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The diameter of the acyclic Birkhoff polytope

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Abstract

In this work we give an interpretation of vertices and edges of the acyclic Birkhoff polytope, $\mathfrak{T}_n = \Omega_n(T)$, where T is a tree with n vertices, in terms of graph theory. We generalize a recent result relatively to the diameter of the graph $G(\mathfrak{T}_n)$.

AMS classification: 05A15; 15A51

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1. Introduction

A real square matrix with nonnegative entries and all rows and columns sums equal to one is said to be doubly stochastic. This denomination is associated to probability distributions and it is amazing the diversity of branches of mathematics in which doubly stochastic matrices arise (geometry, combinatorics, optimization theory, graph theory and statistics).

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Doubly stochastic matrices have been studied quite extensively, especially in their relation with the van der Waerden conjecture for the permanent (cf. [2–6]). In 1946, Birkhoff published a remarkable result asserting that a matrix in the polytope of $n \times n$ nonnegative doubly stochastic matrices, Ω_n, is a vertex if and only if it is a permutation matrix (cf. [1]). In fact, Ω_n is the convex hull of all permutation matrices of order n (cf. [7,13]).

It is a well known fact that Ω_n is a closed bounded convex polyhedron in Euclidean n^2-space whose dimension is $(n-1)^2$. In particular, a compact convex polyhedron with a finite number of vertices is a polytope. In fact, the *Birkhoff polytope* Ω_n is one of the most important polytopes in many dimensions, and it is also known as *transportation polytope* and *doubly stochastic matrices polytope*.

Denoting the graph (the "skeleton") of a polytope $\mathscr{P}$ by $G(\mathscr{P})$, the vertices of $G(\mathscr{P})$ are the vertices (extreme points) of $\mathscr{P}$ that is the 0-dimensional faces of $\mathscr{P}$. Two vertices are joined by an edge in $G(\mathscr{P})$ if and only if they are the vertices of a one-dimensional face of $\mathscr{P}$. In this paper we investigate some geometrical properties of the set $\mathfrak{T}_n = \Omega_n(T)$, when T is a tree.

2. Motivation

In [10], Dahl discussed the subclass of Ω_n consisting of the tridiagonal doubly stochastic matrices and the corresponding subpolytope

$$\Omega_n^t = \{A \in \Omega_n : A \text{ is tridiagonal}\},$$

the so-called *tridiagonal Birkhoff polytope*, and studied the facial structure of Ω_n^t. Dahl stated that Ω_n^t is a polytope in $\mathbb{R}^{n\times n}$ of dimension $n-1$ with f_{n+1} vertices, where f_{n+1} denotes the $(n+1)$th Fibonacci number, and the vertex set consists of all tridiagonal permutation matrices. In fact, each vertex can be written as a direct sum

$$A = A_1 \oplus \cdots \oplus A_p,$$

for some positive integer p, where each matrix A_i is equal to

$$J = [1] \quad \text{or} \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Dahl also established an adjacency relation between the vertices of Ω_n^t.

Recently, da Fonseca and Marques de Sá (cf. [11]), established a closer connection between vertex counting in Ω_n^t and Fibonacci numbers. In particular, the main results on alternating parity sequences – a strictly increasing sequence of integers, with a finite numbers of entries, such that any two adjacent entries have opposite parities – are applied to determine the number of vertices of an arbitrarily given face of Ω_n^t. An expression for the number of edges of Ω_n^t is also provided.

Definition 2.1. The acyclic Birkhoff polytope, $\mathfrak{T}_n$, is the set of matrices whose support correspond to (some subset of) the edges (including loops) of a fixed tree.

In this work, for any tree with n vertices, T, we firstly determine the number of vertices of the *acyclic Birkhoff polytope* associated to T. The diameter of $G(\mathfrak{T}_n)$ which is defined as the maximum of $d(u, v)$ taken over all pairs u, v of vertices, where $d(u, v)$ is the smallest number of edges in a path between u and v in $G(\mathfrak{T}_n)$.

In [10], Dahl established the following result:

Theorem 2.1. [10] *The diameter of* $G(\Omega_n^t)$ *equals to* $\lfloor \frac{n}{2} \rfloor$.

Here, $\lfloor x \rfloor$ represents the largest integer less than or equal to x.

Led by Theorem 2.1 we establish, in the last section a more general result for the diameter for the graph of the acyclic Birkhoff polytope, $\mathfrak{T}_n$, where T is a tree. As the text develops some illustrative examples are provided.

