Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Topology and its Applications 153 (2006) 3085–3100
Topology and its Applications
www.elsevier.com/locate/topol

Topological protomodular algebras

F. Borceux [a,*,1], Maria Manuel Clementino [b,2]

[a] *Université de Louvain, Belgium*
[b] *Universidade de Coimbra, Portugal*

Received 15 January 2004; accepted 21 December 2004

Abstract

Topological groups have very striking properties, which have already been generalized to weaker "group like" structures, like various kinds of loops. This paper intends to show evidence that this generalization holds for a much wider class of theories, known as the protomodular theories, and which admit both an elegant categorical characterization and an easy description in universal algebra terms. Thus we propose a synthetic approach which allows to prove in a unique framework that the most striking properties of topological groups hold as well for loops or even semi-loops, rings with or without unit, associative algebras with or without unit, Lie algebras, Jordan algebras, Boolean algebras, Heyting algebras, Boolean rings, Heyting semi-lattices, and so on.

MSC: 54B30; 08A30; 18C10; 18B30; 54H13

Keywords: Topological algebra; Algebraic theory; Protomodular category

0. Introduction

Protomodular categories have been introduced in [5] as a formal context in which many properties characteristic of groups remain valid. Among the protomodular categories, there

* Corresponding author.
E-mail addresses: borceux@math.ucl.ac.be (F. Borceux), mmc@mat.uc.pt (M.M. Clementino).
1 Research supported by FNRS Grant 1.5.096.02.
2 Research supported by the "Centro de Matemática da Universidade de Coimbra/FCT".

0166-8641/$ – see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.topol.2004.12.010

are in particular the semi-Abelian categories, introduced in [12]. They include of course all Abelian categories, but there are many other examples, such as the category of all groups, loops or even semi-loops, rings with or without unit, associative algebras with or without unit, Lie algebras, Jordan algebras, Boolean algebras, Heyting algebras, Boolean rings, Heyting semi-lattices, and so on.

The algebraic theories $\mathbb{T}$ whose category $\mathsf{Set}^{\mathbb{T}}$ of models is protomodular have been characterized in [7]. Using this characterization, in this paper we study topological $\mathbb{T}$-algebras for such an algebraic theory $\mathbb{T}$, showing that the results obtained in [3] for the special case of topological semi-Abelian algebras can be extended to topological protomodular algebras.

In the case of topological groups, the addition by an element x is a homeomorphism, with inverse $- + (-x)$. When performing the quotient by a normal subgroup, this homeomorphism transforms the equivalence class of the unit in the equivalence class of x. The protomodular theories do not give rise to such homeomorphisms and our first task is to develop some alternative tools which will turn out to play a key role in the generalization of most of the classical results known for topological groups.

We first present protomodular algebraic theories: their characterization, some examples and elementary properties. Then we introduce topological protomodular algebras and the general properties which will be essential for their subsequent study. We emphasize here the presentation of a topological protomodular algebra A as a retract of a power A^n described in Proposition 6 and Metatheorem 8, which are effective substitutes of the homeomorphisms $- + x$ of topological groups.

The main part of the paper studies the topological properties of these algebras. It is divided in two parts. First we show that they are regular spaces and present characterizations which show that topological properties can be simply detected locally, at some special points. The second part deals with the study of quotients based on the study of their respective congruences. It is shown in particular that Hausdorff, discrete, compact, connected, totally disconnected protomodular algebras are closed under extensions (Theorem 31) and easy descriptions of the Hausdorff and the totally disconnected reflections are obtained.

The last section is devoted to the categorical properties of the categories studied. We show that $\mathsf{Top}^{\mathbb{T}}$, as well as its subcategories of Hausdorff, compact, compact Hausdorff, locally compact Hausdorff and totally disconnected algebras are regular and protomodular.

The references of this paper present a wide list of papers devoted to the study of some special classes of topological algebras (cf. [15,18,19,10,9,11]), but most of our general results—particularized to those special concrete cases—seem to remain original, at least at our best knowledge. It would of course be fastidious to repeat each of our results in each particular example that we present in this paper, and it would be even more fastidious to list for each such example all the known consequences of our results, like all the striking properties of regular protomodular categories (see our Theorem 34). But it is a matter of fact that our results imply in particular that topological Boolean algebras, topological Heyting algebras, topological rings, and so on, satisfy all the following properties: the non-pointed version of the famous 3×3-lemma of homological algebra (see [6]); every simplicial object is necessarily a Kan complex (see [8]); every reflexive relation is an equivalence relation and every internal category is a groupoid (see [2]), and so on. And the same comment applies thus to the Hausdorff, compact Hausdorff, locally compact Hausdorff and totally disconnected cases. We simply hope that the wide range of new results that this paper already provides in the domain of topological algebras will allow to investigate further consequences in all the mentioned examples.

1. Protomodular algebras

Protomodular categories have been introduced by Bourn in [5] as a formal context in which many properties characteristic of groups remain valid. We will postpone the presentation of the definition of protomodular category until the last section, since here we are interested only in the algebraic theories whose category $\mathsf{Set}^{\mathbb{T}}$ of models is protomodular. These algebraic theories were characterized by Bourn and Janelidze [7] as follows.

Theorem 1. *An algebraic theory* $\mathbb{T}$ *has a protomodular category* $\mathsf{Set}^{\mathbb{T}}$ *of models precisely when, for some natural number* $n \in \mathbb{N}$, $\mathbb{T}$ *contains*

(1) *constants* $e_1, \ldots, e_n$;
(2) *binary operations* $\alpha_1(X, Y), \ldots, \alpha_n(X, Y)$ *satisfying* $\alpha_i(X, X) = e_i$;
(3) *an* $(n+1)$*-ary operation* $\theta(X_1, \ldots, X_{n+1})$ *satisfying*

$$\theta\big(\alpha_1(X, Y), \ldots, \alpha_n(X, Y), Y\big) = X.$$

We shall in general refer to such an algebraic theory $\mathbb{T}$ as a *protomodular theory*. The corresponding $\mathbb{T}$-algebras will be called *protomodular algebras*.

