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Critical operators for the degree of the minimal polynomial of derivations restricted to Grassmann spaces

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Abstract

Let V be a finite dimension vector space. For a linear operator on V, f, $D(f)$ denotes the restriction of the derivation associated with f to the mth Grassmann space of V. In [J.A. Dias da Silva, Y.O. Hamidoune, Cyclic spaces for Grassmann derivatives and aditive theory, Bull. London Math. Soc., 26 (1994) 140–146] Dias da Silva and Hamidoune obtained a lower bound for the degree of the minimal polynomial of $D(f)$, over an arbitrary field. Over a field of zero characteristic that lower bound is given by

$$\deg(P_{D(f)}) \geqslant m(\deg(P_f) - m) + 1.$$

Using additive number theory results, results on the elementary divisors of $D(f)$ and methods presented by Marcus and Ali in [Marvin Marcus, M. Shafqat Ali, Minimal polynomials of additive commutators and Jordan products, J. Algebra 22 (1972) 12–33] we obtain a characterization of equality cases in the former inequality, over a field of zero characteristic, whenever m does not exceed the number of distinct eigenvalues of f.

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1. Introduction

Let $\mathbb{F}$ be a field of zero characteristic and let V be a finite dimension vector space over $\mathbb{F}$ such that $\dim V \geqslant m \geqslant 2$, where m is an integer. Let S_m be the symmetric group of degree m. For $\sigma \in S_m$, $P(\sigma)$ denotes the unique linear operator on the mth tensor power product of V, $\otimes^m V$, such that

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$$

for all $v_1, v_2, \ldots, v_m \in V$.

Let ε be the alternating character on S_m and consider the symmetrizer defined on $\otimes^m V$ by

$$T_\varepsilon = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) P(\sigma).$$

The mth Grassmann space of V is $\wedge^m V = T_\varepsilon(\otimes^m V)$. For $v_1, v_2, \ldots, v_m \in V$, $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ denotes $T_\varepsilon(v_1 \otimes v_2 \otimes \cdots \otimes v_m)$.

For a linear operator, g, on a vector space over $\mathbb{F}$, P_g denotes the minimal polynomial of g and $\deg(P_g)$ denotes its degree. The spectrum of g, i.e., the set of all eigenvalues of g in the algebraic closure of $\mathbb{F}$, is denoted by $\sigma(g)$.

We are going to use the well known fact that, for a simple structure linear operator, the degree of its minimal polynomial is equal to the cardinality of its spectrum.

Let f be a linear operator on V. The derivation associated with f is the linear operator on $\otimes^m V$,

$$f \otimes I_V \otimes \cdots \otimes I_V + I_V \otimes f \otimes \cdots \otimes I_V + \cdots + I_V \otimes I_V \otimes \cdots \otimes f.$$

The derivation associated with f commutes with T_ε [2, Section 3.2]. Hence, $\wedge^m V$ is an invariant subspace of the derivation associated with f. Let $D(f)$ denote the restriction of the derivation associated with f to $\wedge^m V$. In [1] Dias da Silva and Hamidoune obtained a lower bound for the degree of the minimal polynomial of $D(f)$, over an arbitrary field. Over a field of zero characteristic that lower bound is given by

$$\deg(P_{D(f)}) \geqslant m(\deg(P_f) - m) + 1. \tag{1}$$

Using additive number theory results, results on the elementary divisors of $D(f)$ and methods presented in [3] we shall obtain a characterization of equality cases in (1) (for zero characteristic), whenever m does not exceed the number of distinct eigenvalues of f.

2. Additive number theory results

Let k and r be positive integers. By $Q_{k,r}$ we denote the set of all strictly increasing maps from $\{1, \ldots, k\}$ into $\{1, \ldots, r\}$. If $\alpha \in Q_{k,r}$ we use the k-tuple notation for α, that is, $\alpha = (\alpha(1), \ldots, \alpha(k))$.

Let $A = \{a_1, a_2, \ldots, a_r\}$ be a finite non-empty subset of $\mathbb{F}$, such that $|A| = r \geqslant m$, where $|A|$ denotes the cardinality of A.

By $\wedge^m A$ we denote the set of sums of m distinct elements in A, that is,

$$\wedge^m A = \left\{ \sum_{i=1}^{m} a_{\alpha(i)} : \alpha \in Q_{m,r} \right\}.$$

In [1] Dias da Silva and Hamidoune obtained a lower bound for the cardinality of $\wedge^m A$, for A subset of an arbitrary field. In zero characteristic that lower bound is given by

$$|\wedge^m A| \geqslant m(|A| - m) + 1. \tag{2}$$

For subsets of $\mathbb{Q}$ it is well known a characterization of equality cases in (2).

Lemma 1 [6, Theorem 1.10]. *Let A be a finite subset of $\mathbb{Q}$ such that $|A| \geqslant m \geqslant 2$. Then*

$$|\wedge^m A| = m(|A| - m) + 1$$

if and only if one of the following conditions holds:

(1) $|A| \in \{m, m+1\}$;
(2) *A is an arithmetic progression*;
(3) $m = 2$, $|A| = 4$ *and there exist* $a \in \mathbb{Q}$, $q, q' \in \mathbb{Q} \setminus \{0\}$ *such that* $q \neq q'$, $q + q' \neq 0$ *and*

$$A = a + \{0, q, q', q + q'\}.$$

Next lemma will be used to adjust the proof of Lemma 1 in [6] to the case of an arbitrary field of zero characteristic. It is a straightforward generalization of Lemma 2.1 from [3].

Lemma 2. *Let $m \geqslant 2$ and let V be an n-dimensional vector space over a field of zero characteristic, $\mathbb{F}$. Let $r \in \mathbb{N}$ and let $u_1, \ldots, u_r \in V$ be distinct. Then there exists a basis $\{g_1, \ldots, g_n\}$ of V^*, such that, for each $j \in \{1, \ldots, n\}$, $g_j(u_1), \ldots, g_j(u_r)$ are r distinct elements in $\mathbb{F}$ and*

$$\left|\left\{\sum_{i=1}^{m} u_{\alpha(i)} : \alpha \in Q_{m,r}\right\}\right| \geqslant |\wedge^m \{g_j(u_1), \ldots, g_j(u_r)\}| \geqslant m(r - m) + 1.$$

Proposition 1. *Let $\mathbb{F}$ be a field of zero characteristic and let A be a finite subset of $\mathbb{F}$ such that $|A| \geqslant m \geqslant 2$. Then*

$$|\wedge^m A| = m(|A| - m) + 1$$

if and only if one of the following conditions holds:

(1) $|A| \in \{m, m+1\}$;
(2) *A is an arithmetic progression*;
(3) $m = 2$, $|A| = 4$ *and there exist* $a \in \mathbb{F}$, $q, q' \in \mathbb{F} \setminus \{0\}$ *such that* $q \neq q'$, $q + q' \neq 0$ *and*

$$A = a + \{0, q, q', q + q'\}.$$

Proof. The sufficient condition's proof is obvious, so we include only the necessary condition's proof. Suppose $A = \{a_1, \ldots, a_r\}$, where $r = |A| \geqslant m + 2 \geqslant 4$, and $|\wedge^m A| = m(r - m) + 1$.