Recall that the fractional matching polytope for a graph G is the polyhedron, in fact polytope, where nonnegative variables are associated with edges in G and the sum of variables of edges incident to a vertex is 1. So, the nonzeros of matrices in $\mathfrak{T}_n$ correspond to vertices and edges in the fixed tree T. Therefore, the acyclic Birkhoff polytope may be viewed as the (fractional) matching polytope associated with the tree T.

3. Definitions

In this section we recall further more-or-less standard definitions on graph theory, which will be used in the sequel. A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set of *vertices*, and $E(G)$ is a set of unordered pair of vertices called *edges*; $e = ij$ denotes the edge containing the vertices i and j; we say that i and j are adjacent, and we denote this by writing $i \sim j$; we also say that e is incident both on i and j. The vertices i and j are the *endpoints* of the edge ij.

A subgraph of a graph G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If S is a subset of $E(G)$ then by $G \backslash S$ we denote the graph with vertex set $V(G)$ and edge set $E(G) \backslash S$.

Given two graphs G and H the union of G and H, denoted by $G \cup H$, corresponds to the graph $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$.

If G is a graph and $i \in V(G)$, $G \backslash i$ denotes the subgraph of G obtained from G deleting all the edges incident on i.

A path of length k, $k \geqslant 0$ is a sequence $(v_0, v_1, \ldots, v_k)$ of distinct vertices such that $v_{i-1}v_i \in E(G)$, for $i = 1, \ldots, k$. The vertices v_0 and v_k are said to be *joined* by the path. From now on, the vertices of a graph G are simply denoted by $1, 2, \ldots, n$.

A graph is *connected* if every two vertices are joined by a path. A path with n vertices is denoted by P_n.

By a *generalized star* (or a *treelike star*) we mean a tree T having at most one vertex of degree greater than two. We call to this vertex the central vertex. Note that this definition also includes the particular case of stars; recall that a star with n vertices is a tree with all branches of length one in which there is a vertex of degree $n - 1$. We will denote a generalized star with k branches of lengths $\ell_1, \ldots, \ell_k$ by $S_{\ell_1,\ldots,\ell_k}$. For further concepts and definitions the reader is referred to [8,12].

4. Acyclic Birkhoff polytope

We start analyzing the shape of an element of the $n \times n$ acyclic polytope $\mathfrak{T}_n = \Omega_n(T)$, where T is a tree.

Theorem 4.1. *Given a tree T with n vertices, each matrix $A = [a_{ij}]$ in $\mathfrak{T}_n$ is symmetric and*

$$a_{ii} = 1 - \sum_{j \sim i} a_{ij}$$

for $i = 1, \ldots, n$.

Proof. Since the case $n = 2$ is trivial, let us proceed by induction on n. Without loss of generality, suppose that the vertex n is of degree one and it is adjacent (only) to $n-1$. Then

$$A = \left[\begin{array}{cccc|c} & & & & 0 \\ & \widetilde{A} & & & \vdots \\ & & & & 0 \\ & & & & a_{n-1,n} \\ \hline 0 & \cdots & 0 & a_{n,n-1} & a_{nn} \end{array}\right],$$

since A is doubly stochastic, we get $a_{nn} = 1 - a_{n-1,n}$ and henceforth $a_{n-1,n} = a_{n,n-1}$. We have also

$$\widetilde{A} + \text{diag}(0, \ldots, 0, a_{n-1,n}) \in \Omega_{n-1}(T\backslash n).$$

Therefore, by induction hypothesis, $\widetilde{A}$ is symmetric, so A is symmetric as well, $a_{ii} = 1 - \sum_{j\sim i} a_{ij}$, for $i = 1, \ldots, n-1$ and $a_{n-1,n-1} + a_{n-1,n} = 1 - \sum_{n\neq j\sim i} a_{ij}$. $\square$

The affine variety of $\mathbb{R}^{n\times n}$ generated by $\mathfrak{T}_n$ has dimension $n-1$. Remind that $\mathfrak{T}_n$ is a face of Ω_n, and thus the vertices of $\mathfrak{T}_n$ are the $n \times n$ permutation matrices whose graph is T.

