\bar{\alpha}, \bar{\beta}) & \text{if } (\alpha, \beta) \leq (\bar{\alpha}, \bar{\beta}). \end{cases} \tag{I$_n$}$$

In doing so, we shall use the following sets:

$$A_n = \{\alpha_r : r > r_n\}, \qquad B_n = \{\beta_r : r < r_n\},$$
$$C_n = \{\gamma_{r_i} : r_i < r_n, i < n\}, \qquad D_n = \{\gamma_{r_i}^* : r_i > r_n, i < n\}.$$

Now we proceed inductively. For $n = 2$, if $r < r_1 < s$, then, since α is monotone and β is antitone, one has

$$\alpha_s \vee \bigwedge B_1 \geq \alpha_s \vee \beta_{r_1} = 1 \quad \text{and} \quad \beta_r \vee \bigwedge A_1 \geq \beta_r \vee \alpha_{r_1} \geq 1.$$

Hence we have $(a, b) \in \mathcal{U}_L$ with a and b being monotone enumerations of A_1 and B_1, respectively. Using Proposition 4.9, we put $\gamma_{r_1} = \Gamma(\alpha, \beta)_{r_1} = \Upsilon(a, b)$.

Assume we have constructed $\{\gamma_{r_i} : i < n\}$ satisfying (I_n). Let c and d be monotone enumerations of $A_n \cup D_n$ and $B_n \cup C_n$, respectively. As above, we check that $(c, d) \in \mathcal{U}_L$. Indeed, if $r < r_i < r_j < s$ $(i, j < n)$, then

$$\begin{aligned} \alpha_s \vee \bigwedge(B_n \cup C_n) &= \left(\alpha_s \vee \bigwedge B_n\right) \wedge \left(\alpha_s \vee \bigwedge C_n\right) \\ &\geq (\alpha_s \vee \beta_{r_n}) \wedge \bigwedge_{r_j < r_n} (\alpha_s \vee \gamma_{r_j}) = 1 \end{aligned}$$

and, similarly,

$$\gamma_{r_i}^* \vee \bigwedge(B_n \cup C_n) \geq (\gamma_{r_i}^* \vee \beta_{r_n}) \wedge \bigwedge_{r_i < r_j} (\gamma_{r_i}^* \vee \gamma_{r_j}) = 1.$$

Analogously, $\beta_r \vee \bigwedge(A_n \cup D_n) = 1 = \gamma_{r_i} \vee \bigwedge(A_n \cup D_n)$. Thus, using Proposition 4.9 again, we define $\gamma_{r_n} = \Upsilon(c, d)$ and (I_{n+1}) holds true.

(2) $\Rightarrow$ (1): As in the proof of Proposition 4.9. $\square$

5. Monotone localic Katětov–Tong insertion theorem

Given $f \in \mathrm{USC}(L)$ and $g \in \mathrm{LSC}(L)$, we define

$$f \lhd_{u,l} g \Leftrightarrow f(-,q) \vee g(p,-) = 1 \quad \text{for all } p < q \text{ in } \mathbb{Q},$$

and

$$g \lhd_{l,u} f \Leftrightarrow f(-,p) \wedge g(p,-) = 0 \quad \text{for all } p \in \mathbb{Q}.$$

Remark 5.1. It is easy to see that there is a one-to-one correspondence between $\mathrm{C}(L)$ and the set

$$\mathcal{C}_L = \{(f,g) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : f \lhd_{u,l} g \text{ and } g \lhd_{l,u} f\}.$$

Indeed, given an $h \in \mathrm{C}(L)$ and restricting it to $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ yields the pair $(h_{|\mathfrak{L}_l(\mathbb{R})}, h_{|\mathfrak{L}_u(\mathbb{R})}) \in \mathcal{C}_L$. Conversely, any $(f,g) \in \mathcal{C}_L$ gives rise to an $h \in \mathrm{C}(L)$ defined by $h(p,q) = f(-,q) \wedge g(p,-)$. In such a case we shall write $h = \langle f, g \rangle$.