Consider the vector space over $\mathbb{Q}$,

$$W = \left\{\sum_{i=1}^{r} \beta_i a_i : \beta_i \in \mathbb{Q}\right\}$$

and let $n = \dim_{\mathbb{Q}} W \leqslant r$. From Lemma 2 there exists a basis of W^*, $\{g_1, \ldots, g_n\}$, such that, for $t = 1, \ldots, n$,

$$|\{g_t(a_1), \ldots, g_t(a_r)\}| = r.$$

Without loss of generality we assume that $a_1, \ldots, a_r$ are ordered in such way that

$$g_1(a_1) < g_1(a_2) < \cdots < g_1(a_r).$$

We consider the elements in $\wedge^m A$ given by

$$\begin{aligned} b_{i,1} &= a_1 + \cdots + a_{m-1} + a_i, \quad i = m, \ldots, r, \\ b_{i,j} &= \underbrace{a_1 + \cdots + a_{m-j}}_{m-j} + a_i + \underbrace{a_{r-j+2} + \cdots + a_r}_{j-1}, \\ &\quad i = m - j + 2, \ldots, r - j + 1, \quad j = 2, \ldots, m \end{aligned}$$

and the m subsets of $\wedge^m A$ given by

$$\begin{aligned} B_1 &= \{b_{i,1} : i = m, \ldots, r\}, \\ B_j &= \{b_{i,j} : i = m - j + 2, \ldots, r - j + 1\}, \quad j = 2, \ldots, m. \end{aligned}$$

Since $g_1(a_1) < g_1(a_2) < \cdots < g_1(a_r)$, we have

$$g_1(b_{m,1}) < g_1(b_{m+1,1}) < \cdots < g_1(b_{r,1}) \tag{3}$$

and

$$g_1(b_{r-j+2,j-1}) < g_1(b_{m-j+2,j}) < g_1(b_{m-j+3,j}) < \cdots < g_1(b_{r-j+1,j}) < g_1(b_{m-j+1,j+1}), \quad j = 2, \ldots, m. \tag{4}$$

Hence the sets $B_1, B_2, \ldots, B_m$ are pairwise disjoint and, from $|\wedge^m A| = m(r - m) + 1$, it follows that

$$\wedge^m A = \bigcup_{j=1}^{m} B_j. \tag{5}$$

Let $j \in \{1, \ldots, m - 1\}$. For $i = m - j + 2, \ldots, r - j$ let

$$c_{i,j} = \underbrace{a_1 + \cdots + a_{m-j-1}}_{m-j-1} + a_{m-j+1} + a_i + \underbrace{a_{r-j+2} + \cdots + a_r}_{j-1}.$$

Suppose $j \geqslant 2$. Since $c_{i,j} \in \wedge^m A$ and $g_1(b_{m-j+2,j}) < g_1(c_{i,j}) < g_1(b_{m-j+1,j+1})$, it follows that $c_{i,j} \in B_j \setminus \{b_{m-j+2,j}\}$.

Therefore, from $g_1(c_{m-j+2,j}) < g_1(c_{m-j+3,j}) < \cdots < g_1(c_{r-j,j})$ and (4), we have $c_{i,j} = b_{i+1,j}$. Hence

$$a_{m-j+1} + a_i = a_{m-j} + a_{i+1}, \quad i = m - j + 2, \ldots, r - j, \quad j = 2, \ldots, m - 1.$$

Next we prove that this is also true for $j = 1$. For $m + 1 \leqslant i \leqslant r - 1$ we have

$$g_1(b_{m+1,1}) < g_1(c_{i,1}) < g_1(b_{m,2})$$

and so $c_{i,1} \in B_1 \setminus \{b_{m,1}, b_{m+1,1}\}$. From $g_1(c_{m+1,1}) < g_1(c_{m+2,1}) < \cdots < g_1(c_{r-1,1})$ and (3), we have $c_{i,1} = b_{i+1,1}$, that is, $a_m + a_i = a_{m-1} + a_{i+1}$.

Thus we have proved that

$$a_{t+1} - a_t = a_{m-j+1} - a_{m-j}, \quad j = 1, \ldots, m - 1, \quad t = m - j + 2, \ldots, r - j. \tag{6}$$

(I) $r \geqslant m+3$
First suppose $m = 2$. From (6) we have

$$a_{i+1} - a_i = a_2 - a_1, \quad i = 3, \dots, r-1. \tag{7}$$

Since $r \geqslant 5$ and

$$g_1(a_1 + a_t) < g_1(a_3 + a_{r-1}) < g_1(a_{t+1} + a_r), \quad t = 2, \dots, r-1,$$

from (5) it follows that $a_3 + a_{r-1} \in \{a_1 + a_r, a_2 + a_r\}$.
Then $a_3 + a_{r-1} = a_2 + a_r$, since, from (7), $a_1 + a_r = a_2 + a_{r-1} \neq a_3 + a_{r-1}$.
Hence, for $m = 2$ we have

$$a_{i+1} - a_i = a_2 - a_1, \quad i = 1, 2, \dots, r-1.$$

Next we prove that this is also true for $m \geqslant 3$. Suppose $m \geqslant 3$. For $i \in \{1, \dots, m-2\}$, taking $j = i$ and $t = m - i + 2$ in (6) we obtain $a_{m-i+3} - a_{m-i+2} = a_{m-i+1} - a_{m-i}$.
Taking $j = i+1$ and $t = m - (i+1) + 3 \leqslant r - (i+1)$ in (6) we obtain $a_{m-i+3} - a_{m-i+2} = a_{m-i} - a_{m-i-1}$.
Then $a_{m-i+1} - a_{m-i} = a_{m-i} - a_{m-i-1}$, for $i = 1, \dots, m-2$.
Hence

$$a_{i+1} - a_i = a_2 - a_1, \quad i = 1, \dots, m-1.$$

Taking $j = 2$ and $t = m$ in (6) we get $a_{m+1} - a_m = a_{m-1} - a_{m-2} = a_2 - a_1$.
For $i = m+1, \dots, r-1$, taking $j = 1$ and $t = i$ in (6) we have $a_{i+1} - a_i = a_m - a_{m-1} = a_2 - a_1$.
Thus

$$a_{i+1} - a_i = a_2 - a_1, \quad i = 1, \dots, r-1,$$

that is, A is an arithmetic progression with first term a_1 and difference $a_2 - a_1$.

