Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 319 (2008) 1116–1131
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Multiplicative invariant lattices in $\mathbb{R}^n$ obtained by twisting of group algebras and some explicit characterizations

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Received 26 September 2006

Communicated by Efim Zelmanov

Abstract

Let G be a finite group and $\mathbb{R}G$ be its group algebra defined over $\mathbb{R}$. If we define in G a 2-cochain F, then we can consider the algebra $\mathbb{R}_F G$ which is obtained from $\mathbb{R}G$ deforming the product, $x \cdot_F y = F(x,y)xy$, $\forall x, y \in G$. Examples of $\mathbb{R}_F(\mathbb{Z}_2)^n$ algebras are Clifford algebras and Cayley algebras like octonions. In this paper we consider generalizations of lattices with complex multiplication in the context of these twisted group algebras. We explain how these induce the natural algebraic structure to endow any arbitrary finite-dimensional lattice whose real components stem from any finite algebraic field extension over $\mathbb{Q}$ with a multiplicative closed structure. Furthermore, we develop some fully explicit characterizations in terms of generalized trace and norm functions.

Keywords: Twisted group algebras; Lattices; Algebraic number fields; Generalized norm and trace functions

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[1] Financial support by CMUC–FCT is gratefully acknowledged.
[2] Financial support from FWO project G.0335.08 and from CMUC–FCT is gratefully acknowledged.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.11.019

1. Introduction

As is well known, an n-dimensional lattice in $\mathbb{R}^n$ is a set of points of the form $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. Here $\omega_1, \ldots, \omega_n$ are some $\mathbb{R}$-linear independent vectors from $\mathbb{R}^n$. A priori, such a lattice is only endowed with the algebraic structure of a $\mathbb{Z}$-module. That means, if $\omega, \eta \in \Omega$, then $\omega \pm \eta \in \Omega$ and $\alpha\Omega \subseteq \Omega$ for any $\alpha \in \mathbb{Z}$.

However, if one defines a further multiplication operation on the underlying vector space $\mathbb{R}^n$, then *special* classes of lattices have the additional property, that there are also elements $a \in \mathbb{R}^n \setminus \mathbb{Z}$ such that $a \cdot \Omega \subset \Omega$. Such special lattices are then called lattices with multiplication. In the particular case where $\omega \cdot \eta \subset \Omega$ for all $\omega, \eta \in \Omega$, the whole lattice has a closed multiplicative structure. In fact, these lattices are exactly the $\mathbb{Z}$-orders in an associated n-dimensional $\mathbb{R}$-algebra.

The simplest non-trivial explicit examples are lattices with complex multiplication. The two-dimensional vector space $\mathbb{R}^2$ can be endowed with the multiplicative structure of the complex numbers. This is done by identifying a vector $(x_0, x_1)^T \in \mathbb{R}^2$ with $x_0 + x_1 i \in \mathbb{C}$ where $i^2 = -1$. Then a two-dimensional lattice of the normalized form $\mathbb{Z} + \mathbb{Z}\tau$ ($\Im(\tau) > 0$) has complex multiplication, if and only if $\tau \in \mathbb{Q}[\sqrt{-D}\,]$. Here D is supposed to be any positive square-free integer. Square-free means that no prime number appears more than once in the prime factorization. In the case where $\tau \in \mathbb{Z}[\sqrt{-D}\,]$, one even has $\omega \cdot \eta \in \Omega$ for all $\omega, \eta \in \Omega$ when $\cdot$ is the complex multiplication operator. Conversely, $\mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathbb{Q}[\sqrt{-D}\,]$ are the only two-dimensional lattices with complex multiplication. Lattices with complex multiplication are extremely well studied by numerous authors. For their basic properties, we refer the reader for example to the textbooks [8,16,18] in which their important role in analytic number theory is described. The division values of the modular function j associated to these particular lattices lie in finite Galois field extensions of an imaginary quadratic number field $\mathbb{Q}[\sqrt{-D}\,]$ and play a key role for Hilbert's twelfth problem [12].

In view of getting explicit analogous constructions for more general algebraic number fields, we are motivated to revisit the problem of complex lattice multiplication within a more general context. The simplest canonical non-trivial higher-dimensional associative examples of lattices with multiplication in dimension 2^n are lattices with Clifford algebra multiplication. These were first considered in [9,10] in the quaternionic setting. Later, these were more extensively studied in [6,14] and in [15, Chapter 2.7] in the context of the general Clifford algebras $Cl_{0,n}$.

Just for convenience we recall that the real Clifford algebra $Cl_{0,n}$ is the free algebra generated over the vector space $\mathbb{R}^n$ with basis e_i, $i = 1, \ldots, n$ in which the multiplication rules $e_i^2 = -1$, $i = 1, 2, \ldots, n$, and $e_i e_j = -e_j e_i$ for $i \neq j$ are valid. Each element a of the Clifford algebra $Cl_{0,n}$ can be written as $a = \sum_{A \in P(1,2,\ldots,n)} a_A e_A$. In this representation the expressions a_A are uniquely defined real numbers and the elements e_A are products of the basis vectors from the vector space $\mathbb{R}^n$ of the form $e_A = e_{l_1} \ldots e_{l_r}$ where $1 \leqslant l_1 < \cdots < l_r < n$ and where $e_\emptyset := 1$. $P(1, 2, \ldots, n)$ denotes the set of all possible subsets of $\{1, 2, \ldots, n\}$.

As a vector space, $Cl_{0,n}$ is isomorphic to $\mathbb{R}^{2^n}$. In the case $n = 1$ the associated Clifford algebra is isomorphic to the complex number field. The Clifford algebra $Cl_{0,2}$ is isomorphic to the skew field of Hamilton's quaternions. Now one can identify $\mathbb{R}^{2^n}$ with the Clifford algebra $Cl_{0,n}$. Let $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2^n}$ be a lattice where the generators w_i ($i = 1, \ldots, 2^n$) have the form

$$\omega_i = \omega_0^{(i)} + \sum_{j=1}^{n} \omega_j^{(i)} \sqrt{D_j}\, e_j + \sum_{j,k \in 1,\ldots,n,\ j<k} \omega_{jk}^{(i)} \sqrt{D_j D_k}\, e_j e_k + \cdots$$
$$+ \omega_{12\ldots n}^{(i)} \sqrt{D_1 D_2 \cdots D_n}\, e_{12\ldots n}.$$

If each $\omega_A^{(i)}$ ($A \subset P(1,2,\ldots,n)$) is an integer and $D_1,\ldots,D_n$ are mutually distinct positive square-free integers, then Ω has a non-trivial $Cl_{0,n}$ multiplication. Here again one can show that the class of 2^n-dimensional lattices that have Clifford multiplication are those whose real components of the generators stem up to conjugation from the multiquadratic number fields $\mathbb{Q}[\sqrt{D_1},\ldots,\sqrt{D_n}]$. See [15] for details.

In this paper we now deal with a class of twisted group algebras that contains the complex number field, Hamilton's quaternionic skew field and all Clifford algebras as very particular cases. Furthermore, we give some explicit algebraic characterizations in terms of generalized norm and trace functions.