Let

$$\mathrm{UL}(L) = \{(f,g) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : f \lhd_{u,l} g\}.$$

This set has the componentwise order inherited from $\mathrm{USC}(L)^{op} \times \mathrm{LSC}(L)$, i.e.,

$$(f_1, g_1) \le (f_2, g_2) \Leftrightarrow f_2 \le f_1 \text{ and } g_1 \le g_2.$$

A *scale* (*descending trail* in [1]) in L is a map $\tau : \mathbb{Q} \to L$ such that $\tau(r) \vee \tau^*(s) = 1$ whenever $r < s$, and $\bigvee \tau(\mathbb{Q}) = 1 = \bigvee \tau^*(\mathbb{Q})$ where $\tau^* = (\cdot)^* \circ \tau$.

Lemma 5.2 (*[1, Lemma 2]*). *Each scale τ in L generates an $h \in \mathrm{C}(L)$ defined by $h(p,q) = \bigvee\{\tau(r) \wedge \tau^*(s) : p < r < s < q\}$.* $\square$

Remark 5.3. If τ_1 and τ_2 are scales with $\tau_1 \le \tau_2$ in $L^{\mathbb{Q}}$ and h_1 and h_2 are the corresponding frame homomorphisms, then $h_1 \le h_2$.

We shall need the *characteristic maps* $\chi_a^u \in \mathrm{USC}(L)$ and $\chi_a^l \in \mathrm{LSC}(L)$, for any $a \in L$, defined by

$$\chi_a^u(-,q) = \begin{cases} 0, & \text{if } q \le 0, \\ a, & \text{if } 0 < q \le 1, \\ 1, & \text{if } q > 1, \end{cases} \quad \text{and} \quad \chi_a^l(p,-) = \begin{cases} 1, & \text{if } p < 0, \\ a, & \text{if } 0 \le p < 1, \\ 0, & \text{if } p \ge 1. \end{cases}$$

We are eventually in a position to give a monotone version of the localic Katětov–Tong theorem of [22] (the reader should consult [22] for a criticism of the localic insertion theorem of [19] which has not been a true generalization of the Katětov–Tong insertion theorem; see also Remark 2.1 and [11]). When applied to $L = \mathcal{O}X$ it yields the monotone insertion theorem of [16]. When $(f,g) \in \mathrm{UL}(L)$ and $h \in \mathrm{C}(L)$ we shall simply write $f \le h \le g$ whenever $f \le h_{|\mathfrak{L}_l(\mathbb{R})}$ in $\mathrm{USC}(L)$ and $h_{|\mathfrak{L}_u(\mathbb{R})} \le g$ in $\mathrm{LSC}(L)$.

Theorem 5.4. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *There exists a monotone function $\Lambda : \mathrm{UL}(L) \to \mathrm{C}(L)$ such that $f \le \Lambda(f,g) \le g$ for all $(f,g) \in \mathrm{UL}(L)$.*

Proof. (1) $\Rightarrow$ (2): Let $(f,g) \in \mathrm{UL}(L)$ and let $\varphi, \gamma \in L^{\mathbb{Q}}$ be defined by $\varphi(r) = f(-,r)$ and $\gamma(r) = g(r,-)$. Then $(\varphi,\gamma) \in \mathcal{S}_L$. Let $\Gamma : \mathcal{S}_L \to L^{\mathbb{Q}}$ be the monotone function given by Proposition 4.10. Then the map $\Gamma(\varphi,\gamma) : \mathbb{Q} \to L$ is a scale that generates the required $\Lambda(f,g) \in \mathrm{C}(L)$.

We first observe that Λ is monotone. Indeed, $(f,g) \le (f_1,g_1)$ in $\mathrm{UL}(L)$ if and only if $(\varphi,\gamma) \le (\varphi_1,\gamma_1)$ in $\mathcal{S}_L$. Thus, $\tau \le \tau_1$ in $L^{\mathbb{Q}}$ and, consequently (Remark 5.3), $\Lambda(f,g) \le \Lambda(f_1,g_1)$.