(II) $r = m+2$
In this case, from (6), we have

$$a_{m-j+3} - a_{m-j+2} = a_{m-j+1} - a_{m-j}, \quad j = 1, \dots, m-1.$$

That is,

$$a_{m+2} - a_{m+1} = a_m - a_{m-1} = \cdots = \begin{cases} a_2 - a_1, & \text{if } m \text{ is even,} \\ a_3 - a_2, & \text{if } m \text{ is odd} \end{cases}$$

and

$$a_{m+1} - a_m = a_{m-1} - a_{m-2} = \cdots = \begin{cases} a_3 - a_2, & \text{if } m \text{ is even,} \\ a_2 - a_1, & \text{if } m \text{ is odd.} \end{cases}$$

Let

$$d = \begin{cases} a_2 - a_1, & \text{if } m \text{ is even,} \\ a_3 - a_2, & \text{if } m \text{ is odd,} \end{cases} \quad \text{and} \quad d' = \begin{cases} a_3 - a_2, & \text{if } m \text{ is even,} \\ a_2 - a_1, & \text{if } m \text{ is odd.} \end{cases}$$

If $m = 2$ then $r = 4$ and condition (2) or condition (3) holds according to $d = d'$ or $d \neq d'$.
Suppose $m \geqslant 3$. Since $r = m+2$, we have

$$B_1 = \{b_{m,1}, b_{m+1,1}, b_{m+2,1}\} = b_{m,1} + \{0, d', d + d'\}$$

and

$$B_2 = \{b_{m,2}, b_{m+1,2}\} = b_{m,1} + \{2d + d', 2d + 2d'\}.$$

Let $z = a_1 + \cdots + a_{m-3} + a_{m-1} + a_m + a_{m+1} = b_{m,1} + d + 2d' \in \wedge^m A$.
From $g_1(z) < g_1(b_{m-1,3})$ it follows that $z \in B_1 \dot{\cup} B_2$.
Then $d + 2d' \in \{0, d', d + d', 2d + d', 2d + 2d'\}$. Analyzing the five possibilities we conclude that only $d + 2d' = 2d + d'$ is admissible. Then $d = d' = a_2 - a_1$ and A is an arithmetic progression with first term a_1 and difference $a_2 - a_1$. □

3. Elementary divisors

Let $m \geqslant 2$, let $\mathbb{F}$ be a field of zero characteristic and let V be a finite dimension vector space over $\mathbb{F}$ such that $\dim V \geqslant m$. Let f be a linear operator on V. The following characterization of the elementary divisors of $D(f)$ is well known [4,5].

Let

$$(X - \mu_i)^{n_i}, \quad i = 1, 2, \ldots, \ell$$

be the elementary divisors of f, where $\mu_1, \ldots, \mu_\ell \in \overline{\mathbb{F}}$ are not necessarily distinct. Let $k_1, k_2, \ldots, k_\ell$ be nonnegative integers such that

$$k_1 + k_2 + \cdots + k_\ell = m \quad \text{and} \quad k_i \leqslant n_i, \quad i = 1, 2, \ldots, \ell. \tag{8}$$

Let $r_1, r_2, \ldots, r_\ell$ be nonnegative integers such that

$$2r_i \leqslant k_i(n_i - k_i), \quad i = 1, 2, \ldots, \ell. \tag{9}$$

For $s \in \{1, 2, \ldots, \ell\}$ define

$$E_s = k_s(n_s - k_s) - 2r_s + 1 \quad \text{and} \quad \mathscr{E}_s = \sum_{i=1}^{s} E_i.$$

For $q_1, q_2, \ldots, q_{\ell-1}$ integers such that

$$1 \leqslant q_s \leqslant \min\{\mathscr{E}_s - 2(q_1 + \cdots + q_{s-1}) + s - 1, E_{s+1}\}, \quad s = 1, \ldots, \ell - 1, \tag{10}$$

define

$$\eta(r_1, \ldots, r_\ell, q_1, \ldots, q_{\ell-1}) = \mathscr{E}_\ell - 2(q_1 + q_2 + \cdots + q_{\ell-1}) + \ell - 1.$$

Let $s \in \{1, 2, \ldots, \ell\}$. For each positive integer j we denote by $p_{s,j}$ the number of partitions of j into not more than k_s parts, each part at most $n_s - k_s$ and define $p_{s,0} = 1$.

For each $s \in \{1, 2, \ldots, \ell\}$ let

$$c_s = \begin{cases} 1, & \text{if } r_s = 0, \\ p_{s,r_s} - p_{s,r_s-1}, & \text{if } r_s > 0. \end{cases}$$

Theorem 1 [4,5]. *The elementary divisors of $D(f)$ are*

$$\left(X - \sum_{s=1}^{\ell} k_s\mu_s\right)^{\eta(r_1,\ldots,r_\ell,q_1,\ldots,q_{\ell-1})}, \quad c_1c_2\cdots c_\ell \text{ times},$$

when $k_1, \ldots, k_\ell, r_1, \ldots, r_\ell, q_1, \ldots, q_{\ell-1}$ run over the sets of nonnegative integers satisfying (8)–(10).

Remark 1. For $k_1, \dots, k_\ell, r_1, \dots, r_\ell, q_1, \dots, q_{\ell-1}$ satisfying (8)–(10), we have

$$\eta(r_1, \dots, r_\ell, q_1, \dots, q_{\ell-1}) \leqslant \mathscr{E}_\ell - \ell + 1 \leqslant \sum_{s=1}^{\ell} k_s(n_s - k_s) + 1.$$

Remark 2. If we consider $r_1 = \cdots = r_\ell = 0$ and $q_1 = \cdots = q_{\ell-1} = 1$, we obtain $c_1 = \cdots = c_\ell = 1$ and

$$\eta(\underbrace{0, \dots, 0}_{\ell}, \underbrace{1, \dots, 1}_{\ell-1}) = \sum_{s=1}^{\ell} k_s(n_s - k_s) + 1.$$

It follows that, if $k_1 + \cdots + k_\ell = m$ and $0 \leqslant k_i \leqslant n_i$, $i = 1, \dots, \ell$, then

$$\left(X - \sum_{s=1}^{\ell} k_s \mu_s\right)^{\sum_{s=1}^{\ell} k_s(n_s - k_s)+1}$$

is an elementary divisor of $D(f)$.

The following well known results can be obtained as corollaries from Theorem 1.

Corollary 1. *If $a_1, \dots, a_r \in \overline{\mathbb{F}}$ are the distinct eigenvalues of f and*

$$(X - a_i)^{n_{i,j}}, \quad j = 1, 2, \dots, s_i, \quad i = 1, \dots, r$$

are the elementary divisors of f then

$$\sigma(D(f)) = \left\{ \sum_{i=1}^{r} m_i a_i : m_1 + \cdots + m_r = m, m_i \in \mathbb{N}_0 \text{ and } m_i \leqslant \sum_{j=1}^{s_i} n_{i,j}, i = 1, \dots, r \right\}.$$

Corollary 2. *If f is of simple structure then also $D(f)$ is of simple structure.*

Corollary 3

1. $\wedge^m \sigma(f) \subseteq \sigma(D(f))$;
2. *If* $\dim V = |\sigma(f)|$ *then* $\wedge^m \sigma(f) = \sigma(D(f))$.

For $m = 2$ there is a considerably simpler characterization for the elementary divisors of $D(f)$.