2. Graded $\mathbb{R}_F G$ algebras

Let us consider a finite group G and its group algebra defined over $\mathbb{R}$. To get started we recall

Definition 1. Let G be a finite group. A 2-cochain F in G is a map $F: G\times G \to \mathbb{R}^*$ satisfying $F(e,x)=F(x,e)=1$ for all $x\in G$, where e is the neutral element in G.

After having defined a 2-cochain in G we can consider the algebra $\mathbb{R}_F G$ that is obtained from $\mathbb{R}G$ by deforming the product,

$$x \cdot_F y = F(x,y)xy, \quad \forall x,y\in G.$$

Examples of $\mathbb{R}_F(\mathbb{Z}_2)^n$ algebras are Clifford algebras [2] and Cayley algebras like octonions [1]. See also [17]. Here it was proved that if we write the cochain F in the form $F(x,y)=(-1)^{f(x,y)}$, then we obtain:

1. The 'complex number' algebra by considering

$$G=\mathbb{Z}_2, \qquad f(x,y)=xy, \quad x,y\in\mathbb{Z}_2.$$

   Here we identify G as the additive group $\mathbb{Z}_2$ but also make use of its product.
2. The quaternionic algebra by taking

$$G=\mathbb{Z}_2\times\mathbb{Z}_2, \qquad f(x,y)=x_1y_1+(x_1+x_2)y_2$$

   where $x=(x_1,x_2)\in G$ is a vector notation.
3. The octonionic algebra by considering

$$G=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2, \qquad f(x,y)=\sum_{1\leqslant i\leqslant j\leqslant 3} x_iy_j + y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3.$$

Within a similar context we have studied in [2] Clifford algebras as $\mathbb{R}_F(\mathbb{Z}_2)^n$ algebras, where the cochain F is defined by the expression

$$F(x,y)=(-1)^{\sum_{j<i} x_iy_j}\prod_{i=1}^n q_i^{x_iy_i}$$

where $x=(x_1,x_2,\dots,x_n)$, $y=(y_1,y_2,\dots,y_n)$ are elements from $(\mathbb{Z}_2)^n$ and $q_i=e_i^2$ where e_i are the canonical elements of $\mathbb{Z}_2^n$.

If we consider a group G with n elements, say $G=\{g_1,\dots,g_n\}$, then we can identify each element $a_1g_1+\dots+a_ng_n, a_i\in\mathbb{R}$, of the algebra $\mathbb{R}_FG$ with the element $(a_1,\dots,a_n)\in\mathbb{R}^n$. With this identification the multiplication defined in $\mathbb{R}_FG$ will introduce a special multiplication on $\mathbb{R}^n$. This is called the multiplication of $\mathbb{R}^n$ induced by the group G using the cochain F. In this case we say that $\mathbb{R}^n$ is embedded in $\mathbb{R}_FG$.

As a consequence we can multiply the points of an n-dimensional lattice in $\mathbb{R}^n$ with each other and we obtain another element from $\mathbb{R}^n$. In the general case the resulting vector does not always belong again to the lattice. Nevertheless, this will happen in some interesting cases. We introduce:

Definition 2. Let G be a group with n elements and F be a cochain in G. Further, let $\omega_1,\dots,\omega_n$ be linearly independent vectors from $\mathbb{R}^n$. Let

$$\Omega=\mathbb{Z}\omega_1+\dots+\mathbb{Z}\omega_n$$

be the associated n-dimensional lattice embedded in the algebra $\mathbb{R}_FG$. Then we say that Ω has an $\mathbb{R}_FG$ multiplication from the left (respectively from the right) if there exists an $a\in\mathbb{R}^n\setminus\mathbb{Z}$ such that $a\cdot\omega\in\Omega$ (respectively $\omega\cdot a\in\Omega$) for all $\omega\in\Omega$. Furthermore, we say that the lattice Ω is closed under the multiplication of $\mathbb{R}_FG$ if for all $\omega,\eta\in\Omega$ holds $\omega\cdot\eta\in\Omega$. Here $\cdot$ is the multiplication induced by G using the cochain F.

For simplicity we will write $a\omega$ for $a\cdot\omega$ when no ambiguity can occur in all that follows.

In the simplest case $G=\mathbb{Z}_2$ we have only two possibilities for the algebras $\mathbb{R}_F\mathbb{Z}_2$: The complex numbers $\mathbb{C}$ and the group algebra of $\mathbb{Z}_2$ which is denoted by $\mathbb{R}\mathbb{Z}_2$. As mentioned at the beginning, lattices in $\mathbb{R}^2$ with multiplication in the context of the complex numbers are known and completely classified. To study lattices with Clifford and Cayley algebra multiplication, let us briefly say a few words on the other case where we have a lattice in $\mathbb{R}^2$ endowed with a multiplication operation that is defined by the group algebra of $\mathbb{Z}_2$.

Let $\Omega=\mathbb{Z}\tau+\mathbb{Z}$ ($\tau=x_0+e_1x_1\in\mathbb{R}\mathbb{Z}_2,\ x_1>0$) be a lattice in $\mathbb{R}^2$ with $\mathbb{R}\mathbb{Z}_2$ multiplication. Let $R(\tau)$ be the set of multiplicators of Ω. This is the set of $\mathbb{R}\mathbb{Z}_2$-elements a that satisfy $a\Omega\subseteq\Omega$. In close analogy to the complex case, see [13, pp. 84–87], we can establish:

Proposition 1. *Let $\tau=x_0+x_1e_1\in\mathbb{R}\mathbb{Z}_2$ with $x_1>0$. For an element $\lambda\in\mathbb{R}\mathbb{Z}_2$ the following assertions are equivalent*:

1. $\lambda\in R(\tau)$.
2. *There exist $a,b,c,d\in\mathbb{Z}$ such that $\lambda\tau=a\tau+b$ and $\lambda=c\tau+d$.*
3. $w:=\binom{\tau}{1}$ *satisfies $\lambda w=Mw$ with* $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{Mat}(2,\mathbb{Z})$.

Proof. (1) $\Leftrightarrow$ (2): $\lambda\in R(\tau)$ is equivalent to $\lambda\cdot 1\in\mathbb{Z}\tau+\mathbb{Z}$ and $\lambda\cdot\tau\in\mathbb{Z}\tau+\mathbb{Z}$, so $\lambda\cdot 1=c\tau+d$ and $\lambda\cdot\tau=a\tau+b$.

(2) $\Leftrightarrow$ (3): Evident:

$$\lambda\cdot w=\lambda\begin{pmatrix}\tau\\1\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix},\qquad\begin{pmatrix}\lambda\tau\\\lambda\end{pmatrix}=\begin{pmatrix}a\tau+b\\c\tau+d\end{pmatrix}.\qquad\square$$

Proposition 2. *Let $\Omega = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice with $\mathbb{R}\mathbb{Z}_2$-multiplication, i.e. $R(\tau) \neq \mathbb{Z}$. Then $\tau \in \mathbb{Q}[e_1\sqrt{D}]$ for some square-free $D \in \mathbb{N}$ and $R(\tau)$ is a subring of the integral elements of $\mathbb{Q}[e_1\sqrt{D}]$.*

Remark. For the sake of clarity: By $\mathbb{Q}[e_1\sqrt{D}]$ we mean the set of $\mathbb{R}\mathbb{Z}_2$ elements that can be written in the form $r + s\sqrt{D}e_1$ where $r, s \in \mathbb{Q}$. Here $D > 0$ is a positive square-free integer.

