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Journal of Algebra 297 (2006) 361–371
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Morita equivalence of many-sorted algebraic theories

Jiří Adámek [a,*,1], Manuela Sobral [b,2], Lurdes Sousa [c,3]

[a] *Department of Theoretical Computer Science, Technical University of Braunschweig, Postfach 3329, 38023 Braunschweig, Germany*
[b] *Departamento de Matematica da Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal*
[c] *Departamento de Matematica da Escola Superior de Tecnologia de Viseu, Campus Politecnico, 3504-510 Viseu, Portugal*

Received 8 December 2004

Available online 17 February 2006

Communicated by Kent R. Fuller

Abstract

Algebraic theories are called Morita equivalent provided that the corresponding varieties of algebras are equivalent. Generalizing Dukarm's result from one-sorted theories to many-sorted ones, we prove that all theories Morita equivalent to an S-sorted theory $\mathcal{T}$ are obtained as idempotent modifications of $\mathcal{T}$. This is analogous to the classical result of Morita that all rings Morita equivalent to a ring R are obtained as idempotent modifications of matrix rings of R.

* Corresponding author.
E-mail address: adamek@iti.cs.tu-bs.de (J. Adámek).
[1] Research supported by the Czech Grant Agency, Project 201/02/0148.
[2] Partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT.
[3] Partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT and Escola Superior de Tecnologia de Viseu.

0021-8693/$ – see front matter © 2006 Published by Elsevier Inc.
doi:10.1016/j.jalgebra.2006.01.014

1. Introduction

The classical results of Kiiti Morita characterizing equivalence of categories of modules, see [12], have been generalized to one-sorted algebraic theories in several articles. The aim of the present paper is to generalize one of the basic characterizations to many-sorted theories, and to illustrate the result on concrete examples.

Let us first recall the classical results concerning

$$R\text{-}\mathbf{Mod}$$

the category of left R-modules for a given ring R. Two rings R and Q are called *Morita equivalent* if the corresponding categories R-**Mod** and Q-**Mod** are equivalent. (For distinction we speak about *categorical equivalence* whenever the equivalences of categories in the usual sense is discussed.) For Lawvere's algebraic theories $\mathcal{T}$ [9], i.e., small categories having finite products, we have the analogous concept w.r.t. the categories $Alg\,\mathcal{T}$ of $\mathcal{T}$-algebras (i.e., set functors preserving finite products): we call two theories *Morita equivalent* if their categories of algebras are categorically equivalent. For categories of modules K. Morita provided two types of characterizations:

Type 1: Rings R and Q are Morita equivalent iff there exist an R-Q-bimodule M and an Q-R-bimodule M' such that

$$M \otimes M' \cong Q \quad \text{and} \quad M' \otimes M \cong R.$$

This result was fully generalized by F. Borceux and E. Vitale [4] to Lawvere's algebraic theories as follows: given algebraic theories $\mathcal{T}$ and $\mathcal{T}'$, by a $\mathcal{T}$-$\mathcal{T}'$-bimodel M is meant a model of $\mathcal{T}$ in the category of $\mathcal{T}'$-algebras. Two algebraic theories $\mathcal{T}$ and $\mathcal{T}'$ are Morita equivalent iff there exist a $\mathcal{T}$-$\mathcal{T}'$-bimodel M and a $\mathcal{T}'$-$\mathcal{T}$-bimodel M' such that

$$M \otimes M' \cong \mathcal{T}' \quad \text{and} \quad M' \otimes M \cong \mathcal{T},$$

where $\cong$ means natural isomorphism and $\otimes$ is the tensor product corresponding to $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(M', -)$, respectively.

Type 2: Two constructions on a ring R are specified yielding a Morita equivalent ring. Then it is proved that every Morita equivalent ring can be obtained from R by applying successively the two constructions.

(a) *Matrix ring* $R^{[n]}$. This is the ring of all $n \times n$ matrices over R with the usual addition, multiplication, and unit matrix. This ring $R^{[n]}$ is always Morita equivalent to R.
(b) *Idempotent modification* uRu. Let u be an idempotent element of R, $uu = u$, and let uRu be the ring of all elements of the form uxu (i.e., all elements $x \in R$ with $x = uxu$). The addition and multiplication of uRu is that of R, and u is the multiplicative unit. This ring uRu is Morita equivalent to R whenever u is pseudoinvertible, i.e., $eum = 1$ for some elements e and m of R.

