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Fibonacci numbers, alternating parity sequences and faces of the tridiagonal Birkhoff polytope☆

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Abstract

We determine the number of alternating parity sequences that are subsequences of an increasing m-tuple of integers. For this and other related counting problems we find formulas that are combinations of Fibonacci numbers. These results are applied to determine, among other things, the number of vertices of any face of the polytope of tridiagonal doubly stochastic matrices.

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1. Introduction

An *alternating parity sequence* (*a.p. sequence*, for short) is a (strictly) increasing sequence of integers, with a finite number of entries, such that any two adjacent entries have opposite parities. These are well-known objects in combinatorics under the name *alternating subsets of integers*. The first known reference on them goes back to the nineteenth century; since then, many counting results have been obtained involving sequences of this kind and of several of its generalizations. The precise references may be found in the papers [14,17,10]. Most generalizations go over, instead of the ith entry "parity", its residue class modulo a fixed number m, or modulo an m_i depending on the entry's position; and, very often, the counting involves the sequences of a fixed length, and fixed lower and upper bounds. Our approach here is of a different kind: we fix an arbitrary increasing sequence of integers, $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$, and count the a.p. subsequences of τ whose leftmost [rightmost] entry has a prescribed parity.

It is well known that the number of a.p. subsequences of $(1, 2, \ldots, k)$ is $F_{k+3} - 2$, where F_n is the nth Fibonacci number, determined by the usual recursion $F_{n+2} = F_n + F_{n+1}$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Our research on a.p. subsequences of τ lead us to several counting formulas that are sums of products of Fibonacci numbers (cf. Theorems 4.3, 4.4 and 4.5). The methods range from *ad hoc* techniques to the use of the inclusion–exclusion principle.

The motivation for this research was the study of the facial structure of Ω_n, the polytope of all n-by-n nonnegative doubly stochastic matrices, known as *transportation polytope*, or the *Birkhoff polytope*. This has been extensively considered in the literature, lying at the crossroads of several branches of mathematics. For example, the Birkhoff polytope

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arises in the optimal assignment problem which can be seen as a special Hitchcock problem (cf. [9]). Nonnegative doubly stochastic matrices are also connected with probability theory since each row (column) can be identified as a discrete probability law (cf. [15]).

We are particularly concerned with the set $\mathfrak{T}_n$ whose elements are the $n \times n$ tridiagonal doubly stochastic matrices, which is a face of the Birkhoff polytope. The facial structure of Ω_n has been the object of a systematic study in the series of papers [3–6], and also in [2,8]. However, the tridiagonal case has interesting combinatorial peculiarities that deserve further analysis.

In [8] it is proven that $\mathfrak{T}_n$ has F_{n+1} vertices. In this paper, we find a closer connection of vertex counting in $\mathfrak{T}_n$ with Fibonacci numbers. In particular, our results on a.p. subsequences will be applied to determine the number of vertices of an arbitrarily given face of $\mathfrak{T}_n$. We also give an expression for the number of edges of $\mathfrak{T}_n$.

For the general theory of polytopes, and on the number, $f_d(K)$, of faces of dimension d of a polytope K, we refer the reader to [11].

2. The faces of $\mathfrak{T}_n$

As $\mathfrak{T}_n$ is a face of Ω_n, the faces of $\mathfrak{T}_n$ are the faces of Ω_n which are contained in $\mathfrak{T}_n$. According to [3], the faces of $\mathfrak{T}_n$ are in one-to-one correspondence with the $n \times n$ tridiagonal matrices A, with entries 0 or 1 having total support (this means that A is a Boolean sum of $n \times n$, tridiagonal, permutation matrices). In this paper, A will always denote a matrix of this kind. The face of $\mathfrak{T}_n$ corresponding to A is

$$\mathscr{F}_A := \{X \in \mathfrak{T}_n : a_{ij} = 0 \Rightarrow x_{ij} = 0\}.$$

A famous result of G. Birkhoff asserts that the vertices of Ω_n are the $n \times n$ permutation matrices (cf. [1,13]). So the vertices of $\mathfrak{T}_n$ are the tridiagonal permutation matrices; as these matrices are symmetric, all elements of $\mathfrak{T}_n$, and all 0–1 matrices A to be considered in the sequel, are symmetric as well.

Taking a look at the super-diagonal entries of A (*i.e.*, the a_{ij} with $j = i + 1$), we see that A is a direct sum of square blocks, $A = A_1 \oplus \cdots \oplus A_p$, where each A_t is of one of the following types:

Type 1: $A_t = 1$, a one-by-one matrix;

Type 2: $A_t = K$, where K is the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;

Type 3: A_t is not of the previous two types, and all super-diagonal entries of A_t are 1's.

If A_t is of type 3, then its first and last diagonal entries are 1, otherwise A would not have total support. We display a 7×7 example of a type 3 matrix, where unspecified entries are 0:

$$\begin{bmatrix} 1 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 1 \end{bmatrix}. \tag{1}$$

Definition 2.1. An *S-matrix* is a symmetric tridiagonal matrix of 0–1 entries, different from 1 and K, with all super-diagonal entries $=1$, and with first and last diagonal entries $=1$. The blocks A_t of type 3 are called the *S-blocks* of A.

Let B be an S-matrix of order m. We say that b_{ii} is an *inner entry* of B, if $b_{ii} = 1$ and $1 < i < m$. (In example (1) the inner entries are b_{33} and b_{44}.)

Note that $\mathscr{F}_A = \mathscr{F}_{A_1} \times \cdots \times \mathscr{F}_{A_p}$, and $\mathscr{F}_{A_t}$ is a singleton if A_t is a block of type 1 or 2. Now, if we omit these singleton faces from the cartesian product, and reorder the $\mathscr{F}_{A_i}$ corresponding to the S-blocks A_t, we get a polytope that is affinely isometric to $\mathscr{F}_A$ (with respect to the usual inner product for square matrices: $\langle U|V\rangle = \operatorname{tr}(UV^T)$). This implies that only the multi-set of S-blocks of A matters in the study of a single face $\mathscr{F}_A$.

As a simple example of this we determine the dimension of $\mathscr{F}_A$. Recall that an $n \times n$ matrix, with $n > 1$, is *fully indecomposable* if it cannot be brought to the form

$$\begin{bmatrix} * & O \\ * & * \end{bmatrix}$$

by a permutation of its rows and a permutation of its columns, where O is a $p \times q$, nonempty, zero matrix with $p+q=n$.