It remains to show that $f \le \Lambda(f,g)_{|\mathfrak{L}_l(\mathbb{R})}$ and $\Lambda(f,g)_{|\mathfrak{L}_u(\mathbb{R})} \le g$. By Proposition 4.10, if $r < s$ we have

$$1 = \varphi(s) \vee \Gamma(\varphi,\gamma)(r) \le f(-,s) \vee \tau^{**}(r).$$

Hence $\bigvee_{r<s} \tau^*(r) \le f(-,s)$ and this just says that $\Lambda(f,g)_{|\mathfrak{L}_u(L)}(-,s) \le f(-,s)$. The second inequality follows similarly.

(2) $\Rightarrow$ (1): If $(a,b) \in \mathcal{D}_L$, then $(\chi_a^u, \chi_b^l) \in \mathrm{UL}(L)$. Now it suffices to observe that $\Delta : \mathcal{D}_L \to L$ defined by $\Delta(a,b) = \Lambda(\chi_a^u, \chi_b^l)(\frac{1}{2}, -)$ is a monotone normality operator. $\square$

6. Monotone extension property

Recall that an $h \in \mathrm{C}(L)$ is said to be *bounded* if $h(p,q) = 1$ for some $p < q$ (or, equivalently, if $h(-,p) \vee h(q,-) = 0$ for some $p < q$). In the sequel, $\mathrm{C}^*(L)$ stands for all the bounded members of $\mathrm{C}(L)$ such that $h(-,0) \vee h(1,-) = 0$.

Remark 6.1. Let $a \in L$ and $h = \langle f, g\rangle \in \mathrm{C}^*(\uparrow a)$ (recall Remark 5.1). We define $\widehat{f} \in \mathbf{Frm}(\mathfrak{L}_l(\mathbb{R}), L)$ and $\widehat{g} \in \mathbf{Frm}(\mathfrak{L}_u(\mathbb{R}), L)$ by

$$\widehat{f}(-,s) = \begin{cases} 0, & \text{if } s \le 0, \\ f(-,s), & \text{if } 0 < s \le 1, \\ 1, & \text{if } s > 1, \end{cases} \quad \text{and} \quad \widehat{g}(r,-) = \begin{cases} 1, & \text{if } r < 0, \\ g(r,-), & \text{if } 0 \le r < 1, \\ 0, & \text{if } r \ge 1. \end{cases}$$

Moreover, $\widehat{f} \in \mathrm{USC}(L)$ and $\widehat{g} \in \mathrm{LSC}(L)$ since the extra condition defining upper (resp., lower) semicontinuity follows from $\widehat{f}(-,0) = 0$ (resp., $\widehat{g}(1,-) = 0$). Finally, it is easy to check that $\widehat{f} \lhd_{u,l} \widehat{g}$, i.e. $(\widehat{f}, \widehat{g}) \in \mathrm{UL}(L)$.

Note that the construction above is only possible when the continuous function h belongs to $\mathrm{C}^*(L)$.

A function $\Phi_a : \mathrm{C}^*(\uparrow a) \to \mathrm{C}^*(L)$ is called an *extender* if $\Phi_a(h)$ extends h, meaning that $c_{\uparrow a} \circ \Phi_a(h) = h$ for each $h \in \mathrm{C}^*(\uparrow a)$. Thus, $\Phi_a(h)$ extends h whenever

$$\Phi_a(h)(p,q) \vee a = h(p,q)$$

for all $p, q \in \mathbb{Q}$.

We shall say that L has the *monotone bounded extension property* if for each $a \in L$ there exists a monotone extender Φ_a.