K. Morita proved that two rings R and Q are Morita equivalent iff Q is isomorphic to the ring $uR^{[n]}u$ for some pseudoinvertible $n \times n$ matrix u over R.

This result was generalized to one-sorted algebraic theories $\mathcal{T}$ (i.e., categories having as objects natural numbers, and such that every object n is a product of n copies of 1) by J.J. Dukarm [6] who again introduced two constructions yielding from a given one-sorted theory a Morita equivalent theory:

(a) *Matrix theory* $\mathcal{T}^{[n]}$. This is the full subcategory of $\mathcal{T}$ on all objects kn ($k \in \mathbb{N}$).
(b) *Idempotent modification* $u\mathcal{T}u$. Given an idempotent morphism $u : 1 \to 1$, i.e., $u \cdot u = u$, we denote by

$$u^k = u \times u \times \cdots \times u : k \to k$$

the corresponding idempotents of $\mathcal{T}$, and we call u *pseudoinvertible* if there is $k \geqslant 1$ such that

$$eu^k m = \mathrm{id}$$

for some morphisms $1 \xrightarrow{m} k \xrightarrow{e} 1$ of $\mathcal{T}$.

We denote, for every pseudoinvertible idempotent u, by $u\mathcal{T}u$ the theory of all those morphisms $f : p \to q$ of $\mathcal{T}$ which fulfill $f = u^q f u^p$. The composition is as in $\mathcal{T}$, and the identity morphisms are u^n.

J.J. Dukarm proved, again, that whenever $\mathcal{T}$ and $\mathcal{T}'$ are one-sorted algebraic theories, then they are Morita equivalent iff $\mathcal{T}'$ is categorically equivalent to the theory $u\mathcal{T}^{[n]}u$ for some n and some pseudoinvertible idempotent u of $\mathcal{T}^{[n]}$.

Before turning to many-sorted algebraic theories, let us recall a classical result concerning small categories $\mathcal{T}$ and $\mathcal{T}'$ in general, first formulated by M. Bunge [5]: the functor categories $\mathbf{Set}^{\mathcal{T}}$ and $\mathbf{Set}^{\mathcal{T}'}$ are categorically equivalent iff the two categories $\mathcal{T}$ and $\mathcal{T}'$ have the same idempotent completion (see Remark 2.2 below). Consequently, algebraic theories are Morita equivalent iff they have the same idempotent completion. However for one-sorted algebraic theories Dukarm's result is much "sharper" than this general observation. This was nicely demonstrated by R. McKenzie [11] and H.-E. Porst [13] who provided a concrete description of algebras both of matrix theories and idempotent modifications of theories.

The aim of the present paper is to generalize Dukarm's characterization of Morita equivalence to many-sorted theories. By an S*-sorted algebraic theory* we mean one of the following equivalent concepts:

(a) a category with finite products and chosen objects A_s, $s \in S$, such that every object is a finite product $A_{s_1} \times \cdots \times A_{s_n}$ ($s_i \in S$),

or

(b) a category whose objects form the set S^* of all finite words on S, and such that every object $s_1 \cdots s_n$ is a product of $s_1, \ldots, s_n$.

We introduce a concept of idempotent modification of a many-sorted algebraic theory which generalizes the above matrix theory and idempotent modifications (in one step). And we prove that for every S-sorted theory $\mathcal{T}$ all Morita equivalent theories are precisely the idempotent modifications of $\mathcal{T}$.

The result is illustrated by examples of algebraic theories of sets, M-sets for monoids M, and R-modules. For example, **Set** has the obvious list of all one-sorted algebraic theories: just the matrix theories $\mathcal{T}^{[n]}$ of the category $\mathcal{T}$ dual to the one of finite sets. The list of all many-sorted theories (i.e., all S-sorted idempotent modifications of $\mathcal{T}$) is more colorful. We present it at the end of the paper.

2. Morita equivalence of algebraic theories

Notation 2.1. For an *algebraic theory* $\mathcal{T}$, i.e., a small category with finite products, we denote by

$$Alg\,\mathcal{T}$$

the category of algebras, i.e., the full subcategory of $\mathbf{Set}^{\mathcal{T}}$ formed by all functors preserving finite products. For S-sorted algebraic theories $\mathcal{T}$ these categories are (up to categorical equivalence) precisely the S-sorted varieties of algebras, see, e.g., [10].