The proof of Proposition 2 can be done in complete analogy to the calculations of [13, pp. 84–87]. The main difference is that one has to apply in this context the particular multiplication rules in $\mathbb{R}\mathbb{Z}_2$, i.e. $e_1^2 = 1$ instead of applying $e_1^2 = -1$ when adapting the calculations to the context of $\mathbb{R}\mathbb{Z}_2$. Hence, we omit it.

Remark. Since $R(\tau)$ is a submodule of the free $\mathbb{Z}$-module of the integral numbers from $\mathbb{Q}[e_1\sqrt{D}]$, $R(\tau)$ is also free. Since $R(\tau) \neq \mathbb{Z}$ one can readily conclude that $R(\tau)$ has rank 2, i.e. $R(\tau)$ is a lattice in $\mathbb{R}\mathbb{Z}_2$.

3. Lattices in $\mathbb{R}^n$ closed under Clifford and Cayley algebra multiplication

In this section we present a class of lattices in $\mathbb{R}^n$ that are closed under the multiplication induced by Clifford algebras $Cl_{p,q}$ or by Cayley algebras. Here we will call Cayley algebras, the algebras obtained by the Cayley–Dickson process.

Let A be a real finite-dimensional algebra with identity 1 and σ be an involutive automorphism of A. First we introduce

Definition 3. Let G be a group with n elements and F be a cochain in G. Further, let $\omega_1, \ldots, \omega_n$ be some $\mathbb{R}$-linearly independent vectors from $\mathbb{R}^n$. Let

$$\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$$

be the associated lattice embedded in $\mathbb{R}_F G$. Then we say that Ω is stable under the involutive automorphism σ defined in $\mathbb{R}_F G$ if $\sigma(\Omega) = \Omega$.

In the vector space $\tilde{A} = A \oplus Av$, where v is a symbol notation that represents the second copy of A, a new multiplication is defined by

$$(a + bv) \cdot (c + dv) = a \cdot c + \alpha b\sigma(d) + \big(a \cdot d + b\sigma(c)\big) \cdot v$$

for a fixed $\alpha \in \mathbb{R}^*$. Furthermore, a new involutive automorphism is introduced by

$$\tilde{\sigma}(a + bv) = \sigma(a) - \sigma(b)v.$$

We say that $\tilde{A}$ is obtained from A by the Clifford process. In the paper [2] we have shown that if A is an $\mathbb{R}_F G$ algebra then $\tilde{A}$ is an $\mathbb{R}_{\tilde{F}}\tilde{G}$ algebra with $\tilde{G} = G \times \mathbb{Z}_2$. The cochain $\tilde{F}$ is obtained from the cochain F.

Suppose now that $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$ is again an n-dimensional lattice in $\mathbb{R}^n$ embedded in $\mathbb{R}_F G$, where G is a group with n elements. Further, let us consider the lattice $\tilde{\Omega} = \Omega + \Omega v = \mathbb{Z}(\omega_1, 0) + \cdots + \mathbb{Z}(\omega_n, 0) + \mathbb{Z}(0, \omega_1) + \cdots + \mathbb{Z}(0, \omega_n)$ embedded in $\mathbb{R}_{\tilde{F}}\tilde{G}$.

Then we can establish

Theorem 1. *The lattice Ω is closed under multiplication induced by $\mathbb{R}_F G$ and stable under the involutive automorphism σ if and only if the lattice $\tilde{\Omega}$ is closed under multiplication induced by $\mathbb{R}_{\tilde{F}}\tilde{G}$ and stable under the involutive automorphism $\tilde{\sigma}$.*

Proof. If we consider the lattice $\tilde{\Omega} = \Omega + \Omega v$ embedded in $\mathbb{R}_{\tilde{F}}\tilde{G}$, then we can introduce the product of two elements of $\tilde{\Omega}$ in the following way

$$(a+bv)\cdot(c+dv) = \big(a\cdot c + \alpha b\sigma(d)\big) + \big(a\cdot d + b\sigma(c)\big)\cdot v \quad \text{for } a,b,c,d \in \Omega.$$

If $\sigma(x) \in \Omega$, $\forall x \in \Omega$ and $x \cdot y \in \Omega$, $\forall x, y \in \Omega$ then $(a+bv)\cdot(c+dv) \in \tilde{\Omega}$ and $\tilde{\sigma}(a+bv) = \sigma(a) - \sigma(b)v \in \tilde{\Omega}$. The converse statement is obvious. □

Corollary 1. *For any $n \in \mathbb{N}$ we can define a lattice in $\mathbb{R}^{2^n}$ that is closed under the multiplication and the involutive automorphism of the Clifford algebra $Cl_{p,q}$ for any $p,q \in \mathbb{N}_0$ with $p+q=n$.*

Proof. Lattices in $\mathbb{R}^2$ closed under multiplication and conjugation induced by complex numbers or by the group algebra of $\mathbb{Z}_2$ are known. Proposition 2 completes the classification of lattices in $\mathbb{R}^2$ that are closed under multiplication and conjugation by both possible $\mathbb{R}_F\mathbb{Z}_2$ algebras.

As shown in [2], every Clifford algebra $Cl_{p,q}$ can be obtained by the Clifford process from these algebras. Therefore, the statement follows. □

It is therefore not surprising that the lattices in $Cl_{p,q}$ which have Clifford multiplication are those whose components stem from multiquadratic number fields or conjugated ones. This is due to the fact that all the lattices in $\mathbb{R}^2$ which are closed under multiplication and conjugation endowed with the two multiplicative structures $\mathbb{C}$ or $\mathbb{R}\mathbb{Z}_2$, respectively, are those of the form $\mathbb{Z} + \mathbb{Z}\tau$, with $\tau \in \mathbb{Z}[e_1\sqrt{D}\,]$ only.

Remark. The preceding theorem has an analogue if we consider the Cayley–Dickson process. This allow us to obtain, for example, closed lattices under the multiplication induced by the octonions in $\mathbb{R}^8$. The most popular example is the E_8-lattice considered for example in [4]. This is the densest (non-associative) integral domain in the octonions which contains all the eight octonionic units, cf. [5].

4. Lattices with $\mathbb{R}_F\mathbb{Z}_n^m$ multiplication

In this section we now discuss lattices in $\mathbb{R}^{n^m}$ ($n, m \geqslant 1$ arbitrarily) with $\mathbb{R}_F\mathbb{Z}_n^m$ multiplication. Lattices with complex, quaternionic, Clifford and Cayley multiplication fit within this general framework as special cases of lattices in $\mathbb{R}^{2^m}$ (for $m=1$, $m=2$, m arbitrary, respectively).