Proposition 6.2. *Every monotonically normal locale has the monotone bounded extension property.*

Proof. Let L be monotonically normal and let $h = \langle f, g\rangle \in \mathrm{C}^*(\uparrow a)$ with $(f,g) \in \mathcal{C}_{\uparrow a}$ (see Remark 5.1). By Remark 6.1 and Theorem 5.4, $\Phi_a : \mathrm{C}^*(\uparrow a) \to \mathrm{C}^*(L)$ defined by $\Phi_a(h) = \Lambda(\widehat{f}, \widehat{g})$ is as required. In fact, since $\widehat{f} \le \Lambda(\widehat{f}, \widehat{g}) \le \widehat{g}$, it follows that $\Phi_a(h)(-,0) \le \widehat{f}(-,0) = 0$ and $\Phi_a(h)(1,-) \le \widehat{g}(1,-) = 0$, i.e. $\Phi_a(h) \in \mathrm{C}^*(L)$. Also, for each $p, q \in \mathbb{Q}$ we have

$$\Phi_a(h)(p,-) \vee a \le \widehat{g}(p,-) \vee a = g(p,-) = h(p,-)$$

and

$$\Phi_a(h)(-,q) \vee a \le \widehat{f}(-,q) \vee a = f(-,q) = h(-,q).$$

Consequently Φ_a is a monotone extender. $\square$

We would like to emphasize that the previous result could also be stated in terms of unbounded real functions. However, its proof is quite technical and so we omit it.

We now recall the extension theorem of Stares [28, Theorem 2.3] which we enrich with the assertion (3).

Theorem 6.3. *Let X be a topological space. The following are equivalent:*

(1) *X is monotonically normal.*
(2) *For each closed $F \subseteq X$ there exists an extender $\Phi_F : \mathrm{C}(F, [0,1]) \to \mathrm{C}(X, [0,1])$ such that for each two closed $F_1 \subseteq F_0$ and each $f_i \in \mathrm{C}(F_i, [0,1])$ $(i = 0, 1)$ the following is satisfied:*
   (i) *If $f_0|_{F_1} \ge f_1$ and $f_0(x) = 1$ for all $x \in F_0 \setminus F_1$, then $\Phi_{F_0}(f_0) \ge \Phi_{F_1}(f_1)$.*
   (ii) *If $f_0|_{F_1} \le f_1$ and $f_0(x) = 0$ for all $x \in F_0 \setminus F_1$, then $\Phi_{F_0}(f_0) \le \Phi_{F_1}(f_1)$.*

(3) *For each closed $F \subseteq X$ there exists an extender $\Phi_F : \mathrm{C}(F, [0,1]) \to \mathrm{C}(X, [0,1])$ such that for each two non-disjoint closed F, G and $f \in \mathrm{C}(F, [0,1])$, $g \in \mathrm{C}(G, [0,1])$, if $g|_{F\cap G} \geq f|_{F\cap G}$, $g(x) = 1$ for all $x \in G \setminus F$ and $f(x) = 0$ for all $x \in F \setminus G$, then $\Phi_G(g) \geq \Phi_F(f)$.*

Proof. (1) $\Leftrightarrow$ (2): See [28].

(2) $\Rightarrow$ (3): Let F, G be non-disjoint closed subspaces and $f \in \mathrm{C}(F, [0,1])$, $g \in \mathrm{C}(G, [0,1])$ such that $g|_{F\cap G} \geq f|_{F\cap G}$, $g(x) = 1$ for all $x \in G \setminus F$ and $f(x) = 0$ for all $x \in F \setminus G$.

Take $F_1 = F \cap G$, $F_0 = G$, $f_1 = f|_{F\cap G}$ and $f_0 = g$. Then $f_0|_{F_1} \geq f_1$ and $f_0(x) = g(x) = 1$ for all $x \in F_0 \setminus F_1 = G \setminus F$. It follows from (i) of (2) that $\Phi_G(g) = \Phi_{F_0}(f_0) \geq \Phi_{F_1}(f_1)$.