Example 2. Each algebraic theory $\mathbb{T}$ which contains a group operation $+$ is protomodular. This is in particular the case for groups, Abelian groups, Ω-groups, modules on a ring, rings or algebras with or without unit, Lie algebras, Jordan algebras, all these theories with additional sup and/or inf semi-lattice structure.

Proof. In Theorem 1, it suffices to choose $n = 1$ and

$$e_1 = 0, \qquad \alpha_1(X, Y) = X - Y, \qquad \theta(X, Y) = X + Y. \qquad \square$$

Since semi-Abelian theories are in particular protomodular theories, all the examples of semi-Abelian theories given in [3] are examples of protomodular theories: this is in particular the case for loops and, more generally, left or right semi-loops. Notice that the theory of quasi-groups is not protomodular, since it does not contain any constant. Further, we remark that the theory of semigroups is not protomodular.

Example 3. The theory of Heyting algebras and the theory of Boolean algebras are protomodular.

Proof. In the case of Boolean algebras, choose $n = 2$, $e_1 = 0$, $e_2 = 1$, and

$$\alpha_1(X, Y) = X \wedge \neg Y, \qquad \alpha_2(X, Y) = X \vee \neg Y, \qquad \theta(X, Y, Z) = (X \vee Z) \wedge Y.$$

The result for Heyting algebras was obtained by Johnstone in [14], where he proved that the operations of Heyting $\wedge$-semi-lattice suffice already to exhibit the protomodular character: simply put

$$e_1 = 1 = e_2, \qquad \alpha_1(X,Y) = X \Longrightarrow Y,$$
$$\alpha_2(X,Y) = \big((X \Longrightarrow Y) \Longrightarrow Y\big) \Longrightarrow X,$$
$$\theta(X,Y,Z) = (X \Longrightarrow Z) \wedge Y. \qquad \square$$

In general, $\mathbb{T}$ admits other constants and operations rather than simply e_i, α_i and θ: for example the theory of rings with unit contains also the constant 1 and the multiplication $\times$. We point out also that the choice in $\mathbb{T}$ of constants and operations e_i, α_i and θ as indicated is not unique. For example, the operations of Heyting $\wedge$-semi-lattice given in the example above exhibit a second possible choice of constants and operations α_i and θ for Boolean algebras.

In a protomodular theory $\mathbb{T}$, the formula

$$p(X,Y,Z) = \theta\big(\alpha_1(X,Y), \ldots, \alpha_n(X,Y), Z\big)$$

defines a *Mal'cev operation* (as it is shown in the following lemma); that is, the operation p is such that

$$p(X,X,Y) = Y, \qquad p(X,Y,Y) = X.$$

Lemma 4. *Let $\mathbb{T}$ be a protomodular theory. Given elements a, b, c of a $\mathbb{T}$-algebra A, one has*:

$$\big(\forall i\ \alpha_i(a,c) = \alpha_i(b,c)\big) \Longrightarrow (a = b),$$
$$\big(\forall i\ \alpha_i(a,b) = e_i\big) \Longrightarrow (a = b),$$
$$\theta(e_1, \ldots, e_n, a) = a.$$

Proof. If $\alpha_i(a,c) = \alpha_i(b,c)$ for every i, then $a = \theta(\alpha_1(a,c), \ldots, \alpha_n(a,c), c) = \theta(\alpha_1(b,c), \ldots, \alpha_n(b,c), c) = b$. The second case is obtained from the first one by putting $c = b$. The third assertion is obtained by writing $e_i = \alpha_i(a,a)$. $\square$

Notice that the implication

$$\big(\forall i\ \alpha_i(c,a) = \alpha_i(c,b)\big) \Longrightarrow (a = b)$$

is not valid in general.

2. Topological protomodular algebras

Let us now introduce the topic of the present paper:

Definition 5. Given an algebraic theory $\mathbb{T}$, by a *topological* $\mathbb{T}$*-algebra* we mean a topological space A provided with the structure of a $\mathbb{T}$-algebra, in such a way that every operation $\tau : T^n \to T$ of $\mathbb{T}$ induces a continuous mapping

$$\tau_A : A^n \to A, \quad (a_1, \dots, a_n) \mapsto \tau(a_1, \dots, a_n).$$

We write $\mathsf{Top}^{\mathbb{T}}$ for the category of topological $\mathbb{T}$-algebras and continuous $\mathbb{T}$-homomorphisms between them.

If $\mathbb{T}$ is a protomodular theory, the corresponding topological $\mathbb{T}$-algebras will be called *topological protomodular algebras*.

Convention. *Throughout this paper, given a protomodular theory $\mathbb{T}$, the notation e_i, α_i or θ will always indicate constants and operations as above, with $n \in \mathbb{N}$ the corresponding number of operations α_i. Further, when referring to a topological protomodular algebra, we will always assume that it is a topological $\mathbb{T}$-algebra for a given protomodular theory $\mathbb{T}$.*

For example when $\mathbb{T}$ is the theory of groups, $\mathsf{Top}^{\mathbb{T}}$ is the category of topological groups. The theory of topological groups makes a heavy use of the property that, for any element g of a topological group G (written additively), the mapping

$$- + g : G \to G, \quad x \mapsto x + g$$

is a homeomorphism mapping 0 on g. This "homogeneity property" of the topology can be partly recaptured in the case of a protomodular theory, as indicated in the sequel.