Lemma 2.2. *Any S-matrix has total support and is fully indecomposable.*

Proof. For $m > 1$, let H be the $m \times m$ matrix defined by $h_{ij} = 1$ if and only if: $|i-j| = 1$, or $i = j \in \{1, m\}$ (the $m \times m$ S-matrix with minimum number of 1's). Clearly, H has total support, and if we transform a 0 diagonal entry into a 1 we also get a matrix of total support. As the Boolean sum of matrices of total support has total support, we may conclude that any S-matrix has total support.

The matrix H gives rise to a simple bipartite graph, with $2m$ vertices, $u_1, \ldots, u_m, v_1, \ldots, v_m$, and the unordered pair $\{u_i, v_j\}$ is an edge of the graph iff $h_{ij} = 1$. This graph is obviously connected. So any S-matrix gives rise to a connected bipartite graph. Therefore, by [7, Theorem 4.2.7], any S-matrix is fully indecomposable. □

According to [3, Theorem 2.5], if B, of order m, is fully indecomposable, $\dim \mathscr{F}_B = \sigma_B - 2m + 1$, where σ_B is the number of 1's in B. So, for an S-matrix B, $\dim \mathscr{F}_B = 1 + w$, where w is the number of inner entries of B. As $\dim \mathscr{F}_A$ is the sum of the $\dim \mathscr{F}_{A_i}$ for the S-blocks of A, we get:

Lemma 2.3. $\dim \mathscr{F}_A = s_A + \imath_A$, *where s_A is the number of S-blocks of A, and $\imath_A$ is the sum of the numbers of inner entries in the S-blocks of A.*

3. Vertices of $\mathfrak{T}_n$ and alternating parity sequences

We shall give explicit formulas for $f_0(\mathscr{F}_A)$ (cf. [11]), the number of vertices of the face $\mathscr{F}_A$. A vertex of $\mathscr{F}_A$ is a permutation matrix which is entrywise $\leqslant A$, so $f_0(\mathscr{F}_A)$ is the permanent of A (cf. [3]). Clearly, $\mathrm{per}(A)$ is the product of the permanents of the S-blocks of A, so we first determine the permanent of an S-matrix.

For any increasing sequence of integers, $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$, we denote by $N(\tau)$, or N_τ, the number of nonempty a.p. subsequences of τ.

Lemma 3.1. *Let B be an S-matrix. The number of vertices of $\mathscr{F}_B$ is $N_\tau + 2$, where τ is the (strictly) increasing sequence of the positions of the inner entries of B (so $1 < \tau_1 < \cdots < \tau_w < n$, and $b_{\tau_i \tau_i} = 1$).*

Proof. There are exactly 2 permutation matrices $\leqslant B$ with no inner entries. So we have to prove that $\mathscr{M}$, the set of the permutation matrices $\leqslant B$ with at least one inner entry, has cardinality N_τ.

Let $\mathscr{I}(P)$ be the increasing sequence of the positions of the inner entries of $P \in \mathscr{M}$. P is a direct sum of blocks, each of which is either 1 or K. In between two consecutive blocks 1 of P, there are only blocks K (possibly none); so the difference between the positions of these consecutive diagonal entries of P is odd. Therefore, $\mathscr{I}(P)$ is an a.p. sequence.

Next, given an a.p. sequence γ, $1 \leqslant \gamma_1 < \cdots < \gamma_r \leqslant n$, the previous argument makes the converse clear: that there exists a unique permutation matrix P_γ such that $\mathscr{I}(P_\gamma) = \gamma$. Moreover, γ is a nonempty subsequence of τ if and only if $P_\gamma \in \mathscr{M}$. So $\mathscr{M}$ has cardinality N_τ. □

Examples.. For an S-matrix B:

(a) If B has no inner entry, $N_\tau = 0$, and so $\mathscr{F}_B$ has 2 vertices.
(b) If B has only one inner entry, $N_\tau = 1$, and so $\mathscr{F}_B$ is a triangle.
(c) Suppose B has exactly two inner entries, $b_{\tau_1\tau_1} = b_{\tau_2\tau_2} = 1$, $\tau_1 < \tau_2$. We have two cases: (i) $\tau_2 - \tau_1$ is even; then the two singletons, (τ_1), (τ_2), are the only a.p. subsequences of τ. So $\mathscr{F}_B$ has four vertices. (ii) $\tau_2 - \tau_1$ is odd; in this case, $N_\tau = 3$, so $\mathscr{F}_B$ has five vertices.

4. Alternating parity sequences and Fibonacci numbers

In this section we prove closed formulas for N_τ in terms of Fibonacci numbers, where $\tau = (\tau_1, \tau_2, \dots, \tau_w)$ is an arbitrary increasing sequence of integers.

The variable X [Y] may take one of three values, A, E, O, meaning "any", "even", "odd", respectively. So, the expression "X number" means "any number", "even number", or "odd number", according to the current value of X. The symbol $\overline{X}$ denotes the *opposite* of X, *i.e.*,

$$\overline{\mathrm{E}} = \mathrm{O}, \quad \overline{\mathrm{O}} = \mathrm{E}, \quad \overline{\mathrm{A}} = \mathrm{A}.$$

For $X \neq Y$, the symbol $XY(\tau)$ (or just XY_τ) denotes *the number of all* a.p. subsequences of τ started with an X number and ended with a Y number, including the empty sequence; if $X = Y$, $XY(\tau)$ is *the number of nonempty* a.p. subsequences of τ started and ended with an X number. The curly notation $\mathscr{XY}(\tau)$, or $\mathscr{XY}_\tau$, denotes the *set* of the a.p. subsequences of τ started with an X number and ended with an Y number, including the empty sequence in case $X \neq Y$. So XY_τ is the cardinality of $\mathscr{XY}_\tau$.