Theorem 2 [2, Chapter 7, Theorem 2.6]. *Let*

$$(X - \mu_i)^{n_i}, \quad i = 1, 2, \dots, \ell$$

be the elementary divisors of f, where $\mu_1, \dots, \mu_\ell \in \overline{\mathbb{F}}$ are not necessarily distinct. The elementary divisors of the restriction of the derivation associated with f to $\wedge^2 V$ are:

$$(X - 2\mu_i)^k, \quad k = 2n_i - 3, 2n_i - 7, \dots, \begin{cases} 1, & \text{if } n_i \text{ is even}, \\ 3, & \text{if } n_i \text{ is odd}, \end{cases} \quad 1 \leqslant i \leqslant \ell$$

and

$$(X - \mu_i - \mu_j)^{n_i + n_j - 2t + 1}, \quad 1 \leqslant t \leqslant \min\{n_i, n_j\}, \quad 1 \leqslant i < j \leqslant \ell.$$

4. Main result

Theorem 3. *Let $m \geqslant 2$ and let V be a finite dimension vector space over a field of zero characteristic, $\mathbb{F}$, such that* $\dim V \geqslant m$. *Let f be a linear operator on V such that $r := |\sigma(f)| \geqslant m$. Let $D(f)$ be the restriction of the derivation associated with f to $\wedge^m V$. Then*

$$\deg(P_{D(f)}) = m(\deg(P_f) - m) + 1$$

if and only if one of the following conditions holds:

(1) $r = m = \dim V$;
(2) $r = m + 1 = \dim V$;
(3) *The elementary divisors of f are*

$$X - b_1, \ldots, X - b_{m-1}, (X - b_m)^2,$$

where $b_1, \ldots, b_m \in \overline{\mathbb{F}}$ are distinct;
(4) $r \geqslant m + 1$ *and the elementary divisors of f are*

$$X - b_i, \quad s_i \ times, \quad i = 1, \ldots, r,$$

where $b_1, \ldots, b_r$ is an arithmetic progression with first term b_1, $s_1 = \cdots = s_{m-1} = 1$ and $s_{r-m+2} = \cdots = s_r = 1$;
(5) $m = 2$ *and the elementary divisors of f are*

$$X - b, \quad (X - b - q)^2, \quad X - b - 2q,$$

where $b, q \in \overline{\mathbb{F}}$ and $q \neq 0$;
(6) $m = 2$ *and the elementary divisors of f are*

$$X - b, \quad X - b - q, \quad X - b - q', \quad X - b - q - q',$$

where $b \in \overline{\mathbb{F}}$, $q, q' \in \overline{\mathbb{F}} \setminus \{0\}$, $q \neq q'$ and $q + q' \neq 0$;
(7) $m = 2$ *and the elementary divisors of f are*

$$(X - b_1)^2, (X - b_2)^2,$$

where $b_1, b_2 \in \overline{\mathbb{F}}$ and $b_1 \neq b_2$.

Proof
Sufficient condition

(1), (2) and (6) In any of these cases f is of simple structure and $\dim V = |\sigma(f)|$. Then (Corollaries 2, 3 and Proposition 1)

$$\deg(P_{D(f)}) = |\sigma(D(f))| = |\wedge^m \sigma(f)| = m(r - m) + 1 = m(\deg(P_f) - m) + 1.$$

(3) From Corollary 1, the eigenvalues of $D(f)$ are the m elements

$$z_i = b_m + \sum_{\substack{j=1 \\ j \neq i}}^{m} b_j, \quad i = 1, \ldots, m$$

and (Remark 2) $X - z_1, X - z_2, \ldots, X - z_{m-1}, (X - z_m)^2$ are elementary divisors of $D(f)$. Since $\dim \wedge^m V = \binom{m+1}{m} = m + 1$, it follows that

$$P_{D(f)} = (X - z_m)^2 \prod_{i=1}^{m-1} (X - z_i)$$

and $\deg(P_{D(f)}) = m + 1 = m(\deg(P_f) - m) + 1$.

(4) Suppose $b_i = b_1 + (i-1)q$, where $q \in \overline{\mathbb{F}} \setminus \{0\}$. From Corollary 1,

$$\sigma(D(f)) = \left\{ mb_1 + q \sum_{i=1}^{r} m_i(i-1) : m_1 + \cdots + m_r = m \text{ and } 0 \leqslant m_i \leqslant s_i,\ i = 1, \ldots, r \right\}.$$

Since $s_1 = \cdots = s_{m-1} = 1$ and $s_{r-m+2} = \cdots = s_r = 1$,

$$\left\{ \sum_{i=1}^{r} m_i(i-1) : m_1 + \cdots + m_r = m \text{ and } 0 \leqslant m_i \leqslant s_i, i = 1, \ldots, r \right\}$$
$$= \left[\tfrac{m(m-1)}{2}, mr - \tfrac{m(m+1)}{2} \right] \cap \mathbb{N}.$$

Then

$$\sigma(D(f)) = \left\{ mb_1 + qz : z \in \left[\frac{m(m-1)}{2}, mr - \frac{m(m+1)}{2} \right] \cap \mathbb{N} \right\} = \wedge^m \sigma(f).$$

Since f is of simple structure, also $D(f)$ is of simple structure and $\deg(P_{D(f)}) = |\sigma(D(f))| = rm - m^2 + 1 = m \deg(P_f) - m^2 + 1$.

(5) From Theorem 2 the elementary divisors of $D(f)$ are

$$(X - 2b - q)^2, \quad X - 2b - 2q, \quad X - 2b - 2q, \quad (X - 2b - 3q)^2.$$

Then $P_{D(f)} = (X - 2b - 2q)(X - 2b - q)^2(X - 2b - 3q)^2$ and $\deg(P_{D(f)}) = 5 = 2\deg(P_f) - 3$.

(7) In this case $P_{D(f)} = (X - 2b_1)(X - 2b_2)(X - b_1 - b_2)^3$ and $\deg(P_{D(f)}) = 5 = 2\deg(P_f) - 3$.

Necessary condition

Suppose $\deg(P_{D(f)}) = m \deg(P_f) - m^2 + 1$. Let $a_1, \ldots, a_r \in \overline{\mathbb{F}}$ (where $r \geqslant m$) be the distinct eigenvalues of f and let

$$(X - a_i)^{n_{i,j}}, \quad j = 1, 2, \ldots, t_i, \quad i = 1, \ldots, r$$

be the elementary divisors of f, where, for each i, $n_i := n_{i,1} \geqslant n_{i,2} \geqslant \cdots \geqslant n_{i,t_i}$. Then $P_f = (X - a_1)^{n_1} \cdots (X - a_r)^{n_r}$.

Consider the $\mathbb{Q}$-vector space, $W = \left\{ \sum_{i=1}^{r} \beta_i a_i : \beta_i \in \mathbb{Q} \right\}$. Let d be its dimension and let $\{g_1, \ldots, g_d\}$ be a basis of W^* satisfying the conditions in Lemma 2, for the distinct elements in W, $a_1, a_2, \ldots, a_r$.

From Lemma 2, $g_1(a_1), g_1(a_2), \ldots, g_1(a_r)$ are distinct rational numbers. Without loss of generality we assume that $a_1, a_2, \ldots, a_r$ are ordered in such way that

$$g_1(a_1) < g_1(a_2) < \cdots < g_1(a_r). \tag{11}$$

We consider two cases: $r \geqslant m + 1$ and $r = m$.