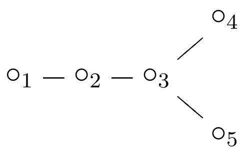
Consider $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ the set of edges of T ordered lexicographically. For $n \geqslant 3$, define the polytope

$$T^n = \left\{ x = (x_{e_1}, x_{e_2}, \ldots, x_{e_{n-1}}) \in \mathbb{R}^{n-1} | x \geqslant 0 \text{ and } \sum_{e\in E_i} x_e \leqslant 1, i \in \{1, \ldots, n-1\} \right\},$$

where each E_i, $i \in \{1, \ldots, n-1\}$, is the set of edges of T incident on a vertex i. It is straightforward that T^n and $\mathfrak{T}_n$ are affinely isomorphic. Here, the components of vector $x \in \mathbb{R}^{n-1}$ are denoted by x_{e_i}, so $x = (x_{e_1}, x_{e_2}, \ldots, x_{e_{n-1}})$. Without loss of misunderstanding, the components of vectors $x \in \mathbb{R}^{n-1}$ are simply denoted by $x = (x_1, x_2, \ldots, x_{n-1})$.

Here, to a bullet circle $\bullet$ and to an open circle $\circ$, we call, respectively, *closed vertex* and *open vertex* of G. We represent a standard graph with open vertices.

Example 4.1. For the tree with five vertices, T_5,



the polytope T^5 is the following set

$$T^5 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x \geqslant 0,\ x_1 + x_2 \leqslant 1,\ \text{and } x_2 + x_3 + x_4 \leqslant 1 \right\}.$$

Bearing in mind Theorem 4.1, and that T^5 and $\mathfrak{T}_5$ are affinely isomorphic we may associate for each vector $x \in T^5$ the following acyclic matrix

$$\begin{bmatrix} 1-x_1 & x_1 & 0 & 0 & 0 \\ x_1 & 1-x_1-x_2 & x_2 & 0 & 0 \\ 0 & x_2 & 1-x_2-x_3-x_4 & x_3 & x_4 \\ 0 & 0 & x_3 & 1-x_3 & 0 \\ 0 & 0 & x_4 & 0 & 1-x_4 \end{bmatrix}.$$

In particular, in a path P with n vertices, respectively, each element of $\Omega_n(P)$ is a tridiagonal matrix and we may state:

Proposition 4.2. [10] *Given a path P with n vertices, each symmetric tridiagonal matrix of $\Omega_n(P)$ is of the form*

$$\begin{bmatrix} 1-x_1 & x_1 & & & & \\ x_1 & 1-x_1-x_2 & x_2 & & & \\ & x_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & x_{n-2} & \\ & & & x_{n-2} & 1-x_{n-2}-x_{n-1} & x_{n-1} \\ & & & & x_{n-1} & 1-x_{n-1} \end{bmatrix}.$$

In the same way, for a star S with n vertices we have:

Proposition 4.3. *Given a star S with n vertices, each symmetric bordered matrix of $\Omega_n(S)$ is of the form*

$$\begin{bmatrix} 1-x_1-\cdots-x_{n-1} & x_1 & \cdots & x_{n-1} \\ x_1 & 1-x_1 & & \\ \vdots & & \ddots & \\ x_{n-1} & & & 1-x_{n-1} \end{bmatrix}.$$

Notice that $\Omega_n(P)$ and $\Omega_n(S)$ are affinely isomorphic to

$$P^n = \{x \in \mathbb{R}^{n-1} | x \geqslant 0 \text{ and } x_i + x_{i+1} \leqslant 1, \quad \text{for } i = 1, \ldots, n-2\}$$

(cf. [10]) and

$$S^n = \{x \in \mathbb{R}^{n-1} | x \geqslant 0 \text{ and } x_1 + \cdots + x_{n-1} \leqslant 1\},$$

respectively.

5. Faces of $\mathfrak{T}_n$

Given a $n \times n$ matrix with 0–1 entries, $A = [a_{i,j}]$, for $n > 1$, A is *fully indecomposable* if cannot be brought to the form

$$\begin{bmatrix} \star & O \\ \star & \star \end{bmatrix}$$

by a permutation of its rows and a permutation of its columns, where O is a $p \times q$, nonempty, zero matrix with $p + q = n$. When $n = 1$, A is fully indecomposable if $A = [1]$. The matrix A is said to have *total support* if there exist permutation matrices P and Q such that

$$PAQ = A_1 \oplus \cdots \oplus A_t,$$

where $A_1, \ldots, A_t$ are fully indecomposable. Equivalently, $A = [a_{ij}]$ has total support if and only if $A \neq O$ and $a_{rs} = 1$ implies that there exists a permutation matrix $P = [p_{ij}]$ with $p_{rs} = 1$ and $P \leqslant A$. Clearly a fully indecomposable matrix has total support.