Proposition 6. *For every element a of a topological protomodular algebra A, the continuous maps*

$$\iota_a : A \rightarrowtail A^n, \quad x \mapsto \big(\alpha_1(x, a), \dots, \alpha_n(x, a)\big), \quad \textit{and}$$
$$\theta_a : A^n \to A, \quad (a_1, \dots, a_n) \mapsto \theta(a_1, \dots, a_n, a)$$

are such that $\theta_a \cdot \iota_a = \mathsf{id}_A$, so that ι_a presents A as a topological section of A^n, which maps the element $a \in A$ into $(e_1, \dots, e_n) \in A^n$.

Notice that the inclusion ι_a is not a $\mathbb{T}$-homomorphism: it does not even preserve the constants e_i.

Corollary 7. *Given an element $a \in A$ of a topological protomodular algebra A:*

(1) *the subsets*

$$\bigcap_{i=1}^{n} \alpha_i(-, a)^{-1}(U_i), \quad U_i \textit{ open neighborhood of } e_i,$$

constitute a fundamental system of open neighborhoods of a;

(2) *the subsets*

$$\theta_a(U_1 \times \cdots \times U_n), \quad U_i \textit{ open neighborhood of } e_i,$$

constitute a fundamental system of neighborhoods of a.

Proof. (1) Neighborhoods of the form $U_1 \times \cdots \times U_n$, with $U_i \subseteq A$ open neighborhood of e_i, constitute a fundamental system of open neighborhoods of $(e_1, \ldots, e_n)$. Hence, the sets

$$\iota_a^{-1}(U_1 \times \cdots \times U_n) = \bigcap_{i=1}^{n} \alpha_i(-, a)^{-1}(U_i)$$

constitute a fundamental system of open neighborhoods of a.

(2) Since

$$\bigcap_{i=1}^{n} \alpha_i^{-1}(-, a)(U_i) \subseteq \theta_a(U_1 \times \cdots \times U_n),$$

the latter is a neighborhood of a, although in general not necessarily open. This, together with the fact that the sets $U_1 \times \cdots \times U_n$ form a fundamental system of neighborhoods of $(e_1, \ldots, e_n)$, gives the result. □

These descriptions of fundamental systems of neighborhoods lead to a key result in the topological study of topological protomodular algebras.

Metatheorem 8. *Let $\mathbb{T}$ be a protomodular theory and P a topological property stable under finite limits, or stable under finite products and images. If the property P is valid at a neighborhood of each constant e_i in a given algebra A, that property P is valid at a neighborhood of every point of A.*

Next we focus on the properties of subalgebras $B \subseteq A$ of a topological protomodular algebra A. Obviously, every subalgebra B of the topological algebra A, provided with the induced topology, is a topological algebra on its own. As usual when we say that the subalgebra B has a topological property we consider B as a topological subalgebra of A.

Straightforward generalizations of the proofs presented in [3] for the semi-Abelian case give:

Proposition 9. *Every open subalgebra $B \subseteq A$ of a topological protomodular algebra A is closed.*

Corollary 10. *Let A be a topological protomodular algebra and $B \subseteq A$ a subalgebra. The following conditions are equivalent*:

(1) *B is a neighborhood of each e_i*;
(2) *B is an open neighborhood of each e_i*;
(3) *B is a closed neighborhood of each e_i.*

Proposition 11. *The closure $\overline{B} \subseteq A$ of every subalgebra $B \subseteq A$ of a topological protomodular algebra A is still a subalgebra.*

3. An overview of topological properties

First, let us immediately observe that

Proposition 12. *Every topological protomodular algebra is a regular topological space.*

Proof. By Metatheorem 8, it suffices to prove that every open neighborhood of each e_i contains the closure of a neighborhood of e_i. Let V be a neighborhood of e $(= e_i)$ in A. Since $\theta : A^{n+1} \to A$ is continuous, there exist $U_1, \dots, U_n$ neighborhoods of $e_1, \dots, e_n$, respectively, and a neighborhood U of e such that $\theta(U_1 \times \cdots \times U_n \times U) \subseteq V$.

Next we show that $\overline{U} \subseteq V$. For $a \in \overline{U}$, since $\bigcap_{i=1}^n \alpha_i^{-1}(-, a)(U_i)$ is an open neighborhood of a, there exists $b \in U \cap \bigcap_{i=1}^n \alpha_i(-, a)^{-1}(U_i)$. But then $a = \theta(\alpha_1(a, b), \dots, \alpha_n(a, b), b)$ and $\alpha_i(a, b) \in U_i$ for all i, hence $a \in \theta(U_1 \times \cdots \times U_n \times U) \subseteq V$. $\square$

Proposition 13. *For a topological protomodular algebra A, the following conditions are equivalent*:

(1) *each point $e_1, \dots, e_n$ is closed* (*respectively open*) *in A*;
(2) *A is a Hausdorff* (*respectively discrete*) *space.*

Proof. To prove (1) $\Rightarrow$ (2), we just observe that, for each $a \in A$, $\{a\}$ is the inverse image of $\{(e_1, \dots, e_n)\}$ along ι_a. The result is then immediate when e_i is open, while, for e_i closed, Hausdorffness follows from the fact that A is then a T_1 regular space. $\square$

Proposition 14. *For a topological protomodular algebra A, the following conditions are equivalent*:

(1) *for each i, the connected component of e_i is reduced to $\{e_i\}$*;
(2) *A is totally disconnected.*

This result is an immediate consequence of the following lemma, where $\Gamma(x)$ denotes the connected component of x.