Two algebraic theories $\mathcal{T}$ and $\mathcal{T}'$ are called *Morita equivalent* provided that the categories $Alg\,\mathcal{T}$ and $Alg\,\mathcal{T}'$ are categorically equivalent.

Remark 2.2. (a) We call a category *idempotent-complete* provided that every idempotent morphism in it splits (i.e., has the form $u = i \cdot e$ where $e \cdot i = \mathrm{id}$). Recall that every category $\mathcal{K}$ has an *idempotent completion* $\mathcal{L}$ (called Cauchy completion in [3]), i.e., $\mathcal{L}$ is an idempotent-complete category containing $\mathcal{K}$ as a full subcategory such that every object of $\mathcal{L}$ is obtained as a splitting of an idempotent of $\mathcal{K}$.

(b) Recall from [1] the concept of a *sifted colimit*. For the proof below all the reader has to know about sifted colimits is the following:

(i) If a category $\mathcal{D}$ has finite coproducts then every diagram with domain $\mathcal{D}$ is sifted.
(ii) A *strongly finitely presentable* object is an object whose hom-functor preserves sifted colimits. In categories $Alg\,\mathcal{T}$ of algebras, strongly finitely presentable objects are precisely the retracts of the "free algebras"

$$YB : \mathcal{T} \to \mathbf{Set} \quad \text{for } B \in \mathcal{T},$$

where $Y : \mathcal{T}^{\mathrm{op}} \to Alg\,\mathcal{T}$ is the Yoneda embedding and B an arbitrary object of $\mathcal{T}$.

Definition 2.3. A collection of idempotent morphisms

$$u_s : B_s \to B_s \quad (s \in S)$$

of an algebraic theory $\mathcal{T}$ is called *pseudoinvertible* provided that for every object $T \in \mathcal{T}$ there exist morphisms

$$T \xrightarrow{m} B_{s_1} \times \cdots \times B_{s_n} \xrightarrow{e} T \quad (s_1 \cdots s_n \in S^*)$$

such that the square

$$\begin{array}{ccc} B_{s_1} \times \cdots \times B_{s_n} & \xrightarrow{u_{s_1} \times \cdots \times u_{s_n}} & B_{s_1} \times \cdots \times B_{s_n} \\ {\scriptstyle m}\uparrow & & \downarrow{\scriptstyle e} \\ T & = & T \end{array}$$

commutes.

Remark 2.4. Given an R-sorted theory $\mathcal{T}$, with chosen objects T_r, $r \in R$, for a verification of the pseudoinvertibility of a collection $u = (u_s)_{s \in S}$ of idempotents it is sufficient to find m and e above for all the chosen objects $T = T_r$, $r \in R$. In particular, in case of one-sorted theories Definition 2.3 coincides with the pseudoinvertibility in the introduction.

Definition 2.5. By an *S-sorted* idempotent modification of an algebraic theory $\mathcal{T}$ is meant the following S-sorted theory $u\mathcal{T}u$, where $u = (u_s)_{s \in S}$ is a pseudoinvertible collection of idempotents $u_s : B_s \to B_s$ of $\mathcal{T}$. Objects of $u\mathcal{T}u$ form the set S^* (see the introduction). The morphisms from $s_1 \cdots s_n$ to $t_1 \cdots t_k$ are precisely those morphisms $f : B_{s_1} \times \cdots \times B_{s_n} \to B_{t_1} \times \cdots \times B_{t_k}$ of $\mathcal{T}$ for which the following square

$$\begin{array}{ccc} B_{s_1} \times \cdots \times B_{s_n} & \xrightarrow{f} & B_{t_1} \times \cdots \times B_{t_k} \\ {\scriptstyle u_{s_1} \times \cdots \times u_{s_n}}\downarrow & & \uparrow{\scriptstyle u_{t_1} \times \cdots \times u_{t_k}} \\ B_{s_1} \times \cdots \times B_{s_n} & \xrightarrow[f]{} & B_{t_1} \times \cdots \times B_{t_k} \end{array} \tag{2.1}$$

commutes. The composition in $u\mathcal{T}u$ is that of $\mathcal{T}$, and the identity morphism of $s_1 \cdots s_n$ is $u_{s_1} \times \cdots \times u_{s_n}$.