(I) $r \geqslant m+1$

As in the proof of Proposition 1 we consider the m subsets of W given by

$$B_1 = \{a_1 + \cdots + a_{m-1} + a_i : i = m, \ldots, r\},$$
$$B_j = \left\{ \underbrace{a_1 + \cdots + a_{m-j}}_{m-j} + a_i + \underbrace{a_{r-j+2} + \cdots + a_r}_{j-1} : i = m-j+2, \ldots, r-j+1 \right\},$$
$$j = 2, \ldots, m.$$

For $j = 1, \ldots, m$ let ϕ_j and Φ_j be, respectively, the minimum and the maximum of $g_1(B_j)$, that is,

$$\phi_1 = g_1(a_1) + \cdots + g_1(a_m),$$
$$\Phi_1 = g_1(a_1) + \cdots + g_1(a_{m-1}) + g_1(a_r),$$
$$\phi_j = \underbrace{g_1(a_1) + \cdots + g_1(a_{m-j})}_{m-j} + g_1(a_{m-j+2}) + \underbrace{g_1(a_{r-j+2}) + \cdots + g_1(a_r)}_{j-1},$$
$$j = 2, \ldots, m,$$
$$\Phi_j = \underbrace{g_1(a_1) + \cdots + g_1(a_{m-j})}_{m-j} + g_1(a_{r-j+1}) + \underbrace{g_1(a_{r-j+2}) + \cdots + g_1(a_r)}_{j-1},$$
$$j = 2, \ldots, m.$$

As we have seen in Proposition 1, $\phi_1 < \Phi_1 < \phi_2 < \Phi_2 < \cdots < \phi_m < \Phi_m$.

Hence the elements in the disjoint union $\bigcup_{j=1}^{m} B_j$ are $m(r-m)+1$ distinct eigenvalues of $D(f)$, with associated elementary divisors

$$\left(X - a_i - \sum_{k=1}^{m-1} a_k\right)^{\sum_{k=1}^{m-1}(n_k-1)+(n_i-1)+1}, \quad i = m, \ldots, r;$$
$$\left(X - a_i - \sum_{k=1}^{m-j} a_k - \sum_{k=r-j+2}^{r} a_k\right)^{\sum_{k=1}^{m-j}(n_k-1)+\sum_{k=r-j+2}^{r}(n_k-1)+(n_i-1)+1},$$
$$i = m-j+2, \ldots, r-j+1, \quad j = 2, \ldots, m.$$

Let $t(X)$ be the product of these elementary divisors. Then

$$\begin{aligned}\deg(t(X)) &= (r-m+1)\left(\sum_{k=1}^{m-1} n_k - m + 1\right) + \sum_{i=m}^{r} n_i \\ &\quad + \sum_{j=2}^{m} \sum_{i=m-j+2}^{r-j+1} \left(\sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k + n_i - m + 1\right) \\ &= (r-m)\sum_{k=1}^{m-1} n_k + \deg(P_f) + (-m+1)(rm - m^2 + 1) \\ &\quad + (r-m-1)\sum_{j=2}^{m}\left(\sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k\right) + \sum_{j=2}^{m}(\deg(P_f) - n_{m-j+1})\end{aligned}$$

$$= (r-m-1)\sum_{k=1}^{m-1} n_k + m\deg(P_f) + (-m+1)(rm-m^2+1)$$
$$+ (r-m-1)\sum_{j=2}^{m}\left(\sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k\right).$$

Since $n_i \geqslant 1$, for all i, we have

$$\begin{aligned}\deg(t(X)) &\geqslant (r-m-1)(m-1) + m\deg(P_f) + (-m+1)(rm-m^2+1)\\ &\quad + (r-m-1)(m-1)^2\\ &\geqslant m\deg(P_f) - m^2 + 1.\end{aligned}$$

From $\deg(P_{D(f)}) = m\deg(P_f) - m^2 + 1$ it follows that $P_{D(f)} = t(X)$ and

$$\sigma(D(f)) = \bigcup_{j=1}^{m} B_j. \tag{12}$$

Suppose that $n_\ell \geqslant 2$ or $t_\ell \geqslant 2$ for some $\ell \in \{1, \ldots, m-1\}$. Then

$$c = 2a_\ell + \sum_{\substack{j=1\\ j\neq\ell}}^{m-1} a_j \in \sigma(D(f)).$$

Since

$$g_1(c) = g_1(a_\ell) + \sum_{i=1}^{m-1} g_1(a_i) < \sum_{i=1}^{m} g_1(a_i) = \phi_1,$$

we obtain a contradiction with (12).

Suppose that $n_\ell \geqslant 2$ or $t_\ell \geqslant 2$, for some $\ell \in \{r-m+2, \ldots, r\}$. Then

$$d = 2a_\ell + \sum_{\substack{j=r-m+2\\ j\neq\ell}}^{r} a_j \in \sigma(D(f))$$

and, from

$$g_1(d) = g_1(a_\ell) + \sum_{j=r-m+2}^{r} g_1(a_j) > \sum_{j=r-m+1}^{r} g_1(a_j) = \Phi_m,$$

we obtain a contradiction with (12).

Hence

$$n_i = t_i = 1 \quad \text{for} \quad i \in \{1, \ldots, m-1\} \cup \{r-m+2, \ldots, r\}. \tag{13}$$

From $|\sigma(D(f))| = m(|\sigma(f)| - m) + 1 \leqslant |\wedge^m \sigma(f)|$ and $\wedge^m\sigma(f) \subseteq \sigma(D(f))$ we conclude that

$$\sigma(D(f)) = \wedge^m\sigma(f) \tag{14}$$

and $|\wedge^m \sigma(f)| = m(|\sigma(f)| - m) + 1$.

Then (Proposition 1) one of the following conditions holds:

(a) $r = m + 1$:
If $m \geqslant 3$ then, from (13), we have $n_i = t_i = 1, i = 1, \ldots, r$. Condition (2) holds. If $m = 2$ then $r = 3$ and, from (13), $n_1 = n_3 = t_1 = t_3 = 1$. If $n_2 = t_2 = 1$ then condition (2) holds. Suppose $n_2 \geqslant 2$ or $t_2 \geqslant 2$. Then, from (14) and Corollary 1, we have

$$2a_2 \in \sigma(D(f)) = \{a_1 + a_2, a_1 + a_3, a_2 + a_3\}.$$

Therefore $2a_2 = a_1 + a_3$ and $\sigma(f)$ is an arithmetic progression with first term a_1 and difference $a_2 - a_1$. If $n_2 = 1$ condition (4) holds.
Suppose $n_2 \geqslant 2$. From $\deg(P_f) = n_2 + 2$ it follows that $\deg(P_{D(f)}) = 2n_2 + 1$.
Hence

$$P_{D(f)} = (X - a_1 - a_2)^{n_2}(X - a_1 - a_3)(X - a_2 - a_3)^{n_2}.$$

Since $2a_2 = a_1 + a_3$ and $(X - 2a_2)^{2n_2-3}$ is an elementary divisor of $D(f)$ we get $n_2 = 2$. Suppose $t_2 \geqslant 2$. Then $(X - a_2)^2$ and $(X - a_2)^{n_{2,2}}$ are elementary divisors of f and $(X - 2a_2)^{n_{2,2}+1} = (X - a_1 - a_3)^{n_{2,2}+1}$ is an elementary divisor of $D(f)$ and this leads to a contradiction. Then $t_2 = 1$ and condition (5) holds.