In this section we present a more specific notation.

Definition 5.1. By a bicolored (vertex) subgraph of G we will mean a subgraph G' of G such that $G' = (V(G'), E(G'))$ with $E(G') \subseteq E(G)$ and the vertex set is a subset of $V(G)$, where some vertices can be closed, i.e., $V(G')$ can be partitioned in $V_\bullet \oplus V_\circ$.

In the literature the concept of bicolored graph is also known as 2-stratified graph i.e., a graph where the vertex set is partitioned into two subsets (cf. [9]).

Using the results presented in [11], we present a different approach for the structure of the faces of the acyclic Birkhoff polytope, $\mathfrak{T}_n$. We will establish a correspondence relation between the faces of dimension $n-1$ of the acyclic Birkhoff polytope, $\mathfrak{T}_n$, and the union of a finite number of bicolored subgraphs of the three following types:

Type 1. A closed vertex $\bullet$.

To this type of subgraph we associate an one-by-one matrix $A = [1]$.

Type 2. An *open edge*, $\circ$–$\circ$.

To an open edge we associate the "adjacency" matrix $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Type 3. This type is not one of the previous two types, and is a bicolored subgraph obtained from any connected bicolored subgraph of T, with all endpoints closed.

Definition 5.2. A T-component is a bicolored subgraph of T of Type 3. An inner entry of a T-component is a closed vertex which is not terminal.

To a T-component we may associate an "adjacency" matrix such that to an inner entry corresponds 1 in the respective diagonal entry. Such matrices are called *T-blocks*.

Notice that a T-block is fully indecomposable and has total support.

As $\mathfrak{T}_n$ is a face of Ω_n, the faces of $\mathfrak{T}_n$ are the faces of Ω_n which are contained in $\mathfrak{T}_n$. The faces of $\mathfrak{T}_n$ are in one-to-one correspondence with the $n \times n$ matrices A, of 0–1 entries having total support.

The face of $\mathfrak{T}_n$ corresponding to A is denoted by

$$\mathscr{F}_A = \{X \in \mathfrak{T}_n | a_{ij} = 0 \Rightarrow x_{ij} = 0\}.$$

Since [3, Theorem 2.5], if A is fully indecomposable,

$$\dim \mathscr{F}_A = \sigma_A - 2n + 1,$$

where σ_A is the number of $1'$s in A. So, for a T-component, and considering its T-block B,

$$\dim \mathscr{F}_B = \theta_B - 1 + w,$$

where w and θ_B are, respectively, the number of inner entries and closed endpoints of the T-component.

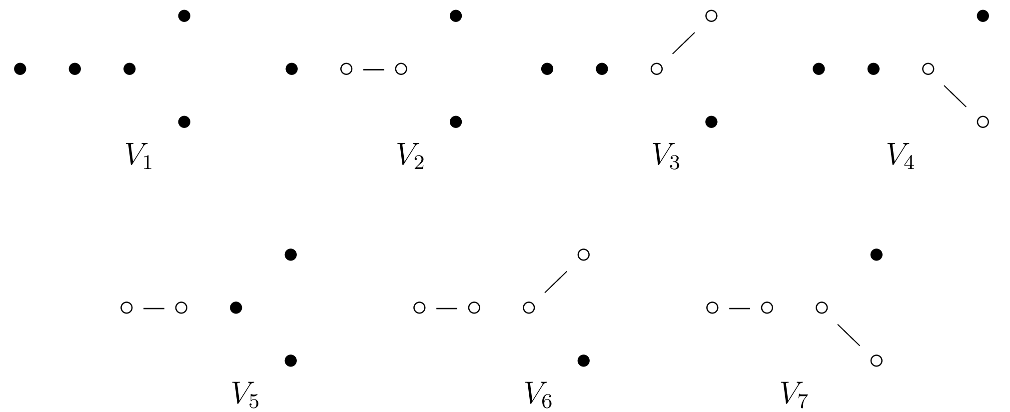
Bearing in mind that $\dim \mathscr{F}_A$ is the sum of $\dim \mathscr{F}_{A_i}$ for the T-blocks A_i of A, we may state the following proposition:

Proposition 5.1. *Let t_A be the number of T-components of the bicolored subgraph of T corresponding to $\mathscr{F}_A$. Let θ_A and ι_A be, respectively, the sum of all closed endpoints and the number of inner entries in all T-components of the same bicolored subgraph of T. Then*

$$\dim \mathscr{F}_A = \theta_A + \iota_A - t_A.$$

Here, each vertex (0-face) of the polytope $\mathfrak{T}_n$ will be identified as a bicolored subgraph of T whose diameter is at most one. In this case we only have the union of bicolored subgraphs of Type 1 and bicolored subgraphs of Type 2.