For example, $\mathscr{EO}(\tau)$ is the set of a.p. subsequences of τ started with an even number and ended with an odd number, and $\emptyset \in \mathscr{EO}(\tau)$; $\mathscr{AA}_\tau$ is the set of all nonempty a.p. subsequences of τ, and $\mathrm{AA}_\tau = N_\tau$. Note that, for $X \in \{\mathrm{O}, \mathrm{E}\}$:

$$\mathrm{A}X_\tau = \mathrm{O}X_\tau + \mathrm{E}X_\tau, \quad X\mathrm{A}_\tau = X\mathrm{O}_\tau + X\mathrm{E}_\tau, \quad \mathrm{AA}_\tau = \mathrm{AO}_\tau + \mathrm{AE}_\tau - 2. \tag{2}$$

(The "-2" in the last equation comes from our conventions on the empty sequence.) In the sequel, we denote by L_τ [R_τ] the parity of the leftmost [resp., rightmost] entry of τ.

Given a subsequence of τ, say $\kappa = (\kappa_1, \dots, \kappa_r)$, and an integer z, the *z-reverse of* κ, denoted κ^z, is defined by $\kappa_i^z := z + 1 - \kappa_{r+1-i}$, for $i = 1, \dots, r$. If z is odd [even], then z reversion preserves [resp., reverses] parity, in the sense that κ_i and κ_{r+1-i}^z have the same parity [resp., opposite parity], for all i. Clearly, z-reversion is an involution that maps the set of a.p. subsequences of τ, onto the set of a.p. subsequences of τ^z. In the sequel, we shall use the following identities with no further comment: if z is odd, $[\mathscr{XY}(\tau)]^z = \mathscr{YX}(\tau^z)$; and, if z is even, $[\mathscr{XY}(\tau)]^z = \overline{\mathscr{Y}}\,\overline{\mathscr{X}}(\tau^z)$.

To determine $XY(\tau)$ in case τ is the sequence $(1, 2, \dots, w)$, we use the notation $XY_w := XY(1, 2, \dots, w)$. The sequences OO_w, OE_w satisfy the following recursions and initial conditions:

$$\mathrm{OO}_1 = 1; \quad \mathrm{OO}_2 = 1; \quad \mathrm{OE}_1 = 1; \quad \mathrm{OE}_2 = 2; \tag{3}$$

$$\mathrm{OO}_w = \mathrm{OE}_{w-1} + \mathrm{OO}_{w-2}, \quad \mathrm{OE}_w = \mathrm{OE}_{w-1} \quad \text{for odd } w > 2; \tag{4}$$

$$\mathrm{OE}_w = \mathrm{OO}_{w-1} + \mathrm{OE}_{w-2}, \quad \mathrm{OO}_w = \mathrm{OO}_{w-1} \quad \text{for even } w > 2. \tag{5}$$

The initial conditions (3) are trivial to check. To prove the first equation in (4), note that any γ in the set $\mathscr{OO}(1, 2, \dots, w)$ is of one of the following mutually exclusive types: (i) γ ends up with w; (ii) γ ends with an odd number $\leqslant w - 2$. Clearly, there are OE_{w-1} sequences of type (i); and OO_{w-2} sequences of type (ii). The second identity in (4) is obvious. To prove (5) we argue in a similar manner.

The recursion (3)–(5) determines uniquely the OO_w and OE_w; if we replace OO_w and OE_w by the following values

$$\mathrm{OO}_w = F_{w+1}, \quad \mathrm{OE}_w = F_w \quad \text{for odd } w \geqslant 1, \tag{6}$$

$$\mathrm{OO}_w = F_w, \quad \mathrm{OE}_w = F_{w+1} \quad \text{for even } w \geqslant 1, \tag{7}$$

then (3)–(5) are satisfied for all $w \geqslant 1$; therefore, OO_w and OE_w are given by (6)–(7). To determine EE_w, EO_w, EA_w, we z-reverse (6)–(7) to get: $\mathrm{EO}_w = F_w$, for odd $w \geqslant 1$; and $\mathrm{EE}_w = F_w$ for even $w \geqslant 1$. Then we use the identities $\mathrm{EO}_w = \mathrm{EO}_{w-1}$, for even w, and $\mathrm{EE}_w = \mathrm{EE}_{w-1}$ for odd w, to get the remaining values of EE_w and EO_w. And we get

$$\mathrm{EE}_w = F_{w-1}, \quad \mathrm{EO}_w = F_w \quad \text{for odd } w \geqslant 1,$$

$$\mathrm{EE}_w = F_w, \quad \mathrm{EO}_w = F_{w-1} \quad \text{for even } w \geqslant 1.$$

Now the numbers XA_w, AX_w and AA_w are obtained at once from (2):

$$\mathrm{OA}_w = F_{w+2} \quad \mathrm{EA}_w = F_{w+1},$$

$$\mathrm{AO}_w = F_{w+2} \quad \mathrm{AE}_w = F_{w+1}, \quad \text{for odd } w \geqslant 1,$$

$$\mathrm{AO}_w = F_{w+1} \quad \mathrm{AE}_w = F_{w+2}, \quad \text{for even } w \geqslant 1,$$

and $\mathrm{AA}_w = F_{w+3} - 2$. From this we get, with an easy proof:

Theorem 4.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_w)$ be an a.p. sequence. Recall L_α $[R_\alpha]$ is the parity of the leftmost [resp., rightmost] entry of α. For $X, Y \in \{\mathrm{E}, \mathrm{O}\}$, we have*

$$XY_\alpha = \begin{cases} F_{w+1} & if\ X = L_\alpha\ and\ Y = R_\alpha, \\ F_w & if\ X \neq L_\alpha\ and\ Y = R_\alpha, \\ F_w & if\ X = L_\alpha\ and\ Y \neq R_\alpha, \\ F_{w-1} & if\ X \neq L_\alpha\ and\ Y \neq R_\alpha, \end{cases} \tag{8}$$

$$X\mathrm{A}_\alpha = \begin{cases} F_{w+2} & if\ X = L_\alpha, \\ F_{w+1} & if\ X \neq L_\alpha, \end{cases}$$

$$\mathrm{A}Y_\alpha = \begin{cases} F_{w+2} & if\ Y = R_\alpha, \\ F_{w+1} & if\ Y \neq R_\alpha. \end{cases} \tag{9}$$

4.1. Homogeneous formulas

We now fix two values, L and R, in the set {E, O}, and seek a formula for LR_τ, the cardinality of $\mathscr{L}\mathscr{R}_\tau$ (recall this is the set of the a.p. subsequences of τ beginning [ending] with an L [resp., R] number).