Lemma 15. *Let A be a topological protomodular algebra. For every point $a \in A$,*

$$\Gamma(a) = \theta_a\big(\Gamma(e_1) \times \cdots \times \Gamma(e_n)\big).$$

Proof. It follows directly from the inequalities

$$\iota_a\big(\Gamma(a)\big) \subseteq \Gamma(e_1, \dots, e_n) = \Gamma(e_1) \times \cdots \times \Gamma(e_n)$$

and

$$\theta_a\big(\Gamma(e_1) \times \cdots \times \Gamma(e_n)\big) \subseteq \Gamma(a). \qquad \square$$

Proposition 16. *For a topological protomodular algebra A, the following conditions are equivalent*:

(1) *each point* $e_1, \dots, e_n$ *has a compact neighborhood*;
(2) *each* e_i *has a fundamental system of compact neighborhoods*;
(3) *A is locally compact.*

Proof. Using Metatheorem 8, we only have to verify that (1) ⇒ (2), and this follows easily from regularity of A. □

Proposition 17. *If A is a Hausdorff protomodular algebra, then every locally compact subalgebra B of A is closed.*

Proof. Given $a \in \overline{B}$, we must prove that $a \in B$. For this we choose, for each index i, a compact neighborhood K_i of e_i in B, which has thus the form $K_i = U_i \cap B$ for some neighborhood U_i of e_i in A. The continuous image of the compact $U_i \cap B \subseteq B$ in A is compact, thus closed. In other words, $K_i = U_i \cap B$ is closed in A. We choose further an open neighborhood $V_i \subseteq U_i$ of e_i in A. We consider then the open subset

$$V = \bigcap_{i=1}^{n} \alpha_i(a, -)^{-1}(V_i)$$

which is a neighborhood of $a \in \overline{B}$, thus meets B:

$$\exists b \in B\ \forall i\ \alpha_i(a,b) \in V_i.$$

Let us prove now that $\alpha_i(a,b) \in B$ for each index i. For this it suffices to prove that

$$\alpha_i(a,b) \in V_i \cap \overline{B} \subseteq \overline{V_i \cap B} \subseteq \overline{U_i \cap B} = U_i \cap B \subseteq B,$$

where the first inclusion holds because V_i is open. By choice of b, $\alpha_i(a,b) \in V_i$. Since $a, b \in \overline{B}$, $\alpha_i(a,b) \in \overline{B}$ by Proposition 11.

One concludes now that

$$a = \theta\big(\alpha_1(a,b), \dots, \alpha_n(a,b), b\big) \in B$$

since b and all the $\alpha_i(a,b)$ are in the subalgebra B. □

4. Quotients of topological protomodular algebras

We first prove an important property of congruences on algebras equipped with a Mal'cev operation, i.e. a ternary operation $p : A^3 \to A$ such that $p(X,X,Y) = Y$ and $p(X,Y,Y) = X$.

Proposition 18. *If* $\mathbb{T}$ *is an algebraic theory containing a Mal'cev operation p and A is a* $\mathbb{T}$*-algebra, every* $\mathbb{T}$*-subalgebra R of* $A \times A$ *containing the diagonal* Δ_A *is a congruence.*

Proof. We have to check that R is an equivalence relation. To check its symmetry, let $(x,y) \in R$. Then, since $(x,x), (x,y), (y,y) \in R$,

$$\big(p(x,x,y), p(x,y,y)\big) = (y,x) \in R.$$

Finally to check its transitivity, we pick (x, y) and (y, z) in R. Then, since (x, y), (y, y) and (y, z) belong to R, also $(p(x, y, y), p(y, y, z)) = (x, z) \in R$. $\square$

Now, let us observe that in contrast with the case of topological spaces:

Proposition 19. *Let $\mathbb{T}$ be a protomodular theory. In $\mathsf{Top}^{\mathbb{T}}$, the closure of a congruence $R \subseteq A \times A$ is another congruence on A.*

Proof. Every protomodular theory is a Mal'cev theory. The topological closure $\overline{R} \subseteq A \times A$ is a $\mathbb{T}$-subalgebra (see Proposition 11) which contains the diagonal of A, since it contains R. It is thus a congruence, by Proposition 18. $\square$

Let us also describe more precisely the quotient by a congruence. We do not include the proof since it is analogous to the proof of Proposition 57 of [3].

Proposition 20. *Let $\mathbb{T}$ be a protomodular theory and $R \subseteq A \times A$ a congruence on A. Given an arbitrary subset $X \subseteq A$, the saturation $\widetilde{X}$ of X for the corresponding quotient $q : A \twoheadrightarrow A/R$ is given by*

$$\begin{aligned}
\widetilde{X} &= q^{-1}\big(q(X)\big) \\
&= \big\{a \in A \mid \exists x \in X\ \forall i\ \alpha_i(a, x) \in [e_i]\big\} \\
&= \big\{a \in A \mid \exists b_1 \in [e_1],\ \ldots,\ b_n \in [e_n]\ \theta(b_1, \ldots, b_n, a) \in X\big\} \\
&= \big\{\theta(b_1, \ldots, b_n, x) \mid b_1 \in [e_1],\ \ldots,\ b_n \in [e_n],\ x \in X\big\} \\
&= \big\{a \in A \mid \exists (u, v) \in R\ \theta\big(\alpha_1(u, v), \ldots, \alpha_n(u, v), a\big) \in X\big\} \\
&= \big\{\theta\big(\alpha_1(u, v), \ldots, \alpha_n(u, v), x\big) \mid (u, v) \in R,\ x \in X\big\}.
\end{aligned}$$

Hence

$$\widetilde{X} = \theta\big([e_1] \times \cdots \times [e_n] \times X\big) = \bigcup_{x \in X} \iota_x^{-1}\big([e_1] \times \cdots \times [e_n]\big).$$

In particular, for every $x \in A$,

$$[x] = \theta_x\big([e_1] \times \cdots \times [e_n]\big) = \iota_x^{-1}\big([e_1] \times \cdots \times [e_n]\big).$$

As already observed in 1954 by Mal'cev, when a theory $\mathbb{T}$ contains a Mal'cev operation, then every quotient map $q : A \twoheadrightarrow Q$ in $\mathsf{Top}^{\mathbb{T}}$ is open (see [16,13,3]).