(b) $\sigma(f)$ is an arithmetic progression:
Let b and d be, respectively, the first term and the difference of that arithmetic progression. Since $b, b + d \in W$, then also $d \in W$ and $g_1(b), g_1(b + d), \ldots, g_1(b + (r - 1)d)$ is an arithmetic progression in $\mathbb{Q}$ with difference $g_1(d) \neq 0$ (from (11)).
If $g_1(d) > 0$ then $g_1(b) < g_1(b + d) < \cdots < g_1(b + (r - 1)d)$ and so, from (11), we have $a_i = b + (i - 1)d$, for $i = 1, \ldots, r$.
If $g_1(d) < 0$ then $a_i = b + (r - i)d$, for $i = 1, \ldots, r$.
From (13) we have $n_i = t_i = 1$ for all $i \in \{1, \ldots, m - 1\} \cup \{r - m + 2, \ldots, m\}$. If f is of simple structure then condition (4) holds.
Suppose f is not of simple structure. Then, from (13), it follows that $r - m + 1 \geqslant m$ and $n_\ell \geqslant 2$ for some $\ell \in \{m, \ldots, r - m + 1\}$. Let ℓ be the smallest element in $\{m, \ldots, r - m + 1\}$ such that $n_\ell \geqslant 2$. Notice that $\ell \leqslant r - m + 1 \leqslant r - 1$. Suppose $\ell \leqslant r - 2$ and consider

$$\begin{aligned} x_i &= \sum_{j=1}^{m-1} a_j + a_i, \quad i = m, \ldots, \ell; \\ y_i &= \sum_{j=1}^{m-2} a_j + a_i + a_r, \quad i = \ell + 1, \ldots, r - 1; \\ v_i &= \sum_{j=1}^{m-2} a_j + a_\ell + a_i, \quad i = m, \ldots, r. \end{aligned} \tag{15}$$

Since $g_1(x_m) < g_1(x_{m+1}) < \cdots < g_1(x_\ell) < g_1(v_m) < g_1(v_{m+1}) < \cdots < g_1(v_r) < g_1(y_{\ell+1}) < \cdots < g_1(y_{r-1}) < \phi_3$, the elements in (15) are $2r - 2m + 1$ distinct eigenvalues of $D(f)$, not in $\bigcup_{j=3}^{m} B_j$.
From (13) and $n_1 = \cdots = n_{\ell-1} = 1$, we conclude that

$$(X - x_i), \quad i = m, \ldots, \ell - 1;$$

$$(X - x_\ell)^{n_\ell};$$
$$(X - y_i)^{n_i}, \quad i = \ell + 1, \ldots, r - 1;$$
$$(X - v_i)^{n_\ell + n_i - 1}, \quad i = m, \ldots, r, \quad i \neq \ell;$$
$$(X - v_\ell)^{2n_\ell - 3}$$

are elementary divisors of $D(f)$.
Then

$$\begin{aligned} m \deg(P_f) - m^2 + 1 &\geqslant \ell - m + n_\ell + \sum_{i=\ell+1}^{r-1} n_i + \sum_{\substack{i=m \\ i \neq \ell}}^{r} (n_\ell + n_i - 1) \\ &\quad + 2n_\ell - 3 + \sum_{j=3}^{m} \sum_{i=m-j+2}^{r-j+1} n_i \\ \Rightarrow m \deg(P_f) - m^2 + 1 &\geqslant \ell - m - 3 + \sum_{i=\ell}^{r-1} n_i + \sum_{i=m}^{r} n_i + (r - m)(n_\ell - 1) \\ &\quad + n_\ell + \sum_{j=3}^{m} \sum_{i=m-j+2}^{r-j+1} n_i \\ \Rightarrow m \deg(P_f) - m^2 + 1 &\geqslant \ell - m - 3 + (\deg(P_f) - \ell) + (\deg(P_f) - m + 1) \\ &\quad + (r - m + 1)n_\ell - r + m + \sum_{j=3}^{m} \sum_{i=m-j+2}^{r-j+1} n_i. \end{aligned}$$

For $3 \leqslant j \leqslant m$ we have $m - j \leqslant m - 3$ and $r - j \geqslant r - m$. So, if $i \leqslant m - j + 1$ or $i \geqslant r - j + 2$ then $n_i = 1$. Hence $\sum_{i=m-j+2}^{r-j+1} n_i = \deg(P_f) - m$ and

$$\begin{aligned} m \deg(P_f) - m^2 + 1 &\geqslant 2 \deg(P_f) - m - 2 - r + (r - m + 1)n_\ell \\ &\quad + (m - 2)(\deg(P_f) - m) \\ \Rightarrow -m^2 + 1 &\geqslant (r - m + 1)n_\ell - r - m - m^2 + 2m - 2 \\ \Rightarrow (r - m + 1)n_\ell &\leqslant r - m + 3. \end{aligned}$$

From the last inequality, since we are assuming that $n_\ell \geqslant 2$, we have $r = m + 1$ and, from $\ell \leqslant r - 2 = m - 1$, we obtain a contradiction with (13). Then $\ell = r - 1$ and, from $\ell \leqslant r - m + 1$, it follows that $m = 2$.
So, if f is not of simple structure then $m = 2$, $n_{r-1} \geqslant 2$ and

$$n_i = t_i = 1 \quad \text{for } i \in \{1, \ldots, r\} \setminus \{r - 1\}.$$

In this case,

$$x_i = a_1 + a_i, \quad i = 2, \ldots, r - 1;$$
$$v_i = a_{r-1} + a_i, \quad i = 2, \ldots, r$$

are $2r-3$ distinct eigenvalues of $D(f)$. Since $n_i = 1$ for $i \neq r-1$ we obtain

$$2\deg(P_f) - 3 \geqslant \sum_{i=2}^{r-1} n_i + \sum_{\substack{i=2\\ i\neq r-1}}^{r} (n_{r-1} + n_i - 1) + 2n_{r-1} - 3$$
$$\Rightarrow 2\deg(P_f)-3 \geqslant \deg(P_f)-2+(r-2)(n_{r-1}-1)+\deg(P_f)-n_{r-1}-1+2n_{r-1}-3$$
$$\Rightarrow 2 \geqslant (r-1)(n_{r-1}-1).$$

Since $r \geqslant m+1 = 3$, it follows that $r = 3$ and $n_2 = 2$. From (13) we have $t_1 = t_3 = 1$. Suppose $t_2 \geqslant 2$. Then $(X-a_2)^2$ and $(X-a_2)^{n_{2,2}}$ are elementary divisors of f and

$$(X-a_1-a_2)^2, \quad (X-a_2-a_3)^2, \quad (X-2a_2)^{n_{2,2}+1}$$

are elementary divisors of $D(f)$ associated to distinct eigenvalues. Hence $5 = \deg(P_{D(f)}) \geqslant 5 + n_{2,2}$, which leads to a contradiction. Then $t_2 = 1$ and condition (5) holds.