We may represent τ as a concatenation

$$\tau = \alpha^1 \alpha^2 \cdots \alpha^m, \tag{10}$$

where α^i is a nonempty a.p. subsequence of τ, such that the concatenation $\alpha^i \alpha^{i+1}$ is not an a.p. sequence, for $1 \leqslant i < m$. So α^i is made up of consecutive entries of τ of alternating parities, and it is maximal under these conditions. The α^i's are uniquely determined, and called *a.p. components of* τ. The length of α^i will be denoted by r_i. The formulas given in Theorems 4.3 and 4.4 for LR_τ are homogeneous in the sense that they are sums of m-fold products of Fibonacci numbers.

Any $\gamma \in \mathscr{L}\mathscr{R}_\tau$ will be represented, according to the a.p. decomposition (10) of τ, as a concatenation

$$\gamma = \gamma^1 \gamma^2 \cdots \gamma^m,$$

where γ^i is a, possibly empty, subsequence of α^i. To each such γ we associate a sequence of $m+1$ parities,

$$P_\gamma = (Z_1, Z_2, \ldots, Z_{m+1}),$$

satisfying the condition

$$\gamma^i \in \mathscr{Z}_i \overline{\mathscr{Z}}_{i+1}(\alpha^i) \quad \text{for } 1 \leqslant i \leqslant m. \tag{11}$$

In case γ^i is nonempty, (11) implies that Z_i $[\overline{Z}_{i+1}]$ is the parity of the first [resp., last] entry of γ^i. But if γ^i is empty, (11) only says that $Z_i = Z_{i+1}$. In any case the "boundary conditions"

$$Z_1 = L \quad \text{and} \quad Z_{m+1} = \overline{R} \tag{12}$$

together with (11), *determine P_γ in a unique way*. To show this, suppose p and q are integers such that γ^i is empty, for $p < i < q$, and γ^p and γ^q are nonempty. Then $Z_i = Z_{i+1}$ for $p < i < q$. On the other hand, $\overline{Z}_{p+1}$ is the parity, R_{γ^p}, of the rightmost entry of γ^p; so $\overline{Z}_i = R_{\gamma^p}$, for $p < i \leqslant q$. Note that the value $Z_q = \overline{R}_{\gamma^p}$ agrees with the fact that the last entry of γ^p and the first entry of γ^q, are consecutive entries of γ. The case when γ^1 [γ^m] is empty is similarly treated, taking (12) into account. Note also that, if γ is empty, then R is opposite to L, and $Z_i = L$ for all i.

Let us denote by $\mathscr{S}$ the set of all sequences of parities

$$S = (X_1, \dots, X_{m+1}), \tag{13}$$

and by $\mathscr{S}(L, R)$ the set of the elements of $\mathscr{S}$ satisfying $X_1 = L$ and $X_{m+1} = \overline{R}$. The mapping defined above, $\gamma \mapsto P_\gamma$, maps $\mathscr{L}\mathscr{R}_\tau$ onto $\mathscr{S}(L, R)$. Given (13), how many $\gamma \in \mathscr{L}\mathscr{R}_\tau$ satisfy $P_\gamma = S$? The answer is, obviously, $\prod_{i=1}^m X_i \overline{X}_{i+1}(\alpha^i)$. This implies the following formula

$$LR_\tau = \sum_{S\in\mathscr{S}(L,R)} \prod_{i=1}^{m} X_i \overline{X}_{i+1}(\alpha^i), \tag{14}$$

where the sum is extended to all sequences (13) in $\mathscr{S}(L, R)$. In the next theorem, we express LR_τ as a combination of Fibonacci numbers.

Definition 4.2. Let Ψ be the set of m-tuples $\omega = (\omega_1, \dots, \omega_m)$, with entries in $\{-1, 0, 1\}$, whose nonzero entries occur, from left to right, with alternating signs (example: $(0, 0, -1, 0, 1, -1, 0, \dots, 0, 1)$). Four subsets of Ψ will be distinguished, denoted by $\Psi(u, v)$, with u, v in $\{1, -1\}$. By definition, $\Psi(u, v)$ contains all nonzero $\omega \in \Psi$ whose first nonzero entry is u, and whose last nonzero entry is v; besides, by convention, the zero m-tuple belongs to $\Psi(1, -1)$ and to $\Psi(-1, 1)$, but *does not belong* to either $\Psi(1, 1)$ or $\Psi(-1, -1)$.

Clearly $\Psi(1, -1) = -\Psi(-1, 1)$ and $\Psi(1, 1) = -\Psi(-1, -1)$. Note that these four sets have cardinality 2^{m-1} (hint: $\Psi(1, 1)$ [$\Psi(1, -1)$] is in natural bijective correspondence with the set of subsets of $\{1, \dots, m\}$ of odd [resp., even] cardinality).

Theorem 4.3. *For any increasing integer sequence τ, with m a.p. components of lengths $r_1, \dots, r_m$, we have*

$$LR_\tau = \sum_{\omega\in\Psi(u,v)} \prod_{i=1}^{m} F_{r_i+\omega_i},$$

where $u = 1$ [$u = -1$] if $L = L_\tau$ [resp., $L \neq L_\tau$], and $v = 1$ [$v = -1$] if $R = R_\tau$ [resp., $R \neq R_\tau$] .

Proof. For any $S = (X_1, \dots, X_{m+1})$, Theorem 4.1 implies

$$X_i \bar{X}_{i+1}(\alpha^i) = F_{r_i+\varepsilon_i(S)},$$

where the coefficients $\varepsilon_i(S)$ are given according to the table (8):

$$\varepsilon_i(S) = \begin{cases} 1 & \text{if } X_i = L_{\alpha^i} \text{ and } X_{i+1} \neq L_{\alpha^{i+1}}, \\ 0 & \text{if } X_i \neq L_{\alpha^i} \text{ and } X_{i+1} \neq L_{\alpha^{i+1}}, \\ 0 & \text{if } X_i = L_{\alpha^i} \text{ and } X_{i+1} = L_{\alpha^{i+1}}, \\ -1 & \text{if } X_i \neq L_{\alpha^i} \text{ and } X_{i+1} = L_{\alpha^{i+1}}. \end{cases} \tag{15}$$