Example 5.1. For the graph T defined in Example 4.1, the seven vertices of $\mathfrak{T}_5$ are:



These vertices correspond respectively to the following "adjacency" matrices:

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{bmatrix},$$

$$\begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{bmatrix},\begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{bmatrix}.$$

Recalling the structure of the faces of $\mathfrak{T}_n$ previously presented, for example the vertex V_4 is a 0-face which is the union of three bicolored subgraphs of Type 1 and one bicolored subgraph of Type 2.

Here, a connection to the matching polytope can be done: the closed vertices correspond to loop variables that are 1, the edges that are indicated simply show the matching edges (except loops) and the closed vertices are the unsaturated vertices (those not incident to any matching edge).

6. Counting vertices and edges

We are now able to establish a recurrence relation to count the number of vertices of $\mathfrak{T}_n$, for a given tree T. In general, we denote by $f_0(T)$ the number of vertices (0-faces) of the polytope $\mathfrak{T}_n$ and by $f_{0,ij}(T)$ the number of bicolored subgraphs of T that contains the edge ij and whose diameter is at most one. Note that the number of vertices of $\mathfrak{T}_n$ is the number of matchings in T.

Let ij be any edge of the tree T. We have

$$f_0(T) = f_0(T\backslash ij) + f_{0,ij}(T\backslash\{i, j\} \cup ij), \tag{6.1}$$

with initial conditions $f_0(\emptyset) = f_0(v) = 1$, where v is a vertex of T.

Dahl stated for a path P with n vertices that

$$f_0(P) = f_{n+1},$$

where f_{n+1} is the $(n + 1)$th Fibonacci number. Taking into account the definition of Fibonacci numbers, the previous relation satisfies (6.1).

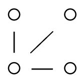
Notice that if

$$T = T_{n_1} \cup T_{n_2} \cup \cdots \cup T_{n_p},$$

with $n_1, n_2, \ldots, n_p$ positive integers, and T_{n_j} are disjoint trees, for $j \in \{1, \ldots, p\}$,

$$f_0(T) = f_0(T_{n_1}) \times f_0(T_{n_2}) \times \cdots \times f_0(T_{n_p}).$$

Example 6.1. Let $S = S_{1,1,1}$ be the star with four vertices presented below:

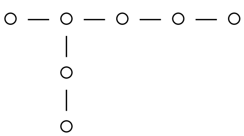


Let ij be any edge of S. The number of vertices of $\Omega_4(S)$ is $f_0(S) = f_0(S\backslash ij) + f_{0,ij}(S\backslash\{i, j\} \cup ij) = f_0(P_3) + f_{0,ij}(S\backslash\{i, j\} \cup ij) = 3 + 1 = 4$. The vertices are

The first bicolored subgraph and two of the others correspond to the vertices of $\Omega_4(S\backslash ij)$ and the reminder one corresponds to the only vertex of $\Omega_4(S\backslash\{i, j\} \cup ij)$.

Example 6.2. For the graph presented in Example 4.1, the number of vertices of $\mathfrak{T}_5$ is $f_0(T_5) = f_0(T_5 \setminus ij) + f_{0,ij}(T_5\backslash\{i, j\} \cup ij) = 4 + 3 = 7$, where ij is taken as the first edge considered from the left to the right. The vertices were presented in Example 5.1. The first four subgraphs correspond to the vertices of $\Omega_5(T_5 \setminus ij)$ and the last ones correspond to the vertices of $\Omega_5(T_5\backslash\{i, j\} \cup ij)$.

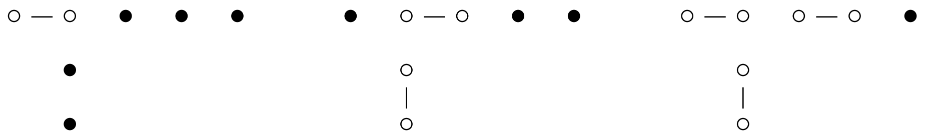
Example 6.3. Let $S' = S_{1,2,3}$ be the starlike tree presented below:



The number of vertices of $\Omega_7(S')$ is $f_0(S') = f_0(S'\backslash ij) + f_{0,ij}(S'\backslash\{i, j\} \cup ij)$ where ij is any edge. Therefore if ij is the first edge considered from the left to the right we obtain

$$f_0(S') = f_0(P_6) + f_0(P_3) \times f_0(P_2) = 13 + 3 \times 2 = 19.$$

We present some of the 19 vertices of S'



Proposition 6.1. *Let $S = S_{q,\ldots,q}$ be a generalized star with n branches, with n, q positive integers. Then*

$$f_0(S) = f_{q+1}^{n-1}(f_{q+1} + nf_q),$$

where f_q and f_{q+1} are the qth and the $(q + 1)$th Fibonacci's numbers, respectively.