Now, with F and g playing the role of F_0 and f_0 of (ii) of (2), we get $\Phi_F(f) = \Phi_{F_0}(f_0) \leq \Phi_{F_1}(f_1)$.

Consequently $\Phi_G(g) \geq \Phi_{F_1}(f_1) \geq \Phi_F(f)$.

(3) $\Rightarrow$ (2): Let $F_1 \subseteq F_0$ be closed subspaces and $f_i \in C(F_i, [0,1])$ $(i = 0, 1)$. If $f_0|_{F_0} \geq f_1$ and $f_0(x) = 1$ for all $x \in F_0 \setminus F_1$, then (2) applies with $F = F_1$, $f = f_1$, $G = F_0$ and $g = f_0$ and so $\Phi_G(g) \geq \Phi_F(f)$, i.e. $\Phi_{F_0}(f_0) \geq \Phi_{F_1}(f_1)$.

Also, if $f_0|_{F_0} \leq f_1$ and $f_0(x) = 0$ for all $x \in F_0 \setminus F_1$, then (2) applies with $F = F_0$, $f = f_0$, $G = F_1$ and $g = f_1$ and so $\Phi_G(g) \geq \Phi_F(f)$, i.e. $\Phi_{F_1}(f_1) \geq \Phi_{F_0}(f_0)$. $\square$

The particular case $L = \mathcal{O}X$ of the next theorem is the pointfree counterpart of Theorem 6.3 and hence implies Theorem 2.3 of [28]. Our proof, when applied to the case $L = \mathcal{O}X$, provides another proof of Theorem 2.3 of [28]. It is worth mentioning that the proof in [28] depends upon the T_1-axiom, while our argument is free of it. For historical reasons, it may also be remarked that Theorem 2.3 of [28] could have already been obtained as a consequence of the (topological) insertion theorem of [16].

Theorem 6.4. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *For every $a \in L$ there exists an extender $\Phi_a : \mathrm{C}^*(\uparrow a) \to \mathrm{C}^*(L)$ such that for each $a_1, a_2 \in L$ and $h_i = \langle f_i, g_i \rangle \in \mathrm{C}^*(\uparrow a_i)$ $(i = 1, 2)$ with $(\widehat{f_1}, \widehat{g_1}) \leq (\widehat{f_2}, \widehat{g_2})$ one has $\Phi_{a_1}(h_1) \leq \Phi_{a_2}(h_2)$.*

Proof. (1) $\Rightarrow$ (2): For each $a \in L$ let $\Phi_a : \mathrm{C}^*(\uparrow a) \to \mathrm{C}^*(L)$ be the extender given by Proposition 6.2, i.e., $\Phi_a(h) = \Lambda(\widehat{f}, \widehat{g})$ where $(f, g) \in \mathcal{C}_{\uparrow a}$ generates h (see Remark 5.1). Let $a_2, a_1 \in L$ and $h_i = \langle f_i, g_i \rangle$ $(i = 1, 2)$ with $(\widehat{f_1}, \widehat{g_1}) \leq (\widehat{f_2}, \widehat{g_2})$. Then, $\Phi_{a_1}(h_1) \leq \Phi_{a_2}(h_2)$.

(2) $\Rightarrow$ (1): We shall exhibit a function $\Sigma : \mathcal{D}_L \to L$ satisfying condition (2) of Proposition 4.5. For each $(a, b) \in \mathcal{D}_L$ consider the characteristic maps of a and b that take values in the locale $\uparrow(a \wedge b)$ (rather than in L), i.e., $\chi_a^u \in \mathrm{USC}(\uparrow(a \wedge b))$ and $\chi_b^l \in \mathrm{LSC}(\uparrow(a \wedge b))$.

Since $a \vee b = 1$, it follows that $\chi_a^u \lhd_{u,l} \chi_b^l$, while $\chi_b^l \lhd_{l,u} \chi_a^u$ follows from the fact that $0_{\uparrow(a\wedge b)} = a \wedge b$. By Remark 5.1, $h_{ab} = \langle \chi_a^u, \chi_b^l \rangle \in \mathrm{C}^*(\uparrow(a \wedge b))$.