Proposition 21. *For every protomodular theory $\mathbb{T}$, the regular epimorphisms in $\mathsf{Top}^{\mathbb{T}}$ are precisely the surjective open maps.*

From now on, for a topological property P, we will say that a congruence R satisfies *essentially* P if every equivalence class of R, as a subspace of $A \times A$, satisfies P.

Proposition 22. *Let $\mathbb{T}$ be a protomodular theory and A a topological $\mathbb{T}$-algebra. For a congruence $R \subseteq A \times A$, the following conditions are equivalent*:

(1) *R is essentially closed* (*respectively open*);
(2) *the equivalence classes* $[e_1], \dots, [e_n]$ *are closed* (*respectively open*) *in* A;
(3) *R is closed* (*respectively open*) *in* $A \times A$;
(4) *the quotient topological* $\mathbb{T}$*-algebra* A/R *is Hausdorff* (*respectively discrete*).

Proof. (4) $\Rightarrow$ (3) is well known: the diagonal of A/R is closed (respectively open) by Hausdorffness (respectively discreteness). Writing $q : A \twoheadrightarrow A/R$ for the quotient map, R is the inverse image of this diagonal along $q \times q$.

(3) $\Rightarrow$ (2) is also classical since the equivalence class of $e_i \in A$ is the inverse image of R along the continuous mapping

$$(e_i, \mathsf{id}_A) : A \to A \times A, \quad x \mapsto (e_i, x).$$

(2) $\Rightarrow$ (1) follows from the equality $[a] = \iota_a^{-1}([e_1] \times \cdots \times [e_n])$.

Finally if $[a]$ is closed (respectively open) in A, its image $[a] \in A/R$ is a closed (respectively open) point because $[a]$ is saturated and the quotient map q is open. Therefore (1) $\Rightarrow$ (4). □

Although in Top the construction of the Hausdorff reflection needs a transfinite argument, in the case of protomodular topological algebras this reflection is easily described. This generalizes again the well-known construction for topological groups.

Corollary 23. *Let* $\mathbb{T}$ *be a protomodular theory. For any topological* $\mathbb{T}$*-algebra A, the quotient* $q : A \twoheadrightarrow A/\overline{\Delta_A}$ *is the Hausdorff reflection of A.*

Proof. By Lemma 19, $\overline{\Delta_A}$ is a congruence on A; the algebra $A/\overline{\Delta_A}$ is Hausdorff by Proposition 22. To check that $q : A \twoheadrightarrow A/\overline{\Delta_A}$ is the reflection is pure routine. □

Proposition 24. *If A is a topological protomodular algebra, then, for a congruence* $R \subseteq A \times A$, *the following conditions are equivalent*:

(1) *R is essentially compact* (*respectively connected*);
(2) *the equivalence classes* $[e_1], \dots, [e_n]$ *are compact* (*respectively connected*) *in A.*

Proof. Since $[a] = \theta_a([e_1] \times \cdots \times [e_n])$, the nontrivial implication follows from finite productivity and closure under images of compact and connected spaces, according to Metatheorem 8. □

Proposition 25. *Let A be a topological protomodular algebra. For a congruence* $R \subseteq A \times A$, *the following conditions are equivalent*:

(1) *R is essentially totally disconnected*;
(2) *the equivalence classes* $[e_1], \dots, [e_n]$ *are totally disconnected in A.*

Proof. Since totally disconnected spaces are finitely productive and closed under subspaces, the identity $[a] = \iota_a^{-1}([e_1] \times \cdots \times [e_n])$ guarantees that (2) $\Rightarrow$ (1). □

As in the case of topological groups, the reflection of a topological protomodular algebra into the subcategory of totally disconnected protomodular algebras is easily described.

Lemma 26. *Let $\mathbb{T}$ be a protomodular theory and A a topological $\mathbb{T}$-algebra. The set $R = \{(a,b) \in A \times A \mid \Gamma(a) = \Gamma(b)\}$ is a congruence on A.*

Proof. Since the continuous image of a connected subset is connected, R is a $\mathbb{T}$-subalgebra which, moreover, contains the diagonal. Therefore, it is a congruence, by Proposition 18. □

Proposition 27. *Let $\mathbb{T}$ be a protomodular theory and A a topological $\mathbb{T}$-algebra. The quotient $q : A \twoheadrightarrow A/R$ of A by the congruence $R = \{(a,b) \mid \Gamma(a) = \Gamma(b)\}$ is the reflection of A into the subcategory of totally disconnected $\mathbb{T}$-algebras.*

Proof. For each index i, in the following pullback diagram,

$$\begin{tikzcd}
C \arrow[r, two heads, "p"] \arrow[d, tail, "s"'] & \Gamma([e_i]) \arrow[d, tail, "t"] \\
A \arrow[r, two heads, "q"'] & A/R
\end{tikzcd}$$

by Proposition 21 p is an open surjection, hence a topological quotient, with connected codomain. Since the fibres of p are connected, we conclude that C is connected as well, by q-reversibility of connected spaces (see [1]). Hence $C = \Gamma(e_i)$ and then $\Gamma([e_i]) = [e_i]$. One concludes that A/R is totally disconnected by Proposition 14. To check that q is the reflection is now straightforward. □

Here are now some interesting properties of quotients.