Proof. The proof follows by induction on n. In fact, for $n = 1$, S is a path with $q + 1$ vertices, and by the recurrence relation presented in (6.1) the result follows easily.

If $S = S_{q,\ldots,q}$ is a star with n branches, then by induction hypothesis $f_0(S) = f_{q+1}^{n-1}(f_{q+1} + nf_q)$.

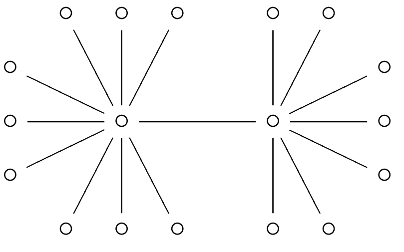
Let now S' be a star with $n + 1$ branches of length q. We have

$$\begin{aligned} f_0(S') &= f_0(S) \times f_0(P_q) + f_0(P_{q-1}) \times (f_0(P_q))^n \\ &= f_{q+1}^{n-1}(f_{q+1} + nf_q)f_{q+1} + f_q \times f_{q+1}^n \\ &= f_{q+1}^n(f_{q+1} + (n + 1)f_q). \qquad \square \end{aligned}$$

In the particular case of $q = 1$ we have $f_0(S) = n + 1$.

Given two generalized stars, $S_{p,\ldots,p}$ and $S_{q,\ldots,q}$, with m and n branches, respectively, a *double generalized star*, $G = G(S_{p,\ldots,p}, S_{q,\ldots,q})$, is the tree resulting from joining the central vertices of $S_{p,\ldots,p}$ and $S_{q,\ldots,q}$ by an edge.

Example 6.4. Consider the double generalized star



We have $f_0(G) = 10 \times 8 + 1$.

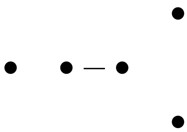
Proposition 6.2. *Let $G = G(S_{p,\ldots,p}, S_{q,\ldots,q})$ be a double generalized star with m and n branches, respectively. Then*

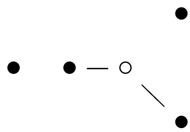
$$f_0(G) = f_{p+1}^{m-1}(f_{p+1} + mf_p) \cdot f_{q+1}^{n-1}(f_{q+1} + nf_q) + f_p^m \cdot f_q^n.$$

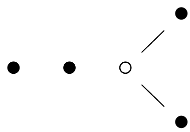
In the particular case of $p = q = 1$ we have $f_0(G) = (m + 1) \times (n + 1) + 1$.

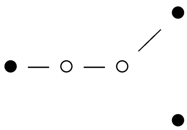
Since an edge (1-face) of the acyclic Birkhoff polytope $\mathfrak{T}_n$, is the union of bicolored subgraphs of Type 1, Type 2, and exactly one bicolored subgraph of Type 3, without inner entries and two closed endpoints we can also describe the edges of $\mathfrak{T}_n$. Next we provide some examples of the 15 edges of $\mathfrak{T}_5$.

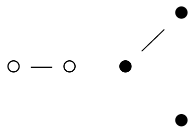
Example 6.5. For the graph T defined in Example 4.1, some of the 15 edges of $\mathfrak{T}_5$ are

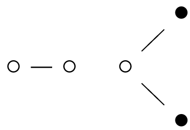

V_1V_2


V_2V_4


V_3V_4


V_2V_6

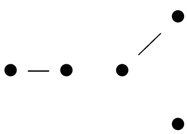

V_5V_6

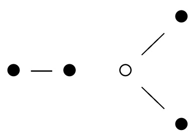

V_6V_7

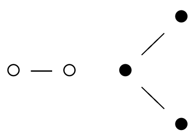
By Proposition 5.1, a 2-face of the acyclic Birkhoff polytope $\mathfrak{T}_n$ is, for example, the union of bicolored subgraphs of Type 1, Type 2 and one bicolored subgraph of Type 3 with one inner entry and two closed endpoints; or the union of bicolored subgraphs of Type 1, Type 2 and one bicolored subgraph of Type 3 with three closed enpoints and without inner entries; or the union of bicolored subgraphs of Type 1, Type 2 and two bicolored subgraphs of Type 3 each one with two closed endpoints and without inner entries.