Note also that $\widehat{\chi_a^u} = \chi_a^u$ and $\widehat{\chi_b^l} = \chi_b^l$ in $\mathrm{C}^*(L)$ (see Remark 6.1).

By hypothesis, there exists an extender $\Phi_{a\wedge b} : \mathrm{C}^*(\uparrow(a \wedge b)) \to \mathrm{C}^*(L)$ satisfying the condition of (2). Define $\Sigma : \mathcal{D}_L \to L$ by

$$\Sigma(a, b) = \Phi_{a\wedge b}(h_{ab})\left(\frac{1}{2}, -\right) \wedge \Phi_{a\wedge b}(h_{ba})\left(-, \frac{1}{2}\right).$$

Claim 1: $\Sigma \wedge \Sigma^{op} = 0$. Indeed,

$$\Sigma(a, b) \wedge \Sigma(b, a) \leq \Phi_{a\wedge b}(h_{ab})\left(\frac{1}{2}, -\right) \wedge \Phi_{a\wedge b}(h_{ab})\left(-, \frac{1}{2}\right) = 0.$$

Claim 2: $\Sigma(a, b) \leq b$ and $a \vee \Sigma(a, b) = 1$. Indeed, since $\Phi_{a\wedge b}$ is an extender, one has $\Phi_{a\wedge b}(h) \vee (a \wedge b) = h$. Thus

$$(a \wedge b) \vee \Sigma(a, b) = h_{ab}\left(\frac{1}{2}, -\right) \wedge h_{ba}\left(-, \frac{1}{2}\right) = \chi_b^l\left(\frac{1}{2}, -\right) \wedge \chi_a^u\left(-, \frac{1}{2}\right) = b.$$

In particular, $\Sigma(a, b) \leq b$ and $a \vee \Sigma(a, b) \geq a \vee b = 1$

Claim 3: $\Sigma(a, b) \le \Sigma(a, c)$ whenever $(a, b) \le (a, c)$ in $\mathcal{D}_L$. Indeed, we have $h_{ca} = \langle \chi^u_c, \chi^l_a \rangle$, $h_{ba} = \langle \chi^u_b, \chi^l_a \rangle$, and since

$$(\widehat{\chi^u_c}, \widehat{\chi^l_a}) = (\chi^u_c, \chi^l_a) \le (\chi^u_b, \chi^l_a) = (\widehat{\chi^u_b}, \widehat{\chi^l_a}),$$

it follows that $\Phi_{a\wedge c}(h_{ca}) \le \Phi_{a\wedge b}(h_{ba})$.

Dually, $h_{ac} = \langle \chi^u_a, \chi^l_c \rangle$, $h_{ab} = \langle \chi^u_a, \chi^l_b \rangle$, and since

$$(\widehat{\chi^u_a}, \widehat{\chi^l_c}) = (\chi^u_a, \chi^l_c) \le (\chi^u_a, \chi^l_b) = (\widehat{\chi^u_a}, \widehat{\chi^l_b}),$$

it follows that $\Phi_{a\wedge c}(h_{ac}) \ge \Phi_{a\wedge b}(h_{ab})$.

Finally

$$\begin{aligned}\Sigma(a, c) &= \Phi_{a\wedge c}(h_{ac})\left(\frac{1}{2}, -\right) \wedge \Phi_{a\wedge c}(h_{ca})\left(-, \frac{1}{2}\right)\\ &\ge \Phi_{a\wedge b}(h_{ab})\left(\frac{1}{2}, -\right) \wedge \Phi_{a\wedge b}(h_{ba})\left(-, \frac{1}{2}\right) = \Sigma(a, b). \quad \square\end{aligned}$$