Remark 2.6. (1) If $\mathcal{T}$ is a one-sorted theory, and $S = \{s\}$ has just one element, i.e., a single idempotent $u : n \to n$ is given, then $u\mathcal{T}u$ of Definition 2.5 is the category $u\mathcal{T}^{[n]}u$ of the introduction, with the difference that in Definition 2.5 we call the objects words $s \cdots s$ (of length k) rather than the corresponding natural numbers kn.

(2) The matrix theory $\mathcal{T}^{[n]}$ of the introduction has the obvious S-sorted generalization: given a collection $D = \{B_s;\ s \in S\}$ of objects of $\mathcal{T}$, we consider the full subcategory $\mathcal{T}^{[D]}$ of $\mathcal{T}$ on all finite products of these objects. This is a special case of $u\mathcal{T}u$: choose $u_s = \mathrm{id}_{B_s}$, for $s \in S$. Pseudoinvertibility means here that all objects are retracts of products $B_{s_1} \times \cdots \times B_{s_n}$.

Theorem 2.7. *Let $\mathcal{T}$ be an algebraic theory. Then an S-sorted algebraic theory is Morita equivalent to $\mathcal{T}$ iff it is categorically equivalent to an S-sorted idempotent modification of $\mathcal{T}$.*

Proof. (1) Sufficiency: let

$$u_s : B_s \to B_s \quad (s \in S)$$

be a pseudoinvertible collection of idempotents. We will find a category $\mathcal{T}^{\langle u \rangle}$ Morita equivalent to $\mathcal{T}$ which is categorically equivalent to $u\mathcal{T}u$—then $u\mathcal{T}u$ is, obviously, also Morita equivalent to $\mathcal{T}$. Denote by

$$Y : \mathcal{T}^{\mathrm{op}} \to Alg\,\mathcal{T}$$

the Yoneda embedding. Since $Alg\,\mathcal{T}$ is complete, the idempotent Yu_s has a splitting

$$Yu_s \circlearrowright YB_s \underset{\mu_s}{\overset{\varepsilon_s}{\rightleftarrows}} A_s$$

in $Alg\,\mathcal{T}$: let μ_s be an equalizer of Yu_s and id, and $\varepsilon_s : YB_s \to A_s$ be the unique morphism with

$$\mu_s \varepsilon_s = Yu_s \quad \text{and} \quad \varepsilon_s \mu_s = \mathrm{id} \quad \text{in } Alg\,\mathcal{T}. \tag{2.2}$$

Denote by

$$\mathcal{T}^{\langle u \rangle} \subseteq (Alg\,\mathcal{T})^{\mathrm{op}} \tag{2.3}$$

the S-sorted algebraic theory which is the full subcategory of $(Alg\,\mathcal{T})^{\mathrm{op}}$ on all objects which are, in $(Alg\,\mathcal{T})^{\mathrm{op}}$, finite products of the algebras A_s $(s \in S)$.

(1a) We prove that $\mathcal{T}$ and $\mathcal{T}^{\langle u \rangle}$ are Morita equivalent. The closure $\mathcal{C}$ of $\mathcal{T}^{\langle u \rangle}$ under retracts in the (idempotent-complete) category $(Alg\,\mathcal{T})^{\mathrm{op}}$ is an idempotent completion of $\mathcal{T}^{\langle u \rangle}$. It is sufficient to prove that

$$YB_s \in \mathcal{C}$$

for every $s \in S$: in fact, we then have $YT \in \mathcal{C}$ for every $T \in \mathcal{T}$ because T is a retract of a finite product $B_{s_1} \times \cdots \times B_{s_n}$ (use m and $\bar{e} = e \cdot (u_{s_1} \times \cdots \times u_{s_n})$ in Definition 2.3). Therefore, $Y^{\mathrm{op}}[\mathcal{T}]$ is contained in $\mathcal{C}$. Moreover, since A_s is a retract of YB_s (use (2.2) above), we conclude that $\mathcal{C}$ is an idempotent completion of $Y^{\mathrm{op}}[\mathcal{T}] \cong \mathcal{T}$, thus, $\mathcal{T}$ and $\mathcal{T}^{\langle u \rangle}$ are Morita equivalent.