Proposition 28. *Let $\mathbb{T}$ be a protomodular theory and A a topological $\mathbb{T}$-algebra. When $R \subseteq A \times A$ is an essentially compact congruence on A, the quotient $q : A \twoheadrightarrow A/R$ is a closed map.*

Proof. For a closed subset $C \subseteq A$, its saturation $\widetilde{C} = q^{-1}(q(C))$ can be described as

$$\widetilde{C} = \big\{a \in A \mid \exists b_1 \in [e_1],\ \ldots,\ b_n \in [e_n]\ \ \theta(b_1, \ldots, b_n, a) \in C\big\},$$

by Proposition 20. Considering the continuous mappings

$$A \xleftarrow{p_A} [e_1] \times \cdots \times [e_n] \times A \overset{\iota}{\rightarrowtail} A^{n+1} \xrightarrow{\theta} A$$

where ι is the canonical inclusion, we have thus

$$\widetilde{C} = p_A\big(\iota^{-1}\big(\theta^{-1}(C)\big)\big).$$

Since C is closed, $\iota^{-1}(\theta^{-1}(C))$ is closed as well. Since $[e_1] \times \cdots \times [e_n]$ is compact, the projection p_A is a closed map (see [4]) and therefore $\widetilde{C}$ is closed. □

Lemma 29. *If A is a topological protomodular algebra and $R \subseteq A \times A$ is an essentially connected congruence on A, then every clopen $U \subseteq A$ is R-saturated.*

Proof. Similar to the proof of Lemma 42 of [3]. □

We are now in a position to discuss closure under extensions of several topological properties, as defined next.

Definition 30. Let $\mathcal{P}$ be a given property. We say that $\mathcal{P}$ is *closed under extensions* if, given a short exact sequence

$$R \underset{v}{\overset{u}{\rightrightarrows}} A \overset{q}{\twoheadrightarrow} A/R,$$

that is, q is the coequalizer of its kernel pair (u, v), if A/R and each R-equivalence class have the property $\mathcal{P}$ then A has the property $\mathcal{P}$.

Theorem 31. *For a protomodular theory $\mathbb{T}$, Hausdorff, discrete, compact, connected, totally disconnected $\mathbb{T}$-algebras are closed under extensions.*

Proof. The Hausdorff and the discrete case have a similar proof. Since

$$a \rightarrowtail [a] \rightarrowtail A$$

and, by hypothesis, a is closed (respectively open) in $[a]$, which is closed (respectively open) in A, the result follows from Proposition 13.

To check this property for compactness, we use Proposition 28. The quotient $q : A \twoheadrightarrow A/R$ is a closed continuous map with compact fibres $[a]$; thus q is a proper map and therefore, reflects compact subspaces (see [4]). In particular, $A = q^{-1}(A/R)$ is compact.

To conclude that A is connected whenever R is essentially connected and A/R is connected, let U be a clopen subset of A. By Lemma 29, U is saturated, thus $q(U)$ is a clopen subset of A/R. This forces $q(U) = \emptyset$ or $q(U) = A/R$, that is, $U = \emptyset$ or $U = A$.

Finally we want to check closure under extensions for totally disconnected algebras. Since $q(\Gamma(e_i))$ is connected and contains $[e_i]$, it is reduced to that element, because A/R is totally disconnected. This implies $\Gamma(e_i) \subseteq [e_i]$, which is totally disconnected. Hence $\Gamma(e_i) = \{e_i\}$ and the conclusion follows from Proposition 25. □

Using an argument analogous to the argument used in the above proof for compactness, one can prove that:

Proposition 32. *Let A be a topological protomodular algebra and $R \subseteq A \times A$ a congruence on A. If R is essentially compact and A/R is locally compact, A is locally compact.*

To finalize the topological study of quotients of topological protomodular algebras, we list some observations concerning the topological properties of congruences.

It was shown in Proposition 22 that, if R is a congruence on A, then R is essentially closed (respectively open) if and only if it is closed (respectively open) in $A \times A$.

This is not the case for the other properties studied; that is, R essentially Hausdorff is not equivalent to R being Hausdorff, and the same holds for compact, discrete, connected, totally disconnected. Indeed, for any topological algebra A, if $R = \Delta_A$ then R is essentially P while R has property P if and only if A has it.

However, one can deduce from our results that:

Proposition 33. *If R has essentially P and A/R has property P, then R itself has property P, for P Hausdorff, discrete, compact, connected or totally disconnected.*

Proof. For Hausdorff, discrete and totally disconnected, this statement is immediate, since:

- from R essentially Hausdorff (respectively discrete) and A/R Hausdorff (respectively discrete) it follows that A is Hausdorff (respectively discrete) by Theorem 31, and A is Hausdorff (respectively discrete) if and only if R is Hausdorff (respectively discrete);
- analogously, if R is essentially totally disconnected and A/R is totally disconnected, then A is totally disconnected by Theorem 31, and then R is totally disconnected.

For compactness, if we assume that R is essentially compact and A/R is compact, then we conclude that A is compact and that $q : A \twoheadrightarrow A/R$ is proper; hence also $q \times q : A \times A \twoheadrightarrow A/R \times A/R$ is proper, and then $R = (q \times q)^{-1}(\Delta_{A/R})$, as the inverse image of a compact subset along a proper map, is compact.