The simplest first essentially different case to those considered before, is the case where $G = \mathbb{Z}_3$. In contrast to the $Cl_{0,2}$-multiplication, the three-dimensional vector space $\mathbb{R}^3$ is closed under the $\mathbb{R}_F\mathbb{Z}_3$ multiplication. Let us first consider the simplest case where $F=1$. In this case the multiplication is commutative and associative. The canonical examples of three-dimensional lattices $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3$ which have an $\mathbb{R}\mathbb{Z}_3$-multiplication turn out to be lattices with generators whose real components stem from cubic number fields.

We prove

Proposition 3. *Suppose that ω_i $(i = 1, 2, 3)$ are $\mathbb{R}$-linearly independent vectors in $\mathbb{R}^3$ of the form*

$$\omega_i = a_0^{(i)} e_0 + a_1^{(i)} \sqrt[3]{D}\, e_1 + a_2^{(i)} \sqrt[3]{D^2}\, e_2 \tag{1}$$

where the elements $a_0^{(i)}$, $a_1^{(i)}$, $a_2^{(i)}$ are some integers and where D is a cubic-free positive integer. Then the associated lattice $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3$ has an $\mathbb{R}\mathbb{Z}_3$-multiplication.

Proof. If we multiply two generators ω_i and ω_j with each other then we again get a number of the same structure:

$$\begin{aligned}\omega_i \cdot \omega_j = {} & \big(a_0^{(i)} a_0^{(j)} + a_1^{(i)} a_2^{(j)} D + a_2^{(i)} a_1^{(j)} D\big) e_0 + \big(a_0^{(i)} a_1^{(j)} + a_1^{(i)} a_0^{(j)} + a_2^{(i)} a_2^{(j)} D\big) \sqrt[3]{D}\, e_1 \\ & + \big(a_0^{(i)} a_2^{(j)} + a_1^{(i)} a_1^{(j)} + a_2^{(i)} a_0^{(j)}\big) \sqrt[3]{D^2}\, e_2, \end{aligned} \tag{2}$$

after having applied the multiplication rules in $\mathbb{R}\mathbb{Z}_3$. This element can belong to Ω or not.

In fact, the equations

$$\begin{aligned}\omega_1 &= a_0^{(1)} e_0 + a_1^{(1)} \sqrt[3]{D}\, e_1 + a_2^{(1)} \sqrt[3]{D^2}\, e_2, \\ \omega_2 &= a_0^{(2)} e_0 + a_1^{(2)} \sqrt[3]{D}\, e_1 + a_2^{(2)} \sqrt[3]{D^2}\, e_2, \\ \omega_3 &= a_0^{(3)} e_0 + a_1^{(3)} \sqrt[3]{D}\, e_1 + a_2^{(3)} \sqrt[3]{D^2}\, e_2\end{aligned}$$

can be re-written in matrix form as follows

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = A \begin{pmatrix} e_1\\ e_2\\ e_3\end{pmatrix}$$

with

$$A = \begin{pmatrix} a_0^{(1)} & a_1^{(1)} \sqrt[3]{D} & a_2^{(1)} \sqrt[3]{D^2} \\ a_0^{(2)} & a_1^{(2)} \sqrt[3]{D} & a_2^{(2)} \sqrt[3]{D^2} \\ a_0^{(3)} & a_1^{(3)} \sqrt[3]{D} & a_2^{(3)} \sqrt[3]{D^2} \end{pmatrix}.$$

Since the generators $\omega_1, \omega_2, \omega_3$ are supposed to be linearly independent, the matrix A is invertible. As a consequence the basis elements e_0, e_1, e_2 of $\mathbb{R}\mathbb{Z}_3$ can uniquely be expressed as a linear combination in terms of $\omega_1, \omega_2, \omega_3$ of the following form:

$$\begin{aligned} e_0 &= \frac{1}{\det A}\big(b_1^{(0)}\omega_1 + b_2^{(0)}\omega_2 + b_3^{(0)}\omega_3\big), \\ e_1 &= \frac{\sqrt[3]{D^2}}{\det A}\big(b_1^{(1)}\omega_1 + b_2^{(1)}\omega_2 + b_3^{(1)}\omega_3\big), \\ e_2 &= \frac{\sqrt[3]{D}}{\det A}\big(b_1^{(2)}\omega_1 + b_2^{(2)}\omega_2 + b_3^{(2)}\omega_3\big) \end{aligned}$$

where the elements $b_j^{(i)}$ $(j = 1, 2, 3,\ i = 0, 1, 2)$ are uniquely defined integers.

Consider the subset $\mathcal{A}$ of Ω formed by the elements of the form

$$\det A\alpha\omega_1 + \det A\beta\omega_2 + \det A\gamma\omega_3, \quad \alpha, \beta, \gamma \in \mathbb{Z}.$$

Then $a\Omega \subset \Omega$ for all $a \in \mathcal{A}$. This proves that a lattice of the form (1) has $\mathbb{R}\mathbb{Z}_3$ multiplication. $\square$

As a direct consequence of these observations we readily obtain

Corollary 2. *Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3$ be a lattice with generators of the form*

$$\omega_i = a_0^{(i)} e_0 + a_1^{(i)} \sqrt[3]{D}\, e_1 + a_2^{(i)} \sqrt[3]{D^2}\, e_3, \quad i = 1, 2, 3,$$

where $a_j^{(i)} \in \mathbb{Z}$ for all $i = 1, 2, 3$ and $j = 0, 1, 2$ and where D is a cubic-free integer. Let

$$A = \begin{pmatrix} a_0^{(1)} & a_1^{(1)} \sqrt[3]{D} & a_2^{(1)} \sqrt[3]{D^2} \\ a_0^{(2)} & a_1^{(2)} \sqrt[3]{D} & a_2^{(2)} \sqrt[3]{D^2} \\ a_0^{(3)} & a_1^{(3)} \sqrt[3]{D} & a_2^{(3)} \sqrt[3]{D^2} \end{pmatrix}.$$

If $\det A = \pm 1$*, then Ω is closed under the $\mathbb{R}\mathbb{Z}_3$ multiplication.*

Remark. In the other cases where $\det A \neq \pm 1$, the lattice Ω contains non-trivial subrings $\mathcal{A} \subset \Omega$ such that $a\Omega \subset \Omega$ for all $a \in \mathcal{A}$.

These lattices can be regarded as the natural analogues of lattices with complex multiplication (whose generators stem from imaginary quadratic number fields) within the framework of cubic number fields over $\mathbb{Q}$.