We may then write (14) as

$$LR_\tau = \sum_{S\in\mathscr{S}(L,R)} \prod_{i=1}^{m} F_{r_i+\varepsilon_i(S)}, \tag{16}$$

and we are left with the proof that $\Psi(u, v)$ is precisely the set of m-tuples $(\varepsilon_1(S), \varepsilon_2(S), \dots, \varepsilon_m(S))$ occurring on the right-hand side of (16), and that these m-tuples occur with no repetition. So we examine in detail the mapping

$$\varepsilon(S) = (\varepsilon_1(S), \varepsilon_2(S), \dots, \varepsilon_m(S)).$$

Assume that $\varepsilon_i = 1$, that is, $X_i = L_{\alpha^i}$ and $X_{i+1} \neq L_{\alpha^{i+1}}$. Therefore ε_{i+1} is either 0 or -1; if it is 0, then ε_{i+2} is either 0 or -1; and if ε_{i+2} is also 0, then ε_{i+3} is either 0 or -1; etc. So, by induction, we see that if $\varepsilon_i = 1$, the next nonzero ε_j equals -1. Similarly we obtain: if $\varepsilon_i = -1$, the next nonzero ε_j equals 1. This means that ε maps $\{\mathrm{E}, \mathrm{O}\}^{m+1}$ into Ψ.

To determine the kernel of ε, let $\varepsilon(S)=0$. From (15) we have the alternative: (a) $X_1 \neq L_{\alpha^1}$ and $X_2 \neq L_{\alpha^2}$; or (b) $X_1 = L_{\alpha^1}$ and $X_2 = L_{\alpha^2}$. In case (a), $X_2 \neq L_{\alpha^2}$ and $\varepsilon_2(S)=0$ imply $X_3 \neq L_{\alpha^3}$; in this way, we may prove by induction that $X_i \neq L_{\alpha^{i+1}}$, for all i; therefore, $S=(\overline{L}_{\alpha^1}, \dots, \overline{L}_{\alpha^{m+1}})$. In case (b) a similar argument proves $S=(L_{\alpha^1}, \dots, L_{\alpha^{m+1}})$. By now, it is obvious that ε transforms $(\overline{L}_{\alpha^1}, \dots, \overline{L}_{\alpha^{m+1}})$ and $(L_{\alpha^1}, \dots, L_{\alpha^{m+1}})$ into 0. So these two m-tuples form the kernel of ε.

Now let ω be a nonzero element of Ψ, and let $\{i_1, \dots, i_t\}$ be the support of ω, $i_1 < \dots < i_t$. Partition the integer interval $]0, m+1]$ into $t+1$ subintervals, $J_k :=]i_k, i_{k+1}]$, $k = 0, \dots, t$ (with $i_0 := 0$, $i_{t+1} := m+1$). Now suppose that $\omega_{i_1} = 1$ (the case $\omega_{i_1} = -1$ is analogous). Define $Z_i = L_{\alpha^i}$ $[Z_i = \overline{L}_{\alpha^i}]$ for i inside the intervals J_k with odd [resp., even] k. It is easy to check that $\varepsilon(Z_1, \dots, Z_{m+1}) = \omega$. So ε is onto Ψ.

Note that Ψ has $2^{m+1}-1$ elements, one less than $\{\mathrm{E}, \mathrm{O}\}^{m+1}$. As the kernel of ε has two elements, ε has to be injective outside its kernel.

Let $S=(X_1, \dots, X_{m+1}) \in \mathscr{S}(L, R)$. By definition (13) have $X_1 = L$, $X_{m+1} = \overline{R}$. The proof here splits into four cases, according to the value of the pair (u, v). These cases are quite similar to each other, and so we chose to consider only one, namely: $(u, v) = (-1, 1)$, that is, the parity L is opposed to the parity of the first entry of τ, and R is the parity of the last entry of τ. So, in table (15), we enter $X_1 \neq L_{\alpha^1}$ and $X_{m+1} \neq L_{\alpha^{m+1}}$. We get: $\varepsilon_1(S)$ is either 0 or -1; if it is 0, then $\varepsilon_2(S)$ is either 0 or -1; etc. So, either $\varepsilon(S)=0$ or the first nonzero entry of $\varepsilon(S)$ is -1. Now enter $X_{m+1} \neq L_{\alpha^{m+1}}$, to obtain: $\varepsilon_m(S)$ is either 0 or 1; if it is 0, then $\varepsilon_{m-1}(S)$ is either 0 or 1; etc. So, either $\varepsilon(S)=0$ or the last nonzero entry of $\varepsilon(S)$ is 1. So we proved $\varepsilon(S)$ belongs to $\Psi(-1, 1)$. And in general we have $\varepsilon(S) \in \Psi(u, v)$, for the prescribed (u, v).

Finally, note that one of the kernel elements of ε, determined above, may lie in $\mathscr{S}(L, R)$, but not both (for fixed L, R). Therefore, ε is one-to-one from $\mathscr{S}(L, R)$ into $\Psi(u, v)$; it is in fact onto $\Psi(u, v)$, because $\mathscr{S}(L, R)$ and $\Psi(u, v)$ both have 2^{m-1} elements. This ends the proof. $\square$

Theorem 4.4. *With the notation of Theorem* 4.3, *we have*

$$L\mathrm{A}_\tau = \sum_{\omega \in \Psi(u,1) \cup \Psi(u,-1)} \prod_{i=1}^{m} F_{r_i+\omega_i}, \quad \mathrm{A}R_\tau = \sum_{\omega \in \Psi(1,v) \cup \Psi(-1,v)} \prod_{i=1}^{m} F_{r_i+\omega_i}$$

and

$$N_\tau = \prod_{i=1}^{m} F_{r_i} + \sum_{\omega \in \Psi} \prod_{i=1}^{m} F_{r_i+\omega_i} - 2.$$

Proof. The formulas are direct consequences of Theorem 4.3, combined with (2), and the fact that $\Psi(u,1) \cup \Psi(u,-1)$ and $\Psi(1,v) \cup \Psi(-1,v)$ are disjoint unions. For the last formula, take into account that these two sets have union Ψ, and have exactly one element in common: the zero m-tuple. $\square$

4.2. Inclusion–exclusion formulas

Other formulas for LR_τ, LA_τ, AR_τ and N_τ may be obtained based on the well-known inclusion–exclusion theorem. Let us go back to (10), the a.p. decomposition of τ, where the a.p. component α^i has length r_i. The number $r_1 + \dots + r_m + m - 1$ is denoted by M; the numbers $c_0, \dots, c_m$, given by

$$c_k := r_1 + r_2 + \dots + r_k + k,$$

are called the *gaps* of τ (note that $c_0 = 0$, $c_m = M+1$).