In the following we shall provide an argument showing that the extension theorem of Stares [28] indeed follows from our pointfree extension theorem. For this purpose, let us recall the relationship between $\mathrm{C}(F, [0, 1])$ and $\mathrm{C}^*(\uparrow(X \setminus F))$ given by:

$$\mathrm{C}(F, [0, 1]) \ni f \longmapsto h_f \in \mathrm{C}^*(\uparrow(X \setminus F)) : h_f(p, q) = f^{-1}(p, q) \cup (X \setminus F).$$

Then we have the following:

Proposition 6.5. *Let X be a topological space, F, G be non-disjoint closed and $f \in \mathrm{C}(F, [0, 1])$, $g \in \mathrm{C}(G, [0, 1])$. Then $g|_{F\cap G} \ge f|_{F\cap G}$, $g(x) = 1$ for all $x \in G \setminus F$ and $f(x) = 0$ for all $x \in F \setminus G$ if and only if*

$$\widehat{h_g} = \langle \widehat{h^1_g}, \widehat{h^2_g} \rangle \ge \langle \widehat{h^1_g}, \widehat{h^2_g} \rangle = \widehat{h_f}.$$

Proof. $\Rightarrow$: $g|_{F\cap G} \ge f|_{F\cap G}$ implies that $f^{-1}(p, +\infty) \cap F \cap G \subseteq g^{-1}(p, +\infty)$ for all $p \in \mathbb{Q}$. Also, since $g(x) = 1$ for all $x \in G \setminus F$, it follows that $G \setminus F \subseteq g^{-1}(p, +\infty)$ whenever $p < 1$. Consequently

$$\widehat{h^1_g}(p, -) = g^{-1}(p, +\infty) \cup (X \setminus G) \supseteq f^{-1}(p, +\infty) \cup (X \setminus F) = \widehat{h^1_f}(p, -).$$

On the other hand, $g|_{F\cap G} \ge f|_{F\cap G}$ implies that $g^{-1}(-\infty, q) \cap F \cap G \subseteq f^{-1}(-\infty, q)$ for all $q \in \mathbb{Q}$. Also, since $f(x) = 0$ for all $x \in F \setminus G$, it follows that $F \setminus G \subseteq f^{-1}(-\infty, q)$ whenever $q > 0$. Consequently

$$\widehat{h^1_g}(-, q) = g^{-1}(-\infty, q) \cup (X \setminus G) \subseteq f^{-1}(-\infty, q) \cup (X \setminus F) = \widehat{h^1_f}(-, q).$$

It follows that $\widehat{h_g} = \langle \widehat{h^1_g}, \widehat{h^2_g} \rangle \ge \widehat{h_f} = \langle \widehat{h^1_f}, \widehat{h^2_f} \rangle$.

$\Leftarrow$: Let $x \in F \cap G$. If $g(x) < f(x)$, take $q \in \mathbb{Q}$ such that $g(x) < q < f(x)$, and since $x \in \widehat{h^1_g}(-, q) \subseteq \widehat{h^1_f}(-, q)$ we conclude that $x \in f^{-1}(-\infty, q) \cup (X \setminus F)$, a contradiction.

On the other hand, if $g(x) < 1$ for some $x \in G \setminus F$, then there exists $p \in \mathbb{Q}$ such that $g(x) < p < 1$ and, since $x \in X \setminus F \subseteq \widehat{h^1_f}(p, -) \subseteq \widehat{h^1_g}(p, -)$ we conclude that $x \in g^{-1}(p, +\infty) \cup (X \setminus G)$, a contradiction.

Dually, if $f(x) > 0$ for some $x \in F \setminus G$, then there exists $q \in \mathbb{Q}$ such that $f(x) > q > 0$ and, since $x \in X \setminus G \subseteq \widehat{h^1_g}(-, q) \subseteq \widehat{h^1_f}(-, q)$ we conclude that $x \in f^{-1}(-\infty, q) \cup (X \setminus F)$, a contradiction. $\square$

After these explanations it is now clear how Stares' extension theorem can be deduced from Theorem 6.3.

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