For connectedness, let R be essentially connected and A/R connected. Then $p : R \twoheadrightarrow A/R$, as a pullback of the quotient $q \times q$, is a quotient, with connected codomain. Moreover, for each $[a] \in A/R$, $p^{-1}([a]) = [a] \times [a]$ is a connected subset of R, hence R is connected by q-reversibility of connected spaces. □

5. Regularity and protomodularity

First, let us recall the notion of a *regular* category. We consider a category $\mathcal{V}$ with finite limits. The kernel pair $u, v : R \rightrightarrows A$ of a morphism $f : A \to B$ is the pullback of f with itself, which in the case of $\mathbb{T}$-algebras is simply the congruence determined by f:

$$R = \{(a, a') \in A \times A \mid f(a) = f(a')\}.$$

An epimorphism $f : A \twoheadrightarrow B$ is regular when it is the quotient of A by its kernel pair. A category $\mathcal{V}$ with finite limits is *regular* when

- the quotient by a kernel pair exists always;
- regular epimorphisms are stable under pulling back along an arbitrary arrow.

One essential property of a regular category is the existence of *images*: every arrow factors uniquely (up to an isomorphism) as a regular epimorphism followed by a monomorphism. Among the examples of regular categories, we find all the categories of models of an algebraic theory $\mathbb{T}$, without any assumption on $\mathbb{T}$. The most celebrated counter-example is the category Top of topological spaces: indeed, topological quotient maps are not stable under pulling back.

Next, we recall the notion of a *protomodular* category. Let $\mathcal{V}$ be a category with finite limits. Given an object $X \in \mathcal{V}$, the category $\mathsf{Split}_X(\mathcal{V})$ of split epimorphisms over X has as objects the triples (A, p, s) in $\mathcal{V}$

$$p : A \to X, \qquad s : X \to A, \quad p \circ s = \mathsf{id}_X.$$

A morphism $f : (A, p, s) \to (B, q, t)$ is a morphism of $\mathcal{V}$ such that

$$f : A \to B, \qquad q \circ f = p, \qquad f \circ s = t.$$

Every arrow $v : Y \to X$ in $\mathcal{V}$ induces by pullback an *inverse image functor*

$$v^* : \mathsf{Split}_X(\mathcal{V}) \to \mathsf{Split}_Y(\mathcal{V}).$$

The category $\mathcal{V}$ is *protomodular* (see [5]) when all these inverse image functors v^* reflect isomorphisms.

To grasp the intuition behind that definition, consider the case where $\mathcal{V}$ has a zero object $\mathbf{0}$ and write $\alpha_X : \mathbf{0} \to X$ for the unique arrow. Given an arrow $v : Y \to X$, the equality $v \circ \alpha_Y = \alpha_X$ implies $\alpha_Y^* \circ v^* = \alpha_X^*$. Since each functor (and in particular α_Y^*) preserves isomorphisms, the category $\mathcal{V}$ is protomodular if and only if each functor α_X^* reflects isomorphisms. But pulling back an arrow $p : A \to X$ along α_X is just taking its kernel. Thus the protomodularity reduces to the following condition: given a commutative diagram

$$\begin{tikzcd}
\mathbf{0} \arrow[r] & \mathsf{Ker}\, p \arrow[r, "k"] \arrow[d, "g"'] & A \arrow[r, shift right, "p"'] \arrow[d, "f"'] & X \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, equal] & \mathbf{0} \\
\mathbf{0} \arrow[r] & \mathsf{Ker}\, p' \arrow[r, "k'"] & A' \arrow[r, shift right, "p'"'] & X \arrow[l, shift right, "s'"'] \arrow[r] & \mathbf{0}
\end{tikzcd}$$

where $p \circ s = \mathsf{id}_X = p' \circ s'$, $k = \mathsf{Ker}\, p$, $k' = \mathsf{Ker}\, p'$, if g is an isomorphism, then f is an isomorphism as well. This is a special "split" case of the classical "short five lemma". The protomodularity axiom is thus a generalization of this "split short five lemma" to the context of a category without a zero object. Let us finally mention that in a regular category with a zero object, the split short five lemma implies the general form of the short five lemma (see [5]).

Let us now conclude this paper observing that:

Theorem 34. *Let $\mathbb{T}$ be a protomodular theory. The categories of topological, Hausdorff, compact Hausdorff, locally compact Hausdorff or totally disconnected $\mathbb{T}$-algebras are all regular and protomodular.*

Proof. All the categories of the statement are closed under finite products in the category of topological $\mathbb{T}$-algebras. To get closure under all finite limits, it remains to prove that they are also closed under equalizers of two parallel arrows $f, g : A \rightrightarrows B$. Since

$$\mathsf{Ker}\,(g, h) = \big\{ a \in A \mid f(a) = g(a) \big\},$$

this is clear in the Hausdorff and totally disconnected cases. For the compact Hausdorff and locally compact Hausdorff cases, observe further that $\mathsf{Ker}\,(f,g)$ is closed as the inverse image of the diagonal of $B \times B$ along the mapping $(g,h): A \to B \times B$.

It is well known that $\mathsf{Top}^{\mathbb{T}}$ is complete and cocomplete, without any assumption on the algebraic theory $\mathbb{T}$. Since every continuous open surjection is necessarily a topological quotient, regular epimorphisms in $\mathsf{Top}^{\mathbb{T}}$ coincide with open continuous surjections (see Proposition 9) and these are stable under pulling back along an arbitrary arrow. Thus the category $\mathsf{Top}^{\mathbb{T}}$ is regular.

Observe now that the kernel pair of a map $f: A \to B$ is the inverse image of the diagonal of $B \times B$ along $f \times f$: thus it is closed in $A \times A$ as soon as B is Hausdorff. By Proposition 11, this proves that in the Hausdorff case, the quotient of A by the kernel pair of f is computed as in $\mathsf{Top}^{\mathbb{T}}$. Therefore the category of Hausdorff $\mathbb{T}$-algebras is regular as well. When moreover the algebra A is compact, the quotient also is compact and again this forces the regularity of the category of compact $\mathbb{T}$-algebras.