(c) $m = 2, r = 4$ and $\sigma(f) = a + \{0, q, q', q+q'\}$, for some $a \in \overline{\mathbb{F}}$, $q, q' \in \overline{\mathbb{F}} \setminus \{0\}$ such that $q \neq q'$ and $q + q' \neq 0$.
First we prove that f is of simple structure. From (13) $n_1 = n_4 = 1$. Hence $P_f = (X - a_1)(X-a_2)^{n_2}(X-a_3)^{n_3}(X-a_4)$ and $\deg(P_{D(f)}) = 2n_2 + 2n_3 + 1$.
On the other hand, since $a_2 + a_3 \in \wedge^2\sigma(f) = \sigma(D(f))$ and

$$\sigma(D(f)) = B_1 \cup B_2 = \{a_1+a_2, a_1+a_3, a_1+a_4, a_2+a_4, a_3+a_4\},$$

it follows that $a_2 + a_3 = a_1 + a_4$ and, from Theorem 2, we have

$$P_{D(f)} = (X-a_1-a_2)^{n_2}(X-a_1-a_3)^{n_3}(X-a_2-a_3)^{n_2+n_3-1}(X-a_2-a_4)^{n_2}(X-a_3-a_4)^{n_3}.$$

Then $n_2 = n_3 = 1$ and f is of simple structure.
From $\sigma(f) = a + \{0, q, q', q+q'\}$ it follows that

$$\sigma(D(f)) = \wedge^2\sigma(f) = 2a + \{q, q', q+q', 2q+q', q+2q'\}.$$

Let

$X - a$, s_1 times
$X - a - q$, s_2 times
$X - a - q'$, s_3 times
$X - a - q - q'$, s_4 times

be the elementary divisors of f. From (13) we know that, at least, two of the numbers s_1, s_2, s_3, s_4 are equal to 1.
If $s_1 = s_2 = s_3 = s_4 = 1$ then condition (6) holds.
Suppose $s_1 \geqslant 2$. Then $2a \in \sigma(D(f))$. Hence $2q + q' = 0$ or $q + 2q' = 0$. Then $\sigma(f)$ is an arithmetic progression. Similarly, if $s_i \geqslant 2$ for some $i \in \{2, 3, 4\}$, then $\sigma(f)$ is an arithmetic progression. As we have seen in (b), condition (4) holds.

(II) $r = m$:
First we assume that f is not of simple structure. Then $n_i \geqslant 2$ for some $i \in \{1, \ldots, m\}$. Let ℓ be the greatest element in $\{1, \ldots, m\}$ such that

$$n_\ell = \max\{n_i : i = 1, \ldots, m\} \geqslant 2.$$

Let

$$z_i = a_\ell + \sum_{\substack{j=1\\ j\neq i}}^{m} a_j, \quad i = 1, \ldots, m.$$

$z_1, \ldots, z_m$ are distinct eigenvalues of $D(f)$ and, since $n_\ell \geqslant 2$, $(X - z_\ell)^{\sum_{j=1}^{m}(n_j-1)+1}$ and

$$(X - z_i)^{\sum_{\substack{j=1\\ j\neq i\\ j\neq \ell}}^{m}(n_j-1)+2(n_\ell-2)+1}, \quad i = 1, \ldots, m, \quad i \neq \ell$$

are elementary divisors of $D(f)$.

Then, for some monic polynomial $q(X) \in \overline{\mathbb{F}}[X] \setminus \{0\}$,

$$P_{D(f)} = q(X)(X - z_\ell)^{\sum_{j=1}^{m}(n_j-1)+1} \prod_{\substack{i=1\\ i\neq \ell}}^{m} (X - z_i)^{\sum_{\substack{j=1\\ j\neq i\\ j\neq \ell}}^{m}(n_j-1)+2(n_\ell-2)+1} \tag{16}$$

and

$$\begin{aligned}
\deg(q(X)) &= m \deg(P_f) - m^2 + 1 - \deg(P_f) + m - 1 - \sum_{\substack{i=1\\ i\neq \ell}}^{m} \sum_{\substack{j=1\\ j\neq \ell\\ j\neq i}}^{m} (n_j - 1) - (m-1)(2n_\ell - 3)\\
&= (m-1)\deg(P_f) - m^2 + m - \sum_{\substack{i=1\\ i\neq \ell}}^{m} \sum_{\substack{j=1\\ j\neq \ell\\ j\neq i}}^{m} n_j + (m-1)(m-2) - (m-1)(2n_\ell - 3)\\
&= (m-1)\deg(P_f) - m^2 - \sum_{\substack{i=1\\ i\neq \ell}}^{m} (\deg(P_f) - n_i - n_\ell) - 2(m-1)n_\ell + m^2 + m - 1\\
&= \deg(P_f) - mn_\ell + m - 1.
\end{aligned} \tag{17}$$

We consider two subcases:

(i) $n_i = 1$, for all $i \neq \ell$:
In this case $\deg(P_f) = n_\ell + m - 1$ and, from (17), we obtain $0 \leqslant \deg(q(X)) = (n_\ell - 2)(1 - m)$. Then $n_\ell = 2$, $\deg(q(X)) = 0$ and

$$P_{D(f)} = (X - z_\ell)^2 \prod_{\substack{i=1\\ i\neq \ell}}^{m} (X - z_i).$$

Suppose $t_q \geqslant 2$ for some $q \in \{1, \ldots, m\} \setminus \{\ell\}$. Then, for $i = 1, \ldots, m$, $y_i = a_q + \sum_{\substack{j=1\\ j\neq i}}^{m} a_j$ is an eigenvalue of $D(f)$ and $g_1(y_1) > g_1(y_2) > \cdots > g_1(y_m)$. Since $\sigma(D(f)) = \{z_1, \ldots, z_m\}$ and $g_1(z_1) > g_1(z_2) > \cdots > g_1(z_m)$, it has to be $z_i = y_i$, for all i, which contradicts $a_q \neq a_\ell$. Then $t_q = 1$, for all $q \in \{1, \ldots, m\} \setminus \{\ell\}$.

Now suppose $t_\ell \geqslant 2$. Then $(X-a_\ell)^2$ and $(X-a_\ell)^{n_{\ell,2}}$ are elementary divisors of f. If $\ell \geqslant 2$ then

$$(X-z_1)^{\sum_{\substack{j=2\\ j\neq \ell}}^{m}(n_j-1)+(n_\ell-1)+(n_{\ell,2}-1)+1}$$

is an elementary divisor of $D(f)$, with degree $n_\ell + n_{\ell,2} - 1 \geqslant 2$ and we obtain a contradiction. Then $\ell = 1$ and

$$(X-z_2)^{\sum_{j=3}^{m}(n_j-1)+(n_1-1)+(n_{1,2}-1)+1}$$

is an elementary divisor of $D(f)$ with degree $n_1 + n_{1,2} - 1 \geqslant 2$. Once more, we obtain a contradiction. Then $t_i = 1$ for all $i \in \{1, 2, \ldots, m\}$ and condition (3) holds.