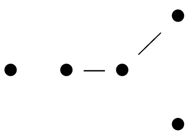
In a 2-face (or simply a *face*) we have at least one bicolored subgraph of Type 3.

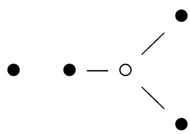
Example 6.6. Some 2-faces of $\mathfrak{T}_5$

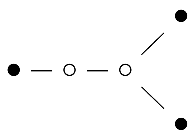

$V_1V_3V_5$


$V_3V_4V_6$

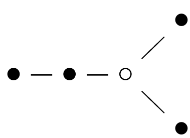

$V_5V_6V_7$


$V_1V_2V_3$


$V_2V_3V_4$


$V_2V_6V_7$

Example 6.7. A 3-face (a *cell*) of $\mathfrak{T}_5$



7. Adjacency of vertices of $\mathfrak{T}_n$

Let $G_1 = (V(G_1), E(G_1))$, $G_2 = (V(G_2), E(G_2))$ be two bicolored subgraphs of order p, $p \geqslant 1$ of G, where $V(G_1) = V_\bullet^1 \oplus V_\circ^1$ and $V(G_2) = V_\bullet^2 \oplus V_\circ^2$. We define *bicolored sum of subgraphs* G_1 *and* G_2 as the bicolored subgraph of G, such that

$$G_1 \boxplus G_2 = (V_\circ \oplus V_\bullet, E(G_1) \cup E(G_2)),$$

where $V_\circ = V_\circ^1 \cap V_\circ^2$ and $V_\bullet = V_\bullet^1 \cup V_\bullet^2$, i.e.,

$$\begin{aligned} &\circ^1 \boxplus \circ^2 = \circ,\\ &\bullet^1 \boxplus \circ^2 = \bullet,\\ &\circ^1 \boxplus \bullet^2 = \bullet,\\ &\bullet^1 \boxplus \bullet^2 = \bullet, \end{aligned}$$

where $\circ^1$, $\circ^2$, $\bullet^1$, $\bullet^2$ denote, respectively, the open and closed vertices of the bicolored subgraphs G_1 and G_2.

The cell presented in Example 6.7 is obtained from the faces $V_3V_4V_6$ and $V_2V_3V_4$ or from the faces $V_2V_6V_7$ and $V_2V_3V_4$.

Next, we establish an adjacency criterium for the vertices of the acyclic Birkhoff polytope, $\mathfrak{T}_n$.

Definition 7.1. Given the path P_n with n vertices

$$\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ \cdots,$$

the bicolored subgraphs

$$\bullet \quad \circ\!-\!\circ \quad \circ\!-\!\circ \cdots,$$

and

$$\circ\!-\!\circ \quad \circ\!-\!\circ \quad \circ \cdots,$$

are said *complementary* (in P_n).

Remark 7.1. If P_n is of odd order then

$$\bullet \quad \circ\!-\!\circ \quad \cdots \quad \circ\!-\!\circ,$$

and

$$\circ\!-\!\circ \quad \cdots \quad \circ\!-\!\circ \quad \bullet,$$

are complementary. If P_n is of even order then

$$\bullet \quad \circ\!-\!\circ \quad \cdots \quad \circ\!-\!\circ \quad \bullet,$$

and

$$\circ\!-\!\circ \quad \circ\!-\!\circ \quad \cdots \quad \circ\!-\!\circ$$

are complementary.

Let H_1 and H_2 be any bicolored subgraphs of T and P_n and P_n' be two paths with n vertices. Let V_1, V_2 two vertices of $\mathfrak{T}_n$. Suppose that

$$V_1 = H_1 \dotplus P_n \dotplus H_2$$

and

$$V_2 = H_1 \dotplus P'_n \dotplus H_2.$$

If there exist, the edge that contains V_1 and V_2 is

H_1 ●–○–○ · · · ○–●H_2.

We point out that the bicolored sum of the bicolored subgraphs corresponding to the 0-faces V_1 and V_2 gives only one T-component with two closed endpoints and no inner entries.