Theorem 4.5. *With the notation just introduced, we have*

$$LR_\tau = F_{M+u+v-1} + \sum_{t=1}^{m-1} (-1)^t \sum_{0<k_1<\dots<k_t<m} F_{c_{k_1}+u-1} F_{M-c_{k_t}+v} \prod_{0<i<t} F_{c_{k_{i+1}}-c_{k_i}},$$

$$L\mathrm{A}_\tau = F_{M+u+1} + \sum_{t=1}^{m-1} (-1)^t \sum_{0<k_1<\dots<k_t<m} F_{c_{k_1}+u-1} F_{M-c_{k_t}+2} \prod_{0<i<t} F_{c_{k_{i+1}}-c_{k_i}},$$

$$\mathrm{A}R_\tau = F_{M+v+1} + \sum_{t=1}^{m-1} (-1)^t \sum_{0<k_1<\cdots<k_t<m} F_{c_{k_1}+1} F_{M-c_{k_t}+v} \prod_{0<i<t} F_{c_{k_{i+1}}-c_{k_i}},$$

$$N_\tau = F_{M+3} + \sum_{t=1}^{m-1} (-1)^t \sum_{0<k_1<\cdots<k_t<m} F_{c_{k_1}+1} F_{M-c_{k_t}+2} \prod_{0<i<t} F_{c_{k_{i+1}}-c_{k_i}} - 2,$$

where $u = 1$ *if* $L = L_\tau$, $u = 0$ *if* $L \neq L_\tau$, $v = 1$ *if* $R = R_\tau$, $v = 0$ *if* $R \neq R_\tau$.

Proof. To prove the theorem we assume, without loss of generality, that τ_1 is odd. (If τ_1 is even, then the proof goes the same way with appropriate reversion of parities.) And also assume, without loss of generality, that the entries of τ are the smallest positive integers compatible with the conditions that τ is an increasing sequence, and the a.p. components of τ have lengths $r_1, \ldots, r_m$ (from left to right), that is

$$\alpha^i = (c_{i-1} + 1, c_{i-1} + 2, \ldots, c_{i-1} + r_i).$$

So, in this context, M is the maximum entry of τ, and the gaps of τ are the elements of $\{0, 1, \ldots, M + 1\}$ that are not entries of τ.

We only prove the formula for LR_τ; the others may be obtained from this one as in Corollary 4.4.

For $0 < k_1 < \cdots < k_t < m$, we let $W_{k_1\cdots k_t}$ be the number of elements of $\mathscr{L}\mathscr{R}(1, 2, \ldots, M)$ having all t gaps $c_{k_1}, \ldots, c_{k_t}$ among its entries; and we denote by $W(t)$ the sum of all $W_{k_1\cdots k_t}$ for a fixed t. For example, $W(0)$ is the number of elements of $\mathscr{L}\mathscr{R}(1, \ldots, M)$; table (8) gives its value

$$W(0) = F_{M+u+v-1}, \tag{17}$$

where u and v are as given in the theorem's statement.

According to the inclusion–exclusion formula [16, Theorem 1.2, p. 19], the number of elements of $\mathscr{L}\mathscr{R}(1, 2, \ldots, M)$ is given by

$$LR_\tau = W(0) - W(1) + W(2) - \cdots + (-1)^{m-1} W(m - 1). \tag{18}$$

We now prove the following formula:

$$W_{k_1\ldots k_t} = F_{c_{k_1}+u-1} \left(\prod_{1<i<t} F_{c_{k_{i+1}}-c_{k_i}} \right) F_{M-c_{k_t}+v}, \tag{19}$$

for positive t. It is easy to describe how one can generate all elements of $\mathscr{L}\mathscr{R}(1, \ldots, M)$ that have $c_{k_1}, \ldots, c_{k_t}$ among their entries. Such a sequence γ has the following structure:

$$\gamma = \Gamma^0 (c_{k_1}) \Gamma^1 (c_{k_2}) \Gamma^2 \cdots (c_{k_t}) \Gamma^t, \tag{20}$$

i.e., γ is a concatenation of $t + 1$ sequences Γ^i, and the t singletons (c_{k_i}). To specify the parities of the extreme entries of each Γ^i, let X_i be the parity of $c_i + 1$, for $0 < i < m$. For notational reasons, we define $k_0 = 0$, $k_{t+1} = m$, $X_0 = L$ and $X_m = R$. Then Γ^i is an arbitrary a.p. sequence in the integer interval $[c_{k_i} + 1, c_{k_{i+1}} - 1]$, beginning with an X_{k_i} number, and ending with an $X_{k_{i+1}}$ number. Therefore, the number of possible Γ^i's is

$$X_{k_i} X_{k_{i+1}} (c_{k_i} + 1, c_{k_i} + 2, \ldots, c_{k_{i+1}} - 1). \tag{21}$$

Let ℓ_i denote the length of the a.p. sequence $(c_{k_i} + 1, c_{k_i} + 2, \ldots, c_{k_{i+1}} - 1)$. Clearly, $\ell_i = c_{k_{i+1}} - c_{k_i} - 1$. To apply Theorem 4.1 we consider three cases:

Case 1: $0 < i < t$. We are in the first instance of (8), therefore (21)$=F_{\ell_i+1}$.

Case 2: $i = 0$. We have to determine $LX_{k_1}(1, 2, \ldots, \ell_0)$; table (8) yields (21)$=F_{\ell_0+1}$ if L is odd, and (21)$=F_{\ell_0}$ if L is even. As we are assuming L_τ is odd, we get (21)$=F_{\ell_0+u}$ with u as given in the statement of the theorem.

Case 3: $i = t$. We determine $X_{k_t} R(c_{k_t} + 1, c_{k_t} + 2, \ldots, c_{k_t} + \ell_t)$; as X_{k_t} is the parity of $c_{k_t} + 1$, (8) yields (21)$=F_{\ell_t+1}$ if $R = R_\tau$, and (21)$=F_{\ell_t}$ if $R \neq R_\tau$. So we get (21)$=F_{\ell_0+v}$ with v as in the statement of the theorem.