More generally, we obtain

Proposition 4. *Let n be an arbitrary positive integer and F be an arbitrary integer valued cochain in $\mathbb{Z}_n$.*

Let $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$ be an n-dimensional lattice embedded in the twisted group algebra $\mathbb{R}_F\mathbb{Z}_n$, where the generators are each of the form

$$\omega_i = a_0^{(i)} e_0 + a_1^{(i)} \sqrt[n]{D}\, e_1 + \cdots + a_{n-1}^{(i)} \sqrt[n]{D^{n-1}}\, e_{n-1}$$

where $a_0^{(i)}, \ldots, a_{n-1}^{(i)}$ are integers and D is an n-power free integer, that means each prime appears at most $n-1$ times in the prime factorization. Then Ω has an $\mathbb{R}_F\mathbb{Z}_n$-multiplication. Let

$$A := \begin{pmatrix} a_0^{(1)} & a_1^{(1)} \sqrt[n]{D} & & a_{n-1}^{(1)} \sqrt[n]{D^{n-1}} \\ a_0^{(2)} & a_1^{(2)} \sqrt[n]{D} & & a_{n-1}^{(2)} \sqrt[n]{D^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{(n)} & a_1^{(n)} \sqrt[n]{D} & & a_{n-1}^{(n)} \sqrt[n]{D^{n-1}} \end{pmatrix}.$$

If $\det A = \pm \frac{1}{\sqrt[n]{D^S}}$*, where $S := \sum_{i=1}^{n-1} i$, then Ω is closed under the $\mathbb{R}_F\mathbb{Z}_n$-multiplication.*

Proof. Suppose that the generators of Ω are of the form

$$\omega_i = a_0^{(i)} e_0 + a_1^{(i)} \sqrt[n]{D}\, e_1 + \cdots + a_{n-1}^{(i)} \sqrt[n]{D^{n-1}}\, e_{n-1} \tag{3}$$

for $i = 1, \ldots, n$. Applying the product definition of $\mathbb{R}_F\mathbb{Z}_n$, forming the product of ω_i with another element of the same form, say with

$$\omega_j = a_0^{(j)} e_0 + a_1^{(j)} \sqrt[n]{D}\, e_1 + \cdots + a_{n-1}^{(j)} \sqrt[n]{D^{n-1}}\, e_{n-1},$$

turns out to be again

$$\begin{aligned}
\omega_i \cdot \omega_j = {} & \Bigg[a_0^{(i)} a_0^{(j)} + \sum_{p,q>0,\, p+q=n} F(e_p, e_q) a_p^{(i)} a_q^{(j)} D \Bigg] e_0 \\
& + \Bigg[\sum_{p+q=1} F(e_p, e_q) a_p^{(i)} a_q^{(j)} + \sum_{p,q>1,\, p+q=n+1} F(e_p, e_q) a_p^{(i)} a_q^{(j)} D \Bigg] \sqrt[n]{D}\, e_1 \\
& + \Bigg[\sum_{p+q=2} F(e_p, e_q) a_p^{(i)} a_q^{(j)} + \sum_{p,q>2,\, p+q=n+2} F(e_p, e_q) a_p^{(i)} a_q^{(j)} D \Bigg] \sqrt[n]{D^2}\, e_2 \\
& \vdots \\
& + \Bigg[\sum_{p+q=n-1} F(e_p, e_q) a_p^{(i)} a_q^{(j)} \Bigg] \sqrt[n]{D^{n-1}}\, e_{n-1}.
\end{aligned}$$

Under the assumption that F is integer-valued, the element $\omega_i \cdot \omega_j$ again has the form as in (3). With the same reasoning as in the three-dimensional case, we can again conclude that the elements of the subset $\mathcal{A}$ formed by the elements

$$\det A \sqrt[n]{D^S}\, \alpha_1 \omega_1 + \det A \sqrt[n]{D^S}\, \alpha_2 \omega_2 + \cdots + \det A \sqrt[n]{D^S}\, \alpha_n \omega_n, \quad \alpha_1, \ldots, \alpha_n \in \mathbb{Z},$$

where $S := \sum_{i=1}^{n-1} i$, satisfy $a\Omega \subset \Omega$. Hence, lattices of the above stated form have an $\mathbb{R}_F\mathbb{Z}_n$ multiplication. The second statement also now follows immediately. $\square$

These are the simplest n-dimensional analogues of the class of lattices with complex multiplication within the context of the number fields $\mathbb{Q}[\sqrt[n]{D}\,]$ for arbitrary $n \in \mathbb{N}$. We see that the structure of the $\mathbb{R}_F\mathbb{Z}_n$ algebras is indeed a very natural one to endow a lattice with components from the number fields $\mathbb{Q}[\sqrt[n]{D}\,]$ with a closed multiplication structure.

As mentioned previously one can endow lattices with components from multiquadratic number fields with a multiplicative structure when applying the Clifford or the Cayley–Dickson process. One considers $G = \mathbb{Z}_2^n$. Similarly, we can endow lattices with generators whose components stem from multi-n-power fields of the form $\mathbb{Q}[\sqrt[n]{D_1}, \ldots, \sqrt[n]{D_m}\,]$ with a multiplicative structure, when we take more generally $G = \mathbb{Z}_n^m$. This again can easily be verified by a similar direct calculation.

Combining this observation with the statement of Proposition 4, one obtains:

Theorem 2. *Let $k \in \mathbb{N}$ and $n_1, \ldots, n_k, m_1, \ldots, m_k$ be some positive integers. Let $D_1^{(n_j)}, \ldots, D_{m_j}^{(n_j)}$ be n_j-power free positive integers for all $j = 1, \ldots, k$. Suppose that $F_1, \ldots, F_k$ are integer valued cochains. Lattices with generators whose components stem from the algebraic field*

$$\mathbb{Q}\Big[\sqrt[n_1]{D_1^{(n_1)}}, \ldots, \sqrt[n_1]{D_{m_1}^{(n_1)}}, \ldots, \sqrt[n_k]{D_1^{(n_k)}}, \ldots, \sqrt[n_k]{D_{m_k}^{(n_k)}}\Big]$$

have an $\mathbb{R}_{F_1}\mathbb{Z}_{n_1}^{m_1} \times \cdots \times \mathbb{R}_{F_k}\mathbb{Z}_{n_k}^{m_k}$ multiplication.

5. Generalized Brandt-algebras

Directly related to the problem of lattice multiplication is the problem of the explicit description of integral domains in so-called *Brandt-algebras*. Quaternionic Brandt-algebras first appeared in an early work of Brandt (1920), see [3]. Their study was intensively continued by Fueter in the 1930s and 1940s, cf. [9,11]. Generalizations to the setting of Clifford algebras appear for instance in works of Elstrodt et al. (1987), see [6,7] (however, under a different viewpoint) and more close to the spirit of Brandt and Fueter in [14,15].