For the proof of $YB_s \in \mathcal{C}$ apply Definition 2.3 to $T = B_s$ and consider the following morphisms of $Alg\,\mathcal{T}$:

$$\tilde{e} \equiv YB_s \xrightarrow{Ye} YB_{s_1} + \cdots + YB_{s_n} \xrightarrow{\varepsilon_{s_1} + \cdots + \varepsilon_{s_n}} A_{s_1} + \cdots + A_{s_n}$$

and

$$\tilde{m} \equiv A_{s_1} + \cdots + A_{s_n} \xrightarrow{\mu_{s_1} + \cdots + \mu_{s_n}} YB_{s_1} + \cdots + YB_{s_n} \xrightarrow{Ym} YB_s.$$

Since (2.2) implies $\tilde{m} \cdot \tilde{e} = Ym \cdot Y(u_{s_1} \times \cdots \times u_{s_n}) \cdot Ye = Y[e \cdot (u_{s_1} \times \cdots \times u_{s_n}) \cdot m] = \mathrm{id}$, we see that YB_s is a retract of $A_{s_1} \times \cdots \times A_{s_n}$ in $(Alg\,\mathcal{T})^{\mathrm{op}}$, thus, it lies in $\mathcal{C}$.

(1b) We prove next that $\mathcal{T}^{\langle u \rangle}$ is categorically equivalent to $u\mathcal{T}u$—thus, by (1a), $u\mathcal{T}u$ is Morita equivalent to $\mathcal{T}$.

Define a functor

$$E : u\mathcal{T}u \to \mathcal{T}^{\langle u \rangle}$$

on objects by

$$E(s_1 \cdots s_n) = A_{s_1} \times \cdots \times A_{s_n}$$

and on morphisms $f : s_1 \cdots s_n \to t_1 \cdots t_k$ (which, recall, are special morphisms $f : B_{s_1} \times \cdots \times B_{s_n} \to B_{t_1} \times \cdots \times B_{t_k}$ of $\mathcal{T}$) by the commutativity of the following square in $Alg\,\mathcal{T}$:

$$\begin{array}{ccc} A_{s_1} + \cdots + A_{s_n} & \xleftarrow{\quad Ef \quad} & A_{t_1} + \cdots + A_{t_k} \\ \uparrow{\scriptstyle \varepsilon_{s_1} + \cdots + \varepsilon_{s_n}} & & \downarrow{\scriptstyle \mu_{t_1} + \cdots + \mu_{t_k}} \\ YB_{s_1} + \cdots + YB_{s_n} = Y(B_{s_1} \times \cdots \times B_{s_n}) & \xleftarrow[Yf]{} & Y(B_{t_1} \times \cdots \times B_{t_k}) = YB_{t_1} + \cdots + YB_{t_k} \end{array} \tag{2.4}$$

It is easy to verify that E is well defined, let us prove that it is an equivalence functor.

E is faithful because Y is faithful, and we have

$$\begin{aligned} Yf &= Y(u_{s_1} \times \cdots \times u_{s_n}) \cdot Yf \cdot Y(u_{t_1} \times \cdots \times u_{t_k}) \quad \text{see (2.1)} \\ &= (\mu_{s_1} + \cdots + \mu_{s_n}) \cdot (\varepsilon_{s_1} + \cdots + \varepsilon_{s_n}) \\ &\quad \cdot Yf \cdot (\mu_{t_1} + \cdots + \mu_{t_k}) \cdot (\varepsilon_{t_1} + \cdots + \varepsilon_{t_k}) \quad \text{see (2.2)} \\ &= (\mu_{s_1} + \cdots + \mu_{s_n}) \cdot Ef \cdot (\varepsilon_{t_1} + \cdots + \varepsilon_{t_k}) \quad \text{see (2.4).} \end{aligned}$$

E is full because Y is full: given $h : A_{t_1} + \cdots + A_{t_k} \to A_{s_1} + \cdots + A_{s_n}$ in $Alg\,\mathcal{T}$, we have $f : B_{s_1} \times \cdots \times B_{s_n} \to B_{t_1} \times \cdots \times B_{t_k}$ in $\mathcal{T}$ with

$$Yf = (\mu_{s_1} + \cdots + \mu_{s_n}) \cdot h \cdot (\varepsilon_{t_1} + \cdots + \varepsilon_{t_k}). \tag{2.5}$$

From (2.2) we conclude that

$$Yf = Y\big[(u_{t_1} \times \cdots \times u_{t_k}) \cdot f \cdot (u_{s_1} \times \cdots \times u_{s_n})\big],$$

hence f is a morphism of $u\mathcal{T}u$ (recall that Y is faithful). From (2.2), (2.4) and (2.5) we conclude $Ef = h$.

Since E is surjective on objects, it is an equivalence functor.