(ii) $n_i \geqslant 2$, for some $i \neq \ell$:
Let k be the greatest element in $\{1, \ldots, m\} \setminus \{\ell\}$ such that

$$n_k = \max\{n_i : i = 1, \ldots, \ell-1, \ell+1, \ldots, m\}.$$

From the definition of k, $n_\ell \geqslant n_k \geqslant 2$ and $\deg(P_f) \leqslant n_\ell + (m-1)n_k$. Then $0 \leqslant \deg(q(X)) \leqslant (m-1)(n_k - n_\ell + 1)$ and so $n_k \in \{n_\ell, n_\ell - 1\}$. Suppose $n_k = n_\ell - 1$. Then $\deg(q(X)) = 0$ and

$$\sigma(D(f)) = \{z_1, \ldots, z_m\}. \tag{18}$$

If $k < \ell$ then

$$w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{\substack{j=1\\ j\neq k}}^{m-1} a_j$$

is an eigenvalue of $D(f)$ such that $g_1(w_1) < g_1(z_m) < \cdots < g_1(z_1)$ and this contradicts (18).
If $k > \ell$ then

$$w_2 = a_k + a_2 + \cdots + a_m = 2a_k + \sum_{\substack{j=2\\ j\neq k}}^{m} a_j$$

is an eigenvalue of $D(f)$ such that $g_1(w_2) > g_1(z_1) > \cdots > g_1(z_m)$ and this contradicts (18).
Then $n_k = n_\ell \geqslant 2$ and, from the definitions of k and ℓ, we have $k < \ell$. Also in this case

$$w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{\substack{j=1\\ j\neq k}}^{m-1} a_j$$

is an eigenvalue of $D(f)$ not in $\{z_1, \ldots, z_m\}$. Therefore

$$(X-w_1)^{\sum_{\substack{j=1\\ j\neq k}}^{m-1}(n_j-1)+2(n_k-2)+1}$$

divides $q(X)$ and, from (17), it follows that

$$\sum_{\substack{j=1\\ j\neq k}}^{m-1}(n_j-1) + 2(n_k-2) + 1 \leqslant \deg(P_f) - mn_\ell + m - 1,$$

that is,

$\deg(P_f) - n_k - n_m - m + 2 + 2n_k - 3 \leqslant \deg(P_f) - mn_\ell + m - 1.$

Since $n_k = n_\ell \geqslant n_m$, we obtain $mn_\ell \leqslant 2m$ and $n_k = n_\ell = 2$. Then $m + 2 \leqslant \deg(P_f) \leqslant 2m$.

If $m = 2$ then $P_f = (X - a_1)^2(X - a_2)^2$, $\sigma(D(f)) = \{2a_1, 2a_2, a_1 + a_2\}$ and (Theorem 2) $(X - a_1 - a_2)^3$ is an elementary divisor of $D(f)$. Since $\deg(P_{D(f)}) = 5$ we have $P_{D(f)} = (X - 2a_1)(X - 2a_2)(X - a_1 - a_2)^3$. Suppose $t_1 \geqslant 2$. Then $(X - a_1)^{n_{1,2}}$ is another elementary divisor of f associated with a_1. Hence $(X - 2a_1)^{2+n_{1,2}-1}$ is an elementary divisor of $D(f)$ and this contradicts $n_{1,2} \geqslant 1$. Then $t_1 = 1$ and, similarly, $t_2 = 1$. Condition (7) holds.

Assume now that $m \geqslant 3$. Suppose $n_q = 2$ for some $q \in \{1, \ldots, m\} \setminus \{\ell, k\}$. Then $\deg(P_f) \geqslant m + 3$. From the definitions of ℓ and k we have $q < k < \ell$. Then

$$w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{\substack{j=1 \\ j \neq k}}^{m-1} a_j$$

and

$$w_3 = a_q + a_1 + \cdots + a_{m-1} = 2a_q + \sum_{\substack{j=1 \\ j \neq q}}^{m-1} a_j$$

are eigenvalues of $D(f)$ such that $g_1(w_3) < g_1(w_1) < g_1(z_m) < \cdots < g_1(z_1)$. Therefore,

$$(X - w_1)^{\sum_{\substack{j=1 \\ j \neq k}}^{m-1}(n_j-1)+2(n_k-2)+1} \qquad (X - w_3)^{\sum_{\substack{j=1 \\ j \neq q}}^{m-1}(n_j-1)+2(n_q-2)+1}$$

has degree, at most, equal to the degree of $q(X)$, that is,

$2\deg(P_f) - 2m - 2n_m + n_k + n_q - 2 \leqslant \deg(P_f) - mn_\ell + m - 1,$

which contradicts $\deg(P_f) \geqslant m + 3$, since $n_\ell = n_k = n_q = 2$ and $n_m \leqslant 2$.

So, for $r = m \geqslant 3$ and $n_k \geqslant 2$ it must be $n_k = n_\ell = 2$ and $n_i = 1$ for $i \in \{1, \ldots, m\} \setminus \{\ell, k\}$. Then $\deg(P_f) = m + 2$ and $\deg(q(X)) = 1$. From $w_1 \in \sigma(D(f)) \setminus \{z_1, \ldots, z_m\}$, it follows that $q(X) = X - w_1$. Since

$$(X - w_1)^{\sum_{\substack{j=1 \\ j \neq k}}^{m-1}(n_j-1)+2(n_k-2)+1}$$

is an elementary divisor of $D(f)$ it follows that $\ell = m$ and, from (16), we have

$$P_{D(f)} = (X - w_1)(X - z_k)(X - z_m)^3 \prod_{\substack{i=1 \\ i \neq k}}^{m-1} (X - z_i)^2.$$

If $k \leqslant m - 2$, then

$$w_4 = a_k + a_1 + \cdots + a_{m-2} + a_m = 2a_k + a_m + \sum_{\substack{j=1 \\ j \neq k}}^{m-2} a_j$$

is also an eigenvalue of $D(f)$, and again we have a contradiction, since $g_1(w_1) < g_1(w_4) < g_1(z_m) < \cdots < g_1(z_1)$.
Then $k = m - 1$. If $m \geqslant 4$ then $w_5 = a_3 + \cdots + a_{m-2} + 2a_{m-1} + 2a_m$ is also an eigenvalue of $D(f)$ and, from $g_1(w_1) < g_1(z_m) < \cdots < g_1(z_1) < g_1(w_5)$, we have a contradiction.
Then $m = 3$, $\ell = 3$, $k = 2$, $P_f = (X - a_1)(X - a_2)^2(X - a_3)^2$ and

$$P_{D(f)} = (X - z_3)^3(X - z_1)^2(X - z_2)(X - w_1).$$

Since $(X - 2a_2 - a_3)^2$ is an elementary divisor of $D(f)$, $2a_2 + a_3 \in \{z_1, z_3\} = \{a_2 + 2a_3, a_1 + a_2 + a_3\}$, and, once more, we obtain a contradiction.
So if $r = m$ and f is not of simple structure then conditions (3) or (7) hold.
For $r = m$ it remains to consider the case f is of simple structure.

Suppose $t_\ell \geqslant 2$ for some $\ell \in \{1, \ldots, m\}$. Then $z_1, \ldots, z_m$, defined as before, are m distinct eigenvalues of $D(f)$, to which $X - z_i$, $i = 1, \ldots, m$, are associated elementary divisors. Then $m \deg(P_f) - m^2 + 1 \geqslant m$ and this contradicts $\deg(P_f) = m$. It follows that $t_1 = \cdots = t_m = 1$ and condition (1) holds. □

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