When the sequence of disjoint open edges is empty we have the following particular case:

H_1 ● ● H_2

H_1 ○–○ H_2

The previous observations lead to the main theorem of this section:

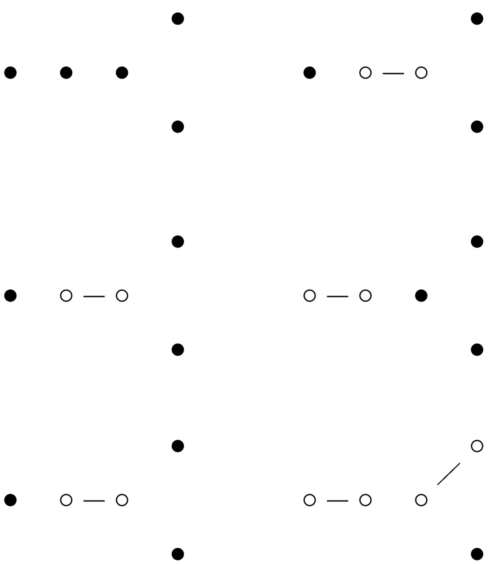
Theorem 7.1. *Let H_1, H_2 be any bicolored subgraphs of T and P_n, P'_n bicolored paths with n vertices. Let V_1, V_2 be two vertices of $\mathfrak{T}_n$. Suppose that*

$$V_1 = H_1 \dotplus P_n \dotplus H_2 \quad \text{and} \quad V_2 = H_1 \dotplus P'_n \dotplus H_2.$$

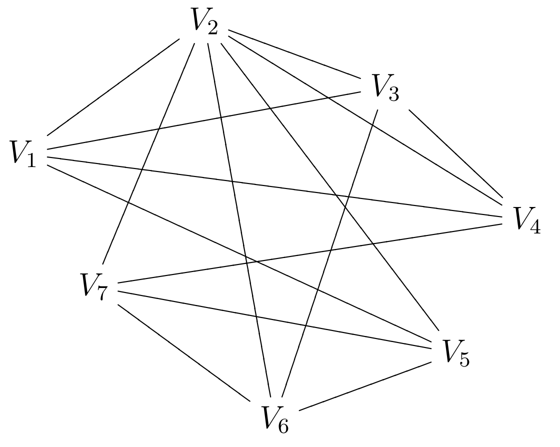
Then V_1 and V_2 are adjacent if and only if P_n and P'_n are complementary.

The adjacency criterium given by Dahl, in a matricial form, in Theorem 1 of [10], for paths, follows straightforward from the previous theorem.

Example 7.1. Let us go back to Example 5.1. By Theorem 7.1 the following pairs of vertices of the polytope $\mathfrak{T}_5$ are adjacent



From the previous criterium and the vertices of the Example 4.1 we can establish the adjacency relations of all vertices of the polytope $\mathfrak{T}_5$, and obtain the edges and faces of the respective polytope



8. The diameter of $G(\mathfrak{T}_n)$

A set of pairwise disjoint edges in a graph G is called a *matching* in G. The matching number of G, $\beta(G)$, is the cardinality of a matching of maximum cardinality.

Theorem 8.1. *Given a tree T with n vertices, the diameter of $G(\mathfrak{T}_n)$ is equal to $\beta(T)$.*

Proof. We start proving that there are two vertices V and V' such that $d(V, V') = \beta(T)$. In fact, let V the vertex of $\mathfrak{T}_n$ that is only union of subgraphs of Type 1, i.e., closed vertices, and V' the vertex that is composed by a maximum matching in T (i.e., closed vertices and open edges). Then, since V' is a maximum matching, and taking into account Theorem 7.1, we have

$$d(V, V') = \beta(T).$$

Next, given two any vertices V and V' of $\mathfrak{T}_n$, such that

$$V = H_1 \dot{+} H_2$$

and

$$V' = H_1' \dot{+} H_2'.$$

If $H_1 = H_1' \neq \emptyset$, then, by induction,

$$d(V, V') = d(H_2, H_2') \leqslant \beta(T) - 1.$$

If H_1 and H_1' are complementary bicolored subgraphs in V and V', in some path in T, then let

$$\bar{V} = H_1' \dot{+} H_2.$$

By Theorem 7.1, V and $\bar{V}$ are adjacent. From previous case we get $d(\overline{V}, V') \leqslant \beta(T) - 1$. Therefore

$$d(V, V') \leqslant \beta(T). \qquad \square$$

When the tree is a path with n vertices,

$$\beta(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$$

and, therefore, Theorem 2.1 follows from Theorem 8.1.

Acknowledgments

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