(2) Necessity: given an S-sorted algebraic theory $\mathcal{T}'$ with chosen objects C_s ($s \in S$), and given an equivalence functor

$$F : Alg\,\mathcal{T}' \to Alg\,\mathcal{T},$$

we find a pseudoinvertible collection $u = (u_s)_{s\in S}$ of idempotents with $\mathcal{T}'$ categorically equivalent to $u\mathcal{T}u$. Denote the corresponding Yoneda embeddings by $Y_{\mathcal{T}} : \mathcal{T}^{\mathrm{op}} \to Alg\,\mathcal{T}$ and $Y_{\mathcal{T}'} : \mathcal{T}'^{\,\mathrm{op}} \to Alg\,\mathcal{T}'$. The $\mathcal{T}$-algebras

$$A_s = F(Y_{\mathcal{T}'}C_s) \quad (s \in S)$$

are strongly finitely presentable (since $Y_{\mathcal{T}'}C_s$ are, see Remark 2.2(b)). Thus, each A_s is a retract of some $Y_{\mathcal{T}}B_s$ for $B_s \in \mathcal{T}$. Choose homomorphisms

$$Y_{\mathcal{T}}B_s \underset{\mu_s}{\overset{\varepsilon_s}{\rightleftarrows}} A_s \quad \text{with } \varepsilon_s\mu_s = \mathrm{id} \text{ (in } Alg\,\mathcal{T}).$$

Then the idempotent $\mu_s\varepsilon_s$ has the form $Y_{\mathcal{T}}u_s$ for a unique idempotent $u_s : B_s \to B_s$ of $\mathcal{T}^{\mathrm{op}}$. And the codomain restriction of $(F \cdot Y_{\mathcal{T}'})^{\mathrm{op}} : \mathcal{T}' \to (Alg\,\mathcal{T})^{\mathrm{op}}$ yields an equivalence functor between $\mathcal{T}'$ and $\mathcal{T}^{\langle u\rangle}$, see (2.3) above. As in (1b), we deduce that $u\mathcal{T}u$ is categorically equivalent to $\mathcal{T}^{\langle u\rangle}$. It remains to show that u is pseudoinvertible.

For every object $T \in \mathcal{T}$ we will prove that $Y_{\mathcal{T}}T$ is a retract of an object of $(\mathcal{T}^{\langle u\rangle})^{\mathrm{op}}$ in $Alg\,\mathcal{T}$, i.e., that there exist homomorphisms $\bar{e} : A_{s_1} + \cdots + A_{s_n} \to Y_{\mathcal{T}}T$ and $\bar{m} : Y_{\mathcal{T}}T \to A_{s_1} + \cdots + A_{s_n}$ with $\bar{e} \cdot \bar{m} = \mathrm{id}$ in $Alg\,\mathcal{T}$. This will prove the pseudoinvertibility: we have unique morphisms m and e in $\mathcal{T}$ with

$$Y_{\mathcal{T}}e = Y_{\mathcal{T}}T \xrightarrow{\bar{m}} A_{s_1} + \cdots + A_{s_n} \xrightarrow{\mu_{s_1}+\cdots+\mu_{s_n}} Y_{\mathcal{T}}(B_{s_1} \times \cdots \times B_{s_n})$$

and

$$Y_{\mathcal{T}}m = Y_{\mathcal{T}}(B_{s_1} \times \cdots \times B_{s_n}) \xrightarrow{\varepsilon_{s_1}+\cdots+\varepsilon_{s_n}} A_{s_1} + \cdots + A_{s_n} \xrightarrow{\bar{e}} Y_{\mathcal{T}}T.$$

The desired square in Definition 2.3 follows from the fact that $Y_{\mathcal{T}}$ is faithful:

$$\begin{array}{ccccc}
Y_{\mathcal{T}}T & & \xrightarrow{Y_{\mathcal{T}}[e(u_{s_1}\times\cdots\times u_{s_n})m]=\mathrm{id}} & & Y_{\mathcal{T}}T \\
\downarrow{\scriptstyle \bar{m}} & & & & \uparrow{\scriptstyle \bar{e}} \\
A_{s_1}+\cdots+A_{s_n} & & & & A_{s_1}+\cdots+A_{s_n} \\
\downarrow{\scriptstyle \mu_{s_1}+\cdots+\mu_{s_n}} & \searrow^{\mathrm{id}} & & \nearrow^{\mathrm{id}} & \uparrow{\scriptstyle \varepsilon_{s_1}+\cdots+\varepsilon_{s_n}} \\
Y_{\mathcal{T}}B_{s_1}+\cdots+Y_{\mathcal{T}}B_{s_n} & \xrightarrow[\varepsilon_{s_1}+\cdots+\varepsilon_{s_n}]{} & A_{s_1}+\cdots+A_{s_n} & \xrightarrow[\mu_{s_1}+\cdots+\mu_{s_n}]{} & Y_{\mathcal{T}}B_{s_1}+\cdots+Y_{\mathcal{T}}B_{s_n}
\end{array}$$

To prove that $Y_{\mathcal{T}}T$ is a retract of an object of $(\mathcal{T}^{\langle u\rangle})^{\mathrm{op}}$, observe that since the algebras $Y_{\mathcal{T}'}C_s$ $(s \in S)$ are dense in $Alg\,\mathcal{T}'$, it follows that A_s $(s \in S)$ are dense in $Alg\,\mathcal{T}$. And so is their closure $(\mathcal{T}^{\langle u\rangle})^{\mathrm{op}}$ under finite coproducts. Therefore, $Y_{\mathcal{T}}T$ is a canonical colimit of the diagram D of all homomorphisms $A \to Y_{\mathcal{T}}T$ with $A \in (\mathcal{T}^{\langle u\rangle})^{\mathrm{op}}$. The domain of this diagram, i.e., the comma-category $(\mathcal{T}^{\langle u\rangle})^{\mathrm{op}}/Y_{\mathcal{T}}T$, has finite coproducts (being closed under them in $Alg\,\mathcal{T}/Y_{\mathcal{T}}T$), thus, the diagram is sifted, see Remark 2.2(b). Since $Y_{\mathcal{T}}T$ is strongly finitely presentable, it follows that one of the colimit morphisms of D is a split epimorphism. □

3. Examples

Example 3.1. Modules. For one-sorted theories K. Morita covered the whole spectrum: there exist no other one-sorted theories of R-**Mod** than those canonically derived from Morita equivalent rings.

More detailed:

(i) Each R^n $(n \in \mathbb{N})$ has a natural structure of a left R-module. The full subcategory

$$\mathcal{T}_R = \{R^n;\ n \in \mathbb{N}\}$$

of $(R\text{-}\mathbf{Mod})^{\mathrm{op}}$ is a one-sorted algebraic theory of R-**Mod**.

(ii) Consequently, for every ring Q Morita equivalent to R, we have an algebraic theory $\mathcal{T}_S$ of R-**Mod**.

(iii) The above are, up to categorical equivalence, all one-sorted algebraic theories of R-**Mod**. In fact, let $\mathcal{T}$ be a one-sorted algebraic theory with an equivalence functor

$$E : Alg\,\mathcal{T} \to R\text{-}\mathbf{Mod}.$$

Then $\mathcal{T}$ is categorically equivalent to $\mathcal{T}_Q$ for a ring Q Morita equivalent to R: indeed, following [7], $Alg\,\mathcal{T}$ is equivalent to Q-**Mod**, with $Q = \mathcal{T}(1,1)$. Moreover, the composition of the Yoneda embedding $Y : \mathcal{T}^{\mathrm{op}} \to Alg\,\mathcal{T}$ with the equivalence $Alg\,\mathcal{T} \to Q$-**Mod** sends an object n to $\mathcal{T}(n,1)$ which, by additivity, is isomorphic to $\mathcal{T}(1,1)^n = Q^n$. This shows that $\mathcal{T}$ is equivalent to $\mathcal{T}_Q$, with Q Morita equivalent to R.

Remark 3.2. There are, of course, additional algebraic theories of R-**Mod** which are not one-sorted. For example, in $\mathbf{Ab} = \mathbb{Z}$-**Mod** the theory $\mathcal{T}'$ generated by $\mathbb{Z}$ and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is certainly Morita equivalent to $\mathcal{T}_{\mathbb{Z}}$, but it is not categorically equivalent to $\mathcal{T}_Q$ for any Morita equivalent ring Q (e.g., $\mathcal{T}'$ contains an object with a finite hom).