For convenience, let us recall the definition and some of the fundamental properties. Following [15, Chapter 2.7], every element

$$x = x_0 + x_1 e_1 + \cdots + x_n e_n$$

from the so-called paravector space $\mathbb{R} \oplus \mathbb{R}^n \cong \mathbb{R}^{n+1}$ satisfies a quadratic equation of the form

$$x \cdot x - S(x)x + N(x) = 0.$$

Here $\cdot$ is the $Cl_{0,n}$ multiplication operation as defined in the introductory section. Furthermore, $S(x) = 2x_0$ and $N(x) = \sum_{i=0}^{n} x_i^2$ denote the trace and the norm of x, respectively. Using the Clifford algebra conjugation anti-automorphism which is defined for each $a, b \in Cl_{0,n}$ by $\overline{(ab)} = \bar{b}\bar{a}$ and $\overline{e_i} = -e_i$ for $i = 1, 2, \ldots, n$, one can also write $S(x) = x + \bar{x}$ and $N(x) = x \cdot \bar{x} = \bar{x}x$. Now we introduce

Definition 4. A subset $B_{\mathbb{Q}}$ of $\mathbb{R}^n$ is called a rational $Cl_{0,n}$ Brandt-algebra if

$$S(a+b), N(a+b), S(a \cdot b), N(a \cdot b) \in \mathbb{Q}, \quad \forall a, b \in B_{\mathbb{Q}}.$$

A subset $B_{\mathbb{Z}}$ of $\mathbb{R}^n$ is called an integral $Cl_{0,n}$ Brandt-algebra if $S(a+b), N(a+b), S(a \cdot b), N(a \cdot b) \in \mathbb{Z}$ for all $a, b \in B_{\mathbb{Z}}$.

In the case $Cl_{0,2}$ we are dealing with the classical quaternionic rational and integral Brandt-algebras, respectively.

As shown in [15], the latter conditions are satisfied if and only if

$$S(a), S(b), N(a), N(b), 2\langle a, b\rangle \in \mathbb{Z}.$$

Here $\langle a, b\rangle := \sum_{i=0}^{n} a_i b_i$ denotes the Euclidean scalar product on the paravector space $\mathbb{R} \oplus \mathbb{R}^n$.

In the quaternionic case, the integral Brandt-algebras coincide precisely with the four-dimensional lattices which have quaternionic multiplication. In the more general case, they coincide with the lattices in $\mathbb{R}^{n+1}$ that have paravector multiplication. This is proved in [14,15].

An interesting question is to analyze how one can extend the theory of Brandt-algebras to the more general context of $\mathbb{R}_F G$ algebras and how these generalizations are related to lattices with $\mathbb{R}_F G$ multiplication that we introduced in the previous section.

In this paper we focus on the simplest case $G = \mathbb{Z}_3$ exclusively. The more general cases will be discussed in our follow-up work.

For simplicity let us denote in all that follows the integer $F(e_i, e_j)$ by F_{ij}. We start by noting

Lemma 1. *Each element $x = x_0e_0 + x_1e_1 + x_2e_2 \in \mathbb{R}_F\mathbb{Z}_3$ satisfies the cubic equation*

$$x^3 - T(x)x^2 + S(x)x - N(x)e_0 = 0$$

where for x^3 we mean $x^2 \cdot x$. Here, T is a linear form called the trace, S is a quadratic form and N is a cubic form called the norm. These expressions are uniquely defined by $T(x) = 3x_0$, $S(x) = 3x_0^2 - \delta x_1x_2$ and $N(x) = x_0^3 + x_1^3F_{11}F_{21} + x_2^3F_{22}F_{12} - x_0x_1x_2\delta$ where $\delta = F_{12} + F_{21} + F_{11}F_{22}$.

Proof. Consider two elements of $\mathbb{R}_F\mathbb{Z}_3$, say

$$x = x_0e_0 + x_1e_1 + x_2e_2,$$
$$y = y_0e_0 + y_1e_1 + y_2e_2.$$

Multiplying them with each other in terms of the $\mathbb{R}_F\mathbb{Z}_3$ multiplication one obtains

$$\begin{aligned} x \cdot y = {} & (x_0y_0 + F_{12}x_1y_2 + F_{21}x_2y_1)e_0 + (x_0y_1 + x_1y_0 + F_{22}x_2y_2)e_1 \\ & + (x_0y_2 + x_2y_0 + F_{11}x_1y_1)e_2. \end{aligned}$$

In particular we get

$$x^2 = \bigl(x_0^2 + (F_{12} + F_{21})x_1x_2\bigr)e_0 + \bigl(2x_0x_1 + F_{22}x_2^2\bigr)e_1 + \bigl(2x_0x_2 + F_{11}x_1^2\bigr)e_2.$$

Furthermore,

$$\begin{aligned} x^3 = x^2 \cdot x = {} & \bigl[x_0^3 + 3(F_{12} + F_{21})x_0x_1x_2 + F_{12}F_{22}x_2^3 + F_{11}F_{21}x_1^3\bigr]e_0 \\ & + \bigl[3x_0^2x_1 + (F_{12} + F_{21})x_1^2x_2 + 3F_{22}x_0x_2^2 + F_{11}F_{22}x_1^2x_2\bigr]e_1 \\ & + \bigl[3x_0^2x_2 + (F_{12} + F_{21})x_1x_2^2 + 3F_{11}x_0x_1^2 + F_{22}F_{11}x_1x_2^2\bigr]e_2 \end{aligned}$$

for all $x \in \mathbb{R}_F\mathbb{Z}_3$. After applying direct calculations we obtain

$$x^3 - T(x)x^2 + S(x)x - N(x)e_0 = 0$$

with the uniquely determined expressions $T(x) = 3x_0$, $S(x) = 3x_0^2 - \delta x_1x_2$ and $N(x) = x_0^3 + x_1^3F_{11}F_{21} + x_2^3F_{22}F_{12} - x_0x_1x_2\delta$ where $\delta = F_{12} + F_{21} + F_{11}F_{22}$. $\square$

The expression $T(x)$ evidently provides a generalization of the trace expression. The expression $N(x)$ generalizes the norm to the context of $\mathbb{R}_F\mathbb{Z}_3$ algebras. Notice that $N(e_0)=1$, $N(e_1)=F_{11}F_{21}$ and $N(e_2)=F_{22}F_{12}$. Cubic algebras with $N(x)=0$ were studied by S. Walcher in [19]. As we will show next, this expression serves as invertibility indicator, similarly as the classical norm expression does for the case of Clifford algebras:

Lemma 2. *An element $x=x_0e_0+x_1e_1+x_2e_2\in\mathbb{R}_F\mathbb{Z}_3$ has a left inverse in $\mathbb{R}_F\mathbb{Z}_3$ if and only if $N(x)\neq 0$. In this case, the $\mathbb{R}_F\mathbb{Z}_3$-left inverse element has the form $x^{-1}=\frac{\bar{x}}{N(x)}$ where the $\mathbb{R}_F\mathbb{Z}_3$-conjugate of x is*

$$\bar{x}=x^2-T(x)x+S(x)=\left(x_0^2-F_{11}F_{22}x_1x_2\right)e_0+\left(F_{22}x_2^2-x_0x_1\right)e_1+\left(F_{11}x_1^2-x_0x_2\right)e_2.$$

Proof. It is enough to note that $(x^2-T(x)x+S(x))x=N(x)e_0$ and we can illustrate the algebraic meaning of $N(x)$ as a determinant of a matrix. Indeed, an element $x\in\mathbb{R}_F\mathbb{Z}_3$ has a left inverse if and only if there exists an element $y\in\mathbb{R}_F\mathbb{Z}_3$ such that $y\cdot x=1$. This can be determined as solution to the linear system of equations

$$\begin{cases} x_0y_0+F_{12}y_1y_2+F_{21}y_2x_1=1,\\ y_0x_1+y_1x_0+F_{22}y_2x_2=0,\\ y_0x_2+x_0y_2+F_{11}y_1x_1=0. \end{cases}$$

This system has a solution if and only if the expression

$$\det\begin{pmatrix} x_0 & F_{12}x_2 & F_{21}x_1\\ x_1 & x_0 & F_{22}x_2\\ x_2 & F_{22}x_1 & x_0 \end{pmatrix}=N(x)$$

differs from zero. □

Remark. Notice that for elements $x\in\mathbb{R}_F\mathbb{Z}_3$ satisfying the equality $x^2\cdot x=x\cdot x^2$ we have that $\bar{x}\cdot x=x\cdot\bar{x}=N(x)$. In this case, the left inverse coincides with the right inverse.