In the case of totally disconnected $\mathbb{T}$-algebras, the quotient of A by the kernel pair of f is first computed in $\mathsf{Top}^{\mathbb{T}}$, let us say $q: B \twoheadrightarrow Q$, and next it is composed with the quotient of Q described in Proposition 27. Thus the regular epimorphisms of totally disconnected $\mathbb{T}$-algebras are again continuous open surjections, as composites of two such maps, proving the regularity of the corresponding category.

To prove the protomodularity, consider an arbitrary category $\mathcal{C}$ with finite limits. Being a $\mathbb{T}$-model in $\mathcal{C}$ is a finite limit statement. Being protomodular is a finite limit statement as well. Since the Yoneda embedding

$$Y_{\mathcal{C}}: \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}], \quad C \mapsto \mathcal{C}(-, C)$$

preserves and reflects limits, and limits are computed pointwise in the category of functors $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$, a finite limit statement holds in $\mathcal{C}$ as soon as it holds in Set. In particular, the category of $\mathbb{T}$-models in $\mathcal{C}$ is protomodular as soon as the category of $\mathbb{T}$-models in Set is protomodular. Applying this observation to the categories of topological, Hausdorff, compact Hausdorff, locally compact Hausdorff or totally disconnected spaces, we obtain that the various categories of the statement are protomodular. (In fact, the category of compact Hausdorff $\mathbb{T}$-algebras is not only regular, but exact, without any assumption on the theory $\mathbb{T}$; see [17].) □

The concise formulation of Theorem 34 hides hundreds of deep results, all inherited from the general theory of regular protomodular categories. For example: every simplicial object of topological Boolean algebras is a Kan complex, the homological 3×3 lemma for Hausdorff loops, every reflexive relation of locally compact Hausdorff Heyting algebras is an equivalence relation, every internal category of totally disconnected rings is a groupoid, and so on (see [8,6,2]). And of course, all the more specific topological results proved in Sections 2–4 yield corresponding particularizations in all the examples mentioned in Section 1.

References

[1] A. Arhangels'skiĭ, R. Wiegandt, Connectedness and disconnectedness in topology, Gen. Topology Appl. 5 (1975) 9–33.
[2] F. Borceux, D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Kluwer Academic, Dordrecht, 2004.
[3] F. Borceux, M.M. Clementino, Topological semi-Abelian algebras, Adv. Math. 190 (2005) 425–453.
[4] N. Bourbaki, Éléments de Mathématique, Topologie Générale, Hermann, Paris, 1971 (Chapters 1–4).
[5] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, in: Lecture Notes in Math., vol. 1448, Springer, Berlin, 1991, pp. 43–62.
[6] D. Bourn, The denormalized 3×3 lemma, J. Pure Appl. Algebra 177 (2003) 113–129.
[7] D. Bourn, G. Janelidze, Characterization of protomodular varieties of universal algebra, Theory Appl. Categ. 11 (2002) 143–147.
[8] A. Carboni, G.M. Kelly, M.C. Pedicchio, Some remarks on Maltsev and Goursat categories, Appl. Categ. Structures 1 (1993) 385–421.
[9] J.P. Coleman, Separation in topological algebras, Algebra Universalis 35 (1996) 72–84.
[10] H.P. Gumm, Topological implications in n-permutable varieties, Algebra Universalis 19 (1984) 319–321.
[11] K. Kearnes, L. Sequeira, Hausdorff properties of topological algebras, Algebra Universalis 47 (2002) 343–366.
[12] G. Janelidze, L. Márki, W. Tholen, Semi-Abelian categories, J. Pure Appl. Algebra 168 (2002) 367–386.
[13] M. Hušek, Productivity of properties of topological spaces, Topology Appl. 44 (1992) 189–196.
[14] P.T. Johnstone, A note on the semiabelian variety of Heyting semilattices, in: Galois Theory Hopf Algebras and Semiabelian Categories, Fields Inst. Commun. 43 (2004) 317–318.
[15] I. Kaplansky, Topological rings, Amer. J. Math. 69 (1947) 153–183.
[16] A.I. Mal'cev, On the general theory of algebraic systems, Mat. Sb. (N.S.) 35 (1954) 3–20.
[17] E. Manes, A triple theoretic construction of compact algebras, in: Lecture Notes in Math., vol. 80, Springer, Berlin, 1969, pp. 91–118.
[18] R.S. Pierce, Topological Boolean algebras, in: Proc. Conf. on Universal Algebra, Queen's Univ., Kingston, Ont., 1969, 1970, pp. 107–130.
[19] W. Taylor, Varieties of topological algebras, J. Austral. Math. Soc. Ser. A 23 (1977) 207–241.

Further reading

[20] M. Barr, Exact categories, in: Lecture Notes in Math., vol. 236, Springer, Berlin, 1971, pp. 1–120.
[21] F. Borceux, Handbook of Categorical Algebra, vols. 1–3, Cambridge University Press, Cambridge, 1994.
[22] D. Bourn, Mal'cev categories and fibrations of pointed objects, Appl. Categ. Structures 4 (1996) 302–327.
[23] P.J. Higgins, An Introduction to Topological Groups, London Math. Soc. Lecture Notes Ser., vol. 15, Cambridge University Press, Cambridge, 1974.
[24] J. Kelley, General Topology, Springer, Berlin, 1971 (from Van Nostrand, 1955).
[25] S.A. Morris, Pontryagin Duality and the Structure of Locally Compact Abelian Groups, London Math. Soc. Lecture Note Ser., vol. 29, Cambridge University Press, Cambridge, 1977.