Example 3.3. All algebraic theories of **Set**. The one-sorted theories are well known to be just the theories

$$\mathcal{T}^{[n]} \quad (n = 1, 2, 3, \ldots),$$

where $\mathcal{T} \subseteq \mathbf{Set}^{\mathrm{op}}$ is the full subcategory on all natural numbers, and $\mathcal{T}^{[n]}$ is the matrix theory, i.e., the full subcategory of $\mathcal{T}$ on $0, n, 2n, \ldots$. And these theories are, obviously, pairwise categorically nonequivalent.

We now describe all many-sorted theories: they are precisely the matrix theories $\mathcal{T}^{[D]}$, see 2.6(2), for finite sets

$$D \subseteq \mathbb{N}$$

which are *sum-irreducible*, i.e., no number of D is a sum of more than one member of D. Recall that

$$\mathcal{T}^{[D]}$$

is the dual of the full subcategory of **Set** on all finite sums of members of D. Then we know that $\mathcal{T}^{[D]}$ is an algebraic theory of **Set**. We are going to prove that these are precisely all of them:

(a) Every algebraic theory $\mathcal{T}'$ is categorically equivalent to $\mathcal{T}^{[D]}$ for some finite sum-irreducible $D \subseteq \mathbb{N}$. In fact, consider a pseudoinvertible collection $u_s : B_s \to B_s$ $(s \in S)$ of idempotents in $\mathcal{T}$ with $\mathcal{T}'$ categorically equivalent to $u\mathcal{T}u$, where u_s has precisely r_s fixed points. Without loss of generality we can assume $u_s \neq \mathrm{id}_\emptyset$ for every s, i.e., $r_s \geqslant 1$. Let K be the subsemigroup of the additive semigroup $\mathbb{N}$ generated by $\{r_s\}_{s\in S}$. (That is, K is the set of all numbers of fixed points of the morphisms $u_{s_1} \times \cdots \times u_{s_n}$ in $\mathbf{Set}^{\mathrm{op}}$.) Then $u\mathcal{T}u$ is categorically equivalent to K as a full subcategory of $\mathbf{Set}^{\mathrm{op}}$. Recall that every subsemigroup K of the additive semigroup of natural numbers is finitely generated (see [14]). Therefore, if D is a minimum set of generators of K, then D is finite, sum-irreducible and K is categorically equivalent to $\mathcal{T}^{[D]}$.

(b) The theories $\mathcal{T}^{[D]}$ are pairwise nonequivalent categories. In fact, every element $n \in D$ defines an object of $\mathcal{T}^{[D]}$ which is product-indecomposable and has n^n endomorphisms—this determines D categorically.

Example 3.4. M-sets. For monoids M the question of Morita equivalence (that is, given a monoid M' when are M-**Set** and M'-**Set** equivalent categories) was studied by B. Banaschewski [2] and U. Knauer [8]. The main result is formally very similar to that of K. Morita: let us say that an idempotent $u \in M$ is *pseudoinvertible* if there exist $e, m \in M$ with $eum = 1$. It follows that the monoid

$$uMu = \{uxu:\ x \in M\}$$

whose unit is u and multiplication is as in M is Morita equivalent to M. And these are all monoids Morita equivalent to M, up to isomorphism.

Unlike Example 3.1, this does *not* describe all one-sorted theories of M-**Set**. In fact, if $M = \{1\}$ is the trivial one-element monoid, then M-**Set** = **Set** has infinitely many pairwise nonequivalent one-sorted theories, as we saw in Example 3.3, although there are no nontrivial monoids Morita equivalent to $\{1\}$.

Remark 3.5. We saw above that all algebraic theories of **Set** are finitely-sorted (i.e., have finitely many objects whose finite products form all objects). This is not true for M-sets, in general. In fact, whenever M is a commutative monoid with uncountably many idempotents, then the "standard" algebraic theory $\mathcal{T}$ (dual to the category of all free M-sets on finitely many generators) has an idempotent completion $\mathcal{T}'$ which has uncountably many pairwise nonisomorphic objects. In fact, every idempotent m of M yields an idempotent endomorphism $m \cdot - : M \to M$ in $\mathcal{T}$, and the splittings of these endomorphisms produce pairwise nonisomorphic objects A_m of $\mathcal{T}'$: indeed, whenever A_m is isomorphic to A_n, then for every element x of M we see that $m \cdot x = x$ iff $n \cdot x = x$. By choosing $x = n$ and $x = m$ we conclude $m = n$. Consequently, $\mathcal{T}'$ is an algebraic theory of M-sets which is not finitely-sorted.

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