As the Γ^i's may vary independently of each other, the number of all a.p. sequences (20) is the product of these $t+1$ Fibonacci numbers, namely

$$F_{\ell_0+u}\left[\prod_{0<i<t} F_{\ell_i+1}\right] F_{\ell_t+v}.$$

As $\ell_0 = c_{k_1} - 1$ and $\ell_t = M - c_{k_t}$, this is precisely the right-hand side of (19) in a different notation. The first formula of the theorem results by entering (17) and (19) in (18). □

Remark 4.6. To have the flavor of the preceding results, we exhibit the formulas for N_τ, for small values of m. For $m = 1$, we have

$$N_\tau = F_{r_1} + F_{r_1} + F_{r_1+1} + F_{r_1-1} - 2 \tag{22}$$
$$= F_{r_1+3} - 2. \tag{23}$$

Formula (22) is what we get from Theorem 4.4, and (23) is from Theorem 4.5. For $m = 2$, we also give the two formulas for N_τ, according to Theorems 4.4 and 4.5, respectively:

$$\begin{aligned} N(\tau) &= F_{r_1}F_{r_2} + F_{r_1}F_{r_2} + F_{r_1+1}F_{r_2} + F_{r_1}F_{r_2+1} \\ &\quad + F_{r_1-1}F_{r_2} + F_{r_1}F_{r_2-1} + F_{r_1+1}F_{r_2-1} + F_{r_1-1}F_{r_2+1} - 2 \\ &= F_{r_1+r_2+4} - F_{r_1+2}F_{r_2+2} - 2. \end{aligned}$$

For $m = 3$, Theorems 4.4 and 4.5 offer the following two formulas:

$$\begin{aligned} N(\tau) &= F_{r_1}F_{r_2}F_{r_3} + F_{r_1}F_{r_2}F_{r_3+1} + F_{r_1+1}F_{r_2-1}F_{r_3} + F_{r_1}F_{r_2+1}F_{r_3-1} \\ &\quad + F_{r_1}F_{r_2}F_{r_3} + F_{r_1-1}F_{r_2}F_{r_3} + F_{r_1+1}F_{r_2}F_{r_3-1} + F_{r_1}F_{r_2-1}F_{r_3+1} \\ &\quad + F_{r_1+1}F_{r_2}F_{r_3} + F_{r_1}F_{r_2-1}F_{r_3} + F_{r_1-1}F_{r_2+1}F_{r_3} + F_{r_1+1}F_{r_2-1}F_{r_3+1} \\ &\quad + F_{r_1}F_{r_2+1}F_{r_3} + F_{r_1}F_{r_2}F_{r_3-1} + F_{r_1-1}F_{r_2}F_{r_3+1} + F_{r_1-1}F_{r_2+1}F_{r_3-1} - 2 \\ &= F_{r_1+r_2+r_3+5} - F_{r_1+2}F_{r_2+r_3+3} - F_{r_1+r_2+3}F_{r_3+2} + F_{r_1+2}F_{r_2+1}F_{r_3+2} - 2. \end{aligned}$$

Later on we need the following instances of Theorem 4.5 when $m = 2$ and the first entry of τ is odd (note that R_τ is, therefore, the parity of $r_1 + r_2 + 1$. So $v = 0$ [$v = 1$] if R has [resp., has not] the same parity as $r_1 + r_2$):

$$\mathrm{AO}(\alpha^1\alpha^2) = \begin{cases} F_{r_1+r_2+2} - F_{r_1+2}F_{r_2} & \text{if } r_1 + r_2 \text{ is odd}, \\ F_{r_1+r_2+3} - F_{r_1+2}F_{r_2+1} & \text{if } r_1 + r_2 \text{ is even}, \end{cases} \tag{24}$$

$$\mathrm{AE}(\alpha^1\alpha^2) = \begin{cases} F_{r_1+r_2+3} - F_{r_1+2}F_{r_2+1} & \text{if } r_1 + r_2 \text{ is odd}, \\ F_{r_1+r_2+2} - F_{r_1+2}F_{r_2} & \text{if } r_1 + r_2 \text{ is even}. \end{cases} \tag{25}$$

We may change slightly the indices of "F" in (24)–(25), by using the following well-known identities on Fibonacci numbers (see, *e.g.*, [18]):

$$\begin{aligned} &F_{r_1+r_2+2} - F_{r_1+2}F_{r_2} = F_{r_1+r_2+1} - F_{r_1}F_{r_2-2}, \\ &F_{r_1+r_2+3} - F_{r_1+2}F_{r_2+1} = F_{r_1+r_2+2} - F_{r_1}F_{r_2-1}. \end{aligned} \tag{26}$$

4.3. An upper bound to N_τ

Assume, for a moment, that τ is not an a.p. sequence. Then it may be represented as

$$\tau = \alpha^1\alpha^2\gamma. \tag{27}$$

We are assuming that τ is not an a.p. sequence; α^1 and α^2 are the first two a.p. components of τ [cf. (10)], and γ is the (eventually empty) concatenation of all other a.p. components. Let τ^* be the sequence obtained from τ by adding 1 to all entries of α^1, *i.e.*, $\tau^* = \alpha^*\alpha^2\gamma$, where $\alpha^* := (\tau_1 + 1, \ldots, \tau_{r_1} + 1)$. Note that $\alpha^*\alpha^2$ is an a.p. sequence of length $r_1 + r_2$, and τ^* has one less a.p. component than τ.