It remains to analyze the role of the other expression $S(x)$. It turns out to be a dual counterpart of the generalized trace function whenever the algebra $\mathbb{R}_F\mathbb{Z}_3$ is associative.

More precisely, we obtain

Proposition 5. *If $\mathbb{R}_F\mathbb{Z}_3$ is an associative algebra then for each $x\in\mathbb{R}_F\mathbb{Z}_3$ we have $S(x)=T(\bar{x})$.*

Proof. If the algebra $\mathbb{R}_F\mathbb{Z}_3$ is associative then $F(x,y)F(xy,z)=F(y,z)F(x,yz)$, $\forall x,y,z\in\mathbb{Z}_3$. Therefore, $F_{11}F_{21}=F_{11}F_{12}$ and $F_{12}=F_{21}$. On the other hand, $F_{12}F_{02}=F_{22}F_{11}$ and $F_{12}=F_{22}F_{11}$. As a consequence we obtain $\delta=3F_{12}$ and $S(x)=T(\bar{x})$. □

Using this approach we want to give criterions for the closure under conjugation and multiplication of a given lattice in $\mathbb{R}_F\mathbb{Z}_3$. To proceed in this direction let us consider two elements $z=x_0e_0+x_1e_1+x_2e_2$ and $w=y_0e_0+y_1e_1+y_2e_2$ from $\mathbb{R}_F\mathbb{Z}_3$. Next we introduce the following bilinear forms:

$$\langle z, w\rangle_0 = x_0 y_0 + F_{12} x_1 y_2 + F_{21} x_2 y_1;$$
$$\langle z, w\rangle_0^\delta = x_0 y_0 + (\delta/3)(x_1 y_2 + x_2 y_1);$$
$$\langle z, w\rangle_1 = x_0 y_1 + x_1 y_0 + F_{22} x_2 y_2;$$
$$\langle z, w\rangle_2 = x_0 y_2 + x_2 y_0 + F_{11} x_1 y_1.$$

Now we can write the product of the two elements z and w in the form

$$zw = \langle z, w\rangle_0 e_0 + \langle z, w\rangle_1 e_1 + \langle z, w\rangle_2 e_2.$$

If $\mathbb{R}_F\mathbb{Z}_3$ is associative, then as mentioned in the proof of Proposition 5, it follows that $F_{21} = F_{12} = F_{11}F_{22}$. In this case we thus obtain $\langle z, w\rangle_0^\delta = \langle z, w\rangle_0$.

Lemma 3. *Suppose that $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3$ is a three-dimensional lattice embedded in the algebra $\mathbb{R}_F\mathbb{Z}_3$. Assume that $T(\omega_i), S(\omega_i) \in \mathbb{Z}$ for all $i = 1, 2, 3$ and that $3\langle \omega_i, \omega_j\rangle_0^\delta \in \mathbb{Z}$ for all $i, j = 1, 2, 3$; $i \neq j$. Then $z^2 \in \Omega$ if and only if $\bar{z} \in \Omega$.*

Proof. Let us consider an element from Ω, say $z = a\omega_1 + b\omega_2 + c\omega_3$ with $a, b, c \in \mathbb{Z}$. Then

$$T(z) = aT(\omega_1) + bT(\omega_2) + cT(\omega_3);$$
$$S(z) = a^2 S(\omega_1) + b^2 S(\omega_2) + c^2 S(\omega_3) + ab\bigl(T(\omega_1)T(\omega_2) - 3\langle \omega_1, \omega_2\rangle_0^\delta\bigr)$$
$$+ ac\bigl(T(\omega_1)T(\omega_3) - 3 < \omega_1, \omega_3 >_0^\delta\bigr) + cb\bigl(T(\omega_3)T(\omega_2) - 3 < \omega_3, \omega_2 >_0^\delta\bigr).$$

In view of the relation $\bar{z} = z^2 + T(z)z - S(z)$ we obtain the desired result. □

Now we give a necessary and sufficient condition for a class of lattices in $\mathbb{R}_F\mathbb{Z}_3$ to be closed under multiplication and under the conjugation. To do so let us suppose that $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^3$ are $\mathbb{R}$-linearly independent vectors with integral coordinates with respect to the canonical basis, i.e.

$$w_i = w_0^{(i)} e_0 + w_1^{(i)} e_1 + w_2^{(i)} e_2 \quad (i = 1, 2, 3).$$

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3$ be the associated lattice in $\mathbb{R}_F\mathbb{Z}_3$. Further, let $M = (m_{kj})$ be the 3×3 matrix where the kj-entry is defined by $m_{kj} = w_{k-1}^{(j)}$ for all $k, j = 1, 2, 3$. Let x, z be two arbitrary points of the lattice. Further, we denote by $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ the vector with the coordinates $x_k = \langle z, x\rangle_{k-1}$ $(k = 1, 2, 3)$ in the standard basis. Finally $Y = (y_1, y_2, y_3)^T \in \mathbb{R}^3$ denotes the vector with the coordinates $y_k = \langle z, z\rangle_{k-1}$ in the standard basis.

Using these notations we formate the following

Lemma 4. *Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3$ be a lattice in $\mathbb{R}^3$ embedded in $\mathbb{R}_F\mathbb{Z}_3$ generated by elements with integral coefficients in the standard basis. Let x, z be two arbitrary lattice points.*

Then $z \cdot x \in \Omega$ if and only if $M^{-1}X$ is an integral vector. Moreover, if

$$T(\omega_i), S(\omega_i), 3\langle \omega_i, \omega_j\rangle_0^\delta \in \mathbb{Z}, \quad \forall i, j = 1, 2, 3,\ i \neq j,$$

then $\bar{z} \in \Omega$ if and only if $M^{-1}Y$ is an integral vector.

Proof. Since w_1, w_2, w_3 are three linearly independent vectors from $\mathbb{R}^3$, we can consider the matrix M that represents the change of the basis $(w_i)_{i=1,2,3}$ to the standard basis $(e_i)_{i=1,2,3}$. The vector $M^{-1}X$ expresses the components of $z \cdot x$ with respect to the basis $(w_i)_{i=1,2,3}$. Finally if $M^{-1}Y$ has only integral components then $z^2 \in \Omega$. Applying Lemma 3 allows us to conclude that $\bar{z} \in \Omega$. □

A meaningful generalization of a rational Brandt-algebra to the context of $\mathbb{R}_F\mathbb{Z}_3$ can be introduced as follows:

Definition 5. A subset $B_{\mathbb{Q}}$ of $\mathbb{R}_F\mathbb{Z}_3$ is called a rational $\mathbb{R}_F\mathbb{Z}_3$ Brandt-algebra if for all elements $a, b \in B_{\mathbb{Q}}$

$$T(a+b), S(a+b), N(a+b), T(a \cdot b), S(a \cdot b), N(a \cdot b) \in \mathbb{Q}.$$

Definition 6. A subset $B_{\mathbb{Z}}$ of $\mathbb{R}_F\mathbb{Z}_3$ is called an integral Brandt-algebra if all $a, b \in B_{\mathbb{Z}}$ satisfy

$$T(a+b), S(a+b), N(a+b), T(a \cdot b), S(a \cdot b), N(a \cdot b) \in \mathbb{Z}.$$

Remark. If moreover all $x \in B_{\mathbb{Z}} \setminus \{0\}$ satisfy $N(x) \neq 0$, then $B_{\mathbb{Z}}$ is an integral domain in the rational Brandt algebra $B_{\mathbb{Q}}$. This is then the division ring of $B_{\mathbb{Z}}$.

In the associative case we have $S(x) = T(\bar{x})$. Then the conditions $S(a), S(b), S(a+b)$, $S(a \cdot b) \in \mathbb{Q}$ (respectively in $\mathbb{Z}$) can be re-expressed equivalently by $T(\bar{a}), T(\bar{b}), T(\overline{a+b})$, $T(\overline{a \cdot b}) \in \mathbb{Q}$ (respectively in $\mathbb{Z}$). So, we may establish

Proposition 6. *Two elements $y = y_0e_0 + y_1e_1 + y_2e_2, x = x_0e_0 + x_1e_1 + x_2e_2 \in \mathbb{R}_F\mathbb{Z}_3$ belong to an integral Brandt-algebra in $\mathbb{R}_F\mathbb{Z}_3$, if and only if the following expressions*

$$T(y), S(y), N(y), T(x), S(x), N(x), 3\langle x, y\rangle_0, 3\langle x, y\rangle_0^\delta,$$
$$3\langle x, y\rangle_0^2 - \delta\langle x, y\rangle_1\langle x, y\rangle_2,$$
$$\langle x, y\rangle_0\langle x, y\rangle_1\langle x, y\rangle_2\delta - \left(\langle x, y\rangle_0^3 + \langle x, y\rangle_1^3 F_{11}F_{21} + \langle x, y\rangle_2^3 F_{22}F_{12}\right),$$
$$3\sum_{i=1}^{3} x_i y_i (x_i + y_i) F_{ii} F_{i^{-1}i} - \delta \sum_{i,j,k=1, i\neq j\neq k, i\neq k} (x_i x_j y_k + y_i y_j x_k)$$

are all elements in $\mathbb{Z}$.

Proof. By a direct calculation we may deduce that $T(x+y) = T(x) + T(y)$ and that indeed $T(xy) = 3\langle x, y\rangle_0$. Furthermore, $S(x+y) = S(x) + S(y) + T(x)T(y) - 3\langle x, y\rangle_0^\delta$ and $S(xy) = 3\langle x, y\rangle_0^2 - \delta\langle x, y\rangle_1\langle x, y\rangle_2$. For the norm of the sum and the product of x with y we obtain

$$N(x+y) = N(x) + N(y) + 3\sum_{i=1}^{3} x_i y_i (x_i + y_i) F_{ii} F_{i^{-1}i} - \delta \sum_{i,j,k=1, \neq j \neq k} (x_i x_j y_k + y_i y_j x_k)$$

as well as

$$N(xy) = -\langle x, y\rangle_0\langle x, y\rangle_1\langle x, y\rangle_2\delta + \big(\langle x, y\rangle_0^3 + \langle x, y\rangle_1^3 F_{11}F_{21} + \langle x, y\rangle_2^3 F_{22}F_{12}\big). \qquad \square$$

Examples of integral domains in rational Brandt algebras of $\mathbb{R}_F\mathbb{Z}_3$ that are closed under multiplication and conjugation are some of the prototypes of the three-dimensional lattices that are described in Section 4.

Theorem 3. *Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \mathbb{Z}\omega_3$ be a lattice in $\mathbb{R}_F\mathbb{Z}_3$. Suppose that the generators have the form*

$$\omega_i = w_1^{(i)} e_0 + w_2^{(i)} \sqrt[3]{D}\, e_1 + w_3^{(i)} \sqrt[3]{D^2}\, e_2 \quad (i = 1, 2, 3)$$

with $w_j^{(i)} \in \mathbb{Z}$ for $i, j = 1, 2, 3$ and where D is a cubic-free positive integer. Then Ω is an integral domain. Let ω be the matrix whose ij-entry is equal to w_{ij}. If ω or $-\omega$ lies in $SL(3, \mathbb{Z})$ then the lattice Ω is closed under multiplication and conjugation.

Proof. Let us consider an arbitrary element from Ω, say $z = me_0 + n\sqrt[3]{D}\, e_1 + p\sqrt[3]{D^2}$ where m, n, p are integers. (Note that the sum and the product of two elements of this type has this form.) Therefore, $T(z) = 3m \in \mathbb{Z}$, $S(z) = 3m^2 - \delta Dnp \in \mathbb{Z}$ and $N(z) = m^3 + n^3 D F_{11}F_{21} + p^3 D^2 F_{22}F_{12} - mnpD\delta \in \mathbb{Z}$. As a consequence Ω is an integral Brandt-algebra in $\mathbb{R}_F\mathbb{Z}_3$. Moreover, one can verify by a direct computation that the norm of any non-zero element of this lattice is different from zero. Furthermore, the product of two arbitrary lattice elements turns out to be zero if and only if at least one of the factors is zero. Hence, this lattice has no zero divisors. It is thus an integral domain (in the more general sense admitting non-associativity).

Consequently every element has a left inverse. This is the inverse element if the algebra $\mathbb{R}_F\mathbb{Z}_3$ is associative. Furthermore, in the associative case the matrix M defined before Lemma 4 is the product of the diagonal matrix $D = \mathrm{diag}(1, \sqrt[3]{D}, \sqrt[3]{D^2})$ with an integral matrix ω that has an inverse which is an integral matrix if $\det\omega = +1$ or $\det\omega = -1$. However, for all $x, y \in \Omega$, their product in the basis e_i can be represented by a column vector that can be expressed as a product of the same diagonal matrix D with a column vector with integer elements. Thus, $x \cdot y$ has integer components with respect to the basis $(w_i)_{i=0,1,2}$ and Ω is closed for multiplication. Since $T(\omega_i)$, $S(\omega_i)$, $\forall i \in \{0, 1, 2\}$ and $\langle\omega_i, \omega_j\rangle_0^\delta$ $(i, j = 1, 2, 3;\ i \neq j) \in \mathbb{Z}$, we can conclude that $\bar{x} \in \Omega$ for all $x \in \Omega$. $\square$

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