Lemma 4.7. *Assume the first entry of τ is odd. Then*

$$N_{\tau^*} - N_\tau = \begin{cases} F_{r_1}F_{r_2-2}\mathrm{EA}_\gamma + F_{r_1}F_{r_2-1}\mathrm{OA}_\gamma & if\ r_1+r_2\ is\ odd, \\ F_{r_1}F_{r_2-1}\mathrm{EA}_\gamma + F_{r_1}F_{r_2-2}\mathrm{OA}_\gamma & if\ r_1+r_2\ is\ even. \end{cases} \tag{28}$$

Proof. We obviously have $N_\tau = \mathrm{AO}(\alpha^1\alpha^2)\mathrm{EA}_\gamma + \mathrm{AE}(\alpha^1\alpha^2)\mathrm{OA}_\gamma - 2$, and $N_{\tau^*} = \mathrm{AO}(\alpha^*\alpha^2)\mathrm{EA}_\gamma + \mathrm{AE}(\alpha^*\alpha^2)\mathrm{OA}_\gamma - 2$ (note that, when γ is empty, $\mathrm{EA}_\gamma = \mathrm{OA}_\gamma = 1$, and these expressions are samples of (2)). Therefore, the difference $N_{\tau^*} - N_\tau$ is given by

$$[\mathrm{AO}(\alpha^*\alpha^2) - \mathrm{AO}(\alpha^1\alpha^2)]\mathrm{EA}_\gamma + [\mathrm{AE}(\alpha^*\alpha^2) - \mathrm{AE}(\alpha^1\alpha^2)]\mathrm{OA}_\gamma. \tag{29}$$

As the rightmost entry of $\alpha^*\alpha^2$ has the parity of $r_1 + r_2 + 1$, (9) yields

$$\mathrm{AO}(\alpha^*\alpha^2) = \begin{cases} F_{r_1+r_2+1} & \text{if } r_1+r_2 \text{ is odd,} \\ F_{r_1+r_2+2} & \text{if } r_1+r_2 \text{ is even,} \end{cases}$$
$$\mathrm{AE}(\alpha^*\alpha^2) = \begin{cases} F_{r_1+r_2+2} & \text{if } r_1+r_2 \text{ is odd,} \\ F_{r_1+r_2+1} & \text{if } r_1+r_2 \text{ is even.} \end{cases}$$

The lemma follows by entering, in (29), the values just obtained for $\mathrm{AO}(\alpha^*\alpha^2)$ and $\mathrm{AE}(\alpha^*\alpha^2)$, and the values of $\mathrm{AO}(\alpha^1\alpha^2)$ and $\mathrm{AE}(\alpha^1\alpha^2)$ given in (24)–(25) and modified according to (26). $\square$

Theorem 4.8. *Let w be the length of τ. Then $N_\tau \leqslant F_{w+3} - 2$, with equality if and only if τ is an a.p. sequence.*

proof. If τ is not an a.p. sequence, the right-hand side of (28) is positive, so $N_\tau < N_{\tau^*}$. We may apply the *star operation*, $\tau \mapsto \tau^*$, repeatedly, and obtain $\tau^*, \tau^{**}, \tau^{***}, \ldots$ until we get (after $m-1$ iterations) an a.p. sequence of length w. This proves the theorem. $\square$

5. Back to the tridiagonal Birkhoff polytope

To apply the preceding results to a general face $\mathscr{F}_A$ of $\mathfrak{T}_n$, we let $B_1, \ldots, B_q$ be the S-blocks of A, and assume B_k has w_k inner entries. We denote by $\tau^{(k)}$ the sequence of the positions of the inner entries of B_k (so $\tau^{(k)}$ is an increasing integer sequence of length w_k). The next theorem follows easily from Lemma 3.1, and Theorems 4.4, 4.5 and 4.8.

Theorem 5.1. *The number of vertices of $\mathscr{F}_A$ is*

$$f_0(\mathscr{F}_A) = [N(\tau^{(1)}) + 2][N(\tau^{(2)}) + 2] \cdots [N(\tau^{(w)}) + 2],$$

where the $N(\tau^{(k)})$ is given by Fibonacci formulas as those of Theorems 4.4 and 4.5. Moreover

$$f_0(\mathscr{F}_A) \leqslant F_{w_1+3}F_{w_2+3}\cdots F_{w_p+3},$$

with equality if and only if all $\tau^{(k)}$ are a.p. sequences.

It is easy to determine the number of edges of $\mathfrak{T}_n$, denoted $f_1(\mathfrak{T}_n)$. An edge is a face $\mathscr{F}_A$ of dimension 1. So, from the dimensional formula of [3, Corollary 2.6], or its specialized form given by Lemma 2.3, we get

Proposition 5.2. *$\mathscr{F}_A$ is an edge of $\mathfrak{T}_n$ if and only if A has only one S-block, and no inner entry.*

It is easy to check that this agrees with the characterization of the pairs of vertices that form an edge of $\mathfrak{T}_n$ given in [8, Theorem 2(iii)]. The matrices A as in Proposition 5.2 are those of the form $U \oplus M_k \oplus V$, where U and V are tridiagonal permutation matrices of orders i and j, respectively, M_k is the S-matrix of order k with no inner entry, and $i + j + k = n$. Here, k runs over $\{2, 3, \ldots, n\}$, and $i, j \geqslant 0$. For each such i, j, k, there exist F_{i+1} possible matrices U, and F_{j+1} possible matrices V (cf. [8]). Therefore,

$$f_1(\mathfrak{T}_n) = \sum_{0 \leqslant i+j \leqslant n-2} F_{i+1}\, F_{j+1}.$$

This may be simplified, or given many different forms, by means of well-known identities involving summations of order two products of Fibonacci numbers (see, *e.g.*, [12,18]). Finally, we briefly consider the determination of $f_2(\mathfrak{T}_n)$, the number of polygons, *i.e.*, 2-dimensional faces $\mathscr{F}_A$, of the tridiagonal Birkhoff polytope. By Lemma 2.3, dim $\mathscr{F}_A = 2$ splits into two cases: ($'$) A has only one S-block, which has 1 inner entry; ($''$) A has two S-blocks, and no inner entries. Accordingly, $f_2(\mathfrak{T}_n) = f_2' + f_2''$, each term corresponding to the respective case. Clearly

$$f_2' = \sum_{0 \leqslant i+j \leqslant n-3} (n-i-j-1) F_{i+1} F_{j+1}$$
$$f_2'' = \sum_{0 \leqslant i+j+k \leqslant n-4} (n-i-j-k+1) F_{i+1} F_{j+1} F_{k+1}.$$

Note that in case ($''$), per$A = 4$, and so $\mathscr{F}_A$ is a quadrilateral (as a matter of fact, it is a square for the standard inner product. Hint: it is enough to consider the 4-by-4 case). In case ($'$), $\mathscr{F}_A$ is a triangle.

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